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ON MEROMORPHIC SOLUTIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract We consider meromorphic solutions of functional-differential equations

$$f^{(k)}(z) = a(f^n \circ g)(z) + bf(z) + c,$$

where n, k are two positive integers. Firstly, using an elementary method, we describe the forms of f and g when f is rational and $a(\neq 0)$, b, c are constants. In addition, by employing Nevanlinna theory, we show that g must be linear when f is transcendental and $a(\neq 0)$, b, c are polynomials in $\mathbb{C}$.

Keywords: functional-differential equation; meromorphic solution; composition; growth

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Secondary 34K06

1. Introduction and main results

The main purpose of this paper is to characterize meromorphic solutions of functional-differential equation

$$f^{(k)}(z) = a(f^n \circ g)(z) + bf(z) + c, \tag{1.1}$$

where n, k are two positive integers and $a(\neq 0)$, b, c are constants, or more generally, polynomials in $\mathbb{C}$. The equation is a generalization of the following equation

$$f'(z) = a(f \circ g)(z) + bf(z) + c, \tag{1.2}$$

which include the pantograph equations $f'(z) = af(\alpha z) + bf(z)$ and the extensively studied equation

$$f'(z) = a(f \circ g)(z). \tag{1.3}$$

These differential equations with proportional delays are usually referred to as pantograph ones, which have been studied for both real and complex variables by many authors both numerically and analytically owing to numerous practical applications in various fields, such as electrodynamics, astrophysics and cell growth. See, for example [12, 18] and the references cited therein. And the special case $f'(z) = af(z-k)$ is the well-known time delay differential equation (DDE), see monograph [1]. The general form of the time DDE is

$$f'(z) = F(z, f(z), (f \circ g)(z)), \tag{1.4}$$

where $F(z, u, w)$ and $g(z)$ are given analytic functions and $f(z)$ is an unknown function. In [15], Utz proposed the problem of determining conditions for the existence of a real function f (not identically zero) satisfying Equation (1.3). The problem was solved by Siu in [13]. Actually, by applying elementary operator theory in Banach spaces, Siu gave existence and uniqueness results for Equation (1.3) under certain conditions. The problems of local existence and uniqueness for the general Equation (1.4) were studied by Oberg in [10] for local real solutions and in [11] for local complex solutions. Corresponding to the global solutions (entire solutions) of Equation (1.3), it was Gross in [5] who deduced that g must be linear when entire functions f and g in $\mathbb{C}$ satisfy Equation (1.3) with a being a constant. The follow-up work was due to van Brunt–Marshall–Wake in [16], who generalized the above result to Equation (1.2) with the condition $c = 0$. For describing meromorphic solutions of more general equations, Gross and Yang in [6] obtained that the entire function g is a polynomial if f is meromorphic. But, the specific form of g is not given there. In 2007, Li in [8] further studied this kind of equation by characterizing entire functions g to Equation (1.2) when f is a meromorphic function in $\mathbb{C}$ and a, b, c are constants or, more generally, polynomials. More precisely, Li proved the following theorem.

Theorem A. *Suppose that f is a non-constant meromorphic function in $\mathbb{C}$ and g is an entire function satisfying the equation*

$$f'(z) = a(f \circ g)(z) + bf(z) + c,$$

where $a(\neq 0)$, b, c are constants. Then,

(i) *g must be linear, if f is transcendental;*

(ii) *g must be a polynomial of degree less than or equal to 2, if f is rational; furthermore, the degree of g is 2 if and only if $f = \frac{\alpha}{z-w_0} + \beta$, $g = w_0 - a(z-w_0)^2$ and $b = a\beta + c = 0$, where $\alpha(\neq 0)$, β, w_0 are complex numbers.*

By studying (ii) of Theorem A, one wants to know whether the same conclusion holds for the equation $f^{(k)}(z) = a(f^n \circ g)(z) + bf(z) + c$ if f is a rational function, where the left side is the k-th derivative of f and the right side is nonlinear item. Unfortunately, the answer is negative, which is shown by the following example.

Consider the functions $g(z) = z^2$ and $f = \frac{1}{z^2}$. Obviously, f and g satisfy the equation $f''(z) = 6(f \circ g)(z)$. However, the form of f does not satisfy the conclusion of Theorem A. Observe that the only zero of g and the only pole of f are coincident, which leads us to ask whether this always happens or not when g is nonlinear. In this present paper, we firstly pay attention to this question. More specifically, using an elementary method, we completely characterize the rational solutions of the following Equation (1.5) when g is nonlinear.

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Theorem 1. *Suppose that f is a non-constant rational function in $\mathbb{C}$ and g is a non-constant entire function satisfying the equation*

$$f^{(k)}(z) = a(f^n \circ g)(z) + bf(z) + c, \tag{1.5}$$

where $a(\neq 0)$, b, c are constant. Then, g is a polynomial. If g is nonlinear, then

$$g(z) = A(z-a_1)^\mu + a_1, \quad f(z) = A_0 + \sum_{j=1}^{q} \frac{A_j}{(z-a_1)^j},$$

where a_1, A, A_0 and A_j are constants, $\mu \in \{2, \ 3, \ \ldots, \ [\frac{k+1}{n}]\}$, $q = \frac{k}{n\mu-1}$ is an integer.

In particular, if $q = \frac{k}{n\mu-1}$ is not an integer for any $\mu \in \{2, \ldots, \ [\frac{k+1}{n}]\}$ or $n > k+1$, then Equation (1.5) does not admit any non-polynomial rational solution.

Remark 1. For the special case $n = k = 1$, from Theorem 1, it follows that $\mu = \frac{k+1}{n} = 2$ and $q = \frac{k}{n\mu-1} = \frac{1}{2-1} = 1$ if g is nonlinear. Further, $g(z) = A(z-a_1)^2 + a_1$ and $f(z) = \frac{\alpha}{z-a_1} + \beta$ with two constants $\alpha(\neq 0)$, β. This is the conclusion (ii) of Theorem A. Therefore, Theorem 1 is an improvement of Theorem 1 when f is a rational function.

Remark 2. We point out that (1.5) may admit more than one rational solution when g is nonlinear, as seen by the following example. Observe that the functions $f(z) = \frac{1}{z^5}$ and $g(z) = z^2$ satisfy an equation of the form (1.5), namely $f^{(15)}(z) = a(f^2 \circ g)(z)$, where $a = -19!/24$. Moreover, the functions $f(z) = \frac{1}{z^3}$ and $g(z) = Az^3$ also satisfy $f^{(15)}(z) = a(f^2 \circ g)(z)$, where $A^6 = \frac{57}{2}$.

Remark 3. From Theorem 1, we observe that f just has one pole when g is nonlinear. The observation is false if g is linear. We offer an example to show this point. Let $t_0 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then $t_0^3 = -1$. Consider the functions $f(z) = \frac{-t_0}{2(t_0+1)} + \frac{t_0}{z+1} - \frac{1}{z-1}$ and $g(z) = -z$. Then, a calculation yields the following equation

$$f'(z) = -t_0(f^2 \circ g)(z) + \left[-\frac{t_0}{t_0+1} + t_0\right] f(z) - \frac{3}{4(t_0+1)^2}.$$

Remark 4. We emphasize that $a(\neq 0)$, b, c cannot be generalized to polynomials, as seen by the following example. Consider $f(z) = \frac{1}{z}$, $g(z) = z^3$ and $a(z) = -z$. Then, the equation $f'(z) = a(z)(f \circ g)(z)$ holds. But f and g cannot satisfy the conclusion of Theorem 1.

Next, we turn our attention to transcendental solutions of the following functional-differential Equation (1.6) with polynomial coefficients. As a matter of fact, we generalize the conclusion (i) of Theorem A to Equation (1.6) as follows.

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Theorem 2. *Suppose that f is a transcendental meromorphic function in $\mathbb{C}$ and g is a non-constant entire function satisfying the equation*

$$f^{(k)}(z) = a(f^n \circ g)(z) + bf(z) + c, \tag{1.6}$$

where $a(\neq 0)$, b, c are polynomials in $\mathbb{C}$. Then g must be linear. Further, Equation (1.6) does not admit any transcendental meromorphic solution if $n \geq 2$ and $\frac{k}{n-1}$ is not an integer.

Remark 5. The conditions that $n \geq 2$ and $\frac{k}{n-1}$ is not integer are needed in Theorem 2. If one of them is invalid, then Equation (1.6) may admit transcendental meromorphic solutions, which is shown by the following examples.

Consider the equation $f^{(k)}(z) = (f^n \circ g)(z)$, where k is a positive integer and $g(z) = z + 2\pi i$. When $n = 1$, the equation admits the solution $f(z) = e^z$. Consider the entire functions $f(z) = A(\sin(az) + \cos(az))$ and $f(z) = A\cosh(az) + B\sin(az))$, which satisfy the equations $f'(z) = a(f \circ g)(z)$ and $f''(z) = a^2(f \circ g)(z)$ respectively, where $g(z) = -z$ and A, B are arbitrary complex constants. These examples can be found in [15]. Consider the equation $f'(z) = -(f^2 \circ g)(z) - f(z)$, where $g(z) = z + 2\pi i$. Clearly, $n = 2$ and $\frac{k}{n-1} = 1$. Moreover, the equation admits the solution $f(z) = \frac{1}{e^z - 1}$.

The same argument in Theorem 2 yields the following corollary. And we omit its proof here.

Corollary 1. *Suppose that f is a transcendental meromorphic function in $\mathbb{C}$ and g is a non-constant entire function satisfying the equation*

$$L(f) = (f^n \circ g)(z), \tag{1.7}$$

where $L(f) = a_k f^{(k)} + \cdots + a_1 f' + a_0$ $(a_k \neq 0)$ is a linear differential polynomial in f with rational coefficients. Then g must be linear. Further, Equation (1.7) does not admit any transcendental meromorphic solution if $n \geq 2$ and $\frac{k}{n-1}$ is not integer.

Proof of Theorem 1. Clearly, g is a polynomial, since f is a rational function. Firstly, suppose that f is a polynomial. Comparing the degrees of both sides of Equation (1.5), one has $n = 1$ and $\deg g = 1$, which means that g is linear. Next, we assume f is a non-polynomial rational function, which means that f has at least one pole. We below characterize the forms of f and g. Set $f(z) = \frac{P_1(z)}{Q(z)}$, where P_1, Q are two co-prime polynomials. By decomposition of rational function, we then can set

$$f(z) = P(z) + R(z) = P(z) + \sum_{i=1}^{s} \sum_{j=1}^{m_i} \frac{A_{ij}}{(z - a_i)^j} \tag{1.8}$$

where $P(z)$ is a polynomial, s and m_i are positive integers and A_{ij} and a_i $(i = 1, \ldots s)$ are constant with $A_{im_i} \neq 0$ for all $i = 1, \ldots, s$. In addition, without loss of generality, we suppose that a_i and a_j are distinct for any $1 \leq i \neq j \leq s$, and $m_1 \leq m_2 \leq m_3 \leq \ldots \leq m_s$. The above form of f implies that f has distinct poles $a_1, a_2, \ldots, a_s$, with orders (or

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multiplicities) $m_1,\ m_2,\ \ldots,\ m_s$ respectively. By differentiating the function $f(z)$ k-times, we obtain

$$f^{(k)}(z) = P^{(k)}(z) + \sum_{i=1}^{s} \sum_{j=k+1}^{m_i+k} \frac{B_{ij}}{(z-a_i)^j}, \tag{1.9}$$

where B_{ij} are constants and $B_{i(m_i+k)} \neq 0$ for $i = 1, \ldots, s$. Combining (1.8) and (1.9) yields

$$\begin{aligned} &\deg[f^{(k)} - bf - c] \\ &\quad = \deg\left[P^{(k)}(z) - bP(z) - c + \sum_{i=1}^{s} \sum_{j=k+1}^{m_i+k} \frac{B_{ij}}{(z-a_i)^j} - b\sum_{i=1}^{s} \sum_{j=1}^{m_i} \frac{A_{ij}}{(z-a_i)^j}\right] \\ &\quad \leq \deg f + ks. \end{aligned}$$

(In fact, if $b \neq 0$, then $\deg[f^{(k)} - bf - c] = \deg f + ks$.) On the other hand, $\deg[(f^n \circ g)] = nm \deg f$, where $m = \deg g$. Plus $s \leq \deg f - \deg P \leq \deg f$, one has that

$$\begin{aligned} nm \deg f &= \deg[(f^n \circ g)] = \deg[f^{(k)} - bf - c] \\ &\leq \deg f + ks \leq \deg f + k \deg f = (1+k) \deg f, \end{aligned} \tag{1.10}$$

which implies that $m \leq \frac{1+k}{n}$. Clearly, if $n > k+1$, the above inequality is invalid and Equation (1.5) does not admit any rational function f. Below, we will prove the theorem under the condition $n \leq k+1$.

From Equation (1.5), we see that all the zeros of $g(z) - a_1$ must be the poles of $f(z)$, which implies that each zero of $g(z) - a_1$ belongs to set $\{a_1,\ a_2,\ \ldots,\ a_s\}$. Without loss of generality, assume that $g(a_t) = a_1$, where t is a fixed integer such that $t \in \{1, \ldots, s\}$. Suppose that a_t is a zero of $g(z) - a_1$ with multiplicity μ. We consider the following cases.

Case 1. $\mu = 1$. Then, we can assume that $g(z) = a_1 + h(z)(z - a_t)$, where $h(z)$ is a polynomial and $h(a_t) \neq 0$. Further, substituting the forms of f and g into Equation (1.5), one has

$$\begin{aligned} &f^{(k)}(z) - bf(z) - c \\ &= P^{(k)}(z) - bP(z) - c + \sum_{i=1}^{s} \sum_{j=k+1}^{m_i+k} \frac{B_{ij}}{(z-a_i)^j} - b\sum_{i=1}^{s} \sum_{j=1}^{m_i} \frac{A_{ij}}{(z-a_i)^j} \\ &= a(f^n \circ g)(z) = a\left[P(g(z)) + \sum_{i=1}^{s} \sum_{j=1}^{m_i} \frac{A_{ij}}{(g(z)-a_i)^j}\right]^n \\ &= a\left[P(g(z)) + \sum_{j=1}^{m_1} \frac{A_{1j}}{(g(z)-a_1)^j} + \sum_{i=2}^{s} \sum_{j=1}^{m_i} \frac{A_{ij}}{(g(z)-a_i)^j}\right]^n \\ &= a\left[P(g(z)) + \sum_{j=1}^{m_1} \frac{A_{1j}}{h^j(z)(z-a_t)^j} + \sum_{i=2}^{s} \sum_{j=1}^{m_i} \frac{A_{ij}}{[a_1 - a_i + h(z)(z-a_t)]^j}\right]^n. \end{aligned} \tag{1.11}$$

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The first line of (1.11) implies that a_t is a pole of $f^{(k)}(z) - bf(z) - c$ with multiplicity $m_t + k$. And the last line of (1.11) yields that a_t is a pole of $f^{(k)}(z) - bf(z) - c = a(f^n \circ g)(z)$ with multiplicity nm_1. Therefore, Equation (1.5) leads to

$$k + m_t = nm_1 \geq k + m_1.$$

The above equality does not hold for $n = 1$. So $n \geq 2$ and $m_1 \geq \frac{k}{n-1}$, which implies that f has poles with multiplicity at least q, where

$$q = \begin{cases} \dfrac{k}{n-1}, & \text{if } \dfrac{k}{n-1} \text{ is an integer} \\ \left[\dfrac{k}{n-1}\right] + 1, & \text{if } \dfrac{k}{n-1} \text{ is not an integer,} \end{cases}$$

where $[x]$ denotes the greatest integer less than or equal to x. Thus, $s \leq \frac{\deg f - \deg P}{q} \leq \frac{\deg f}{q}$. By the first part of (1.10), we then have

$$nm \deg f = \deg f + ks \leq \deg f + k\frac{\deg f}{q} \leq \left(1 + \frac{k}{q}\right) \deg f, \tag{1.12}$$

Clearly, if $\frac{k}{n-1}$ is an integer, then $(1 + \frac{k}{q}) = n$. Together with the above inequality (1.12), one gets that $\deg g = m = 1$. Further, the above inequalities in (1.12) become equations. So, $s = \frac{\deg f}{q}$, which implies that $\deg P = 0$ and P reduces a constant. If $\frac{k}{n-1}$ is not an integer, then $(1 + \frac{k}{q}) < n$. Plus the above inequality (1.12), one gets that $\deg g = m < 1$, a contradiction.

Case 2. $\mu = 2$. We then set

$$g(z) = a_1 + h(z)(z - a_t)^2, \tag{1.13}$$

where $h(z)$ is a polynomial and $h(a_t) \neq 0$ (For the sake of simplicity, we still use this notation h). Similarly as in Case 1, comparing the multiplicities of both sides of equation $f^{(k)}(z) - bf(z) - c = a(f^n \circ g)(z)$ at the pole-point a_t we can get

$$k + m_t = 2nm_1 \geq k + m_1,$$

which implies that $m_1 \geq \frac{k}{2n-1}$. The same argument as in Case 1 yields that f has poles with multiplicity at least p, where

$$p = \begin{cases} \dfrac{k}{2n-1}, & \text{if } \dfrac{k}{2n-1} \text{ is an integer;} \\ \left[\dfrac{k}{2n-1}\right] + 1, & \text{if } \dfrac{k}{2n-1} \text{ is not an integer.} \end{cases}$$

Thus, $s \leq \frac{\deg f - \deg P}{p} \leq \frac{\deg f}{p}$. Again by the first part of (1.10), we have

$$nm \deg f = \deg f + ks \leq \deg f + k\frac{\deg f}{p} \leq \left(1 + \frac{k}{p}\right) \deg f. \tag{1.14}$$

Obviously, if $\frac{k}{2n-1}$ is an integer, then $(1 + \frac{k}{p}) = 2n$. Then, the above inequality (1.14) yields that $\deg g = m \leq 2$. Together with the form of g in (1.13), we have that $\deg g = 2$

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and

$$g(z) = h(z - a_t)^2 + a_1,$$

where h reduces to a constant. Further, $\deg P = 0$ and P reduces to a constant. We claim that $a_t = a_1$. Otherwise, suppose that $a_t \neq a_1$. Below we derive a contradiction. It is pointed out that all the inequalities in (1.14) become equations. Thus, $s = \frac{\deg f}{p}$. Plus the fact f has poles with multiplicity at least $p = \frac{k}{2n-1}$, we derive that any pole of f has multiplicity $p = \frac{k}{2n-1}$. This means $m_1 = m_2 = \cdots = m_s = p$. Observe that all the zeros of $g(z) - a_t$ must be the poles of f. Then, there exists an index $\nu \in \{2, \ldots, t\}$ such that $g(a_\mu) = a_t$. Clearly, a_μ is a simple zero of $g(z) - a_t$. Then, Similarly as above discussion, again comparing the multiplicities of both sides of the equation $f^{(k)}(z) - bf(z) - c = a(f^n \circ g)(z)$ at the pole-point a_μ, we have

$$k + p = k + m_\mu = nm_t = np,$$

which implies that $p = \frac{k}{n-1}$, a contradiction. Therefore, $a_t = a_1$ and $g(z) = h(z - a_1)^2 + a_1$. Below, we will derive that f just has one pole. Without loss of generality, we assume f has another pole a_2 ($\neq a_1$). Then, Equation (1.5) leads to that $g(a_2)$ is also a pole of f. Assume that $g(a_2) = a_\omega$. Obviously, a_2 is a simple zero of $g(z) - a_\omega$ and $\omega \in \{2, \ldots, t\}$. Again comparing the multiplicities of both sides of equation $f^{(k)}(z) - bf(z) - c = a(f^n \circ g)(z)$ at the pole-point a_2, one has

$$k + p = k + m_2 = nm_\omega = np,$$

which implies that $p = \frac{k}{n-1}$, a contradiction. Thus, f has just a pole a_1 with multiplicity $p = \frac{k}{2n-1}$.

If $\frac{k}{2n-1}$ is not an integer, then $(1 + \frac{k}{p}) < 2n$. Plus the above inequality (1.14), one has that $\deg g < 2$, which contradicts the fact $\mu = 2$.

Case 3. $\mu \in \{3, \ldots, [\frac{k+1}{n}]\}$. Then, the same argument as in Case 2 yields that $g(z) = h(z - a_1)^\mu + a_1$, $\frac{k}{n\mu-1}$ is an integer and Q is a constant. At the same time, f has just one pole a_1 with multiplicity $\frac{k}{n\mu-1}$. When $\frac{k}{n\mu-1}$ is not an integer, we can get a contradiction.

Therefore, all the above discussions yield the desired result and the proof is finished. □

To prove Theorem 2, we will employ Nevanlinna theory. For the reader's convenience, we recall some notation and results in Nevanlinna theory. For a meromorphic function f, the Nevanlinna characteristic $T(r, f)$ is defined as

$$T(r, f) = m(r, f) + N(r, f),$$

where

$$m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad \log^+ x = \max\{\log x,\ 0\}$$

and

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

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$n(t, f)$ denotes the number of poles of f (counting multiplicity) in $|z| \le t$. We also need the notation $\overline{N}(r, f)$, which is defined as

$$\overline{N}(r,f) = \int_0^r \frac{\overline{n}(t,f) - \overline{n}(0,f)}{t} dt + \overline{n}(0,f)\log r,$$

$\overline{n}(t, f)$ denotes the number of poles of f (ignoring multiplicity) in $|z| \le t$. Obviously, $\overline{N}(r, f) \le N(r, f)$. Further, for any positive integer k,

$$N(r, f^{(k)}) \le N(r,f) + k\overline{N}(r,f),$$

since $n(t, f^{(k)}) \le n(t, f) + k\overline{n}(t, f)$ for any positive constant t. Further, we recall the following known facts (see, e.g. [8, 14, 17].)

(1) If f (meromorphic) and g (entire) are transcendental, then

$$\limsup_{r\notin E,\ r\to\infty} \frac{T(r, f\circ g)}{T(r,f)} = \infty,$$

where E is a set of finite Lebesgue measure (see [2, Theorem 2] and [6, p. 370]).

(2) The logarithmic derivative lemma $m(r, \frac{f^{(k)}}{f}) = S(r, f)$ for any positive integer k, where $S(r, f)$ denotes any quantity satisfying that $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside a set of r of finite Lebesgue measure.

(3) If f is transcendental, then $lim_{r\to\infty} \frac{T(r,f)}{\log r} = \infty$; And if h is a rational function, then $T(r, h) = O(\log r) = S(r, f)$.

(4) Suppose that $g = a_m z^m + a_{m_1} z^{m-1} + \ldots + a_1 z + a_0$ is a non-constant polynomial, then for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$,

$$T(r, f\circ g) \ge (1-\epsilon_2) T\left(\frac{|a_m|}{2} r^m, f\right),$$

$$T(r,f) \le \frac{1}{m}(1+\epsilon_1) T\left(\frac{|a_m|}{2} r^m, f\right),$$

for large r outside possibly a set of finite Lebesgue measure. The above inequalities can be seen in [6, (19)]) and [8, (2.7)]), respectively.

(5) Combining the above two inequalities yields that

$$T(r,f) \le \frac{1}{m}\left(\frac{1+\epsilon_1}{1-\epsilon_2}\right) T(r, f\circ g) = \frac{1}{m}(1+\varepsilon) T(r, f\circ g),$$

for large r outside possibly a set of finite Lebesgue measure, where $\varepsilon = \frac{\epsilon_1+\epsilon_2}{1-\epsilon_2}$ is a positive constant. The form of ε implies that ε can be chosen for any arbitrary positive number.

(6) $T(r, f) = T(r, \frac{1}{f}) + O(1)$, $T(r, fg) \le T(r, f) + T(r, g)$, $T(r, f + g) \le T(r, f) + T(r, g) + O(1)$. The last two inequalities also hold for $m(r, \cdot)$.

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Proof of Theorem 2. Based on the idea in [8], we will prove the theorem. Assume the functions f and g satisfy Equation (1.6). From Equation (1.6), the facts (2), (3) and (6), one has that

$$\begin{aligned} nT(r, f\circ g) &= T(r, f^n\circ g) \le T(r, a(f^n\circ g)) + T\left(r, \frac{1}{a}\right) = T(r, a(f^n\circ g)) + O(\log r) \\ &= T(r, f^{(k)} - bf - c) + O(\log r) \le T(r, f^{(k)} - bf) + O(\log r) \\ &\le m(r, f^{(k)} - bf) + N(r, f^{(k)}) + O(\log r) \\ &= m\left(r, \frac{f^{(k)} - bf}{f}\right) + m(r, f) + N(r, f^{(k)}) + O(\log r) \\ &\le m\left(r, \frac{f^{(k)}}{f}\right) + m(r, b) + m(r, f) + N(r, f^{(k)}) + O(\log r) \\ &\le m(r, f) + N(r, f) + k\overline{N}(r, f) + o(T(r, f)) \\ &\le T(r, f) + k\overline{N}(r, f) + o(T(r, f)) \\ &\le T(r, f) + kN(r, f) + o(T(r, f)) \\ &\le (1+k)T(r, f) + o(T(r, f)) = (1+k)(1+o(1))T(r, f), \end{aligned} \tag{1.15}$$

outside possibly a set of finite Lebesgue measure. Notice that f is transcendental. We then see that g must be a polynomial, since if g were transcendental, then it is easy to get a contradiction from the fact (1) and (1.15). Suppose that $g = a_m z^m + a_{m_1} z^{m-1} + \cdots + a_1 z + a_0$ with $a_m \neq 0$. Combining the fact (5) and (1.15) yields

$$nT(r, f) \le \frac{1}{m}(1+\varepsilon)nT(r, f\circ g) \le \frac{1+k}{m}(1+\varepsilon)(1+o(1))T(r, f),$$

outside possibly a set of finite Lebesgue measure. This implies that

$$m \le \frac{1+k}{n}(1+\varepsilon)(1+o(1)).$$

If $\frac{1+k}{n} < 2$, then we get that $m = 1$ and g is linear, since ε can be chosen small enough. For the proof of Theorem 2, it suffices to consider the case $\frac{1+k}{n} \ge 2$. Below we consider two cases.

Case 1. For any positive integer q, f just has finitely many poles with multiplicity q.

Then, for any fixed integer q, we can say that f has poles with multiplicity at least q except finitely many points. Further,

$$\overline{N}(r, f) \le \frac{1}{q}N(r, f) + O(\log r) \le \frac{1}{q}T(r, f) + o(T(r, f)).$$

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Substituting the above inequality into (1.15) yields that

$$\begin{aligned} nT(r, f\circ g) &\le T(r,f) + k\overline{N}(r,f) + o(T(r,f)) \\ &\le T(r,f) + \frac{k}{q}N(r,f) + o(T(r,f)) \\ &\le \left(1+\frac{k}{q}\right)T(r,f) + o(T(r,f)) = \left(1+\frac{k}{q}\right)(1+o(1))T(r,f), \end{aligned} \tag{1.16}$$

outside possibly a set of finite Lebesgue measure. Furthermore, by the fact (5) again, one has

$$nT(r,f) \le \frac{1}{m}(1+\varepsilon)nT(r, f\circ g) \le \frac{1+\frac{k}{q}}{m}((1+\varepsilon))(1+o(1))T(r,f),$$

outside possibly a set of finite Lebesgue measure. Let $q\to\infty$, we can derive that $m = n = 1$ and g is linear.

Case 2. There exists a positive integer q such that f has infinitely many poles with multiplicity q.

Suppose that P is an integer such that f has finitely many poles with multiplicity $< P$ and infinitely many poles with multiplicity P. In fact, the number P is the minimum integer which satisfies the assumption of Case 2. We define a set S as follows:

$$S = \{z : z \text{ is a zero of } a,\ b,\ c,\ g' \ or\ z \text{ is a pole of } f \text{ with multiplicity } < P\}.$$

Clearly, S is a finite set, since a, b, c and g' are polynomials. Set $L = \max\{|z| : z\in S\}$.

Assume that $\{z_n\}_{n=1}^{\infty}$ are the poles of f with multiplicity P such that $|z_1| \le |z_2| \le \ldots \le |z_n| \le \ldots$ and $|z_n|\to\infty$ as $n\to\infty$. Note that g is a polynomial. Without loss of generality, we can assume that the modulus of any zero of $g(z)-z_n$ is bigger than L for any $n > N$, where N is an fixed integer. Suppose that $g(a_0) = z_n$ with $n > N$. Then, it follows $a_0 \notin S$, which implies that a_0 is neither the pole of f with multiplicity $< P$ nor the zeros of the coefficients a, b, c and g'. Moreover, a_0 is a simple zero of $g(z)-z_n$. Assume that $g(z)-z_n = h(z)(z-a_0)$, where $h(z)$ is a polynomial with $h(a_0)\ne 0$. Equation (1.6) implies that a_0 is also a pole of f. Suppose that the multiplicity of f at a_0 is s. The fact $a_0\notin S$ yields that $s\ge P$. Then, the Laurent series expansions of $f(z)$ and $f^{(k)}(z)$ at a_0 are as follows

$$f(z) = \frac{\alpha_s}{(z-a_0)^s} + \frac{\alpha_{s-1}}{(z-a_0)^{s-1}} + \cdots, \tag{1.17}$$

$$f^{(k)}(z) = \frac{\beta_s}{(z-a_0)^{s+k}} + \frac{\beta_{s-1}}{(z-a_0)^{s+k-1}} + \cdots, \tag{1.18}$$

where α_i $(i = s,\ \ s-1, \cdots)$ and β_i $(i = s,\ \ s-1,\cdots)$ are finite complex numbers with $\alpha_s\ne 0$ and $\beta_s \ne 0$. In fact, $\beta_s = (-1)^k\alpha_s s(s+1)\ldots(s+k-1)$. Suppose that the

Laurent series expansion of $f(z)$ at z_n is

$$f(z) = \frac{\theta_P}{(z-z_n)^P} + \frac{\theta_{P-1}}{(z-z_n)^{P-1}} + \cdots, \tag{1.19}$$

where θ_i $(i = P,\ P-1, \cdots)$ are finite complex numbers with $\theta_P \neq 0$. Furthermore, by the form of $g(z)$ and (1.19), one has

$$\begin{aligned}(f^n \circ g)(z) &= \left[\frac{\theta_P}{(g(z)-z_n)^P} + \frac{\theta_{P-1}}{(g(z)-z_n)^{P-1}} + \cdots\right]^n \\ &= \left[\frac{\theta_P}{h(z)^P(z-a_0)^P} + \frac{\theta_{P-1}}{h(z)^{P-1}(z-a_0)^{P-1}} + \cdots\right]^n \\ &= \frac{A_1}{(z-a_0)^{nP}} + \frac{A_2}{(z-a_0)^{nP-1}} + \ldots,\end{aligned} \tag{1.20}$$

where $A_1,\ A_2, \ldots$ are constants and $A_1 \neq 0$. In fact, $A_1 = \left[\frac{\theta_P}{h(a_0)}\right]^n$. We rewrite Equation (1.6) as

$$a(f^n \circ g)(z) = f^{(k)}(z) - bf(z) - c. \tag{1.21}$$

Substitute (1.17), (1.18) and (1.20) into (1.21). Then, by comparing the multiplicities of both sides of Equation (1.21) at the pole-point a_0, one has

$$nP = s + k \geq P + k,$$

which is impossible if $n = 1$. When $n \geq 2$, it implies that

$$P \geq t,$$

where

$$t = \begin{cases} \dfrac{k}{n-1}, & \text{if } \dfrac{k}{n-1} \text{ is an integer;} \\ \left[\dfrac{k}{n-1}\right] + 1, & \text{if } \dfrac{k}{n-1} \text{ is not an integer.} \end{cases}$$

The above discussion yields that f has poles with multiplicity at least t except for finitely many points. Then, it follows that

$$\overline{N}(r,f) \leq \frac{1}{t}N(r,f) + O(\log r) \leq \frac{1}{t}N(r,f) + o(T(r,f)).$$

The similar argument as in (1.16) yields

$$\begin{aligned} nT(r, f\circ g) &\leq T(r,f) + k\overline{N}(r,f) + o(T(r,f)) \\ &\leq T(r,f) + \frac{k}{t}N(r,f) + o(T(r,f)) \\ &\leq \left(1 + \frac{k}{t}\right)T(r,f) + o(T(r,f)) = \left(1+\frac{k}{t}\right)(1+o(1))T(r,f), \end{aligned} \tag{1.22}$$

outside possibly a set of finite Lebesgue measure. Furthermore, plus the fact (5) again, one has

$$nT(r, f) \leq \frac{1+\frac{k}{t}}{m}(1+\varepsilon)(1+o(1))T(r, f),$$

outside possibly a set of finite Lebesgue measure. If $\frac{k}{n-1}$ is an integer, then $1+\frac{k}{t}=n$. Together with the above inequality, we get that

$$mn \leq n(1+\varepsilon)(1+o(1)),$$

which implies that $m=1$ and g is linear, since ε can be taken small enough. Moreover, if $\frac{k}{n-1}$ is not an integer, then $t=[\frac{k}{n-1}]+1$ and $1+\frac{k}{t}=\lambda n<n$, where $0<\lambda<1$ is a fixed constant. Again plus the above inequality, one has

$$mn \leq n\lambda(1+\varepsilon)(1+o(1)),$$

By choosing ε small enough such that $\lambda(1+\varepsilon)(1+o(1))<1$, then it follows $m<1$, which is impossible. Thus, when $\frac{k}{n-1}$ is not integer, the equation does not admit any transcendental meromorphic function.

Therefore, the proof is finished. □

2. Growth of meromorphic solutions of FDE

In this section, we turn our attention to the growth of the meromorphic solutions of the general functional-differential equations (FDE). This is motivated by a result of Gross and Yang in [6]. In fact, they proved that

Theorem B. *Let g be a given non-constant entire function and $P(z, y'(z), y''(z), \ldots, y^{(k)}(z))$ be a given polynomial in variables z, $y(z)$, $y'(z)\ldots, y^{(k)}(z)$. If f is a transcendental meromorphic solutions of the equation*

$$P(z, y'(z), y''(z), \ldots, y^{(k)}(z)) = (y\circ g)(z), \tag{2.1}$$

then g must be polynomial. Furthermore, if g is nonlinear, then $T(r, f)=O((\log r)^{\beta})$ as $r\to\infty$ for some constant $\beta>1$.

In Theorem B, we see that the characteristic function $T(r, f)$ is estimated and a lower bound for β is given. So, it is natural to ask whether one can give an upper bound for β. In this section, we consider the problem by deriving an upper bound for β. In order to state the main result, we need the following definitions.

Let f be a non-constant meromorphic function on $\mathbb{C}$, and let $n_0, n_1, \ldots, n_k$ be $k+1$ non-negative integers. We call

$$M[f] = f^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k}$$

a monomial in f of degree $\lambda_M = n_0+n_1+\cdots+n_k$ and weight $\gamma_M = n_0+2n_1+\ldots+(k+1)n_k$, respectively. Now, let $M_1[f]$, $M_2[f]$, $\ldots$, $M_l[f]$ be l monomials in f of degree

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λ_{M_j} and weight γ_{M_j} $j = 1, 2, \ldots, l$ respectively; and let $b_1, b_2, \ldots, b_l$ be l polynomials. We call

$$Q[f] = b_1 M_1[f] + \cdots + b_l M_l[f]$$

is a differential polynomial in f of total degree $\lambda_Q = \max\{\lambda_{M_1}, \lambda_{M_2}, \ldots, \lambda_{M_l}\}$ and total weight $\gamma_Q = \max\{\gamma_{M_1}, \gamma_{M_2}, \ldots, \gamma_{M_l}\}$. More precisely, we get the following.

Theorem 3. *Suppose that f is a transcendental meromorphic function in $\mathbb{C}$ and g is a non-constant entire function satisfying the equation*

$$Q[f](z) = (f^n \circ g)(z), \tag{2.2}$$

where n is a positive integer and $Q[f]$ is a differential polynomial of total degree Λ and total weight Γ, respective. Let $\omega = \max\{\Lambda, \Gamma\}$. Then g must be a polynomial and the following assertions hold.

(i) *If $\omega < 2n$, then g must be linear;*

(ii) *If g is a polynomial with degree $m \geq 2$, then*

$$T(r, f) = O((\log r)^{\beta}),$$

where $\beta = \frac{\log \frac{\omega}{n}(1+\varepsilon)}{\log m}$ and ε is any arbitrary positive number.

Unfortunately, we don't know whether this upper bound is sharp or not.

Proof of Theorem 3. Note that f is transcendental. It is well known that

$$m(r, Q[f]) \leq \Lambda m(r, f) + S(r, f), \;\; N(r, Q[f]) \leq \Gamma N(r, f) + S(r, f),$$

outside possibly a set of finite Lebesgue measure. The fact can be found in [3, Lemma 1] and given by Doeringer. Further, from Equation (2.2), one has that

$$\begin{aligned} nT(r, f \circ g) = T(r, f^n \circ g) &= T(r, Q[f]) \\ &\leq m(r, Q[f]) + N(r, Q[f]) \\ &\leq \Lambda m(r, f) + \Gamma N(r, f) + S(r, f) \\ &\leq \omega T(r, f) + S(r, f), \end{aligned} \tag{2.3}$$

outside possibly a set of finite Lebesgue measure. Then, together with the fact (1), we then see that g must be a polynomial. Still set $g = a_m z^m + a_{m_1} z^{m-1} + \cdots + a_1 z + a_0$ with $a_m \neq 0$. Then, combining the fact (5) and (2.3) yields

$$nT(r, f) \leq \frac{1}{m}(1+\varepsilon) nT(r, f \circ g) \leq \frac{\omega}{m}(1+\varepsilon)(1+o(1))T(r, f),$$

outside possibly a set of finite Lebesgue measure. It implies that

$$m \leq \frac{\omega}{n}(1+\varepsilon).$$

If $\frac{\omega}{n} < 2$, then we see that $m = 1$ and g is linear by choosing ε small enough. Thus, the proof of (i) is finished.

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Now, we prove the conclusion (ii) under the condition $\omega \geq 2n$. With the above discussion and the fact (4), one has

$$\begin{aligned} T\left(\frac{|a_m|}{2}r^m, f\right) &\leq \frac{1}{1-\epsilon_2}T(r, f\circ g) = \frac{1}{1-\epsilon_2}\frac{1}{n}T(r, (f^n\circ g)(z)) \\ &\leq \frac{1}{1-\epsilon_2}\frac{\omega}{n}(1+o(1))T(r,f) \leq \frac{1}{1-\epsilon_2}\frac{\omega}{n}(1+\varepsilon_1)T(r,f) \\ &= \frac{\omega}{n}(1+\varepsilon)T(r,f), \end{aligned} \tag{2.4}$$

outside possibly a set of finite Lebesgue measure, where ε_1 is an arbitrary positive constant and $\varepsilon = \frac{\varepsilon_1+\epsilon_2}{1-\epsilon_2}$. The form of ε implies that ε is also a arbitrary positive number. Next, we need the following result, which is Lemma 1.1.1 in [7].

Lemma 2.1. *Let $g(r), h(r) : (0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside possibly a set of finite Lebesgue measure. Then for any real number $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Set $g(r) = T(\frac{|a_m|}{2}r^m,\ f)$ and $\frac{\omega}{n}(1+\varepsilon)T(r,\ f) = h(r)$. Obviously, $g(r) \leq h(r)$ outside possibly a set of finite Lebesgue measure. By Lemma 2.1, for any $\alpha > 1$, there exists a positive number r_0 such that $g(r) \leq h(\alpha r)$ for all $r > r_0$. That is

$$T(\frac{|a_m|}{2}r^m, f) \leq \frac{\omega}{n}(1+\varepsilon)T(\alpha r, f). \tag{2.5}$$

Set $\alpha r = R$. Then, we rewrite (2.5) as

$$T\left(\frac{|a_m|}{2\alpha^m}R^m, f\right) \leq \frac{\omega}{n}(1+\varepsilon)T(R,f), \text{ for } R \geq \alpha r_0. \tag{2.6}$$

To end the proof, we will employ a result of Goldstein, which can be found in [4, Lemma 3].

Lemma 2.2. *Let $\psi(r)$ be a function of r $(r \geq r_0)$, positive and bounded in every finite interval. Suppose that*

$$\psi(\mu r^m) \leq A\psi(r) + B, \quad r \geq r_0,$$

where $\mu(>0)$, $m(>1)$, $A(>1)$, B are constants. Then

$$\psi(r) = O((\log r)^\alpha), \textit{with}\alpha = \frac{\log A}{\log m}.$$

Applying Lemma 2.2 to the function $T(R,\ f)$, plus the inequality (2.6), we have

$$T(R,f) = O((\log R)^\alpha),$$

where $\alpha = \frac{\log \frac{\omega}{n}(1+\varepsilon)}{\log m}$.
Thus, the proof is finished. □

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To end this section, we give a simple application of Theorem 3 by considering a special functional-differential equation as follows.

It is known that the Weierstrass $\wp$ function is constructed in the most obvious way. Let ω_1 and ω_2 be two complex numbers such that $\mathsf{Im}(\omega_2/\omega_1) \neq 0$. The Weierstrass $\wp$ function is defined by

$$\wp := \frac{1}{z^2} + \sum_{m,\ n} \left\{ \frac{1}{(z+\Omega_{mn})^2} - \frac{1}{\Omega_{mn}^2} \right\},$$

where $\Omega_{mn} = m\omega_1 + n\omega_2$ and $\sum_{m,\ n}$ denotes the sum over all integer m and n excluding $(m,\ n) = (0,\ 0)$. The Weierstrass $\wp$ function is an even meromorphic function with periods ω_1 and ω_2 and satisfies the ODE

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3, \tag{2.7}$$

where $g_2 = 60\sum_{m,\ n}\Omega_{mn}^{-4}$ and $g_3 = 140\sum_{m,\ n}\Omega_{mn}^{-6}$. Naturally, one wants to know whether Weierstrass $\wp$ function satisfies another functional-differential equation as follows, which maybe different from Equation (2.7).

$$(\wp'(z))^2 = a(\wp^3 \circ g)(z) + b\wp(z) + c, \tag{2.8}$$

where $a(\neq 0)$, b, c are three constants. Actually, we below prove that if Equation (2.8) holds, then it reduces to (2.7).

Now, we give the proof. Suppose that Weierstrass $\wp$ function satisfies (2.8). We know that the characteristic function of $\wp$ is $T(r,\ \wp) = \frac{\pi}{A}r^2(1+o(1))$ with a positive constant A. Then, we can get that g is linear, no matter from the conclusions (i) or (ii) of Theorem 3. Assume that $g(z) = \alpha z + \beta$. Substitute the forms of $g(z)$ and $\wp$ into Equation (2.8), we can easily get that

$$\alpha = \pm 1, \quad \beta = m_0\omega_1 + n_0\omega_2,$$

where m_0 and n_0 are two fixed integers. Therefore, $(\wp \circ g)(z) = \wp(\pm z + m_0\omega_1 + n_0\omega_2) = \wp(z)$, since $\wp$ is an even meromorphic function with periods ω_1 and ω_2. Furthermore, we have $a = 4$, $b = g_2$ and $c = g_3$.

Thus, the proof is finished.

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