Proceedings of the Edinburgh Mathematical Society (2020) **63**, 305–313
doi:10.1017/S0013091519000324

VIRTUALLY SPINNING HYPERBOLIC MANIFOLDS

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(Received 3 April 2019; first published online 5 December 2019)

Abstract We give a new proof of a result of Sullivan [Hyperbolic geometry and homeomorphisms, in *Geometric topology* (ed. J. C. Cantrell), pp. 543–555 (Academic Press, New York, 1979)] establishing that all finite volume hyperbolic n-manifolds have a finite cover admitting a spin structure. In addition, in all dimensions greater than or equal to 5, we give the first examples of finite-volume hyperbolic n-manifolds that do not admit a spin structure.

Keywords: hyperbolic manifold; spin structure

2010 *Mathematics subject classification:* Primary 57N65
Secondary 57D15

1. Introduction

Let M be an orientable Riemannian manifold and let $w_2(M)$ denote its second Stiefel–Whitney class. Then M admits a spin structure or is said to be *spinnable* if $w_2(M) = 0$; we refer the reader to §§ 2 and 3 for more details, definitions and background.

It is well known that all compact orientable surfaces and compact orientable 3-manifolds are spinnable; however, the situation in higher dimensions is a good deal more subtle. For example, a well-known consequence of the Hirzebruch signature theorem is that the signature of any closed orientable hyperbolic 4-manifold is 0, and so, according to Rochlin's theorem (see [**9**]), there is no obstruction to having a spin structure. On the other hand, whether a closed hyperbolic manifold of dimension 4 (or more) is spinnable is much harder to establish.

However, on page 553 of his paper [**19**], Sullivan notes that his previous work with Deligne [**5**] can be used to show that if M^n is a finite-volume hyperbolic n-manifold, then M^n has a finite cover that is stably parallelizable and hence spinnable (see Remark 2.2 below).

The aim of this note is to give a simple proof of Sullivan's virtually spinning result that seems not to have been noticed previously. Furthermore, in the setting of arithmetic hyperbolic manifolds of simplest type, we will provide a sharper version of Sullivan's result in many cases (see § 5 for definitions).

Theorem 1.1. *Assume that $n \geq 4$ and let $M^n = \mathbb{H}^n/\Gamma$ be a well-located arithmetic hyperbolic n-manifold of simplest type from the admissible quadratic form f defined over the totally real field k. Then M^n admits a finite cover of degree $C(k, f)$ that is spinnable. The constant $C(k, f)$ is an effectively computable constant depending on k and f.*

The question of whether all orientable hyperbolic manifolds of finite volume in dimensions greater than or equal to 4 are spinnable seems to be open. To that end, in § 8, we point out that, in each dimension $n \geq 5$, there are infinitely many finite-volume (non-compact) orientable hyperbolic n-manifolds that are not spinnable.

Added in proof. It has recently been shown in [**13**] that, for all $n \geq 4$, there exist closed orientable hyperbolic n-manifolds that are not spinnable.

2. Spin groups and spin structures

We assume that $n \geq 4$ throughout this section. General references for what follows are [**9**, **11**]. We will restrict our discussion to spin structures associated with the tangent bundle of a Riemannian manifold, rather than an arbitrary vector bundle. The spin group in dimension n will be denoted by Spin(n). Since $n \geq 4$, Spin(n) is the universal 2-fold covering group of the special orthogonal group SO(n) (with covering map j), and it determines a short exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1.$$

Now let M be a connected orientable Riemannian manifold of dimension greater than or equal to 4 with tangent bundle TM. Denote by $\mathrm{SO}(TM) \to M$ the SO(n)-principal bundle of oriented orthonormal frames on TM. M *admits a spin structure or is spinnable* if there is a principal Spin(n)-bundle $\mathrm{Spin}(TM) \to M$ together with a 2-fold covering map $\eta : \mathrm{Spin}(TM) \to \mathrm{SO}(TM)$ such that $\eta(pg) = \eta(p)j(g)$ for all $p \in \mathrm{Spin}(TM)$ and $g \in \mathrm{Spin}(n)$.

Not every orientable Riemannian manifold admits a spin structure; the obstruction to this is the second Stiefel–Whitney class. With M as above, we let $w_2(M) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ denote the second Stiefel–Whitney class of TM. We summarize what we need in the following proposition.

Proposition 2.1. *M admits a spin structure if and only if $w_2(M) = 0$.*

Remark 2.2. A smooth orientable manifold M^n is stably parallelizable if its tangent bundle is stably trivial: i.e., $TM \oplus E_1 = E_2$, where E_i are trivial vector bundles for $i = 1, 2$. When this is the case, since Stiefel–Whitney classes are invariants of the stable equivalence class of a vector bundle, it follows that $w_2(M) = 0$, and so this proves Sullivan's result mentioned in § 1.

Remark 2.3. If an oriented 4-manifold X is spinnable, then the intersection form of X is even. The converse holds if $H_1(X, \mathbb{Z})$ has no 2-torsion (see [**9**, Chapter II.4]).

https://doi.org/10.1017/S0013091519000324 Published online by Cambridge University Press

3. Hyperbolic manifolds and spin structures

3.1. Some notation

Let J_n be the diagonal matrix associated to the quadratic form $x_0^2 + x_1^2 + \cdots + x_{n-1}^2 - x_n^2$. We identify hyperbolic space $\mathbb{H}^n$ with $\{x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : J_n(x) = -1, x_n > 0\}$ and, by letting

$$\mathrm{O}(n,1) = \{X \in \mathrm{GL}(n+1, \mathbb{R}) : X^t J_n X = J_n\},$$

we can identify $\mathrm{Isom}(\mathbb{H}^n)$ with the connected component of the identity of $\mathrm{O}(n,1)$, denoted by $\mathrm{O}_0(n,1)$. This is also the subgroup of $\mathrm{O}(n,1)$ preserving the hyperboloid $\{x \in \mathbb{R}^{n+1} : J_n = -1, x_{n+1} > 0\}$. Equivalently, $\mathrm{O}_0(n,1) = \mathrm{PO}(n,1)$ (the central quotient of $\mathrm{O}(n,1)$). With this notation, $\mathrm{Isom}^+(\mathbb{H}^n) = \mathrm{SO}_0(n,1)$, the index 2 subgroup in $\mathrm{O}_0(n,1)$ which is the connected component of the identity of $\mathrm{SO}(n,1)$.

Although we will not make explicit use of this, if $M = \mathbb{H}^n/\Gamma$ is a finite-volume orientable hyperbolic n-manifold, then it is a well-known consequence of Mostow–Prasad rigidity that we can conjugate Γ to be a subgroup of $\mathrm{SO}_0(n,1)$, where the elements all have matrix entries in a real number field k (which we can take to be minimal).

3.2. Spin structures on hyperbolic manifolds

The recent paper [**18**] contains a particularly useful discussion of spin structures on hyperbolic manifolds, and we refer the reader there for a fuller discussion. We continue to assume that $n \geq 4$. As with $\mathrm{SO}(n)$, the group $\mathrm{SO}_0(n,1)$ has a universal 2-fold cover, which, following [**18**], we denote by $\mathrm{Spin}^+(n,1)$ (with covering map ϕ) and, as above, there is an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}^+(n,1) \to \mathrm{SO}_0(n,1) \to 1.$$

Now if $M = \mathbb{H}^n/\Gamma$ is an orientable finite-volume hyperbolic manifold, it is well known that the $\mathrm{SO}(n)$-principal bundle of oriented orthonormal frames on TM can be identified with $\Gamma\backslash \mathrm{SO}_0(n,1)$ (see, for example, [**18**, § 2]). Via the exact sequence given above, we can construct an extension $\overline{\Gamma} < \mathrm{Spin}^+(n,1)$ with $\phi(\overline{\Gamma}) = \Gamma$. Note that $\overline{\Gamma}\backslash \mathrm{Spin}^+(n,1) \cong \Gamma\backslash \mathrm{SO}_0(n,1)$ since

$$\Gamma\backslash \mathrm{SO}_0(n,1) \cong (\{\pm 1\}\backslash\overline{\Gamma})\backslash(\{\pm 1\}\backslash \mathrm{Spin}^+(n,1)) \cong \overline{\Gamma}\backslash \mathrm{Spin}^+(n,1).$$

The following is implicit in the proof of [**18**, Theorem 2.1]. We include a brief sketch.

Lemma 3.1. *We follow the notation established above. Let $M = \mathbb{H}^n/\Gamma$ be an orientable finite-volume hyperbolic manifold with $n \geq 4$.*

(1) *Suppose that there is a subgroup $H < \overline{\Gamma}$ of index 2 so that ϕ maps H isomorphically onto Γ (or, equivalently, $H \cap \{\pm 1\} = 1$). Then M is spinnable.*

(2) *Let $D < \overline{\Gamma}$ be of finite index containing an index 2 subgroup D_0 such that $D_0 \cap \{\pm 1\} = 1$. Let $\Delta = \phi(D) < \Gamma$. Then $\mathbb{H}^n/\Delta$ is a finite cover of M that is spinnable.*

Proof. From above, we have an exact sequence

$$1 \to \{\pm 1\} \to \overline{\Gamma} \to \Gamma \to 1.$$

If H is a subgroup of index 2, as claimed, then $H\backslash\mathrm{Spin}^+(n,1)$ is a principal $\mathrm{Spin}(n)$-bundle that double covers $\overline{\Gamma}\backslash\mathrm{Spin}^+(n,1)$, which, from the discussion above, is $\cong \Gamma\backslash\mathrm{SO}_0(n,1)$. Moreover, as described in the proof of [**18**, Theorem 2.1], the right action of $\mathrm{SO}(n)$ lifts to the right action of $\mathrm{Spin}(n)$: that is, M is spinnable. This proves the first part.

The second part follows from the first part on noting that $\mathbb{H}^n/\Delta$ is a finite cover of M. □

4. A proof of Sullivan's theorem

We prove the following.

Theorem 4.1 (Sullivan). *Let $M^n = \mathbb{H}^n/\Gamma$ be a finite-volume orientable hyperbolic n-manifold. Then M is virtually spinnable.*

The proof of Theorem 4.1 will follow from the next proposition (notation as in § 3.2).

Proposition 4.2. *Let $M^n = \mathbb{H}^n/\Gamma$ be a finite-volume orientable hyperbolic n-manifold and let $\overline{\Gamma} < \mathrm{Spin}^+(n,1)$ with $\phi(\overline{\Gamma}) = \Gamma$. Then $\overline{\Gamma}$ is residually finite.*

Given this, Theorem 4.1 is proved as follows. Residual finiteness implies that there exists a finite quotient $\psi : \overline{\Gamma} \to Q$ so that ψ is injective on $\{\pm 1\}$. Let $D < \overline{\Gamma}$ be the subgroup of finite index given by $\psi^{-1}(\psi((\{\pm 1\})))$. Then D contains a subgroup D_0 of index 2 such that $D_0 \cap \{\pm 1\} = 1$. Let $\Delta = \phi(D)$. Then Δ is a finite index subgroup of Γ and $\mathbb{H}^n/\Delta$ is spinnable by Lemma 3.1. □

The proof of Proposition 4.2 requires some additional material, which is described below.

We follow [**11**, **18**, § 3], which provides a very helpful detailed account of the general framework that we describe below.

Let V be an m-dimensional vector space over $\mathbb{R}$ and let q be a non-degenerate quadratic form on V. The *Clifford algebra* $\mathcal{C}\ell(V,q)$ associated to (V,q) is the associative algebra with 1 obtained from the free tensor algebra on V by adding relations $v \otimes v = -q(v)1$ for each $v \in V$. Note that V embeds naturally into $\mathcal{C}\ell(V,q)$, and $\mathcal{C}\ell(V,q)$ has the structure of a real vector space of dimension 2^m with a basis $\mathcal{B}$ constructed naturally from V (we will not dwell on this). Following [**18**], let $P(V,q)$ denote the multiplicative group of $\mathcal{C}\ell(V,q)$ generated by all $v \in V$ such that $q(v) \neq 0$. Then the *spin group* of (V,q) is the subgroup of $P(V,q)$ defined as

$$\mathrm{Spin}(V,q) = \{v_1 \ldots . v_k : v_i \in V, q(v_i) = \pm 1 \text{ for each } i, \text{ and } k \text{ even}\}.$$

In the case when $q = J_n$ (from § 2), $\mathcal{C}\ell(V,q)$ is denoted by $\mathcal{C}\ell(n,1)$, the group $P(V,q)$ is denoted by $P(n,1)$ and $\mathrm{Spin}(V,q) = \mathrm{Spin}(n,1)$. The group $\mathrm{Spin}^+(n,1)$ is the connected component of the identity in $\mathrm{Spin}(n,1)$.

Now the group $\mathrm{Spin}(n,1)$ acts on the vector space $\mathcal{C}\ell(n,1)$ by left multiplication (on the basis $\mathcal{B}$) thereby determining a *faithful* linear representation of $L : \mathrm{Spin}(n,1) \to \mathrm{GL}(2^{n+1},\mathbb{R})$.

The proof of Proposition 4.2 is now complete since $L(\overline{\Gamma})$ is a finitely generated subgroup of $\mathrm{GL}(2^{n+1},\mathbb{R})$ and hence is residually finite by Malcev's theorem. □

Remark 4.3. Note that what is really important in Proposition 4.2 is that $\overline{\Gamma}$ is finitely generated. However, it is important that the extensions considered above are of arithmetic groups in $\mathrm{SO}_0(n,1)$ since [**15**] constructs arithmetic groups in $\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R})$ that have extensions by $\mathbb{Z}/2\mathbb{Z}$ that are not residually finite.

5. Arithmetic hyperbolic manifolds of simplest type

5.1. Quadratic forms and arithmetic lattices

Let k be a totally real number field of degree d over $\mathbb{Q}$ equipped with a fixed embedding into $\mathbb{R}$, which we refer to as the identity embedding, and denote the ring of integers of k by R_k. Let V be an $(n+1)$-dimensional vector space over k equipped with a non-degenerate quadratic form f, defined over k, which has signature $(n,1)$ at the identity embedding and signature $(n+1,0)$ at the remaining $d-1$ embeddings. Given this, the quadratic form f is equivalent over $\mathbb{R}$ to the quadratic form $x_0^2+x_1^2+\cdots+x_{n-1}^2-x_n^2$, and, for any Galois embedding $\sigma: k\to\mathbb{R}$, the quadratic form f^σ (obtained by applying σ to each entry of f) is equivalent over $\mathbb{R}$ to $x_0^2+x_1^2+\cdots+x_{n-1}^2+x_n^2$. We call such a quadratic form *admissable*.

Let F be the symmetric matrix associated to the quadratic form f and let $\mathrm{O}(f)$ (respectively, $\mathrm{SO}(f)$) denote the linear algebraic groups defined over k described as

$$\mathrm{O}(f)=\{X\in\mathrm{GL}(n+1,\mathbb{C}): X^tFX=F\}\quad\text{and}$$
$$\mathrm{SO}(f)=\{X\in\mathrm{SL}(n+1,\mathbb{C}): X^tFX=F\}.$$

For a subring $L\subset\mathbb{C}$, we denote the L-points of $\mathrm{O}(f)$ (respectively, $\mathrm{SO}(f)$) by $\mathrm{O}(f,L)$ (respectively, $\mathrm{SO}(f,L)$). An *arithmetic subgroup* of $\mathrm{O}(f)$ (respectively, $\mathrm{SO}(f)$) is a subgroup $\Gamma<\mathrm{O}(f)$ commensurable with $\mathrm{O}(f,R_k)$ (respectively, $\mathrm{SO}(f,R_k)$). Note that an arithmetic subgroup of $\mathrm{SO}(f)$ is an arithmetic subgroup of $\mathrm{O}(f)$, and an arithmetic subgroup $\Gamma<\mathrm{O}(f)$ determines an arithmetic subgroup $\Gamma\cap\mathrm{SO}(f)$ in $\mathrm{SO}(f)$.

5.2. Constructing arithmetic groups

To pass to arithmetic subgroups of $\mathrm{O}_0(n,1)$ and $\mathrm{SO}_0(n,1)$, we first note, from § 5.1, that, given an admissable quadratic form defined over k of signature $(n,1)$, there exists $T\in\mathrm{GL}(n+1,\mathbb{R})$ such that $T\mathrm{O}(f,\mathbb{R})T^{-1}=\mathrm{O}(n,1)$.

A subgroup $\Gamma<\mathrm{O}_0(n,1)$ is called *arithmetic of simplest type* if Γ is commensurable with the image in $\mathrm{O}_0(n,1)$ of an arithmetic subgroup of $\mathrm{O}(f)$ (under the conjugation map described above). An arithmetic hyperbolic n-manifold $M=\mathbb{H}^n/\Gamma$ is called *arithmetic of simplest type* if Γ is. The same set-up using special orthogonal groups constructs orientation-preserving arithmetic groups of simplest type (and orientable arithmetic hyperbolic n-manifolds of simplest type).

It is known (see [**20**]) that, when n is even, all arithmetic hyperbolic manifolds are of simplest type. Furthermore, when n is even, the algebraic groups $\mathrm{SO}(f)$ are centreless, and it follows from a result of Borel [**2**] that any arithmetic subgroup of $\mathrm{SO}(f,\mathbb{R})$ commensurable with $\mathrm{SO}(f,R_k)$ is contained in $\mathrm{SO}(f,k)$.

https://doi.org/10.1017/S0013091519000324 Published online by Cambridge University Press

In addition, we will say that an arithmetic hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ of simplest type is *well located* if Γ is conjugate to a subgroup of some $\mathrm{SO}(f, R_k)$ for some admissible quadratic form f, as above.

5.3. Arithmetic subgroups of $\mathbf{Spin}^+(n, 1)$

Now we can repeat much of the discussion in § 4 after replacing $\mathbb{R}$ by a totally real field k and taking for q an admissible quadratic from f. In particular, following the construction of § 4, we can construct a Clifford algebra $\mathcal{C}\ell(V, k, q)$ with a basis $\mathcal{B}$ and we define an algebraic group $\mathrm{Spin}(f)$ over k with k-points $\mathrm{Spin}(f, k)$ and an arithmetic subgroup $\mathrm{Spin}(f, R_k)$. Note that $\mathrm{Spin}(f, k) \to \mathrm{SO}(f, k)$ with kernel $\{\pm 1\}$ and, similarly, $\mathrm{Spin}(f, R_k) \to \mathrm{SO}(f, R_k)$.

In addition, on taking the vector space $V_{\mathbb{R}} = V \otimes_k \mathbb{R}$, we can construct arithmetic subgroups of $\mathrm{Spin}(n, 1)$ since, by admissibility of f, we have that $\mathrm{Spin}(f, \mathbb{R})$ is conjugate to $\mathrm{Spin}(n, 1)$, and so $\mathrm{Spin}(f, R_k)$ can be conjugated into $\mathrm{Spin}(n, 1)$.

Again, referring to the construction in § 4, via the basis $\mathcal{B}$ the following proposition can be proved.

Proposition 5.1. *Let f be an admissible quadratic form of signature $(n, 1)$ defined over the totally real field k. The group $\mathrm{Spin}(f, R_k)$ admits a faithful representation into $\mathrm{GL}(2^{n+1}, R_k)$.*

6. Proof of Theorem 1.1

We assume throughout that $n \geq 4$, and f will be an admissible quadratic form of signature $(n, 1)$ defined over the totally real field k. For $I \subset R_k$ an ideal, we denote the norm of I by NI. We denote the principal ideal $4R_k$ by I_4 and define the constant $N_k = \min\{N\mathcal{P}, NI_4\}$, where $\mathcal{P}$ runs over all prime ideals $\mathcal{P}$ with odd residue class degree. Note that if $[k : \mathbb{Q}] = d$, then $NI_4 \leq 4^d$. Let I denote the ideal of norm N_k.

The key proposition is the following (using the notation established).

Proposition 6.1. *Let $M = \mathbb{H}^n/\Gamma$ be an arithmetic hyperbolic manifold of simplest type that is well located. Then M has a finite cover of degree $\leq |\mathrm{GL}(2^{n+1}, R_k/I)|/2$ that is spinnable.*

Proof. We can assume that M is not spinnable; otherwise, the proof is complete. Following the notation above, we have an extension $\overline{\Gamma}$ of Γ by $\{\pm 1\}$. By Proposition 4.2 and the sharper version, Proposition 5.1, we have the following. Let I denote the ideal of norm N_k, as in the statement, and consider the homomorphism $\psi : \overline{\Gamma} \to \mathrm{GL}(2^{n+1}, R_k/I)$ given by restricting the reduction homomorphism $\mathrm{GL}(2^{n+1}, R_k) \to \mathrm{GL}(2^{n+1}, R_k/I)$. By choice of ideal I (i.e., either it is prime of odd residue class degree or the principal ideal generated by 4), we see that $\{\pm 1\}$ injects under ψ. Hence, $K = \ker \psi$ excludes -1 and has index bounded, as claimed.

Let D denote the preimage of $\{\pm 1\}$ under ψ, so that $[D : K] = 2$. Let $\Delta = \phi(D)$, and let N denote the cover of M given by $\mathbb{H}^n/\Delta$. Then N is a cover of degree $\leq |\mathrm{GL}(2^{n+1}, R_k/I)|/2$ and, by construction, is spinnable by Lemma 3.1. □

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Remark 6.2. By definition, any arithmetic subgroup $\Gamma < \mathrm{SO}(f, \mathbb{R})$ contains a well-located subgroup of finite index. As discussed in § 5.2, when n is even, all arithmetic subgroups commensurable with $\mathrm{SO}(f, R_k)$ are contained in $\mathrm{SO}(f, k)$, and when n is odd, the group $\Gamma^{(2)}$ is contained in $\mathrm{SO}(f, k)$ (see the proof of [**6**, Lemma 10]). Now $[\Gamma : \Gamma^{(2)}] = |H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})|$, and getting a sharper version of Sullivan's result (i.e., bounding the index) in the general arithmetic setting reduces to consideration of $\Gamma < \mathrm{SO}(f, k)$ and getting an effectively computable constant bounding $[\Gamma : \Gamma \cap \mathrm{SO}(f, R_k)]$.

7. Some examples

Example 1. As is shown in [**7**], the Davis manifold D [**4**] is a well-located arithmetic hyperbolic 4-manifold. This was proved to be spinnable in [**17**] by showing that the intersection form of D is even (c.f. Remark 2.3).

Example 2. The closed orientable hyperbolic 4-manifold X of Euler characteristic 16 obtained by Conder and Maclachlan in [**3**] is not known to be spinnable. Although the homology groups are computed in [**3**], its intersection form is not known at present.

It is known to be arithmetic and well located (see [**3**, **8**]) with quadratic form defined over the field $\mathbb{Q}(\sqrt{5})$. Since 2 and 3 are inert in $\mathbb{Q}(\sqrt{5})$, the ideal I of Proposition 6.1 is the ideal $< \sqrt{5} >$. Hence Proposition 6.1 provides a spinnable cover of X of degree at most $|\mathrm{GL}(2^5, \mathbb{F}_5)|/2 = \frac{1}{2}(5^{32} - 1)(5^{32} - 5) \cdots (5^{32} - 5^{31})$. Note that $5^{32} = 23283064365386962890625$.

As is evident in this example, the method given by Proposition 6.1 for producing a spinnable cover gives a gigantic bound for the degree of a spinnable cover!

On the other hand, since D (of Example 1) and X are commensurable [**8**], it seems likely that a smaller degree spinnable cover of X can be constructed using the fact that the Davis manifold is spinnable, and this can be seen to pass to finite sheeted covering spaces.

Remark 7.1. It follows from [**1**, Theorem 1] that if a closed hyperbolic 4-manifold X has even intersection form, then there is a finite cyclic cover that is spinnable. This finite cover is a power of 2 controlled by the size of the two-torsion in $H_1(X, \mathbb{Z})$.

8. Non-spinnable hyperbolic manifolds

As advertised in § 1, we now produce non-spinnable examples of finite volume-orientable hyperbolic manifolds in all dimensions greater than or equal to 5.

Theorem 8.1. *For every $n \geq 5$, there exists infinitely many non-spinnable orientable finite-volume hyperbolic n-manifolds.*

Proof. As shown in [**10**] (see also [**16**]), there exist non-spinnable orientable flat 4-manifolds (although only 3 of the 27 orientable flat, 4-manifolds are not spinnable). Let Y be one of these non-spinnable orientable flat 4-manifolds. By [**12**] and the improvement in [**14**], every flat 4-manifold occurs as some cusp cross-section of a possibly (indeed, likely) multi-cusped arithmetic hyperbolic 5-manifold. Hence Y can be arranged as a

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cusp cross-section of an arithmetic hyperbolic 5-manifold X. X cannot be spin, since it is well known that a spin structure induces a spin structure on a boundary component (see [**11**, Chapter II, Proposition 2.15]).

To get higher-dimensional examples, the manifold $Y_n = Y \times S^1 \times S^1 \times \cdots \times S^1$ (with $n-4$ copies of S^1) is a flat n-manifold. Hence we can repeat the above construction to produce a multi-cusped arithmetic hyperbolic $n+1$-manifold X_n for which Y_n is a cusp cross-section. To see that Y_n is not spinnable, note that $Y \subset Y_5$ as a codimension one orientable submanifold of an orientable manifold, and so an application of [**11**, Chapter II, Proposition 2.15] implies that Y_5 is not spinnable since Y is not spinnable. Repeating this argument and proceeding by induction shows that Y_n is not spinnable for all $n \geq 4$. Hence, it follows, as above, that X_n is not spinnable.

To get infinitely many in each dimension, we note that the fundamental groups $\pi_1(Y_n)$ are separable subgroups of $\pi_1(X_n)$ (see, for example, [**12**]), and hence, for each n, we can construct infinitely many finite sheeted covers for which Y_n occurs as a cusp cross-section. □

Acknowledgements. Both authors were partially supported by the NSF. The second author wishes to thank his former colleagues Dan Freed and Tim Perutz for helpful correspondence on material related to this paper. We also thank Bruno Martelli, for valuable discussions on the topics of this paper, and Mike Davis for pointing out Sullivan's result [**19**].

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