B.A.J. **8**, III, 545-591 (2002)

STOCHASTIC INVESTMENT MODELLING: A MULTIPLE TIME-SERIES APPROACH

By W. S. Chan

ABSTRACT

In this paper we adopt the multiple time-series modelling approach suggested by Tiao & Box (1981) to construct a stochastic investment model for price inflation, share dividends, share dividend yields and long-term interest rates in the United Kingdom. This method has the advantage of being direct and transparent. The sequential and iterative steps of tentative specification, estimation and diagnostic checking parallel those of the well-known Box-Jenkins method in the univariate time-series analysis. It is not required to specify any *a priori* causality as compared to some other stochastic asset models in the literature.

KEYWORDS

ARCH Models; Non-Linear Threshold Models; Wilkie Model; Vector Autoregressive-Moving Average Models; Time-Series Outliers; Price Inflation; Long-Term Interest Rates; Share Dividend Yields; Share Dividends

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1. Introduction

1.1 Stochastic investment modelling has attracted considerable interest from actuaries around the world in recent years.

1.2 Wilkie (1986, 1995) developed linear stochastic asset models for United Kingdom data. Wright (1998) proposed an alternative model based on vector autoregression. Chan & Wang (1998) refined the price inflation component of the Wilkie model by performing a time-series outlier analysis. Whitten & Thomas (1999) suggested a threshold-type non-linear model for U.K. investment series.

1.3 Following Wilkie's footsteps, stochastic investment models have been developed for other countries. They include: Metz & Ort (1993) for Switzerland; Deaves (1993) for Canada; Daykin *et al.* (1994) for Finland; Thomson (1996) for South Africa; Frees *et al.* (1997) for the United States of America; and Sherris *et al.* (1999) for Australia.

https://doi.org/10.1017/S1357321700003822 Published online by Cambridge University Press

1.4 In addition to Wilkie's method, other non-linear or non-Gaussian approaches for building stochastic asset models have been discussed in the literature. They include, among many others: Praetz (1972); Carter (1991); Clarkson (1991); Klein (1993); Finkelstein (1997) and Wong & Li (2000).

1.5 Many stochastic investment models were developed using Box & Jenkins' (1976, Chapter 11) transfer function techniques. Transfer function models combine information of other related (and possible stochastic) time series and an ARIMA (autoregressive-integrated moving average) model of an underlying disturbance to describe the behaviour of a single series.

1.6 However, in many applications, unidirectional assumptions may not be appropriate. For example, it is often difficult to postulate the one-way dynamic relationships between major economic variables using only economic theory. On the contrary, when studying such variables, a primary objective may be to uncover the interdependence among the variables of the system.

1.7 In this paper we consider a general VARMA (vector autoregressive-moving average) model for U.K. investment data. It should be noted that all stochastic models with transfer function structures could be written as restricted VARMA models where no feedback is allowed.

1.8 We shall adopt the multiple time-series modelling approach suggested by Tiao & Box (1981). This method has the advantage of being direct and transparent. The sequential and iterative steps of tentative specification, estimation and diagnostic checking parallel those of the well-known Box-Jenkins method for univariate time-series analysis. Actuarial applications of this approach can be found in Frees *et al.* (1997) and Chan (1998a).

2. Preliminary Data Analysis

2.1 We consider annual observations of force of price inflation $I(t)$, share dividend yield $Y(t)$, force of share dividend growth $K(t)$, and long-term interest rate $C(t)$, from 1923 to 1996 in the U.K. The data used here are the same as those used by Wilkie (see Acknowledgement).

2.2 Following notations in Wilkie (1995) the force of price inflation $I(t)$ is defined as:

$$I(t) = \ln Q(t) - \ln Q(t-1)$$

where $Q(t)$ is the value of a retail price index at year t. The force of share dividend growth $K(t)$ is given by:

$$K(t) = \ln D(t) - \ln D(t-1)$$

where $D(t)$ is the value of a dividend index on ordinary shares at year t.

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Following Wilkie (1995, p857) the *real* part of the long-term interest rate is computed as:

$$\begin{cases} CR(t) = C(t) - CW \cdot CM(t) \\ CM(t) = CD \cdot I(t) + (1 - CD) \cdot CM(t-1) \end{cases}$$

where $CW = 1.0$ and $CD = 0.045$. The starting value required for the $CM(t)$ series includes a carry forward from past inflation:

$$CM(0) = \left(\frac{CD}{1 - (1 - CD)B} \right) I(0)$$

where the backwards step operate B is defined by:

$$B \cdot X(t) = X(t-1).$$

Following Wilkie (1987, p70), we shall use the inflation mean parameter QMU for the neutral value of $CM(0)$. The definition of QMU can be found in Wilkie (1995, p780) or ¶6.2.1 in this paper.

2.3 It is common for the variance of a time series to change as its level changes. To overcome this non-stationary problem, Box & Cox (1964) proposed a method for searching for a proper variance stabilising transformation. A detailed description of the method is available in Wei (1990). After performing the Box & Cox analysis, we conclude that logarithmic transformation is needed for the variables $Y(t)$ and $CR(t)$. Finally, the variables are arranged into a vector form:

$$\mathbf{Z}(t) = \begin{pmatrix} I(t) \\ \ln Y(t) \\ K(t) \\ \ln CR(t) \end{pmatrix}.$$

2.4 Four commonly used types of outliers are considered in this paper. They are additive outlier (AO), innovational outlier (IO), level shift (LS) and temporary change (TC). An additive outlier (AO) affects only the level of the given observation while an innovational outlier (IO) affects all observations beyond the given time through the memory of the underlying outlier-free process. A level shift (LS) is an event that affects a time series at a particular time point whose effect becomes permanent. A temporary change (TC) is an event having an initial impact and whose effect decreases exponentially according to a fixed dampening parameter. For comprehensive discussions on definitions of time-series outliers and detection algorithm refer to Chen & Liu (1993) and Chan (1998b).

Table 2.1. Outlier detection results for U.K. investment data, 1923-1996

		Outlier			
Variable	Year	Size	t-value	Type	Event
$I(t)$	1940	0.167	5.33	IO	World War II
$I(t)$	1948	0.089	3.40	AO	Post-WWII
$I(t)$	1975	0.121	3.85	TC	First oil shock
$I(t)$	1980	0.108	3.46	IO	Second oil shock
$\ln Y(t)$	1933	−0.443	−3.42	TC	World depression
$\ln Y(t)$	1940	0.437	3.98	AO	World War II
$\ln Y(t)$	1974	0.670	5.18	TC	First oil shock
$K(t)$	1931	−0.331	−5.79	IO	World depression
$K(t)$	1941	−0.172	−6.07	AO	World War II
$K(t)$	1980	0.218	7.69	AO	Second oil shock
$\ln CR(t)$		no outlier found			

2.5 Chan & Wang (1998) performed a time-series outlier analysis on U.K. inflation data. In some circumstances, not adjusting for outliers could lead to model mis-specification (Chan, 1992) and biased parameter estimation (Chang *et al*., 1988). This can happen when the outliers are due to poor data readings or where there were disturbances in expected results. In these circumstances, the outliers should not, or are extremely unlikely to, be repeated in the future, and not adjusting for the outliers when constructing the model may then lead to poor forecasts (Ledolter, 1989).

2.6 We have performed outlier detection procedures proposed by Chen & Liu (1993) on each element of $\mathbf{Z}(t)$ and the results are summarised in Table 2.1.

2.7 The consols yield $C(t)$ is decomposed into two components (see ¶2.2):

$$C(t) = CR(t) + CM(t)$$

where $CM(t)$ is an allowance for expected future inflation and $CR(t)$ is the 'real' yield. There is no outlier detected in the $\ln CR(t)$ series (see Table 2.1). It should be noted that both the $C(t)$ and the $CM(t)$ series could contain outliers, but the difference between them (i.e., the real yield series) is not significantly contaminated. For example, if both $C(t)$ and $CM(t)$ were contaminated by an additive outlier of size ω, then:

$$\begin{aligned} CR(t) &= \left[C(t) + \omega\right] - \left[CM(t) + \omega\right] \\ &= C(t) - CM(t) \end{aligned}$$

will not be contaminated. It is often called the cancellation effect.

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Table 2.2. Summary statistics

	Original data				Outlier-adjusted series			
	$I(t)$	$\ln Y(t)$	$K(t)$	$\ln CR(t)$	$I(t)$	$\ln Y(t)$	$K(t)$	$\ln CR(t)$
n	74	74	74	74	74	74	74	74
Mean	0.043	−3.19	0.057	−3.61	0.025	−3.21	0.064	−3.61
Median	0.034	−3.21	0.063	−3.64	0.024	−3.20	0.062	−3.64
Std. dev.	0.056	0.23	0.095	0.57	0.043	0.16	0.064	0.57
Minimum	−0.063	−3.66	−0.333	−4.79	−0.079	−3.53	−0.105	−4.79
Maximum	0.232	−2.62	0.269	−2.18	0.153	−2.83	0.185	−2.18
Skewness	0.91	0.12	−1.70	0.24	0.15	0.03	−0.53	0.24
Kurtosis	4.24	2.69	7.94	2.65	3.31	2.59	3.05	2.65
	Correlation				Correlation			
$I(t)$	1.00				1.00			
$\ln Y(t)$	0.59	1.00			0.22	1.0		
$K(t)$	0.37	0.06	1.00		0.38	−0.08	1.00	
$\ln CR(t)$	0.51	0.57	0.13	1.0	0.61	0.43	0.11	1.00

2.8 Descriptive statistics for both the original series and the outlier-adjusted series are summarised in Table 2.2. A further display of possible interrelationships of the variables using a scatterplot matrix is given in Figure 2.1. As anticipated, the graph shows a positive relationship between the force of inflation and all other variables.

2.9 Whitten & Thomas (1999) suggested a two-regime non-linear model for U.K. force of price inflation:

$$I(t) = \begin{cases} QMU1 + QA1 \cdot (I(t-1) - QMU1) + QSD1 \cdot QZ(t) & \text{if } I(t-1) \leq 10\% \\ QMU2 + QA2 \cdot (I(t-1) - QMU2) + QSD2 \cdot QZ(t) & \text{if } I(t-1) > 10\% \end{cases}$$

where $QMU1$ and $QMU2$ are the long-run averages of price inflation in the lower and upper regimes; $QA1$, $QA2$, $QSD1$ and $QSD2$ are the autoregressive and standard deviation parameters for the corresponding regimes. This class of non-linear time-series models was also proposed by Tong (1983, 1990), and is often referred to as a 'threshold autoregressive' (TAR) model in the literature.

2.10 In order to examine whether threshold behaviour is present in the inflation series, we compute the likelihood ratio test statistic derived by Chan & Tong (1990). The value is 1.41 for the original data and 5.84 for the outlier-adjusted series. They should be compared with the critical value of the test, which is 11.81 at the 5% level. There is insufficient evidence to confirm that TAR processes are appropriate for modelling U.K. price inflation series with this test statistic. Furthermore, there are only three observations in the upper regime of the model for the outlier-adjusted inflation series according to the partition rule ($I(t-1) > 10\%$) proposed by Whitten & Thomas (1999). It is difficult to produce efficient estimates of parameters in that regime.

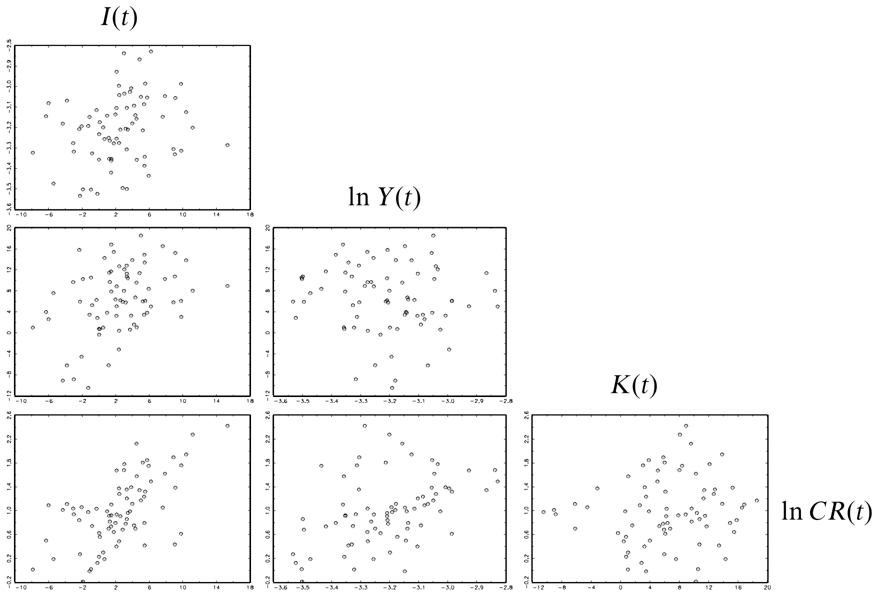

Figure 2.1. Scatterplot matrix for U.K. outlier-adjusted investment data, 1923-1996

2.11 On the basis of the above preliminary analysis, we prefer to use the vector ARMA models rather than cascade-type transfer function models or threshold autoregressive models. Vector ARMA models are closely linked to other econometric models, such as state-space models (see Reinsel, 1997, p53) and simultaneous equation models (see Lütkepohl, 1993, Chapter 10). We will aim to show that VARMA models can lead to more effective characterisations of multivariate time-series (see ¶6.5).

3. Model Building Strategy for VARMA Models

3.1 In this section we review the multiple time-series modelling approach due to Tiao & Box (1981). We shall restrict the discussion to points necessary for describing the applications in this paper. Further details can be found in Tiao & Box (1981) and Reinsel (1997).

3.2 We consider a k-dimension vector time series $\mathbf{Z}(t) = (Z_1(t), Z_2(t), \ldots, Z_k(t))'$ with a VARMA(p, q) representation:

$$\mathbf{Z}(t) = \mathbf{C}_0 + \sum_{i=1}^{p} \mathbf{A}_i \mathbf{Z}(t-i) + \sum_{j=1}^{q} \mathbf{B}_j \mathbf{e}(t-j) + \mathbf{e}(t)$$

where $\mathbf{C}_0$ is a $k \times 1$ intercept vector and $\mathbf{A}_i$s and $\mathbf{B}_j$s are $k \times k$ autoregressive and moving average coefficient matrices, respectively. The residual vectors $\mathbf{e}(t)$ are independently and identically distributed as normal variates with mean zero and variance-covariance matrix $\mathbf{\Sigma}$. For the application in this paper, the vector $\mathbf{Z}$ is given in ¶2.3 and $k = 4$.

3.3 The elements of the autoregressive coefficient matrices $\mathbf{A}_i$ can be denoted by $\{a_{r,s,i}\}$, where $r = 1, \ldots, k$ and $s = 1, \ldots, k$. It should be noted that $a_{r,s,i}$ is a coefficient quantifying the lead-lag relation between the rth element of $\mathbf{Z}(t)$ and the sth element of $\mathbf{Z}(t-i)$. Similarly, $\{b_{r,s,i}\}$, the (r, s) element of $\mathbf{B}_i$, is a coefficient representing the lead-lag relation between the rth element of $\mathbf{e}(t)$ and the sth element of $\mathbf{e}(t-i)$. For example, with the vector $\mathbf{Z}$ given in ¶2.3, the value for $a_{2,3,4}$ would quantify the lead-lag relation between the logarithm of dividend yields ($\ln Y(t)$) and the force of share dividend growth $i = 4$ time periods earlier $K(t-i)$. Similarly, the value for $b_{1,2,4}$ would quantify the lead-lag relation between the residual error for the force of inflation and the residual error for the logarithm of dividend yields $i = 4$ time periods earlier.

3.4 It should be noted that the mean $\boldsymbol{\mu}$ of the vector process $\mathbf{Z}$, $\boldsymbol{\mu} = \mathrm{E}[\mathbf{Z}(t)]$, is related to the intercept term $\mathbf{C}_0$ according to:

$$\boldsymbol{\mu} = [\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 - \ldots - \mathbf{A}_p]^{-1} \mathbf{C}_0$$

where $\mathbf{I}$ is the $k \times k$ identity matrix.

3.5 We define the autoregressive matrix polynomial of order p as:

$$\mathbf{A}(B) = \mathbf{A}_0 - \mathbf{A}_1 B - \mathbf{A}_2 B^2 - \ldots - \mathbf{A}_p B^p$$

where $\mathbf{A}_0 = \mathbf{I}$ and B is the 'lag operator'. The lag operator is defined by the transformation:

$$B^\tau \mathbf{Z}(t) = \mathbf{Z}(t-\tau) \qquad \tau = 1, 2, 3, \ldots.$$

3.6 The vector time-series process $\mathbf{Z}(t)$ is stationary if the zeros of the determinantal polynomial $|\mathbf{A}(B)|$ are all outside the unit circle. Stationarity imposes a behaviour on the series that is without any systematic changes in level (trend), variance or strictly periodic behaviour (Chatfield, 1985). In this paper we only consider stationary processes for $\mathbf{Z}(t)$.

3.7 A stationary VARMA model can be characterised by its cross-correlation matrices, denoted by:

$$\rho(l) = \text{Cor}[\mathbf{Z}(t), \mathbf{Z}(t-l)]$$

for all integers l.

3.8 Analogous to the partial autocorrelation function for univariate time-series (Box & Jenkins, 1976, p64), Tiao & Box (1981) defined the partial autoregression matrix at lag l, denoted by $\mathbf{P}(l)$, as the last coefficient matrix of the following multivariate linear regression:

$$\mathbf{Z}(t+l) = \mathbf{\Phi}_{l,1}\mathbf{Z}(t+l-1) + \mathbf{\Phi}_{l,2}\mathbf{Z}(t+l-2) + \cdots + \mathbf{\Phi}_{l,l}\mathbf{Z}(t) + \mathbf{e}$$

where $\mathbf{e}$ is the error term.

3.9 When $p = 0$, that is $\mathbf{Z}(t)$ is a vector AR(p) process, the partial autoregression matrices $\mathbf{P}(l)$ are zero for $l > p$. On the other hand, the cross-correlation matrices $\rho(l)$ of a vector MA(q) process are zero for $l > q$. These 'cut-off' properties provide very useful information for identifying the order of the underlying VARMA model.

3.10 The orthodox modelling strategy (iterative stages of model identification, estimation and diagnostic checking) proposed by Box & Jenkins (1976) for univariate time-series can be extended and applied to multiple time-series.

3.11 Given a vector time-series of n observations $\mathbf{Z}(1), \mathbf{Z}(2), \ldots, \mathbf{Z}(n)$, we can compute the sample cross-correlation matrix (SCCM):

$$\widehat{\boldsymbol{\rho}}(l) = \{\widehat{\rho}_{ij}(l)\}$$

where the $\widehat{\rho}_{ij}(l)$ are the sample cross-correlations for the ith and jth component series:

$$\widehat{\rho}_{ij}(l) = \frac{\sum_{t=1}^{n-l}(Z_i(t) - \overline{Z}_i)(Z_j(t+l) - \overline{Z}_j)}{[\sum_{t=1}^{n}(Z_i(t) - \overline{Z}_i)^2 \sum_{t=1}^{n}(Z_j(t) - \overline{Z}_j)^2]^{1/2}}$$

and $\overline{Z}_i$ and $\overline{Z}_j$ are the sample averages of the corresponding component series. If the series $\mathbf{e}(t)$ is a white noise, the standard error of each element of the SCCM is approximately $1/\sqrt{n}$.

3.12 The sample partial autoregression matrices (SPAM) $\hat{\mathbf{P}}(l)$ and their standard errors can be obtained by fitting autoregressive models of successively higher order by least squares. Tiao & Box (1981) recommended using the likelihood ratio statistic to test the null hypothesis $\mathbf{P}(l) = 0$ against the alternative $\mathbf{P}(l) \neq 0$. To conduct such a test, we compute:

$$U = |\hat{\mathbf{\Sigma}}(l)|/|\hat{\mathbf{\Sigma}}(l-1)|$$

where $\hat{\mathbf{\Sigma}}(l)$ is the matrix of residual sum of squares and cross products after fitting a vector AR(l) to the data. Using Bartlett's (1938) approximation, the likelihood statistic:

$$M(l) = -\left(n - \frac{3}{2} - l - l \cdot k\right) \ln U$$

is, on the null hypothesis, asymptotically distributed as χ^2 with k^2 degrees of freedom (k is the dimension of the model).

3.13 The SPAM are particularly useful in identifying lower order autoregressive models, as its theoretical counterparts $\mathbf{P}(l)$ are zero beyond lag p for a vector AR(p) process.

3.14 Unfortunately, the SCCM and SPAM are complex when the dimension of the vector is increased. The crowded figures often make recognition of patterns difficult. To alleviate the problem, Tiao & Box (1981) suggested summarising these matrices using indicator symbols $+$, $-$ and $\cdot$, where $+$ denotes a value greater than twice the estimated standard error, $-$ denotes a value less than twice the estimated standard error, and $\cdot$ denotes an insignificant value based on the above criteria.

3.15 After the order of the VARMA model is tentatively selected, asymptotically efficient estimates of the parameters can be determined using the maximum likelihood approach. Approximate standard errors of the estimates of the elements of $\mathbf{A}_i$ and $\mathbf{B}_j$ can also be obtained and used to test for the significance of the parameters. Further gains in the efficiency of the estimates may be achieved by eliminating parameters that are found to be statistically insignificant. Interested readers may refer to Reinsel (1997, Chapter 5) for a detailed discussion of the maximum likelihood estimation for vector ARMA models.

3.16 The maximisation of the likelihood function can be conducted by a conditional likelihood method or an exact likelihood method. The conditional likelihood method is computationally convenient, but may be inadequate if the sample size (n) is not sufficiently large. In this paper we estimate the parameters initially using the conditional likelihood approach and eliminate parameters that are small relative to their standard error. The model is then re-estimated using the exact likelihood method.

3.17 To guard against model mis-specification, a detailed diagnostic analysis of the residuals is required. This includes an examination of the plots of standardised residuals and the SCCM and SPAM of the residuals. At this stage, the $M(l)$ statistic provides a criterion for checking residual serial correlation.

4. The Fitted Model

4.1 The Box & Tiao's (1981) VARMA modelling strategy has been implemented by some time-series computer packages, such as Autobox and SCA. The analysis performed in this paper was carried out using the SCA programming system (Liu & Hudak, 1994).

4.2 We first compute the sample cross-correlation matrices (SCCM) and the sample partial autoregression matrices (SPAM) for U.K. investment series.

4.3 Part (a) of Table 4.1 contains summary information for the SCCM in terms of the $(+, -, \cdot)$ symbols. The criteria used to designate a symbol were discussed in ¶3.14. The (i, j) element of the indicator matrix at lag l summarises the significance of the lag l cross-correlation when the component series $Z_j(t)$ leads the component series $Z_i(t)$. Furthermore, the diagonal elements summarise the significance of the sample autocorrelations for each series.

4.4 Part (b) of Table 4.1 shows the indicator symbols for the SPAM. Like the partial autocorrelation function for the univariate case, the SPAM has the 'cut-off' property for vector AR processes. However, it is important to note that, unlike the univariate partial autocorrelation function, the off-diagonal elements of the SPAM are not proper correlation coefficients (see Wei, 1990, p353 for details). Therefore, in general, SPAM(1) $\neq$ SCCM(1).

4.5 The SCCM clearly does not show a 'cut-off' pattern. The $M(l)$ statistic for the SPAM at the first lag is 202.54, which should be compared with $\chi^2_{16,0.95} = 26.30$ at a 5% level. We also see that SPAM(2) and SPAM(3) have no significant terms, and SPAM(4) has only two out of 16. We tentatively specify a VAR(1) model for the data.

Table 4.1. Indicator matrices for the SCCM and SPAM

lag l: 1, 2, 3, 4, 5

(a) Sample cross-correlation matrices (SCCM)

$$\begin{pmatrix} + & \cdot & + & + \\ + & + & \cdot & + \\ \cdot & \cdot & + & \cdot \\ + & + & \cdot & + \end{pmatrix} \begin{pmatrix} + & \cdot & \cdot & + \\ + & + & \cdot & + \\ \cdot & \cdot & \cdot & \cdot \\ + & + & \cdot & + \end{pmatrix} \begin{pmatrix} + & \cdot & \cdot & + \\ + & \cdot & + & + \\ \cdot & \cdot & \cdot & \cdot \\ + & + & \cdot & + \end{pmatrix} \begin{pmatrix} + & \cdot & \cdot & + \\ + & \cdot & + & + \\ \cdot & \cdot & \cdot & \cdot \\ + & + & \cdot & + \end{pmatrix} \begin{pmatrix} + & \cdot & \cdot & + \\ + & \cdot & + & + \\ \cdot & + & \cdot & \cdot \\ + & + & \cdot & + \end{pmatrix}$$

(b) Sample partial autoregression matrices (SPAM)

$$\begin{pmatrix} + & - & \cdot & + \\ \cdot & + & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & - & \cdot & + \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & - & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & - & \cdot & + \\ \cdot & - & + & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$M(l)$	202.54	14.25	15.01	18.05	33.96

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4.6 We first estimate the specified VAR(1) model using the conditional likelihood method (see Wilson, 1973). All parameters in the model are computed. Imposing zero restrictions on the coefficients that are insignificant, we re-estimate the remaining parameters by the exact likelihood method (see Reinsel, 1997, Chapter 5). The final estimated parameter matrices are given as follows:

$$\hat{\mathbf{C}}_0 = \begin{pmatrix} -0.049 \\ -1.172 \\ 0.047 \\ -1.221 \end{pmatrix} \qquad \hat{\mathbf{A}}_1 = \begin{pmatrix} 0.502 & -0.044 & 0 & 0.021 \\ 0 & 0.646 & 0.551 & 0 \\ 0 & 0 & 0.280 & 0 \\ 0 & -0.328 & 0 & 0.955 \end{pmatrix}$$

and

$$\hat{\mathbf{\Sigma}} = \begin{pmatrix} 0.000820 & & & \\ 0.000136 & 0.014797 & & \\ 0.000579 & -0.000164 & 0.003706 & \\ 0.001710 & 0.003680 & 0.000679 & 0.038967 \end{pmatrix}.$$

4.7 We examine, in Table 4.2, the indicator matrices of the SCCM and SPAM for the residuals from the fitted VAR(1) model. There are some significant cross-correlations left in the residuals. The $M(l)$ statistic for the residual SPAM at the fourth lag is still 27.35, which should be compared with $\chi^2_{16,0.95} = 26.30$ at a 5% level. The results indicate that a simple VAR

Table 4.2. Indicator matrices for the residual SCCM and SPAM of the fitted VAR(1)

	lag l 1	2	3	4	5
(a) Sample cross-correlation matrices (SCCM)	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & + \\ \cdot & - & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & - & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & - & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix}$
(b) Sample partial autoregression matrices (SPAM)	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & - & \cdot \\ \cdot & \cdot & \cdot & - \end{pmatrix}$	$\begin{pmatrix} + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & - \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & - & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & - & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
$M(l)$	16.16	10.80	11.75	27.35	24.28

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model (as suggested by Wright, 1998) cannot efficiently describe the behaviour of the vector series. We need a mixed vector process. Therefore, a simple mixed VARMA(1,1) model is re-specified.

4.8 We estimate all the parameters in the VARMA(1,1) model using the conditional likelihood method. We refer to this as the 'full model'. Imposing zero restrictions on the coefficients that are insignificant, we re-estimate the 'final model' by the exact likelihood method. The estimation results are summarised in Table 4.3.

4.9 The final estimated model in Table 4.3 can be re-written as follows:

$$
\begin{aligned}
I(t) &= 0.074 + 0.506\, I(t-1) + 0.017 \ln CR(t-1) + e_1(t) - 0.063\, e_2(t-1) \\
\ln Y(t) &= -1.583 + 1.067\, I(t-1) + 0.487 \ln Y(t-1) + 0.025 \ln CR(t-1) + e_2(t) \\
K(t) &= 0.087 - 0.342\, K(t-1) + e_3(t) + e_3(t-1) \\
\ln CR(t) &= -0.365 + 0.897 \ln CR(t-1) + e_4(t) - 0.522\, e_2(t-1) + 0.266\, e_4(t-1)
\end{aligned}
$$

with:

$$
\hat{\mathbf{\Sigma}} = \begin{pmatrix} 0.000804 & & & \\ 0.000103 & 0.014178 & & \\ 0.000404 & -0.001131 & 0.002695 & \\ 0.001340 & 0.003279 & 0.000278 & 0.036243 \end{pmatrix}.
$$

4.10 The corresponding residual correlation matrix is computed as:

$$
\begin{pmatrix} 1 & & & \\ 0.031 & 1 & & \\ 0.274 & -0.183 & 1 & \\ 0.248 & 0.145 & 0.028 & 1 \end{pmatrix}.
$$

The two standard error limits $\pm 2/\sqrt{n}$, appropriate for a vector white noise process, can be used as guidelines in assessing the significance of individual residual correlations. With $n = 74$ observations, only values over 0.232 would be significant at a 2.5% level, so four out of these six correlations could reasonably be taken as zero, and the other two are not vastly significant.

4.11 The indicator matrices of the SCCM and SPAM for the residuals are given in Table 4.4. The $M(l)$ statistics for $l = 1, \ldots, 5$ are, respectively, 16.74, 9.86, 8.20, 18.50 and 18.04, which are insignificant at a 5% level (note that the critical value for $\alpha = 5\%$ is $\chi^2_{16,0.95} = 26.30$). Residual checks using the cross-correlation matrices show no significant serial correlation. We conclude that the fitted model is adequate for the vector series.

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Table 4.3. Estimation results* of the VARMA(1,1) model based on outlier-adjusted data

(a) Full model

$$\mathbf{C}_0 = \begin{pmatrix} -0.022 \\ (0.091) \\ -1.431 \\ (0.405) \\ -0.285 \\ (0.258) \\ -0.845 \\ (0.853) \end{pmatrix} \quad \mathbf{A}_1 = \begin{pmatrix} 0.787 & -0.012 & -0.043 & 0.002 \\ (0.157) & (0.032) & (0.109) & (0.010) \\ 0.937 & 0.540 & -0.070 & 0.018 \\ (0.710) & (0.142) & (0.382) & (0.047) \\ 0.582 & -0.111 & -0.359 & -0.000 \\ (0.480) & (0.090) & (0.124) & (0.034) \\ 0.694 & -0.089 & -0.149 & 0.845 \\ (1.539) & (0.299) & (0.577) & (0.106) \end{pmatrix}$$

$$\mathbf{B}_1 = \begin{pmatrix} 0.379 & 0.070 & -0.052 & -0.053 \\ (0.210) & (0.046) & (0.138) & (0.021) \\ 0.443 & -0.062 & -0.704 & -0.106 \\ (0.859) & (0.181) & (0.456) & (0.083) \\ 0.589 & -0.054 & -0.971 & 0.012 \\ (0.401) & (0.065) & (0.089) & (0.031) \\ 0.918 & 0.353 & -0.572 & -0.505 \\ (1.544) & (0.295) & (0.609) & (0.136) \end{pmatrix} \quad \mathbf{\Sigma} \times 10^{-2} = \begin{pmatrix} 0.0749 & & & \\ -0.0036 & 1.2515 & & \\ 0.0496 & -0.1096 & 0.2596 & \\ 0.1272 & 0.2267 & 0.0308 & 3.6611 \end{pmatrix}$$

(b) Final model

$$\mathbf{C}_0 = \begin{pmatrix} 0.074 \\ (0.027) \\ -1.583 \\ (0.268) \\ 0.087 \\ (0.014) \\ -0.365 \\ (0.188) \end{pmatrix} \quad \mathbf{A}_1 = \begin{pmatrix} 0.506 & 0 & 0 & 0.017 \\ (0.091) & & & (0.007) \\ 1.067 & 0.487 & 0 & 0.025 \\ (0.383) & (0.090) & & (0.032) \\ 0 & 0 & -0.342 & 0 \\ & & (0.106) & \\ 0 & 0 & 0 & 0.897 \\ & & & (0.051) \end{pmatrix}$$

$$\mathbf{B}_1 = \begin{pmatrix} 0 & 0.063 & 0 & 0 \\ & (0.027) & & \\ 0 & 0 & 0 & 0 \\ & & & \\ 0 & 0 & -1.000 & 0 \\ & & (0.024) & \\ 0 & 0.522 & 0 & -0.266 \\ & (0.185) & & (0.117) \end{pmatrix} \quad \mathbf{\Sigma} \times 10^{-2} = \begin{pmatrix} 0.0804 & & & \\ 0.0103 & 1.4178 & & \\ 0.0404 & -0.1131 & 0.2695 & \\ 0.1340 & 0.3279 & 0.0278 & 3.6243 \end{pmatrix}$$

*Note: Standard errors of the estimates are given in parentheses.

Table 4.4. Indicator matrices for the residual SCCM and SPAM of the final model

	lag l 1	2	3	4	5
(a) Sample cross-correlation matrices (SCCM)					
	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & - & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
(b) Sample partial autoregression matrices (SPAM)					
	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & - \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$	$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & - & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$
$M(l)$	16.74	9.86	8.20	18.50	18.04

5. Discussion

5.1 *Data-Based vs Theory-Based Modelling Approaches*

5.1.1 Huber & Verrall (1999) argued that a theory-based approach is the most suitable method for actuarial economic modelling. Their paper addresses one of the most fundamental issues in the philosophy of scientific modelling, namely the way that models are formulated.

5.1.2 Analogous to Bayesianism in statistical modelling, theory-based methods take in financial economic theories as prior knowledge. On the other hand, following the frequentist philosophy in statistics, data-based methods mainly rely on observed data.

5.1.3 Actuaries, economists and statisticians have been discussing the appropriateness of these modelling approaches for many years. Feller (1966, p52) stated that:

> ''Theories of this nature [developed purely with an industry of goodness of fit testing, without theoretical reasons why the proposed models are appropriate] are short-lived because they open no new ways, . . ., it may be useful to have an explicit demonstration of how misleading a mere goodness of fit can be.''

On the other hand, Chatfield (1995, p428) quoted Professor John Tukey's comment on this issue from a frequentist's point of view:

> ''we need more honest foundations for data analysis which do not rely on 'assuming that we always know what is in fact we never know'.''

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5.1.4 However, these two approaches of modelling are not always mutually exclusive. In fact, most researchers used a blend of both methods, perhaps with a greater emphasis on one over the other. For examples: Blaug (1992); Hausman (1992); Caldwell (1994); Pemberton (1999) and Huber & Verrall (1999) favoured the Bayesian/theory-based philosophy. Others, like Box (1976); Leamer (1978, Chapter 6); Efron (1986) and Tiao & Tsay (1994) have put a greater emphasis on frequentist/data-based methods.

5.1.5 Huber & Verrall (1999) characterised the original Wilkie (1986, 1995) model as data-based. On the contrary, Wilkie (2000) described his model as strongly theory-based:

> "My method is not like that at all. It is very strongly theory-based, though it is different theory from that of some financial economists. Further, my approach has changed to some extent over the years, from the Report of the Maturity Guarantees Working Party (Ford *et al.,* 1980) through Wilkie (1981) and Wilkie (1986) to Wilkie (1995). As I have learned more, my method has become even more theory-based."

5.1.6 While debating for the philosophical superiority of theory-based or data-based methods is beyond the scope of this paper, it is important to note that we have emphasised the frequentist viewpoint in deriving the final model. Even though the proposed VARMA(1,1) model is very much data-driven, its coefficients are not totally without economic explanation.

5.1.7 The first equation in ¶4.9 indicates that the force of inflation is positively led by the last year's inflation and long-term real interest rates. Fama (1990) has also found that long-term interest rates might be helpful in forecasting the future path of inflation. Furthermore, during a recession, it is well-known that low inflation would lead to fall of corporate profits. The second equation of ¶4.9 has correctly identified a positive coefficient between $\ln Y(t)$ and $I(t-1)$.

5.1.8 Wilkie (1995, p839) specified the connection between $I(t)$ and $K(t)$ through a $DM(t)$ variable. Wilkie (2000) explained the theory behind his formulation. In this article, we do not assume the $DM(t)$ structure as in Wilkie (1995, p835). However, the relationship between $I(t)$ and $K(t)$ has been automatically revealed (and modelled) in the residual correlation matrix (see ¶4.10) of the final model. The corresponding correlation coefficient is 0.274, the highest among all the entries in the matrix.

5.2 *Outliers in the Data*

5.2.1 Whether or not it is appropriate to adjust the data for the outliers depends on the purpose to which the model so derived will be used. If the model will be used in an application for which extreme stochastic fluctuations are less important (e.g. to ensure that premiums are adequate in most, but not extreme, scenarios), then it may be preferable to use a model

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based on outlier-adjusted data. If, however, the model will be used in an application for which extreme stochastic fluctuations are important (such as pricing catastrophe risks or ensuring that investment guarantee reserves are sufficient to keep an insurance company solvent in all but the most extreme scenarios), then a model which is sympathetic to outliers in the data ought to be used.

5.2.2 Various alternative approaches have been proposed for dealing with outliers. For example, one would use the outlier-adjusted data to build the 'skeleton' of the model, and then estimate the parameters from the data with the outliers put back in. For U.K. investment data, the 'skeleton' of the model is specified as VARMA(1,1), using the outlier-adjusted data in ¶4. The estimated parameters from the raw data (i.e., with the outliers put back in) are given as:

$$\hat{\mathbf{C}}_0 = \begin{pmatrix} 0.132 \\ (0.036) \\ -1.311 \\ (0.292) \\ 0.040 \\ (0.016) \\ -0.325 \\ (0.183) \end{pmatrix} \qquad \hat{\mathbf{A}}_1 = \begin{pmatrix} \begin{matrix}0.411\\(0.096)\end{matrix} & 0 & 0 & \begin{matrix}0.029\\(0.009)\end{matrix} \\ \begin{matrix}0.627\\(0.391)\end{matrix} & \begin{matrix}0.487\\(0.095)\end{matrix} & 0 & \begin{matrix}0.099\\(0.040)\end{matrix} \\ 0 & 0 & \begin{matrix}0.289\\(0.182)\end{matrix} & 0 \\ 0 & 0 & 0 & \begin{matrix}0.908\\(0.050)\end{matrix} \end{pmatrix}$$

$$\hat{\mathbf{B}}_1 = \begin{pmatrix} 0 & \begin{matrix}-0.035\\(0.027)\end{matrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{matrix}-0.302\\(0.181)\end{matrix} & 0 \\ 0 & \begin{matrix}0.485\\(0.135)\end{matrix} & 0 & \begin{matrix}-0.308\\(0.118)\end{matrix} \end{pmatrix}$$

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$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 0.001642 & & & \\ 0.002407 & 0.024432 & & \\ 0.001305 & 0.000374 & 0.006383 & \\ 0.000224 & 0.007577 & -0.000748 & 0.034812 \end{pmatrix}.$$

5.2.3 The fitted VARMA(1,1) model using the above parameters can be re-written in equation format as:

$$\begin{aligned} I(t) &= 0.132 + 0.411\, I(t-1) + 0.029 \ln CR(t-1) + e_1(t) + 0.035\, e_2(t-1) \\ \ln Y(t) &= -1.311 + 0.627\, I(t-1) + 0.487 \ln Y(t-1) + 0.099 \ln CR(t-1) + e_2(t) \\ K(t) &= 0.040 + 0.289\, K(t-1) + e_3(t) + 0.302\, e_3(t-1) \\ \ln CR(t) &= -0.325 + 0.908 \ln CR(t-1) + e_4(t) - 0.485\, e_2(t-1) + 0.308\, e_4(t-1). \end{aligned}$$

5.2.4 On the other hand, we may keep the structure and the parameter values exactly as in Table 4.3, in which the outliers are omitted, and then only estimate the residual variance-covariance matrix $\boldsymbol{\Sigma}$ from the data without omitting the outliers. This approach can also be used for dealing with outliers. The estimation result is given as:

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} 0.001653 & & & \\ 0.002672 & 0.026073 & & \\ 0.001336 & 0.002095 & 0.013268 & \\ 0.000922 & 0.011364 & 0.004651 & 0.042972 \end{pmatrix}.$$

5.2.5 An alternative approach for dealing with outliers involves sampling from the empirical distribution of fitted 'errors' instead of sampling from the assumed multivariate normal distribution. This is called 'resampling' or 'bootstrapping' in the literature. The procedure effectively approximates the theoretical distribution of innovations by the empirical distribution of the observed residuals. Thus it is a distribution-free method. An excellent review on the use of computationally intensive methods for complex economic models is given by Veall (1989).

5.2.6 Another approach is to assume a non-Gaussian heavy-tailed error distribution to model the outliers. Classical analysis of non-Gaussian time-series models sometimes requires computationally intensive techniques such as Markov chain Monte Carlo (MCMC) methods. Interested readers may refer to Chambers *et al.* (1976); Finkelstein (1997) and Durbin & Koopman (2000).

6. Comparison with the Wilkie Model

6.1 *Force of Price Inflation*

6.1.1 The Wilkie (1995) model for the force of price inflation series $I(t)$ is:

$$I(t) = QMU + QA(I(t-1) - QMU) + QE(t)$$

$$QE(t) \sim \text{i.i.d. N}(0, QSD^2).$$

6.1.2 The fitted parameters, based on the original series from 1923 to 1996, were computed as:

$$QMU = 0.0459 \qquad QA = 0.6100 \qquad QSD = 0.04229.$$

The values are very similar to the results given by Wilkie (1995, p785) for the period 1923-94.

6.1.3 The estimated parameters for the outlier-adjusted series of the same period were obtained as:

$$QMU^* = 0.0284 \qquad QA^* = 0.6653 \qquad QSD^* = 0.03139.$$

6.1.4 The force of price inflation process obtained by the preliminary VAR(1) model in ¶4.6 is:

$$I(t) = -0.049 + 0.502\, I(t-1) - 0.044 \ln Y(t-1) + 0.021 \ln CR(t-1) + e_1(t)$$

with $QSD^{**} = 0.0286$.

6.1.5 The force of price inflation equation implied by the final fitted VARMA model in Table 4.3, using the outlier-adjusted data is:

$$I(t) = 0.074 + 0.506\, I(t-1) + 0.017 \ln CR(t-1) + e_1(t) - 0.063\, e_2(t-1)$$

with $QSD^{***} = 0.0284$. In addition to its last year's realisation, the price inflation series is led one-year by both the long-term real interest rate and the residual of the corresponding share dividend yield series.

6.1.6 The force of price inflation equation derived from the final fitted

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VARMA model in ¶5.2.2, using the original data (without outlier adjustments), is:

$$I(t) = 0.132 + 0.411\ I(t-1) + 0.029 \ln CR(t-1) + e_1(t) + 0.035\ e_2(t-1)$$

with $QSD^{****} = 0.0405$.

6.1.7 Another model discussed in ¶5.2.4 is to keep the inflation equation exactly as in ¶6.1.5, and then compute the residuals (and QSD) from the original data. The result is $QSD^{*****} = 0.0407$.

6.1.8 We now analyse residuals obtained from different models for price inflation. For convenience, the models described in ¶6.1.2, ¶6.1.3, ¶6.1.4, ¶6.1.5, ¶6.1.6 and ¶6.1.7 are denoted by Model A1, Model B1, Model C1, Model D1, Model E1 and Model F1, respectively.

6.1.9 Normality, independence, homoscedasticity and linearity assumptions for residuals were formally checked by statistical tests. We employ Jarque & Bera's (1981) test for normality, Ljung & Box's (1978) test for serial independence, Engle's (1982) test for homoscedasticity and McLeod & Li's (1983) test for linearity. A brief description of these tests is given in Appendix A. The test results are summarised in Table 6.1. Model B1, Model C1 and Model D1 passed all the tests. However, it should be noted that Models B1 - D1 are based on the outlier-adjusted inflation series, while Model A1 and Models E1 - F1 are for the raw price inflation data.

6.1.10 The price inflation rates can be converted back to the retail price index $Q(t)$:

$$Q(t) = Q(t-1) \cdot exp(I(t)).$$

6.1.11 In Figure 6.1, we show a set of ten simulations of $Q(t)$ using Model B1 from 1997 to 2050, along with the past record since 1950, all on a logarithmic scale. Another ten simulations using Model D1 are plotted in Figure 6.2. The same innovations are used for these two sets of simulations.

6.1.12 The simulations using the Model D1 system fluctuate more widely as compared with those from the outlier adjusted Wilkie model. This is because the variation of future inflation paths is dominated by its $QE(t)$ random series under the Wilkie model. On the other hand, price inflation under the proposed VARMA(1,1) structure has explicit interaction among other variables (see ¶4.9). Its future movements are also affected by the error terms of other variables implicitly through the residual variance-covariance matrix $\hat{\boldsymbol{\Sigma}}$.

6.2 *Share Dividend Yields*

6.2.1 The model proposed by Wilkie (1995, p818) for the share dividend yield (with logarithmic transformation) at time t, $\ln Y(t)$, is as follows:

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$$\ln Y(t) = \ln YMU + YW \cdot I(t) + YN(t)$$
$$YN(t) = YA \cdot YN(t-1) + YE(t)$$
$$YE(t) \sim \text{i.i.d. N}(0, YSD^2).$$

6.2.2 The suggested parameters, based on the experience from 1923 to 1996, were computed as:

$$YMU = 0.0378 \quad YW = 1.7853 \quad YA = 0.5708 \quad YSD = 0.154.$$
$$(0.0089) \qquad (0.4241) \qquad (0.0946)$$

The corresponding standard errors of the estimates are given in parentheses.

6.2.3 There were doubts about including the parameter YW in the model. Wilkie (1995, p821) supported the original model (i.e. YW is

Table 6.1. Diagnostic checking of residuals from different models for inflation

	Model					
	A1	B1	C1	D1	E1	F1
(1) Model characteristics						
Type	AR	AR	VAR	VARMA	VARMA	VARMA
Order	(1)	(1)	(1)	(1,1)	(1,1)	(1,1)
Period	1923-96	1923-96	1923-96	1923-96	1923-96	1923-96
Outlier adjustment	No	Yes	Yes	Yes	No	Yes/No*
(2) Analysis of residuals						
Median	−0.0003	0.0002	−0.0001	0.0014	0.0001	0.0095
Standard deviation	0.0426	0.0313	0.0286	0.0284	0.0408	0.0407
Skewness	1.1311	0.1020	0.1214	0.2327	0.9412	1.1759
Kurtosis	5.1414	3.0635	2.9344	3.0733	4.5771	5.1299
(a) Test for normality						
Test statistic (JB)	29.51	0.14	0.19	0.68	18.34	30.62
p-value	0.0000	0.9324	0.9094	0.7118	0.0001	0.0000
(b) Test for independence						
Test statistic (Q_{15})	8.90	9.92	10.74	8.90	10.90	8.50
p-value	0.8374	0.7680	0.5513	0.6311	0.4517	0.6679
(c) Test for ARCH effects						
Test statistic (LM)	0.1955	0.0283	0.2747	0.1452	0.9908	0.0116
p-value	0.6584	0.8665	0.6002	0.7031	0.0000	0.9141
(d) Test for linearity						
Test statistic (Q^*_{12})	6.1245	8.6526	11.9678	10.8172	7.3551	4.9138
p-value	0.9097	0.7323	0.4483	0.5446	0.8333	0.9608

*Note: the model uses both the original and outlier-adjusted data (see ¶5.2.4)

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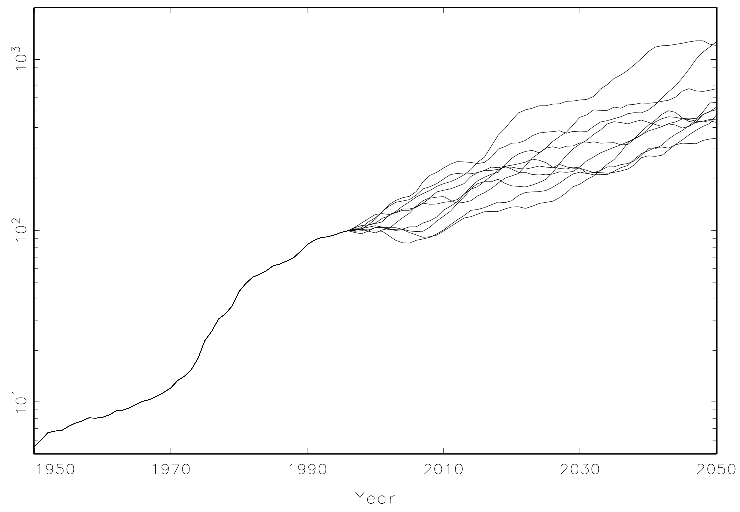

Figure 6.1. Retail prices index, 1950-1996, and simulations, 1997-2050, using Model B1

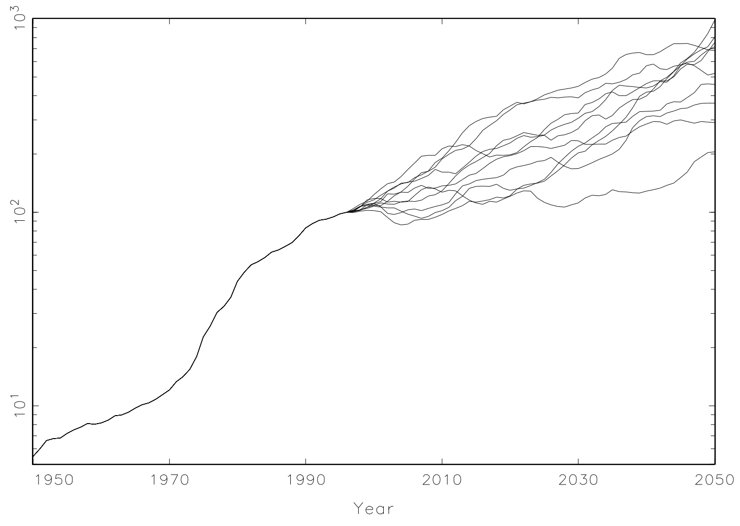

Figure 6.2. Retail prices index, 1950-1996, and simulations, 1997-2050, using Model D1

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retained), and produced two arguments: (i) the parameter estimate for YW is over three standard errors away from zero, so is clearly significant; and (ii) omitting YW worsens the log likelihood function by 6.9 (much more than the commonly used criterion 2.0), so the improvement of the model with YW is highly significant.

6.2.4 The fitted parameters for the outlier-adjusted series were calculated as:

$$\underset{(0.0093)}{YMU^* = 0.0399} \qquad \underset{(0.4922)}{YW^* = 0.3444} \qquad \underset{(0.0935)}{YA^* = 0.6072} \qquad YSD^* = 0.127.$$

6.2.5 It is interesting to note that YW^* is now not significantly different from zero (t-value = 0.70). Excluding YW^* from the model only worsens the log likelihood function by 0.51. It again shows that adjusting the data for outliers could significantly affect the estimation results.

6.2.6 The share dividend yield process obtained by the preliminary VAR(1) model in ¶4.6 is:

$$\ln Y(t) = -1.172 + 0.646 \ln Y(t-1) + 0.551\, K(t-1) + e_2(t)$$

with $YSD^{**} = 0.122$.

6.2.7 The share dividend yield equation implied by the final fitted VARMA model in Table 4.3 is:

$$\ln Y(t) = -1.583 + 1.067\, I(t-1) + 0.487 \ln Y(t-1) + 0.025 \ln CR(t-1) + e_2(t)$$

with $YSD^{***} = 0.119$.

6.2.8 The share dividend yield equation derived from the fitted VARMA model in ¶5.2.2 is:

$$\ln Y(t) = -1.311 + 0.627\, I(t-1) + 0.487 \ln Y(t-1) + 0.099 \ln CR(t-1) + e_2(t)$$

with $YSD^{****} = 0.156$.

6.2.9 An alternative model, discussed in ¶5.2.4, is to keep the share dividend yield equation exactly as in ¶6.2.7, and then compute the residuals from the original data. The resulting YSD^{*****} value is 0.161.

6.2.10 We now analyse residuals obtained from different models for $\ln Y(t)$. For convenience, the models described in ¶6.2.2, ¶6.2.4, ¶6.2.6, ¶6.2.7, ¶6.2.8 and ¶6.2.9 are denoted by Model A2, Model B2, Model C2, Model D2, Model E2 and Model F2, respectively. The results are summarised in Table 6.2.

Table 6.2. Residual checking of different models for share dividend yields

		Model					
		A2	B2	C2	D2	E2	F2
	Median	−0.0193	0.0046	0.0025	0.0006	−0.0086	−0.0138
	Standard deviation	0.1546	0.1277	0.1216	0.1192	0.1574	0.1615
	Skewness	0.2273	0.1328	0.0380	−0.0118	0.4960	0.6812
	Kurtosis	3.1524	2.5285	2.4373	2.7650	4.5591	3.7149
(a)	Test for normality						
	Test statistic (JB)	0.70	0.89	0.98	0.17	10.39	14.59
	p-value	0.7047	0.6408	0.6126	0.9139	0.0055	0.0007
(b)	Test for independence						
	Test statistic (Q_{15})	14.6	11.1	12.2	16.1	13.7	13.8
	p-value	0.2640	0.5204	0.3488	0.1411	0.2500	0.2443
(c)	Test for ARCH effects						
	Test statistic (LM)	0.5497	0.9395	0.0214	0.0005	0.0796	0.0039
	p-value	0.4584	0.3324	0.8837	0.9828	0.7779	0.9500
(d)	Test for linearity						
	Test statistic (Q^*_{12})	17.0459	25.5918	8.6673	10.9147	7.9970	4.4407
	p-value	0.1479	0.0123	0.7311	0.5211	0.7854	0.9741

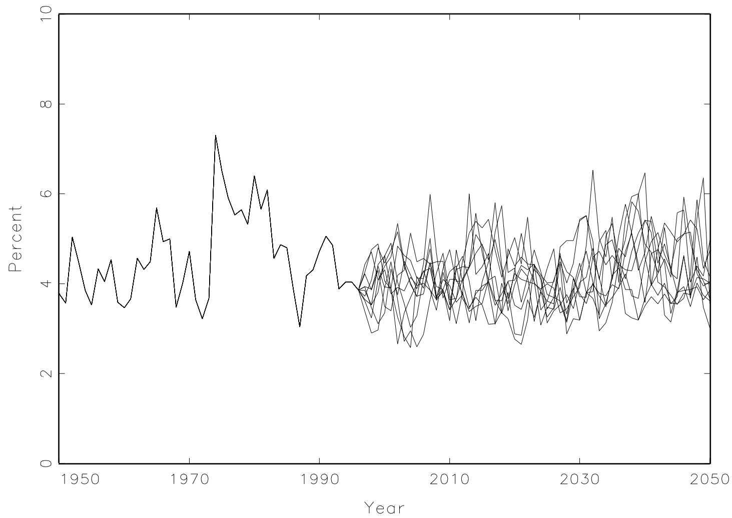

Figure 6.3. Dividend yield, 1950-1996, and simulations, 1997-2050, using Model D2

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6.2.11 The linearity test result with Model B2 (see part (d) of Table 6.2) shows that the outlier adjustments could affect linearity. This is consistent with Chan (1994), who found that standard portmanteau tests are not generally robust to outliers. It therefore appears that the portmanteau-type test for linearity employed in this paper could be affected by time-series outlier adjustments.

6.2.12 In Figure 6.3 we show a set of ten simulations of $Y(t)$ at annual intervals from June 1997 to 2050, along with the past record since 1950, on a linear scale using the proposed VARMA(1,1) model in ¶4.9.

6.3 *Force of Share Dividend Growth*

6.3.1 The original model proposed by Wilkie (1995) for the force of share dividend growth at time t, $K(t)$, is as follows:

$$K(t) = DMU + DI(t) + DY \cdot YE(t-1) + DB \cdot DE(t-1) + DE(t)$$

$$DI(t) = DW \cdot DM(t) + (1 - DW) \cdot I(t)$$

$$DM(t) = DD \cdot I(t) + (1 - DD) \cdot DM(t-1)$$

$$DE(t) \sim \text{i.i.d. N}(0, DSD^2).$$

6.3.2 The estimation results of the above model using observations from 1923 to 1996 are:

$$\underset{(0.0120)}{DMU = 0.0135} \qquad \underset{(0.0427)}{DY = -0.1800} \qquad \underset{(0.0977)}{DB = 0.5495}$$

$$\underset{(0.2146)}{DW = 0.5358} \qquad \underset{(0.0789)}{DD = 0.1528} \qquad DSD = 0.0664.$$

The corresponding standard errors of the estimates are given in parentheses. The values are very similar to the results given by Wilkie (1995, p843) for the period 1923-94.

6.3.3 The estimated parameters for the outlier-adjusted series are shown below:

$$\underset{(0.0086)}{DMU^* = 0.0416} \qquad \underset{(0.0470)}{DY^* = -0.1138} \qquad \underset{(0.1101)}{DB^* = 0.4075}$$

$$\underset{(0.2031)}{DW^* = 0.5575} \qquad \underset{(0.0305)}{DD^* = 0.0578} \qquad DSD^* = 0.0525.$$

6.3.4 The force of share dividend growth process obtained by the fitted VAR(1) model in ¶4.6 is:

$$K(t) = 0.047 + 0.280\ K(t-1) + e_3(t)$$

with $DSD^{**} = 0.0609$.

6.3.5 The force of share dividend growth equation implied by the final fitted VARMA model in Table 4.3 is:

$$K(t) = 0.087 - 0.342\ K(t-1) + e_3(t) + e_3(t-1)$$

with $DSD^{***} = 0.0519$.

6.3.6 The force of share dividend growth equation derived from the fitted VARMA model in ¶5.2.2 is:

$$K(t) = 0.040 + 0.289\ K(t-1) + e_3(t) + 0.302\ e_3(t-1)$$

with $DSD^{****} = 0.0799$.

6.3.7 An alternative model, discussed in ¶5.2.4, is to keep the force of share dividend growth equation exactly as in ¶6.3.5, and then compute the residuals from the original data. The resulting DSD^{*****} value is 0.1152.

6.3.8 We now analyse residuals obtained from different models for $K(t)$. For convenience, the models described in ¶6.3.2, ¶6.3.3, ¶6.3.4, ¶6.3.5, ¶6.3.6 and ¶6.3.7 are denoted by Model A3, Model B3, Model C3, Model D3, Model E3 and Model F3, respectively. The results are summarised in Table 6.3. Only Model D3 passed all the tests.

Table 6.3. Residual checking of different models for the force of share dividend growth

		Model					
		A3	B3	C3	D3	E3	F3
	Median	0.0065	0.0067	−0.0019	0.0028	0.0105	0.0103
	Standard deviation	0.0664	0.0525	0.0609	0.0519	0.0804	0.1152
	Skewness	−0.7776	−0.4004	−0.6470	−0.0332	−0.9186	−1.8904
	Kurtosis	4.0564	3.0053	3.4747	2.7391	4.9680	8.6096
(a)	Test for normality						
	Test statistic (JB)	10.60	1.92	5.70	0.22	22.05	139.2
	p-value	0.0050	0.3829	0.0578	0.8958	0.0000	0.0000
(b)	Test for independence						
	Test statistic (Q_{15})	18.6	24.6	29.2	16.4	14.60	58.27
	p-value	0.0456	0.0062	0.0061	0.2282	0.3330	0.0000
(c)	Test for ARCH effects						
	Test statistic (LM)	5.3564	2.0626	0.0179	0.8001	7.1600	24.0963
	p-value	0.0206	0.1510	0.8935	0.3711	0.0075	0.0001
(d)	Test for linearity						
	Test statistic (Q^*_{12})	11.3753	17.2084	23.7405	19.2093	16.4849	27.7847
	p-value	0.4971	0.1419	0.0221	0.0836	0.1700	0.0059

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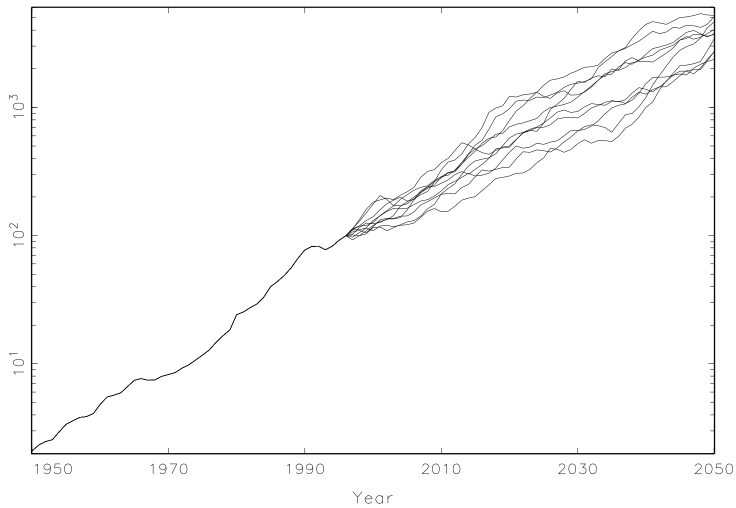

Figure 6.4. Dividend index, 1950-1996, and simulations, 1997-2050, using Model D3

6.3.9 In Figure 6.4 we show a set of simulations of $D(t)$ at annual intervals from June 1997 to 2050, along with the past record since 1950, all on a logarithmic scale using the proposed VARMA(1,1) model in Table 4.3.

6.3.10 The model proposed by Wilkie (1995) for the value of a price index of ordinary shares at time t, $P(t)$, is as follows:

$$P(t) = D(t)/Y(t).$$

6.3.11 In Figure 6.5 we show a set of simulations of $P(t)$ at annual intervals from June 1997 to 2050, along with the past record since 1950, all on a logarithmic scale using the proposed model (Model D).

6.4 *Long-Term Real Interest Rates*

6.4.1 The simplified model for $\ln CR(t)$ proposed by Wilkie (1995, p857) is:

$$\ln CR(t) = \ln CMU + CN(t)$$
$$CN(t) = CA \cdot CN(t-1) + CY \cdot YE(t) + CE(t)$$
$$CE(t) \sim \text{i.i.d. N}(0, CSD^2).$$

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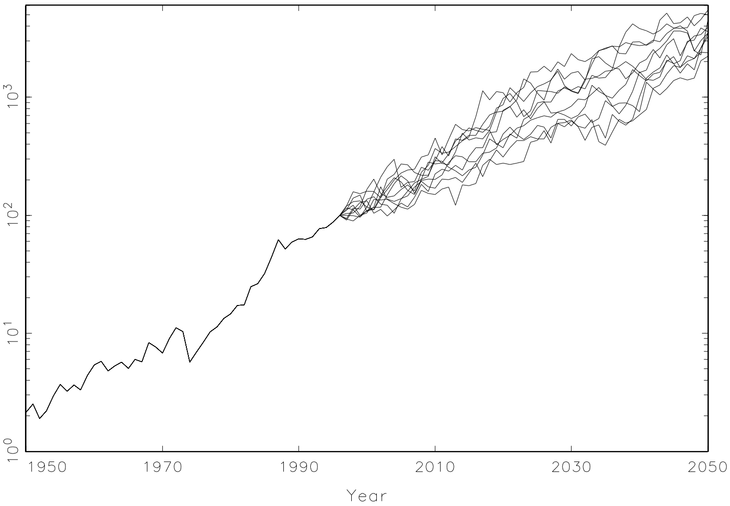

Figure 6.5. Share price index, 1950-1996, and simulations, 1997-2050, using Model D

6.4.2 The fitted parameters, based on the experience from 1923 to 1996, were computed as:

$$CMU = \underset{(0.0131)}{0.0285} \qquad CY = \underset{(0.1596)}{0.3781} \qquad CA = \underset{(0.0432)}{0.9040} \qquad CSD = 0.2027.$$

The corresponding standard errors of the estimates are given in parentheses. The values are very similar to the results given by Wilkie (1995, p861, model (iv)) for the period 1923-94.

6.4.3 There was a large residual in 1974 from the above fitted model. Wilkie (1995, ¶6.3.4) noticed this extreme value, and he employed a dummy variable to accommodate its effect.

6.4.4 Our outlier analysis in Table 2.1 indicates that no outlier was found for the $\ln CR(t)$ series. The extreme residual value in 1974 is unexpected, and it seems worth investigating further. The model for the logarithm of $CR(t)$ is essentially an AR(1) process, plus an additional effect transferred from the current share dividend yield error term $YE(t)$. Figure 6.6 gives a standardised plot of $YE(t)$. There was an outstanding spike in 1974. The extreme residual in 1974 from the original model for $\ln CR(t)$ could be 'imported' from the $YE(t)$ series. The $YE(t)$ series itself 'inherited' the outlier from the price inflation and share dividend yield series through the original model for $\ln Y(t)$, described in ¶6.2.1. It should be noted that large outliers

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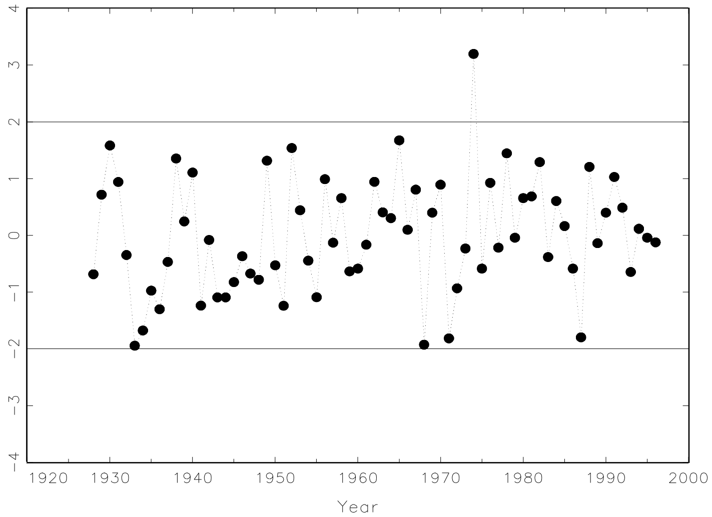

Figure 6.6. Standardised plot of the $YE(t)$ series

were found in 1974 and 1975 for $\ln Y(t)$ and $I(t)$, respectively. For cascade-type transfer function models, effects of contaminated observations could spread from the sources of the 'cascade' down to other variables in the system.

6.4.5 The estimated parameters, based on the outlier-adjusted series from 1923 to 1996, were obtained as:

$$\underset{(0.0148)}{CMU^* = 0.0286} \qquad \underset{(0.1983)}{CY^* = 0.2861} \qquad \underset{(0.0444)}{CA^* = 0.9128} \qquad CSD^* = 0.2077.$$

6.4.6 The long-term real interest rate process obtained from the preliminary VAR(1) model in ¶4.6 is:

$$\ln CR(t) = -1.221 - 0.328 \ln Y(t-1) + 0.955 \ln CR(t-1) + e_4(t)$$

with $CSD^{**} = 0.197$.

6.4.7 The long-term real interest rate equation implied by the final fitted VARMA model in Table 4.3 is:

$$\begin{aligned} \ln CR(t) = -0.365 + 0.897 \ln CR(t-1) + e_4(t) \\ - 0.522 e_2(t-1) + 0.266 e_4(t-1) \end{aligned}$$

with $CSD^{***} = 0.190$.

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6.4.8 The long-term real interest rate equation derived from the fitted VARMA model in ¶5.2.2 is:

$$\ln CR(t) = -0.325 + 0.908 \ln CR(t-1) + e_4(t) - 0.485 e_2(t-1) + 0.308 e_4(t-1)$$

with $CSD^{****} = 0.187$.

6.4.9 An alternative model, discussed in ¶5.2.4, is to keep the long-term real interest rate equation exactly as in ¶6.4.7, and then compute the residuals from the original data. The resulting CSD^{*****} value is 0.207.

6.4.10 Residuals obtained from different models for $\ln CR(t)$ were analysed. For convenience, the models described in ¶6.4.2, ¶6.4.5, ¶6.4.6, ¶6.4.7, ¶6.4.8 and ¶6.4.9 are denoted by Model A4, Model B4, Model C4, Model D4, Model E4 and Model F4, respectively. The results are summarised in Table 6.4. Both Model D4 and Model E4 passed all the tests.

6.4.11 In Figure 6.7 we show a set of ten simulations of $C(t)$ at annual intervals from June 1997 to 2050, along with the past record since 1950, on a linear scale using the proposed Model D.

Table 6.4. Residual checking of different models for long-term real interest rates

		Model					
		A4	B4	C4	D4	E4	F4
	Median	0.0043	0.0002	−0.0023	0.0063	−0.0068	−0.0067
	Standard deviation	0.2027	0.2077	0.1978	0.1907	0.1879	0.2073
	Skewness	−0.6265	−0.4051	−0.4400	−0.1778	−0.1136	−0.4711
	Kurtosis	3.5814	4.3465	3.8191	3.7723	3.5972	4.3156
(a)	Test for normality						
	Test statistic (JB)	5.73	7.41	4.40	2.20	1.24	7.96
	p-value	0.0570	0.0246	0.1108	0.3329	0.5379	0.0187
(b)	Test for independence						
	Test statistic (Q_{15})	22.9	24.8	26.7	15.0	17.0	25.7
	p-value	0.0286	0.0158	0.0029	0.1321	0.0744	0.0042
(c)	Test for ARCH effects						
	Test statistic (LM)	1.6350	0.4695	0.7419	1.7404	0.0750	0.4533
	p-value	0.2010	0.4932	0.3891	0.1871	0.7842	0.5008
(d)	Test for linearity						
	Test statistic (Q^*_{12})	11.0998	9.8855	11.4322	12.7699	13.0076	10.8099
	p-value	0.5204	0.6260	0.4923	0.3860	0.3685	0.5453

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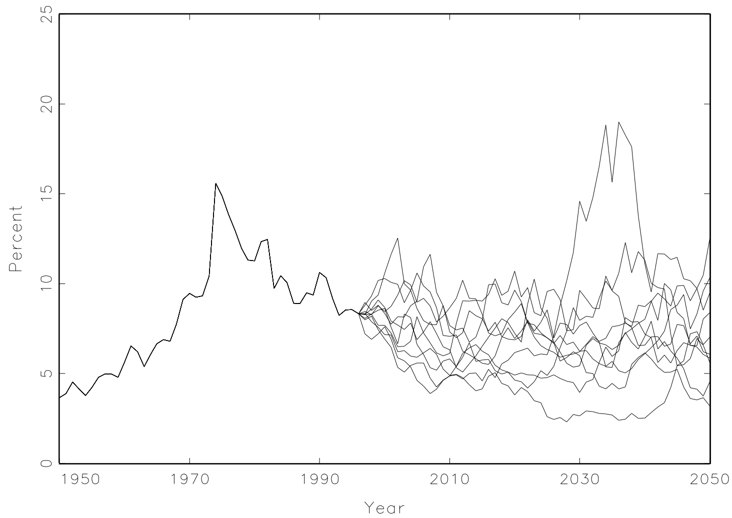

Figure 6.7. Consols yield, 1950-1996, and simulations, 1997-2050, using Model D4

6.5 *Over Parameterisation?*

6.5.1 It should be noted that each model contains a different number of parameters. Therefore, it is not appropriate to focus on only the fit of the models. Akaike (1974) proposed an information criterion to compare alternative models fitted to a data set with different numbers of parameters. The criterion has been called AIC (Akaike Information Criterion) in the literature, and is defined as:

$$AIC = [-2\ln(\text{maximised likelihood}) + 2r]/n$$

where r denotes the number of parameters in the model and n is the sample size.

6.5.2 A simplified version of AIC is considered in this paper (see Reinsel, 1997, p102):

$$AIC_i^* \approx \ln(|\hat{\boldsymbol{\Sigma}}_i|) + (2r_i)/n$$

where $\hat{\boldsymbol{\Sigma}}_i$ is the fitted residual variance-covariance matrix for Model i, and r_i is the number of parameters in Model i. The criterion considers both the model fitting ($\ln(|\hat{\boldsymbol{\Sigma}}|)$) and the model parsimony (r). Under this criterion, one should choose the model with the smallest AIC^*.

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Table 6.5. AIC^* for different fitted models

Model	$\ln(\lvert\hat{\Sigma}\rvert)$	Number of parameters in the model (r)	AIC^*
		(a) The original series	
A	−18.7812	13	−18.4298
E	−18.9140	15	−18.5086
F	−17.9453	15	−17.5399
		(b) The outlier-adjusted series	
B	−20.2569	13	−19.9055
C	−20.4023	12	−20.0780
D	−20.8196	15	−20.4142

6.5.3 Model A and Models E to F are based on the original series, and Models B to D are based on the outlier-adjusted data. We can compare these two groups of models separately using AIC^* values. The results are summarised in Table 6.5. The AIC model selection criterion chooses the proposed VARMA(1,1) model (i.e., Model D) for the outlier-adjusted series and Model E for the raw series.

6.5.4 It can be seen that Model E gives a marginal improvement over Models A and F for explaining the non-adjusted data, while Model D gives a slightly more significant improvement over Models B and C for the outlier-adjusted data. Hence the extra parameters seem justifiable.

6.6 *Summary*

6.6.1 We have examined the Wilkie (1995) composite model for the variables $I(t)$, $\ln Y(t)$, $K(t)$ and $\ln CR(t)$.

6.6.2 The estimated parameters for the original model were computed using observations from 1923 to 1996. The resulting model is denoted by Model A (with component Models A1-A4). The model was also fitted using outlier-adjusted series, and it is represented by Model B (with component Models B1-B4). A pure VAR(1) model was also considered, and it is denoted by Model C (with component Models C1-C4). These three models were compared with the proposed VARMA(1,1) model based on the outlier-adjusted data (denoted by Model D), Model E (the fitted VARMA model using the original series) and Model F (structure of Model D, but standard deviations are estimated from the original data).

6.6.3 Residual checking of these models revealed that there was some violation of residual assumptions for Model A, Model B, Model C, Model E and Model F, as shown in Tables 6.1-6.4. Furthermore, we calculated the residual cross-correlations for Model A, Model B, Model E and Model F. The resulting indicator matrices are summarised in Table 6.6. There are still

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Table 6.6. Indicator matrices for the residual SCCM

lag (l)

1 2 3 4 5

(a) Model A

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ - & \cdot & \cdot & \cdot \end{pmatrix}$$

(b) Model B

$$\begin{pmatrix} \cdot & \cdot & \cdot & + \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & - \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

(c) Model E

$$\begin{pmatrix} \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & - & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & - \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

(d) Model F

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & + \\ \cdot & - & + & \cdot \\ \cdot & - & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & - & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

some significant cross-correlations that cannot be explained by the models, and they are left in the residuals. Model C also suffers a similar problem (see Table 4.2).

6.6.4 Model D passed all the residual checking tests. Its residual cross-correlation matrices are also 'clean' (see part (a) of Table 4.4). It suggests that Model D is an improved representation of the outlier-adjusted data.

6.6.5 For the original series (i.e., without adjusting the outliers), no model passed all the residual checking tests. However, according to the *AIC* analysis performed in Table 6.5, Model E is an improved representation (over Models A and F) for the non-adjusted data.

6.6.6 Finally, Figure 6.8 shows histograms of 10,000 simulated values of $I(t)$, $\ln Y(t)$, $K(t)$ and $\ln CR(t)$ from the two proposed models (Model D and Model E), as well as from the benchmark Wilkie (1995) model (Model A). We have arbitrarily chosen $t = 10$ for illustration. These histograms shed a light on how much the model fitting has changed the probability distribution of the underlying variables.

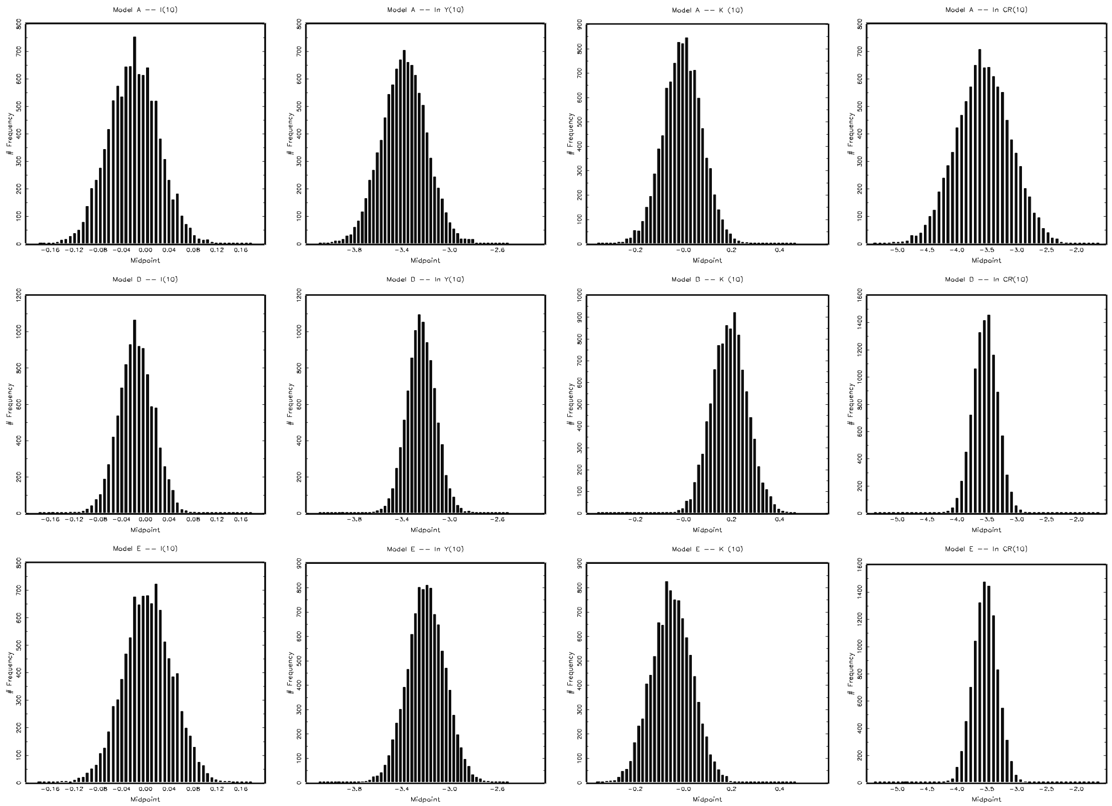

Figure 6.8. Histograms of $I(10)$, ln $Y(10)$, $K(10)$ and ln $CR(10)$ from Model A (first row), Model D (second row) and Model E (third row)

7. Impact of using the Proposed Models

7.1 The Maturity Guarantees Working Party (MGWP) proposed a stochastic investment model (Ford *et al.*, 1980) for simulating equity returns. The model was suggested as the basis for determining valuation reserves for maturity guarantees in the U.K.

7.2 The details of the MGWP methodology are summarised in Appendix B. Some key points to note are:

— the variable $R(t)$ denotes the total annual rate of return in year t of the projection (allowing for both capital gains and dividend income);
— the variable $G(t)$ denotes the annualised geometric rate of return over the t-year period;
— the Working Party assumed an income tax rate of 37.5% on dividends. Tax rates have since changed, and Finkelstein (1997) re-worked their numbers with the simplifying assumption of zero tax. For comparison purposes, I have continued with this gross investor assumption;
— the investment guarantee involved includes a return of premiums upon maturity for a specified portfolio of insurance policies;
— the total guarantee sum assured is, therefore, the total projected premiums maturing; and
— for a policy of term n years and premium £1 p.a., the guarantee claim amount is $GC(n) = \max(n - S(n), 0)$ where $S(n)$ is the accumulated amount of equity assets in which premiums are invested. This can be recognised as the payoff of a call option with a strike of n and where $S(n)$ is the asset underlying the derivative option.

7.3 In addition to the original MGWP model and the Finkelstein-Stable (FS) model (i.e., Model A of Finkelstein (1997)), the Models A-F studied in ¶6.1 to ¶6.4 are also considered. Realisations of $R(t)$ (both for long-term bond investment and equity investment) are generated from each stochastic model with 5,000 replications. It should be noted that the notation for $R(t)$ in this paper (see ¶B.4 of Appendix B) is different from that used by the MGWP.

7.4 We first compare the impact that the alternative models have on various statistics, such as mean annual investment return in the first year of projection E[$R(1)$]; median annual investment return in the first year of projection MED[$R(1)$]; standard deviation of the annual investment return in the first year of projection SD[$R(1)$]; inter-quartile range of the annual investment return in the first year of projection IQR[$R(1)$]; mean long-term (20 years) average investment return E[$G(20)$]; median long-term average investment return MED[$G(20)$]; standard deviation of the long-term average investment return SD[$G(20)$]; and inter-quartile range of the long-term average investment return IQR[$G(20)$]. The results are given in the first two parts of Table 7.1.

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Table 7.1. Simulation results for the MGWP standard liability portfolio with different stochastic investment models (gross investor, i.e. tax basis $\tau = 0.0\%$)

	Model							
	A	B	C	D	E	F	MGWP	FS
(a) Bond investment								
E[R(1)]	0.0990	0.1026	0.0965	0.0904	0.0910	0.0903	NA	NA
SD[R(1)]	0.0754	0.0743	0.0698	0.0671	0.0680	0.0728	NA	NA
E[G(20)]	0.0885	0.0904	0.0909	0.0889	0.0862	0.0887	NA	NA
SD[G(20)]	0.0111	0.0118	0.0232	0.0131	0.0112	0.0130	NA	NA
(b) Equity investment								
E[R(1)]	0.0828	0.1022	0.0984	0.1489	0.1366	0.1445	0.1216	NA
MED[R(1)]	0.0577	0.0842	0.0900	0.1398	0.1200	0.1279	0.0919	0.0733
SD[R(1)]	0.2308	0.1885	0.1473	0.1554	0.1935	0.2114	0.2584	NA
IQR[R(1)]	0.2972	0.2460	0.1972	0.2117	0.2563	0.2811	0.3359	0.2346
E[G(20)]	0.1003	0.1240	0.1092	0.1096	0.1008	0.1105	0.0936	NA
MED[G(20)]	0.0999	0.1237	0.1088	0.1094	0.0996	0.1098	0.0936	0.1880
SD[G(20)]	0.0359	0.0255	0.0219	0.0204	0.0370	0.0432	0.0331	NA
IQR[G(20)]	0.0472	0.0341	0.0290	0.0271	0.0490	0.0572	0.0449	0.0329
(c) Undiscounted reserve for maturity guarantees								
NZ	485	14	23	21	635	671	858	679
V_L(1 : 5000)	33.2	1.1	2.2	0.7	16.7	17.3	16.0	NA
V_L(5 : 5000)	12.6	0.6	0.4	0.2	12.6	15.6	9.1	28
V_L(10 : 5000)	8.5	0.1	0.2	0.1	11.1	12.1	7.9	NA
V_L(25 : 5000)	5.7	0.0	0.0	0.0	7.3	9.4	5.6	15
V_L(50 : 5000)	3.2	0.0	0.0	0.0	4.0	5.7	4.3	9
V_L(100 : 5000)	1.6	0.0	0.0	0.0	2.2	3.0	2.5	NA

7.5 It should be noted that the population means and standard deviations for the investment returns will be infinite under a stable probability framework (Finkelstein, 1997, ¶4.2.3). Furthermore, it is shown that the corresponding sample moments do not fluctuate within any band. The results indicate that these two sample moments are not useful measures of risk and reward under the stable distribution assumption. Therefore, only sample medians and inter-quartile ranges for the FS model are reported in part (b) of Table 7.1. Figures in this table are comparable with those in columns M and A in Table 5.4(a) of Appendix 5 of Finkelstein (1997).

7.6 For the first two parts of Table 7.1, the following observations are made:

— Model A appears to have the most volatile stochastic fluctuations in short-term annual investment performance for bonds.

— Models D, E and F are very similar, with Model F being the most volatile. These three models have the same VARMA structure. The

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parameters of Model D are obtained from the outlier-adjusted data, while the parameters of Model E are derived from the original series. Model F retains exactly the same coefficient parameters as in Model D, but estimates the residual standard deviations from the original data. The differences shown in Table 7.1 reflect the effects of these model parameter variations.

— For equities, E[G(20)] under Model E is slightly greater than (or almost the same as) E[(G20)] under Model A, but E[R(1)] under Model E is much greater than E[R(1)] under Model A. This observation indicates that Model A (Wilkie's model) and Model E (VARMA model) could produce very different average short-term projection statistics. However, it should be noted that both Model A and Model E are linear stationary Gaussian time-series processes. Long-range projections from these models tend to only fluctuate around the historical means. Therefore, the long-term average geometric investment statistics, say E[G(20)], from these two models could be at similar level.

7.7 Next, we illustrate the impact of using alternative stochastic asset models on the simulation results for undiscounted maturity guarantee reserves. Following the MGWP report, reserve calculations are based on a return of premium guarantee when the underlying assets are 100% equities (i.e. no other asset classes are involved).

7.8 Let NZ denote the number of simulations out of 5,000 in which guarantee claims occur (i.e., $L > 0$, see ¶B.8 in Appendix B). Let $V_L(k/5000)$ be the kth largest guarantee claim out of the 5,000 realisations. The quantile $V_L(k/5000)$ represents an estimate for the contingency reserve (undiscounted) needed, as a percentage of total sum assured, to ensure that the probability of ruin is limited to $p = (k/5000)$. The simulation results are summarised in part (c) of Table 7.1. For comparison purpose, relevant figures in column A of Table 5.4(a) of Finkelstein (1997) are also listed in part (c) of Table 7.1.

7.9 It should be noted that Models B, C and D were derived from the outlier-adjusted data. As discussed in ¶5.2.1, these models should not be used for applications for which extreme stochastic fluctuations are important, such as computation of maturity guarantee reserves. They would generate unreasonably small values of V_L. The extent to which these models could underestimate the required reserves is indicated in part (c) of Table 7.1.

7.10 Similar reserve results were obtained by Model E, Model F and Model MGWP. On the other hand, the original Wilkie model (Model A) generated more extreme values of L. Model FS, assuming a stable non-Gaussian distribution, allows even more extreme values of L as compared to Model A. Further comparison of Model FS with other stochastic investment models can be found in Finkelstein (1997).

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8. Problem Areas and Further Work

8.1 A proper understanding of a model's characteristics and limitations is needed in order to decide whether or not it is appropriate to apply the model in a specific circumstance.

8.2 The VARMA(1,1) model (as described in ¶4.10) was proposed for applications not involving extreme stochastic fluctuations. The model was derived from the outlier-adjusted data. On the other hand, if outliers in the data are likely to be repeated in the future, then this feature needs to be incorporated in the model, especially when the application involves very remote ruin probabilities. An alternative model was proposed in ¶5.2.2 for those applications for which extreme fluctuations are important.

8.3 Huber (1995, 1997) showed some concerns on the method of construction of indices and sources of data used by Wilkie (1986, 1995). Since the proposed models in this paper were based on Wilkie's data set, these concerns are also applicable to our results.

8.4 There are only 74 annual observations over the interval 1923-96. It should be noted that the number of parameters in a vector ARMA model increases in a quadratic rate as the dimension of the model increases. Therefore, it is almost impossible, without relying on theory, to extend the proposed models to include the additional five variables (wage inflation, short-term interest rate, property yield, property income and index-linked yield) considered in Wilkie (1995).

8.5 Unlike univariate time-series analysis, the parameter constancy, stationarity and identifiability of the proposed vector models are difficult to examine empirically.

8.6 The proposed VARMA model could easily be extended to include ARCH effects or vector outlier analysis. However, these moves might lead to over parameterisation problems, but in further developments of our work we aim to perform some investigations.

8.7 Multivariate time-series analysis of the term structure of interest rates is another major problem and area for further work. Tiao *et al.* (1993) attempted to study the pattern of Taiwan's interest-rate series for different-term assets using VARMA models with linear transformations. Carrière (2001) considered a Gaussian multivariate factor model of the term structure of interest rates in the U.S.A. In addition to these approaches, other multivariate time-series techniques, such as canonical analyses and co-integration tests (Reinsel, 1997), might be useful in studying the term structure of interest rates. Research in some of these topics is in process.

8.8 It should be noted that the Wilkie model is a mean-reverting process (Kemp, 2000, ¶1.3.3 (c) and Hare *et al.*, 2000, ¶3.3.6), so are the proposed VARMA models in this paper. If an asset price follows a mean-reverting process, then there exists a tendency for the price level to return to its trend path over time. This suggests that future returns are not totally unpredictable, based on historical observations.

8.9 Hare *et al.* (2000, ¶3.3.6 and ¶3.6.4) cautioned the use of the Wilkie model (a kind of autoregression with mean-reverting characteristics) for stochastic projections. In the discussion of the paper, Mr A. C. Smith (Hare *et al.*, 2000, p205) commented:

> "Professor Wilkie made his model autoregressive presumably because he believed that that reflected investment markets, and, therefore, presumably, believed that an investment strategy which switched each year out of the better performing asset class and into the poorer performer would not only be a good strategy in terms of his model, but also in reality."

Hull (2000, p567) has also put forward strong arguments in favour of mean-reverting models, which stated (in the context of modelling interest rates):

> "There are compelling economic arguments in favour of mean reversion. When rates are high, the economy tends to slow down and borrowers require less funds. As a result, rates decline. When rates are low, there tends to be a high demand for funds on the part of borrowers and rates tend to rise."

Recent empirical evidence has lent strong support to the hypothesis of mean-reversion in asset returns (see, for example, Fama & French (1988); Lee (1995); Jorion & Sweeney (1996); Malliaropulos & Priestley (1999) and Nam *et al.* (2001)). Balvers *et al.* (2000) found significant evidence of mean-reversion in annual equity indices for a sample of 18 developed countries (including the U.K.). Fama & French (1988) and Lee (1995) reported that U.S. share prices are mean-reverting with temporary shock components. The proposed models discussed in this paper are also mean-reverting with temporary outlier (shock) components. Statistical testing of the hypothesis of mean-reversion in the Wilkie data set would be an interesting topic for further research.

8.10 Kemp (2000, ¶1.3.3(c)) discussed the applicability of the Wilkie model (a mean-reverting process) to derivatives. The obvious effect of mean-reversion on option values is through its impact on volatility. Here, we cite Professor Fisher Black's famous comment on mean-reversion in share prices and option pricing:

> "If you have a good estimate of a stock's volatility, the stock's expected return won't affect option values. Since the expected return won't affect values, neither will mean reversion." (Black, 1990, p315)

8.11 One relatively easy way to obtain market consistent and arbitrage free valuations of options and guarantees would be to make the models risk neutral. It is known as the principle of risk neutral valuation. The principle of risk neutral valuations says that it is valid to assume that the world of

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investors is risk neutral when valuing options and guarantees, and that the resulting values are valid in all worlds, and not just in those where investors are risk neutral (Hull, 2000, p205 and p249). In a world where investors are risk neutral, the expected return on all securities is the risk free rate. Therefore, to do a risk neutral valuation, the models in this paper would need to be re-parameterised so that all the assets have the same expected returns, that is the risk free rate. If the expected return on the asset classes is the risk free rate, then the risk free rate can be used to obtain the present values of the option and guarantee payoffs (cash flows). Otherwise, an alternative discount rate (so called deflator) will need to be found. This is a non-trivial task (see, e.g., Duffie, 1996, p103), and is put down as an area for further research. Interpolation of the models to continuous time situations would be another topic for further research. Interested readers may refer to Spahr & Schwebach (1998) and Kemp (1997, 2000).

9. Summary and Conclusion

9.1 We have demonstrated that the multiple time-series modelling approach has advantages over transfer function modelling, vector autoregression and non-linear time-series analysis for studying U.K. investment series. The method has the advantage of being direct and transparent. The sequential and iterative steps of tentative specification, estimation and diagnostic checking parallel those of the orthodox Box-Jenkins approach for univariate time-series analysis.

9.2 There are some aberrant observations in the original U.K. investment series. An approach for dealing with aberrant observations is to perform a time-series outlier analysis (see Table 2.1).

9.3 For outlier-adjusted U.K. investment data, a VARMA(1,1) model was proposed. It is denoted by Model D in this paper. The model passed all the residual checking tests (see Tables 6.1-6.4), and it was also suggested by the AIC analysis as an improved representation of the outlier-adjusted data (see Table 6.5). The model is recommended for actuarial applications not involving extreme stochastic fluctuations (see ¶5.2.1).

9.4 On the other hand, Model A (Wilkie's model) and Models E-F (both with a VARMA(1,1) structure) were constructed using the raw (i.e., non-adjusted) series. None of these models passed all the residual checking tests. These indicate that the models are not able to explain totally the extreme observations in the data. However, some effects of outliers have been incorporated in these models through the inflated residual standard deviations. Among these three models, Model E was selected as an improved representation for the original U.K. investment series by the AIC analysis. This model might be useful for actuarial applications for which extreme stochastic fluctuations are important (see ¶5.2.1). An alternative approach to dealing with extreme observations in the data involves using flat tailed or infinite variance distributions, such as the stable distributions used by Finkelstein (1997).

Acknowledgements

The author is grateful to the editor, Professor David Wilkie and two scrutineers for their helpful comments. The data used in this paper have been kindly provided by Professor David Wilkie and by Watson Wyatt Partners, and has been used with their permission. This research is supported by a CRGC grant from the University of Hong Kong.

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APPENDIX A

DIAGNOSTIC TESTS OF RESIDUALS

A.1 *Test for Normality*

A.1.1 Let b_1 and b_2 be the sample coefficients of skewness and kurtosis, respectively, calculated from n residuals.

A.1.2 Jarque & Bera (1981) proposed a composite test for normality. The test statistic is:

$$J = n\left[\frac{b_1^2}{6} + \frac{(b_2 - 3)^2}{24}\right].$$

Under the null hypothesis of normality, J is distributed as χ_2^2.

A.2 *Test for Serial Independence*

A.2.1 Ljung & Box (1978) suggested a useful portmanteau lack of fit test. If the fitted model is adequate for describing the behaviour of the time series, there should not be any serial correlations left in the residuals. The test uses m residual autocorrelations $(\rho_1, \ldots, \rho_m)$ as a unit to check the joint null hypothesis:

$$H_0 : \rho_1 = \rho_2 = \cdots = \rho_m = 0.$$

A.2.2 The test statistic is:

$$Q_m = \sum_{k=1}^{m}\left[\frac{(n)(n-2)}{n-k}\right]\hat{\rho}_k^2$$

where $\hat{\rho}_k$ is the lag-k sample residual autocorrelation function. Under the null hypothesis, the Q_m statistic approximately follows the χ_{m-s}^2 distribution, where s is the number of parameters estimated in the model. We employed $m = 15$ in this paper.

A.3 *Test for ARCH Effects*

A.3.1 Engle (1982) proposed a Lagrange multiplier (LM) test for ARCH disturbances.

A.3.2 Let $\{\hat{e}_1, \ldots, \hat{e}_n\}$ be the residuals computed from the fitted model. The LM test statistic for the first order ARCH process is written as:

$$LM = n\left[\frac{\mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{W}}{\mathbf{W}'\mathbf{W}}\right]$$

where

$$\mathbf{W} = \left(\hat{e}_1^2, \hat{e}_2^2, \cdots, \hat{e}_n^2\right)' \quad \text{and} \quad \mathbf{Z} = \begin{pmatrix} 1, & 1, \cdots, & 1 \\ \hat{e}_0^2, \hat{e}_1^2, & \cdots, & \hat{e}_{n-1}^2 \end{pmatrix}'.$$

The pre-sample value $\hat{e}_0^2$ has been set to zero.

A.3.3 The *LM* statistic has an approximate χ_1^2 distribution under the white-noise null hypothesis.

A.4 *Test for Linearity*

A.4.1 McLeod & Li (1983) proposed a portmanteau test for linearity of the residuals.

A.4.2 Let $\{\hat{e}_1, \ldots, \hat{e}_n\}$ be the fitted residuals from an ARMA model. Let r_k denote the sample autocorrelation of the *squared* residuals, that is:

$$r_k = \frac{\sum_{t=1}^{n-k} (\hat{e}_t^2 - \tilde{\sigma}^2)(\hat{e}_{t+k}^2 - \tilde{\sigma}^2)}{\sum_{t=1}^{n} (\hat{e}_t^2 - \tilde{\sigma}^2)^2}$$

where:

$$\tilde{\sigma}^2 = \frac{1}{n} \sum_{t=1}^{n} \hat{e}_t^2.$$

A.4.3 Analogous to the Ljung-Box portmanteau statistic (see ¶A.2.2), McLeod & Li (1983) derived a new portmanteau statistic:

$$Q_M^* = \sum_{k=1}^{M} \frac{(n)(n-2)}{n-k} r_k^2$$

to detect possible non-linearity in the residuals. Under the null hypothesis (i.e., the residuals are linear white-noise), the Q_M^* statistic approximately follows the χ_M^2 distribution. We employed $M = 12$ in this paper.

APPENDIX B

DETAILS OF THE MGWP METHODOLOGY

B.1 The MGWP model is defined as follows:

$$P(t) = D(t+1)/Y(t)$$
$$\ln D(t) = \mu_D + \ln D(t-1) + \varepsilon_D(t)$$
$$\ln Y(t) = \ln \mu_Y + \phi(\ln Y(t-1) - \ln \mu_Y) + \varepsilon_Y(t)$$
$$\varepsilon_D \sim \text{i.i.d. N}(0, \sigma_D^2) \quad \text{and} \quad \varepsilon_Y \sim \text{i.i.d. N}(0, \sigma_Y^2).$$

B.2 The recommended model parameters by the MGWP are:

$$\phi = 0.60 \quad \mu_Y = 0.05 \quad \sigma_Y = 0.20 \quad Y(0) = 0.05$$
$$\mu_D = 0.04 \quad \sigma_D = 0.13 \quad P(0) = 1.00.$$

B.3 Let $R(t)$ denote the rate of return in year t of the projection; and let $G(t)$ denote the geometric average return over the t-year period, i.e.:

$$G(t) = \left[\prod_{s=1}^{t}\Big(1 + R(t)\Big)\right]^{1/t} - 1.$$

B.4 It should be noted that:

$$R(t) = \left[\frac{1}{C(t)} + (1-\tau)\right]C(t-1) - 1$$

for long-term bond investment, and

$$R(t) = \frac{P(t) + (1-\tau)D(t) - P(t-1)}{P(t-1)}$$

for equity investment, where the rate of tax is τ. In the MGWP report, the tax basis is fixed at $\tau = 0.375$. On the other hand, Wilkie (1986, 1995) and Finkelstein (1997) considered a gross investor (i.e., $\tau = 0$) situation; but with the recent U.K. tax changes, one is not sure what the right tax rate to use is.

B.5 Let $R(t)$ denote the equity return at time t generated by a stochastic investment model; and let $S(n)$ be the accumulated amount of the equity investment of £1 p.a. from $t = 0$ to $t = n - 1$. It should be noted that:

$$S(k) = \left[1 + R(k)\right]\left[1 + S(k-1)\right]$$

for $k = 1, \ldots, n$ and $S(0) = 0$.

B.6 Consider an insurance policy with premium 1 p.a. invested for a term of n years. At maturity, the policyholder is promised that his benefit will be the accumulated amount of his investments, subject to a guaranteed minimum being a return of his total contributions. The guarantee claim amount is, therefore, $GC(n) = \max\{0, n - S(n)\}$.

B.7 The standard reference liability model that was employed by the MGWP will be used in this paper (Ford, *et al.*, 1980, p188 and Finkelstein, 1997, p420). Let $Prem(t)$ denote the premium payable for policies maturing in year t. The standard liability model specifies the values of $Prem(t)$ for $t = 10, \ldots, 30$. It represents a standard cohort of policies written simultaneously at time $t = 0$. It is assumed that there is no mortality (i.e., all policies reach maturity).

B.8 The total sum assured is given by:

$$TSA = \sum_{t=10}^{30}\left[t \times Prem(t)\right]$$

the total guarantee claim is:

$$TGC = \sum_{t=10}^{30}\left[Prem(t) \times GC(t)\right]$$

and the claim ratio (expressed as a percentage of the total sum assured) is defined as:

$$L = \frac{TGC}{TSA} \times 100\%.$$