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FIRST EIGENVALUE CHARACTERISATION OF CLIFFORD HYPERSURFACES AND VERONESE SURFACES

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Abstract

We give a sharp estimate for the first eigenvalue of the Schrödinger operator $L := -\Delta - \sigma$ which is defined on the closed minimal submanifold M^n in the unit sphere $\mathbb{S}^{n+m}$, where σ is the square norm of the second fundamental form.

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1. Introduction

The study of rigidity theorems plays an important role in the theory of minimal submanifolds. There has been extensive research on rigidity theorems for minimal submanifolds in spheres since the pioneering results obtained by Simons [9], Lawson [3] and Chern *et al.* [2]. Let σ denote the square norm of the second fundamental form and let M^n be a compact minimal submanifold in a unit sphere $\mathbb{S}^{n+m}$. From this work, if $0 \leq \sigma \leq n/(2-1/m)$, then either $\sigma = 0$ or $\sigma = n/(2-1/m)$, and M is the Clifford hypersurface or the Veronese surface in $\mathbb{S}^4$. Later, Li [4] and Chen and Xu [1] improved the pinching number $n/(2-1/m)$ to $2n/3$. They showed that if $0 \leq \sigma \leq 2n/3$, then either $\sigma = 0$ or $\sigma = 2n/3$, and M is the Veronese surface in $\mathbb{S}^4$. Recently, Lu generalised this result and proved the following rigidity theorem. Here, λ_2 denotes the second largest eigenvalue of the fundamental matrix (see Definition 2.5).

THEOREM 1.1 (Lu [5]). *Let $0 \leq \sigma + \lambda_2 \leq n$. Then either M is totally geodesic or is one of the Clifford hypersurfaces $M_{r,n-r}$ $(1 \leq r \leq n)$ in $\mathbb{S}^{n+m}$, $m \geq 1$, or a Veronese surface in $\mathbb{S}^{2+m}$, $m \geq 2$.*

REMARK 1.2. Lu suggests that the quantity $\sigma + \lambda_2$ might be the right object for studying pinching theorems.

Using Lu's inequality [5, Lemma 2] (see Lemma 2.4), we investigate the first eigenvalue of the Schrödinger operator $L := -\Delta + q$, where q is a continuous function

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on M. If there is a nonzero $f \in C^\infty(M)$ satisfying $Lf = \mu f$, we call μ an eigenvalue of L. Since Δ is elliptic, so is L and the set of eigenvalues can be written as

$$\mathrm{Spec}(L) = \{\mu_i : \mu_1 < \mu_2 \le \mu_3 \le \cdots\}.$$

We call μ_1 the first eigenvalue of L.

The pinching theorems cited above give a characterisation of Clifford hypersurfaces and Veronese surfaces. The proofs make use of Simons' identity. Similar arguments lead to estimates of the first eigenvalue of the Schrödinger operator, which gives another way of characterising Clifford hypersurfaces and Veronese surfaces. Simons [9] studied the Schrödinger operator $L_I := -\Delta - \sigma$ of minimal hypersurfaces $M^n \to \mathbb{S}^{n+1}$ and proved that its first eigenvalue $\mu_1^I \le -n$ if M is not totally geodesic. Later, Wu [10] and Perdomo [7] independently proved that if $\mu_1^I \ge -n$, then M is either totally geodesic or a Clifford hypersurface. Define $L_{II} := -\Delta - (2 - 1/m)\sigma$ on the minimal submanifold $M^n \to \mathbb{S}^{n+m}$ and $L_{III} := -\Delta - \frac{3}{2}\sigma$ on the minimal submanifold $M^n \to \mathbb{S}^{n+m}, m \ge 2$, and denote by μ_1^{II} and μ_1^{III} their respective first eigenvalues. For L_{II}, Wu [10] proved that $\mu_1^{II} \le -n$ if M is not totally geodesic, and if $\mu_1^{II} \ge -n$, then M is either totally geodesic, or $\mu_1^{II} = -n$ and M is either a Clifford hypersurface or a Veronese surface. Also, for L_{III}, $\mu_1^{III} \le -n$ if M is not totally geodesic, and if $\mu_1^{III} \ge -n$, then M is either totally geodesic, or $\mu_1^{III} = -n$ and M is a Veronese surface. Similar results hold in the Legendrian case. Using a pinching rigidity result in [6], Yin and Qi [11] gave a sharp estimate for the first eigenvalue of the Schrödinger operator defined on a minimal Legendrian submanifold $M^3 \to \mathbb{S}^7$.

Based on the correspondence between pinching theorems and estimates of the first eigenvalue of certain Schrödinger operators, one expects to find the same phenomenon for Lu's rigidity Theorem 1.1. That observation leads to our main theorem. Define the Schrödinger operator $L := -\Delta - \sigma$ and denote the first eigenvalue of L by μ_1.

THEOREM 1.3 (Main Theorem). *Let M^n be a closed minimal submanifold in $\mathbb{S}^{n+m}(1)$. If M is not totally geodesic, then*

$$\mu_1 \le -n + \max_{p\in M} \lambda_2.$$

Moreover, if $\mu_1 \ge -n + \max_{p\in M} \lambda_2$, then either $\mu_1 = 0$ and M is totally geodesic, or $\mu_1 = -n + \max_{p\in M} \lambda_2$ and M is the Clifford hypersurface in $\mathbb{S}^{n+m}(1)$ or the Veronese surface in $\mathbb{S}^{2+m}(1)$.

2. Preliminaries and Lu's inequality

Let M^n be a compact minimal submanifold in a unit sphere $\mathbb{S}^{n+m}$. We shall make use of the following convention on the range of indices:

$$1 \le A, B, C, \ldots \le n+m; \quad 1 \le i, j, k, \ldots \le n; \quad n+1 \le \alpha, \beta, \gamma, \ldots \le n+m.$$

We choose a local field of orthonormal frames $\{e_1, e_2, \ldots, e_{n+m}\}$ in $\mathbb{S}^{n+m}$ such that when restricted to M, $\{e_1, e_2, \ldots, e_n\}$ are tangent to M and $\{e_{n+1}, e_{n+2}, \ldots, e_{n+m}\}$ are normal to M. Also, $\{\omega_1, \ldots, \omega_{n+m}\}$ is the corresponding dual frame. It is well known that

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$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}, \quad h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \quad H = \frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii} e_\alpha,$$

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.1}$$

$$R_{\alpha\beta kl} = \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}), \tag{2.2}$$

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \tag{2.3}$$

where $h, H, R_{ijkl}, R_{\alpha\beta kl}$, are respectively the second fundamental form, the mean curvature vector, the curvature tensor and the normal curvature tensor of M. We define

$$\sigma = |h|^2, \quad A_\alpha = (h^\alpha_{ij})_{n\times n}.$$

Denote by h^α_{ijk} the component of the covariant derivative of h^α_{ij}, defined by

$$h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_l h^\alpha_{il}\omega_{lj} - \sum_l h^\alpha_{lj}\omega_{li} + \sum_\beta h^\beta_{ij}\omega_{\alpha\beta}. \tag{2.4}$$

From the Gauss–Codazzi–Ricci equations (2.1)–(2.3), the well-known Simons identity follows:

$$\sum_{i,j} h^\alpha_{ij}\Delta h^\alpha_{ij} = n\,|A_\alpha|^2 + \sum_\beta \mathrm{Tr}(A_\alpha A_\beta - A_\beta A_\alpha)^2 - \sum_\beta (\mathrm{Tr}\,A_\alpha A_\beta)^2. \tag{2.5}$$

Now, we introduce Lu's inequality [5, Lemma 2] (see Lemma 2.4), which is the main tool in the proof of Theorems 1.1 and 1.3. The proof of Lu's inequality relies on an algebraic inequality [5, Lemma 1]. We use the Lagrange multiplier method to give another proof and find that there are more cases when the equality holds. Consequently, we restate Lu's lemma [5, Lemma 1] as the following lemma.

LEMMA 2.1. *Suppose $\eta_1, \dots, \eta_n$ are real numbers, $\eta_1 + \cdots + \eta_n = 0$ and $\eta_1^2 + \cdots + \eta_n^2 = 1$. Let $r_{ij} \geq 0$ be nonnegative numbers for $i < j$. Then*

$$\sum_{i<j} (\eta_i - \eta_j)^2 r_{ij} \leq \sum_{i<j} r_{ij} + \max(r_{ij}). \tag{2.6}$$

If $\eta_1 \geq \cdots \geq \eta_n$ and r_{ij} are not simultaneously zero, then equality holds in (2.6) only in one of the following cases. Fix an integer k with $k \in \{1, \dots, n-1\}$.

(1) $r_{ij} = 0$ *if* $2 \leq i < j$, $r_{12} = \cdots = r_{1k} = 0, r_{1\,k+1} = \cdots = r_{1n} > 0$,

$$\eta_1 = \frac{\sqrt{n-k}}{\sqrt{n-k+1}}, \quad \eta_2 = \cdots = \eta_k = 0, \quad \eta_{k+1} = \cdots = \eta_n = \frac{-1}{\sqrt{(n-k+1)(n-k)}}.$$

(2) $r_{ij} = 0$ *if* $i < j < n$, $r_{n-1\,n} = \cdots = r_{n-k+1\,n} = 0, r_{n-k\,n} = \cdots = r_{1n} > 0$

$$\eta_n = \frac{-\sqrt{n-k}}{\sqrt{n-k+1}}, \quad \eta_{n-1} = \eta_{n-2} \cdots = \eta_{n-k+1} = 0,$$

$$\eta_{n-k} = \cdots = \eta_1 = \frac{1}{\sqrt{(n-k+1)(n-k)}}.$$

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REMARK 2.2. We prove the lemma in two steps. The first step is the same as Lu's original proof of [5, Lemma 1] which reduces the problem to proving the inequality

$$\sum_{1<j}(\eta_1-\eta_j)^2 r_{1j} \le \sum_{1<j} r_{1j} + \max_{1<j}(r_{1j}).$$

Then, we apply the Lagrange multiplier method to prove this inequality.

PROOF. *First step.* Assume $\eta_1 \ge \cdots \ge \eta_n$. If $\eta_1 - \eta_n \le 1$ or $n = 2$, then (2.6) is trivial. So assume $n > 2$ and $\eta_1 - \eta_n > 1$. Observe that $\eta_i - \eta_j < 1$ for $2 \le i < j \le n-1$. Otherwise,

$$1 \ge \eta_1^2 + \eta_n^2 + \eta_i^2 + \eta_j^2 \ge \tfrac{1}{2}((\eta_1-\eta_n)^2 + (\eta_i-\eta_j)^2) > 1,$$

which is a contradiction.

Using the same reasoning, if $\eta_1 - \eta_{n-1} > 1$, then $\eta_2 - \eta_n \le 1$; and if $\eta_2 - \eta_n > 1$, then $\eta_1 - \eta_{n-1} \le 1$. Replacing $\eta_1, \ldots, \eta_n$ by $-\eta_n, \ldots, -\eta_1$ if necessary, we can always assume that $\eta_2 - \eta_n \le 1$. Thus, $\eta_i - \eta_j \le 1$ if $2 \le i < j$, and (2.6) is implied by the inequality

$$\sum_{1<j}(\eta_1-\eta_j)^2 r_{1j} \le \sum_{1<j} r_{1j} + \max_{1<j}(r_{1j}). \tag{2.7}$$

Before proving (2.7), we observe that if equality holds in (2.6), we must have $\eta_1 - \eta_n > 1$. Otherwise,

$$\sum_{i<j}(\eta_i-\eta_j)^2 r_{ij} \le \sum_{i<j} r_{ij} < \sum_{i<j} r_{ij} + \max(r_{ij}),$$

which is a contradiction.

Notice that when $\eta_1 - \eta_n > 1$, by the discussion above, $\eta_i - \eta_j < 1$ for $2 \le i < j < n$. So, $r_{ij} = 0$ for $2 \le i < j < n$. Otherwise, by (2.7), equality cannot hold in (2.6). Thus, when discussing equality in (2.6), we only need to analyse the inequality (2.7).

Second step. Let $s_j = r_{1j}$, where $j = 2, \ldots, n$. We write (2.7) as

$$\sum_{1<j}(\eta_1-\eta_j)^2 s_j \le \sum_{1<j} s_j + \max_{1<j}(s_j).$$

Write

$$f(\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta_n) = \sum_{1<j}(\eta_1-\eta_j)^2 s_j.$$

We apply the Lagrange multiplier method to f with constraints

$$\eta_1 + \cdots + \eta_n = 0, \quad \eta_1^2 + \cdots + \eta_n^2 - 1 = 0.$$

Consider the function

$$\Phi(\eta_1, \eta_2, \ldots, \eta_{n-1}, \eta_n) = \sum_{1<j}(\eta_1-\eta_j)^2 s_j + \lambda(\eta_1 + \cdots + \eta_n) + \mu(\eta_1^2 + \cdots + \eta_n^2 - 1),$$

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where λ and μ are the Lagrange multipliers. Setting the partial derivatives with respect to each variable to zero gives the equations

$$\begin{aligned}\frac{\partial \Phi}{\partial \eta_1} &= \sum_{1<j} 2(\eta_1 - \eta_j)s_j + \lambda + 2\mu\,\eta_1 = 0,\\ \frac{\partial \Phi}{\partial \eta_j} &= -2(\eta_1 - \eta_j)s_j + \lambda + 2\mu\,\eta_j = 0 \quad \text{for } j = 2, \ldots, n-1, n.\end{aligned} \tag{2.8}$$

Now

$$\sum_{i=1}^{n} \frac{\partial \Phi}{\partial \eta_i} = n\lambda = 0, \qquad \sum_{i=1}^{n} \eta_i \frac{\partial \Phi}{\partial \eta_i} = 2\sum_{1<j} (\eta_1 - \eta_j)^2 s_j + 2\mu = 0,$$

and so

$$\lambda = 0, \qquad \sum_{1<j} (\eta_1 - \eta_j)^2 s_j = -\mu.$$

Hence, the critical values of f are given by $-\mu$.

Assume $-\mu \neq 0$. We can also assume that $\mu + \max_{1<j}(s_j) < 0$. Otherwise,

$$-\mu = \sum_{1<j} (\eta_1 - \eta_j)^2 s_j \leq \max_{1<j}(s_j) < \sum_{1<j} s_j + \max_{1<j}(s_j).$$

Then by (2.8),

$$\eta_j = \frac{\eta_1\, s_j}{\mu + s_j}, \quad j = 2, \ldots, n-1, n. \tag{2.9}$$

Substituting (2.9) into $\eta_1 + \cdots + \eta_n = 0$ gives

$$1 + \sum_{1<j} \frac{s_j}{\mu + s_j} = 0.$$

Hence,

$$0 = 1 + \sum_{1<j} \frac{s_j}{\mu + s_j} \geq 1 + \sum_{1<j} \frac{s_j}{\mu + \max_{1<i}(s_i)}. \tag{2.10}$$

Multiplying both sides of (2.10) by $\mu + \max_{1<i}(s_i)$ gives

$$-\mu = \sum_{1<j} (\eta_1 - \eta_j)^2 s_j \leq \sum_{1<j} s_j + \max_{1<i}(s_i). \tag{2.11}$$

Notice that if $s_j > 0$ for any j, then

$$0 > \frac{s_j}{\mu + s_j} \geq \frac{s_j}{\mu + \max_{1<i}(s_i)}$$

and that equality in (2.11) is equivalent to equality in (2.10). If equality holds in (2.10), then for each $j > 1$,

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$$\frac{s_j}{\mu + s_j} = \frac{s_j}{\mu + \max_{1<i}(s_i)},$$

which means that either $s_j = 0$ or the nonzero $s_j = \max_{1<i}(s_i)$ and so all nonzero s_j are equal.

Thus, from (2.9) and the assumption above, there are $n-1$ cases, one for each k with $k \in \{1, \ldots, n-1\}$, namely,

$$\eta_1 = \frac{\sqrt{n-k}}{\sqrt{n-k+1}}, \quad \eta_2 = \cdots = \eta_k = 0, \quad \eta_{k+1} = \cdots = \eta_n = -\frac{1}{\sqrt{(n-k+1)(n-k)}}.$$

Case (2) in the statement of the lemma is just a permutation of Case (1) under a different assumption at the beginning. This completes the proof. □

REMARK 2.3. The values $k = n-1$ and $k = 1$ in Lemma 2.1(1) correspond to [5, Cases (1) and (2) in Lemma 1], respectively.

The new version of [5, Lemma 1] changes [5, Lemma 2], but Lu's rigidity theorem still holds, as we discuss later.

Define the inner product of two $n \times n$ matrices A, B by $\langle A, B\rangle = \operatorname{Tr} AB^{\top}$ and let $\|A\|^2 = \langle A, A\rangle = \sum_{i,j} a_{ij}^2$, where (a_{ij}) are the entries of A. The next lemma gives the revised version of Lu's inequality [5, Lemma 2].

LEMMA 2.4. *Let A_1 be an $n \times n$ diagonal matrix of norm 1. Let $A_2, \ldots, A_m$ be symmetric matrices such that:*

(1) $\langle A_\alpha, A_\beta\rangle = 0$ *if* $\alpha \neq \beta$*;*
(2) $\|A_2\| \geq \cdots \geq \|A_m\|$.

Then,

$$\sum_{\alpha=2}^{m} \|[A_1, A_\alpha]\|^2 \leq \sum_{\alpha=2}^{m} \|A_\alpha\|^2 + \|A_2\|^2. \tag{2.12}$$

Equality holds in (2.12) if and only if, after an orthonormal base change and up to a sign, and for each integer k with $k \in \{1, \ldots, n-1\}$, A_1 is the diagonal matrix

$$A_1 = \operatorname{diag}\Big(\frac{\sqrt{k}}{\sqrt{k+1}}, -\frac{1}{\sqrt{k(k+1)}}, -\frac{1}{\sqrt{k(k+1)}}, \ldots, -\frac{1}{\sqrt{k(k+1)}}, 0, \ldots, 0\Big), \tag{2.13}$$

with k entries $-1/\sqrt{k(k+1)}$ and $n-k-1$ entries 0, and A_i is μ times the matrix whose only nonzero entries are 1 at the $(1, i)$ and $(i, 1)$ places, where $i = 2, \ldots, k+1$ and $A_{k+2} = \cdots = A_m = 0$.

Next, we briefly review the proof of Lu's rigidity theorem to set up the notation and state some formulae for later use.

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Definition 2.5. The fundamental matrix S of M is an $m \times m$ matrix-valued function defined by $S = (a_{\alpha\beta})$, where

$$a_{\alpha\beta} = \langle A_\alpha, A_\beta \rangle.$$

We denote the eigenvalues of the fundamental matrix S by $\lambda_1 \geq \cdots \geq \lambda_m$. In particular, λ_1 is the largest eigenvalue and λ_2 is the second largest eigenvalue of the matrix S, and r is defined by

$$\lambda_1 = \cdots = \lambda_r > \lambda_{r+1} \geq \cdots \geq \lambda_m.$$

Using this notation, the trace of the fundamental matrix is $\sigma = \lambda_1 + \cdots + \lambda_m$. For a positive integer $p \geq 2$, we define

$$f_p := \mathrm{Tr}\,(S^p) = \sum_{\alpha_1,\ldots,\alpha_p} a_{\alpha_1\alpha_2}\, a_{\alpha_2\alpha_3} \cdots a_{\alpha_p\alpha_1}$$

and $g_p := (f_p)^{1/p}$. Using the Simons identity (2.5) and Lemma 2.4, Lu derived the following inequalities.

Proposition 2.6 (Lu [5]). *With the notation as above,*

$$|\nabla f_p|^2 \leq p^2 f_p \sum_{k,\alpha} \lambda_\alpha^{p-2} (\nabla_{\partial/\partial x_k} a_{\alpha\alpha})^2, \tag{2.14}$$

$$\begin{aligned}
\Delta g_p &= \frac{1}{p} f_p^{1/p-1} \Delta f_p + \frac{1}{p}\Big(\frac{1}{p} - 1\Big) f_p^{1/p-2} |\nabla f_p|^2 \\
&\geq 2 f_p^{1/p-1} \sum_\alpha \Big(\lambda_\alpha^{p-1} \sum_{i,j,k} (h_{ijk}^\alpha)^2 \Big) \\
&\quad + 2 f_p^{1/p-1} \Big(r \|A_1\|^{2p} \Big(n - \|A_1\|^2 - \sum_{\alpha=2}^m \|A_\alpha\|^2 - \lambda_2 \Big) - 3mn\lambda_{r+1}^p \Big).
\end{aligned} \tag{2.15}$$

By integrating both sides of (2.15) and letting $p \to \infty$, since $\lambda_{r+1}^p / f_p \to 0$ as p tends to ∞, Lu derived

$$\int_M \sum_{i,j,k} \sum_{\alpha \leq r} (h_{ijk}^\alpha)^2 + \|A_1\|^2 \Big(n - \|A_1\|^2 - \sum_{\alpha=2}^m \|A_\alpha\|^2 - \lambda_2 \Big) \leq 0. \tag{2.16}$$

If equality holds in (2.16), then equality holds in (2.12), so A_α takes the form in Lemma 2.4. Using the structure equation case by case, Lu proved Theorem 1.1.

Remark 2.7. Although we have found more cases when equality holds in (2.12), we can rule out the new cases using similar arguments to those in the original proof. To be precise, if $n > k + 1, j \geq k + 2$, then from $0 = dh_{1j}^{n+1} = h_{11}^{n+1} \omega_{1j}$, we conclude $\omega_{1j} = 0$. Similarly, by computing dh_{ij}^{n+1} for $i = 2, \ldots, k + 1$, we also have $\omega_{2j} = \cdots = \omega_{k+1\,j} = 0$ for $j \geq k + 2$. By the structure equations, $0 = d\omega_{1j} = \omega_1 \wedge \omega_j$, which is a contradiction if $n > k + 1$. Thus, Theorem 1.1 is still correct.

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3. Proof of the main theorem

Let $g_\epsilon = (g_p + \epsilon)^{1/2}$, where $\epsilon > 0$ is a constant. We first prove the inequality in the main theorem.

PROPOSITION 3.1. *If M^n is a closed nontotally geodesic minimal submanifold in $\mathbb{S}^{n+m}(1)$, then*

$$\mu_1 \leq -n + \max_{p\in M} \lambda_2 - \frac{2}{n+2}\frac{\int_M [\frac{1}{r}\sum_{i,j,k}\sum_{\alpha\leq n+r}(h^\alpha_{ijk})^2]}{\int_M \lambda_1}.$$

PROOF. By direct computation, using (2.15),

$$\begin{aligned}
\Delta g_\epsilon &= \frac{1}{2}(g_p+\epsilon)^{-1/2}\Delta g_p - \frac{1}{4}|\nabla g_p|^2 (g_p+\epsilon)^{-3/2}\\
&= \frac{1}{2}(g_p+\epsilon)^{-1/2}\Big(\frac{1}{p}f_p^{1/p-1}\Delta f_p + \frac{1}{p}\Big(\frac{1}{p}-1\Big)f_p^{1/p-2}|\nabla f_p|^2\Big) - \frac{1}{4}|\nabla g_p|^2(g_p+\epsilon)^{-3/2}\\
&\geq \frac{1}{2}(g_p+\epsilon)^{-1/2}\Big(2f_p^{1/p-1}\sum_\alpha\Big(\lambda_\alpha^{p-1}\sum_{i,j,k}(h^\alpha_{ijk})^2\Big)\\
&\quad + 2f_p^{1/p-1}\Big(r\|A_1\|^{2p}\Big(n+1-\|A_1\|^2 - \sum_{\alpha=2}^m\|A_\alpha\|^2 - \lambda_2\Big)\Big) - 6nm\, f_p^{1/p-1}\lambda_{r+1}^p\Big)\\
&\quad - \frac{1}{4}|\nabla g_p|^2(g_p+\epsilon)^{-3/2}\\
&\geq \underbrace{(g_p+\epsilon)^{-3/2}\Big[(g_p+\epsilon)\,f_p^{1/p-1}\sum_\alpha\Big(\lambda_\alpha^{p-1}\sum_{i,j,k}(h^\alpha_{ijk})^2\Big) - \frac{1}{4}|\nabla g_p|^2\Big]}_{I}\\
&\quad + \underbrace{(g_p+\epsilon)^{-1/2}\Big[f_p^{1/p-1}\Big(r\|A_1\|^{2p}\Big(n-\|A_1\|^2-\sum_{\alpha=2}^m\|A_\alpha\|^2-\lambda_2\Big)\Big) - 3nm\, f_p^{1/p-1}\lambda_{r+1}^p\Big]}_{II}.
\end{aligned}$$

To deal with I, we use the next lemma which follows from [8, (1.9) and (1.11) in Proposition 1].

LEMMA 3.2 (Shen [8]). *If M^n is a closed minimal submanifold in $\mathbb{S}^{n+m}(1)$, then*

$$|\nabla(|A_\alpha|^2)|^2 \leq \frac{4n}{n+2}|A_\alpha|^2\Big[\sum_{i,j,k}(h^\alpha_{ijk})^2\Big].$$

Applying Lemma 3.2 to (2.14) yields

$$|\nabla g_p|^2 = \frac{1}{p^2}f_p^{2/p-2}|\nabla f_p|^2 \leq f_p^{2/p-1}\sum_\alpha \lambda_\alpha^{p-2}|\nabla\lambda_\alpha|^2 \leq f_p^{2/p-1}\sum_\alpha\frac{4n}{n+2}\Big(\lambda_\alpha^{p-1}\sum_{i,j,k}(h^\alpha_{ijk})^2\Big).$$

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Thus,

$$
\begin{aligned}
I &\geq (g_p+\epsilon)^{-3/2}\Big[(g_p+\epsilon)\, f_p^{1/p-1}\sum_\alpha\Big(\lambda_\alpha^{p-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\Big)-\frac{1}{4}\frac{1}{p^2}f_p^{2/p-2}|\nabla f_p|^2\Big]\\
&\geq (g_p+\epsilon)^{-3/2}\Big[(g_p+\epsilon) f_p^{1/p-1}\sum_\alpha\Big(\lambda_\alpha^{p-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\Big)\\
&\qquad -\frac{1}{4}f_p^{2/p-1}\frac{4n}{n+2}\sum_\alpha\Big(\lambda_\alpha^{p-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\Big)\Big]\\
&\geq \frac{2}{n+2}(g_p+\epsilon)^{-1/2}f_p^{1/p-1}\Big[\sum_\alpha\Big(\lambda_\alpha^{p-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\Big)\Big]\\
&\geq 0.
\end{aligned}
$$

Inserting the definition of g_ϵ into $\mu_1=\inf_{f\in C^\infty(M)}\int_M L(f)f/\int_M f^2$, yields

$$
\begin{aligned}
\mu_1\int_M g_\epsilon^2 &\leq \int_M L(g_\epsilon)g_\epsilon=\int_M -g_\epsilon\Delta\, g_\epsilon-\sigma\, g_\epsilon^2=\int_M -g_\epsilon(I+II)-\sigma\, g_\epsilon^2\\
&\leq \int_M -\frac{2}{n+2}f_p^{1/p-1}\Big[\sum_\alpha\Big(\lambda_\alpha^{p-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\Big)\Big]\\
&\quad +\int_M -\Big[f_p^{1/p-1}\Big(r\|A_1\|^{2p}\Big(n-\|A_1\|^2-\sum_{\alpha=2}^m\|A_\alpha\|^2-\lambda_2\Big)\Big)-3nmf_p^{1/p-1}\lambda_{r+1}^p\Big]\\
&\quad -\int_M \sigma\, g_\epsilon^2.
\end{aligned}
$$

Then, letting $p\to\infty$ and $\epsilon\to 0$, and using the fact that $\lambda_{r+1}^p/f_p\to 0$ almost everywhere when $p\to\infty$ completes the proof. □

PROOF OF THEOREM 1.3. From the proof of Proposition 3.1, if

$$
\mu_1\geq -n+\max_{p\in M}\lambda_2,
$$

then either M is totally geodesic so $\mu_1=0$ or $\mu_1=-n+\max_{p\in M}\lambda_2$ and

$$
\frac{1}{r}\sum_{i,j,k}\sum_{\alpha\leq n+r}(h_{ijk}^\alpha)^2=0.
$$

We claim that σ is a constant. By Lemma 2.4, there are two cases.

Case 1. $A_1\neq 0$ and $A_2=A_3=\cdots=A_m=0$. By Lemma 3.2, $\sigma=\|A_1\|^2=\lambda_1$ is a constant.

Case 2. There is a positive integer k with $1\leq k\leq n-1$ such that A_1 is λ times the diagonal matrix in (2.13), A_i is $\mu/\sqrt{k(k+1)}$ times the matrix whose only nonzero entries are 1 at the $(1,i)$ and $(i,1)$ places for $2\leq i\leq k+1$, and $A_{k+2}=\cdots=A_m=0$.

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Since $\sum_{i,j,k}(h_{ijk}^{n+1})^2 = 0$, by (2.4), it follows that λ is constant. Also, μ is constant since $\lambda_2 = \max_{p\in M}\lambda_2$. Thus, σ is constant.

Since σ is constant when $\mu_1 = -n + \max_{p\in M}\lambda_2$ and the first eigenvalue of L is $-\sigma$, it follows that $\sigma + \lambda_2 = n$. Then, by Theorem 1.1, M is either one of the Clifford hypersurfaces or the Veronese surface. □

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