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RESEARCH ARTICLE

The complete separation of the two finer asymptotic ℓ_p structures for $1 \leq p < \infty$

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Abstract
For $1 \leq p < \infty$, we present a reflexive Banach space $\mathfrak{X}^{(p)}_{\mathrm{awi}}$, with an unconditional basis, that admits ℓ_p as a unique asymptotic model and does not contain any Asymptotic ℓ_p subspaces. Freeman *et al., Trans. AMS.* **370** (2018), 6933–6953 have shown that whenever a Banach space not containing ℓ_1, in particular a reflexive Banach space, admits c_0 as a unique asymptotic model, then it is Asymptotic c_0. These results provide a complete answer to a problem posed by Halbeisen and Odell [*Isr. J. Math.* **139** (2004), 253–291] and also complete a line of inquiry of the relation between specific asymptotic structures in Banach spaces, initiated in a previous paper by the first and fourth authors. For the definition of $\mathfrak{X}^{(p)}_{\mathrm{awi}}$, we use saturation with asymptotically weakly incomparable constraints, a new method for defining a norm that remains small on a well-founded tree of vectors which penetrates any infinite dimensional closed subspace.

Contents

1. Introduction

The purpose of this article is to provide an answer to the following problem of Halbeisen and Odell from [20] and is, in particular, the last step towards the complete separation of a list of asymptotic structures from [9]. Given a Banach space X, let $\mathscr{F}_0(X)$ denote the family of normalised weakly null sequences in X and $\mathscr{F}_b(X)$ denote the family of normalised block sequences of a fixed basis, if X has one.

Problem 1. Let X be a Banach space that admits a unique asymptotic model with respect to $\mathscr{F}_0(X)$, or with respect to $\mathscr{F}_b(X)$ if X has a basis. Does X contain an Asymptotic ℓ_p, $1 \leq p < \infty$ or an Asymptotic c_0 subspace?

The following definition from [9] provides a more general setting in which we will describe this problem, as well as other previous separation results. A property of a Banach space is called hereditary if it is inherited by all of its closed and infinite dimensional subspaces.

Definition 1.1. Let (P) and (Q) be two hereditary properties of Banach spaces, and assume that (P) implies (Q).

(i) If (Q) $\not\Rightarrow$(P), that is, there exists a Banach space satisfying (Q) and failing (P), then we say that (P) is separated from (Q).
(ii) If there exists a Banach space satisfying (Q) and whose every infinite dimensional closed subspace fails (P), then we say that (P) is completely separated from (Q) and write (Q)$\not\Rrightarrow$ (P).

We consider properties that are classified into the following three categories: the sequential asymptotic properties, the array asymptotic properties and the global asymptotic properties.

Sequential asymptotic properties are related to the notion of a spreading model from [15], which describes the asymptotic behaviour of a sequence in a Banach space. We say that a Banach space admits a unique spreading model with respect to some family of normalised sequences $\mathscr{F}$, if whenever two sequences from $\mathscr{F}$ generate spreading models, then those must be equivalent. If this equivalence happens with some uniform constant, then we say that the space admits a uniformly unique spreading model.

The category of array asymptotic structures concerns the asymptotic behaviour of arrays of sequences $(x^i_j)_j$, $i \in \mathbb{N}$, in a Banach space. Notions that describe this behaviour are those of asymptotic models from [20] and joint spreading models from [8]. We define the uniqueness of asymptotic models and the uniform uniqueness of joint spreading models in a similar manner to the uniqueness and uniform uniqueness of spreading models, respectively. Although asymptotic models and joint spreading models are not identical notions, they are strongly related. As Sari pointed out, a Banach space X admits a uniformly unique joint spreading model with respect to $\mathscr{F}_b(X)$ or $\mathscr{F}_0(X)$ if and only if it admits a unique asymptotic model with respect to $\mathscr{F}_b(X)$ or $\mathscr{F}_0(X)$, respectively (see, e.g. [6, Remark 4.21] or [9, Proposition 3.12]). Notably, the property that a Banach space X with a basis admits some ℓ_p as a uniformly unique joint spreading model with respect to $\mathscr{F}_b(X)$ can be described by the following statement. The case where this happens with respect to $\mathscr{F}_0(X)$ is given by an easy modification.

Proposition 1.2 (Lemma 3.4). *Let $1 \leq p \leq \infty$. A Banach space X with a basis admits ℓ_p (or c_0 for $p = \infty$) as a uniformly unique joint spreading model with respect to $\mathscr{F}_b(X)$ if and only if there exist constants $c, C > 0$, such that for every $\ell \in \mathbb{N}$, any choice of successive families $(F_j)_j$ of normalised blocks in X with $\#F_j = \ell$, there is an infinite subset of the naturals $M = \{m_1 < m_2 < \ldots\}$, such that for*

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any choice of $x_j \in F_j$, $j \in M$, every $G \subset M$ with $m_k \le G$ and $\#G \le k$, for $k \in \mathbb{N}$, and any choice of scalars a_j, $j \in G$, we have

$$c\|(a_j)_{j\in G}\|_p \le \|\sum_{j\in G} a_j x_j\| \le C\|(a_j)_{j\in G}\|_p.$$

Even though this property is very close to the weaker one that X admits ℓ_p or c_0 as a uniformly unique spreading model, it was shown in [9] that these two properties are in fact completely separated for all $1 \le p \le \infty$.

Finally, global asymptotic properties describe the behaviour of finite block sequences that are chosen sufficiently far apart in a space with a basis. We recall the following definition from [25].

Definition 1.3. Let X be a Banach space with a basis $(e_i)_i$ and $1 \le p \le \infty$. We say that the basis $(e_i)_i$ of X is asymptotic ℓ_p (asymptotic c_0 when $p = \infty$) if there exist positive constants D_1 and D_2, such that for all $n \in \mathbb{N}$, there exists $N(n) \in \mathbb{N}$ with the property that whenever $N(n) \le x_1 < \cdots < x_n$ are vectors in X, then

$$\frac{1}{D_1}(\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} \le \|\sum_{i=1}^{n} x_i\| \le D_2(\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}},$$

where for $p = \infty$, the above inequality concerns the $\|\cdot\|_\infty$. Specifically, we say that $(e_i)_i$ is D-asymptotic ℓ_p (D-asymptotic c_0 when $p = \infty$) for $D = D_1 D_2$.

This definition is given with respect to a fixed basis of the space. The coordinate-free notion of Asymptotic ℓ_p and c_0 spaces was introduced in [24], generalising the aforementioned one to spaces with or without a basis (note the difference between the terms asymptotic ℓ_p and Asymptotic ℓ_p). Moreover, this property is hereditary and any Asymptotic ℓ_p (or c_0) space is asymptotic ℓ_p (respectively, c_0) saturated. Given a Banach space X with a basis, we focus on the following properties, where $1 \le p \le \infty$ and whenever $p = \infty$, then ℓ_p should be replaced with c_0.

(a) $_p$ The space X is Asymptotic ℓ_p.
(b) $_p$ The space X admits ℓ_p as a uniformly unique joint spreading model (or equivalently, a unique asymptotic model, as mentioned above) with respect to $\mathscr{F}_b(X)$.
(c) $_p$ The space X admits ℓ_p as a uniformly unique spreading model with respect to $\mathscr{F}_b(X)$.
(d) $_p$ The space X admits ℓ_p as a unique spreading model with respect to $\mathscr{F}_b(X)$.

Note that it is fairly straightforward to see that the following implications hold for all $1 \le p \le \infty$: (a) $_p \Rightarrow$(b) $_p \Rightarrow$(c) $_p \Rightarrow$(d) $_p$. It is also easy to see that (d) $_p \not\Rightarrow$(c) $_p$ for all $1 \le p < \infty$. In [14] it was shown that (c) $_p \not\Rightarrow$(b) $_p$ for all $1 \le p \le \infty$ and that (b) $_p \not\Rightarrow$(a) $_p$ for all $1 < p < \infty$. The latter was also shown in [8], as well as that (b) $_1 \not\Rightarrow$(a) $_1$ along with an even stronger result, namely, the existence of a Banach space with a basis satisfying (b) $_1$ and, such that any infinite subsequence of its basis generates a non-Asymptotic ℓ_1 subspace. However, it was proved in [12] that (d) $_\infty \Leftrightarrow$(c) $_\infty$ and a remarkable result from [18] states that (b) $_\infty \Leftrightarrow$(a) $_\infty$ for Banach spaces not containing ℓ_1. Towards the complete separation of these properties, it was shown in [9] that (c) $_p \not\Leftarrow$ (b) $_p$ for all $1 \le p \le \infty$ and that (d) $_p \not\Leftarrow$ (c) $_p$ for all $1 \le p < \infty$. Hence, the only remaining open question was whether (b) $_p \not\Leftarrow$ (a) $_p$ for $1 \le p < \infty$. We prove this in the affirmative and, in particular, we show the following.

Theorem 1.4. *For $1 \le p < \infty$, there exists a reflexive Banach space $\mathfrak{X}^{(p)}_{awi}$ with an unconditional basis that admits ℓ_p as a uniformly unique joint spreading model with respect to $\mathscr{F}_b(\mathfrak{X}^{(p)}_{awi})$ and contains no Asymptotic ℓ_p subspaces.*

To construct these spaces, we use a saturation method with asymptotically weakly incomparable constraints. This method, initialised in [8], employs a tree structure, penetrating every subspace of $\mathfrak{X}^{(p)}_{\text{awi}}$, that admits segments with norm strictly less than the ℓ_p-norm. Thus, we are able to prove that no subspace

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of $\mathfrak{X}_{\text{awi}}^{(p)}$ is an Asymptotic ℓ_p space. This saturation method is different from the method of saturation with increasing weights from [9], used to define spaces with no subspaces admitting a unique asymptotic model. It does not seem possible to use the method of increasing weights to construct a space with a unique asymptotic model, that is, it is not appropriate for showing (b) $_p \not\Rightarrow$ (a) $_p$. On the other hand, the method of asymptotically weakly incomparable constraints yields spaces with a unique asymptotic model, and thus it cannot be used to show (c) $_p \not\Rightarrow$ (b) $_p$. This method will be discussed in detail in Part 1.

In the case of $1 < p < \infty$, it is possible to obtain a stronger result. Namely, for every countable ordinal ξ, the space separating the two asymptotic properties additionally satisfies the property that every block subspace contains an ℓ_1-tree of order ω^ξ. This is achieved using the attractors method, which was first introduced in [3] and later also used in [10]. The precise statement of this result is the following.

Theorem 1.5 ([7]). *For every $1 < p < \infty$ and every infinite countable ordinal ξ, there exists a hereditarily indecomposable reflexive Banach space $\mathfrak{X}_\xi^{(p)}$ that admits ℓ_p as a uniformly unique joint spreading model with respect to the family of normalised block sequences and whose every subspace contains an ℓ_1-block tree of order ω^ξ.*

However, in the case of ℓ_1, we are not able to construct a space whose every subspace contains a well-founded tree which is either ℓ_p for some $1 < p < \infty$ or c_0. This case is more delicate, since as we mentioned, the two properties are in fact equivalent in its dual problem for spaces not containing ℓ_1.

The paper is organised as follows: In Section 2, we recall the notions of Schreier families and special convex combinations and prove some of their basic properties, while Section 3 contains the precise definitions of the aforementioned asymptotic structures. In Section 4, we recall certain combinatorial results concerning measures on countably branching well-founded trees from [8], which are a key ingredient in the proof that $\mathfrak{X}_{\text{awi}}^{(p)}$ admits ℓ_p as a unique asymptotic model for $1 \leq p < \infty$. We then split the remainder of the paper into two main parts, each dedicated to the definition and properties of $\mathfrak{X}_{\text{awi}}^{(1)}$ and $\mathfrak{X}_{\text{awi}}^{(p)}$ for $p = 2$, respectively. The construction of $\mathfrak{X}_{\text{awi}}^{(p)}$ for $1 < p < \infty$ and $p \neq 2$ follows as an easy modification of our construction and is omitted. Each of these parts contains an introduction in which we describe the main key points of each construction. Finally, we include two appendices containing variants of the basic inequality, which has been used repeatedly in the past in several related constructions (see, e.g. [3], [9], [10] and [16]).

2. Preliminaries

In this section, we recall some necessary definitions, namely, the Schreier families $(\mathcal{S}_n)_n$ [2] and the corresponding repeated averages $\{a(n, L) : n \in \mathbb{N},\ L \in [\mathbb{N}]^\infty\}$ [11] which we call n-averages, as well as the notion of special convex combinations. For a more thorough discussion of the above, we refer the reader to [13]. We begin with some useful notation.

Notation. By $\mathbb{N} = \{1, 2, \ldots\}$, we denote the set of all positive integers. We will use capital letters, such as $L, M, N, \ldots$ (respectively, lower case letters, such as $s, t, u, \ldots$) to denote infinite subsets (respectively, finite subsets) of $\mathbb{N}$. For every infinite subset L of $\mathbb{N}$, the notation $[L]^\infty$ (respectively, $[L]^{<\infty}$) stands for the set of all infinite (respectively, finite) subsets of L. For every $s \in [\mathbb{N}]^{<\infty}$, by $|s|$, we denote the cardinality of s. For $L \in [\mathbb{N}]^\infty$ and $k \in \mathbb{N}$, $[L]^k$ (respectively, $[L]^{\leq k}$) is the set of all $s \in [L]^{<\infty}$ with $|s| = k$ (respectively, $|s| \leq k$). For every $s, t \in [\mathbb{N}]^{<\infty}$, we write $s < t$ if at least one of them is the empty set, or $\max s < \min t$. Also for $\emptyset \neq s \in [\mathbb{N}]^{<\infty}$ and $n \in \mathbb{N}$, we write $n < s$ if $n < \min s$. We shall identify strictly increasing sequences in $\mathbb{N}$ with their corresponding range, that is, we view every strictly increasing sequence in $\mathbb{N}$ as a subset of $\mathbb{N}$ and, conversely, every subset of $\mathbb{N}$ as the sequence resulting from the increasing order of its elements. Thus, for an infinite subset $L = \{l_1 < l_2 < \ldots\}$ of $\mathbb{N}$ and $i \in \mathbb{N}$, we set $L(i) = l_i$ and, similarly, for a finite subset $s = \{n_1 < \ldots < n_k\}$ of $\mathbb{N}$ and for $1 \leq i \leq k$, we set $s(i) = n_i$.

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Finally, throughout the paper, we follow [23] (see also [1]) for standard notation and terminology concerning Banach space theory. For $x \in c_{00}(\mathbb{N})$, we denote $\mathrm{supp}(x) = \{n \in \mathbb{N} : x(n) \neq 0\}$, and by $\mathrm{range}(x)$, the minimum interval of $\mathbb{N}$ containing $\mathrm{supp}(x)$. Moreover, for $x, y \in c_{00}(\mathbb{N})$, we write $x < y$ to denote that $\max\mathrm{supp}(x) < \min\mathrm{supp}(y)$.

2.1. Schreier families

For a family $\mathcal{M}$ and a sequence $(E_i)_{i=1}^k$ of finite subsets of $\mathbb{N}$, we say that $(E_i)_{i=1}^k$ is $\mathcal{M}$-admissible if there is $\{m_1, \dots, m_k\} \in \mathcal{M}$, such that $m_1 \leq E_1 < m_2 \leq E_2 < \cdots < m_k \leq E_k$. Moreover, a sequence $(x_i)_{i=1}^k$ in $c_{00}(\mathbb{N})$ is called $\mathcal{M}$-admissible if $(\mathrm{supp}(x_i))_{i=1}^k$ is $\mathcal{M}$-admissible. In the case where $\mathcal{M}$ is a spreading family (i.e. whenever $E = \{m_1, \dots, m_k\} \in \mathcal{M}$ and $F = \{n_1 < \dots < n_k\}$ satisfy $m_i \leq n_i$, $i = 1, \dots, k$, then $F \in \mathcal{M}$), a sequence $(E_i)_{i=1}^k$ is $\mathcal{M}$-admissible if $\{\min E_i : i = 1, \dots, k\} \in \mathcal{M}$, and thus a sequence of vectors $(x_i)_{i=1}^k$ in $c_{00}(\mathbb{N})$ is $\mathcal{M}$-admissible if $\{\min\mathrm{supp}(x_i) : i = 1, \dots, k\} \in \mathcal{M}$.

For $\mathcal{M}, \mathcal{N}$ families of finite subsets of $\mathbb{N}$, we define the convolution of $\mathcal{M}$ and $\mathcal{N}$ as follows:

$$\mathcal{M} * \mathcal{N} = \Big\{E \subset \mathbb{N} : \text{ there exists an } \mathcal{M}\text{-admissible finite sequence } (E_i)_{i=1}^k \text{ in } \mathcal{N}, \text{ such that } E = \cup_{i=1}^k E_i\Big\} \cup \big\{\emptyset\big\}.$$

The Schreier families $(\mathcal{S}_n)_{n\in\mathbb{N}}$ are defined inductively as follows:

$$\mathcal{S}_0 = \big\{\{k\} : k \in \mathbb{N}\big\} \cup \big\{\emptyset\big\} \quad \text{and} \quad \mathcal{S}_1 = \big\{E \subset \mathbb{N} : \#E \leq \min E\big\} \cup \{\emptyset\}$$

and if $\mathcal{S}_n$, for some $n \in \mathbb{N}$, has been defined, then

$$\mathcal{S}_{n+1} = \mathcal{S}_1 * \mathcal{S}_n = \Big\{E \subset \mathbb{N} : E = \cup_{i=1}^k E_i \text{ where } E_1 < \dots < E_k \in \mathcal{S}_n \text{ and } k \leq \min E_1\Big\} \cup \{\emptyset\}.$$

It follows easily by induction that for every $n, m \in \mathbb{N}$,

$$\mathcal{S}_n * \mathcal{S}_m = \mathcal{S}_{n+m}.$$

Furthermore, for each $n \in \mathbb{N}$, the family $\mathcal{S}_n$ is regular. This means that it includes the singletons, it is hereditary, that is, if $E \in \mathcal{S}_n$ and $F \subset E$, then $F \in \mathcal{S}_n$, it is spreading and finally it is compact, identified as a subset of $\{0, 1\}^{\mathbb{N}}$.

For each $n \in \mathbb{N}$, we also define the regular family

$$\mathcal{A}_n = \big\{E \subset \mathbb{N} : \#E \leq n\big\}.$$

Then, for $n, m \in \mathbb{N}$, we are interested in the family $\mathcal{S}_n * \mathcal{A}_m$, that is, the family of all subsets of $\mathbb{N}$ of the form $E = \cup_{i=1}^k E_i$, where $E_1 < \dots < E_k$, $\#E_i \leq m$ for $i = 1, \dots, k$ and $\{\min E_i : 1 \leq i \leq k\} \in \mathcal{S}_n$. In fact, any such E is the union of at most m sets in $\mathcal{S}_n$, and moreover, if $m \leq E$, then $E \in \mathcal{S}_{n+1}$, as we show next.

Lemma 2.1. *For every $n, m \in \mathbb{N}$,*

(i) *$\mathcal{S}_n * \mathcal{A}_m \subset \mathcal{A}_m * \mathcal{S}_n$ and*
(ii) *if $E \in \mathcal{S}_n * \mathcal{A}_m$ with $m \leq E$, then $E \in \mathcal{S}_{n+1}$.*

Remark 2.2. Let $k, m \in \mathbb{N}$ and F be a subset of $\mathbb{N}$ with $\#F \leq km$ and $k \leq F$. Set $d = \max\{1, \lfloor \#F/m \rfloor\}$, and define $F_j = \{F(n) : n = (j-1)d + 1, \dots, jd\}$ for each $j = 1, \dots, m-1$ and $F_m = F \setminus \cup_{j=1}^{m-1} F_j$. Then, it is immediate to check that $F_j \in \mathcal{S}_1$ for every $i = 1, \dots, m$.

Proof of Lemma 2.1. Fix $n, m \in \mathbb{N}$. We prove (i) by induction on $n \in \mathbb{N}$. For $n = 1$, let $E \in \mathcal{S}_1 * \mathcal{A}_m$, that is, $E = \cup_{i=1}^k E_i$ with $k \leq E_1 < \dots < E_k$ and $\#E_i \leq m$ for every $i = 1, \dots, k$. Since $\#E \leq km$, Remark 2.2 yields a partition $E = \cup_{j=1}^m F_j$ with $F_j \in \mathcal{S}_1$ for every $j = 1, \dots, m$, and, hence, $E \in \mathcal{A}_m * \mathcal{S}_1$.

Suppose that (i) holds for some $n \in \mathbb{N}$ and let $E \in \mathcal{S}_{n+1} * \mathcal{A}_m$. Then $E = \cup_{i=1}^k E_i$ for an $\mathcal{S}_{n+1}$-admissible sequence $(E_i)_{i=1}^k$ with $\#E_i \leq m$ for every $i = 1, \ldots, m$. Hence, $\{\min E_i : i = 1, \ldots, k\} = \cup_{j=1}^l F_j$, where $F_j \in \mathcal{S}_n$ for every $j = 1, \ldots, l$ and $l \leq F_1 < \cdots < F_l$. Define, for each $j = 1, \ldots, l$,

$$G_j = \cup\{E_i : i = 1, \ldots, k \text{ and } \min E_i \in F_j\},$$

and note that $G_j \in \mathcal{S}_n * \mathcal{A}_m$ since $F_j \in \mathcal{S}_n$. Hence, for every $j = 1, \ldots, l$, the inductive hypothesis implies that $G_j \in \mathcal{A}_m * \mathcal{S}_n$, that is, $G_j = \cup_{i=1}^{m_j} G_i^j$ with $m_j \leq m$ and $G_i^j \in \mathcal{S}_n$ for all $i = 1, \ldots, m_j$. Define

$$H = \{\min G_i^j : j = 1, \ldots, l, \text{ and } i = 1, \ldots, m_j\}.$$

Observe that $H \in \mathcal{S}_1 * \mathcal{A}_m$ and apply Remark 2.2 to obtain a partition $H = \cup_{q=1}^m H_q$, where $H_q \in \mathcal{S}_1$ for every $q = 1, \ldots, m$. Finally, define

$$\Delta_q = \cup\{G_i^j : j = 1, \ldots, l,\ i = 1, \ldots, m_j \text{ and } \min G_i^j \in H_q\},$$

for each $q = 1, \ldots, m$, and observe that $E = \cup_{q=1}^m \Delta_q$ and that $\Delta_q \in \mathcal{S}_1 * S_n = S_{n+1}$ since $H_q \in \mathcal{S}_1$ and $G_i^j \in \mathcal{S}_n$. Thus, we conclude that $E \in \mathcal{A}_m * \mathcal{S}_{n+1}$.

Finally, note that (ii) is an immediate consequence of (i). □

2.2. *Repeated averages*

The notion of repeated averages was first defined in [11]. The notation we use below, however, is somewhat different, and we instead follow the one found in [13], namely, $\{a(n, L) : n \in \mathbb{N},\ L \in [\mathbb{N}]^\infty\}$. The n–averages $a(n, L)$ are defined as elements of $c_{00}(\mathbb{N})$ in the following manner.

Let $(e_j)_j$ denote the unit vector basis of $c_{00}(\mathbb{N})$ and $L \in [\mathbb{N}]^\infty$. For $n = 0$, we define $a(0, L) = e_{l_1}$, where $l_1 = \min L$. Suppose that $a(n, M)$ has been defined for some $n \in \mathbb{N}$ and every $M \in [\mathbb{N}]^\infty$. We define $a(n + 1, L)$ in the following way: We set $L_1 = L$ and $L_k = L_{k-1} \setminus \operatorname{supp}(a(n, L_{k-1}))$ for $k = 2, \ldots, l_1$ and finally define

$$a(n + 1, L) = \frac{1}{l_1}\big(a(n, L_1) + \cdots + a(n, L_{l_1})\big).$$

Remark 2.3. Let $n \in \mathbb{N}$ and $L \in [\mathbb{N}]^\infty$. The following properties are easily established by induction.

(i) $a(n, L)$ is a convex combination of the unit vector basis of $c_{00}(\mathbb{N})$.
(ii) $\|a(n, L)\|_{\ell_1} = 1$ and $a(n, L)(k) \geq 0$ for all $k \in \mathbb{N}$.
(iii) $\operatorname{supp}(a(n, L))$ is the maximal initial segment of L contained in $\mathcal{S}_n$.
(iv) $\|a(n, L)\|_\infty = l_1^{-n}$, where $l_1 = \min L$.
(v) If $\operatorname{supp}(a(n, L)) = \{i_1 < \ldots < i_d\}$ and $a(n, L) = \sum_{k=1}^d a_{i_k} e_{i_k}$, then we have that $a_{i_1} \geq \ldots \geq a_{i_d}$.

A proof of the following proposition can be found in [13].

Proposition 2.4. *Let $n \in \mathbb{N}$ and $L \in [\mathbb{N}]^\infty$. For every $F \in \mathcal{S}_{n-1}$, we have that*

$$\sum_{k \in F} a(n, L)(k) < \frac{3}{\min L}.$$

2.3. *Special convex combinations*

Here, we recall the notion of (n, ε)-special convex combinations, where $n \in \mathbb{N}$ and $\varepsilon > 0$ (see [5] and [13]).

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Definition 2.5. For $n \in \mathbb{N}$ and $\varepsilon > 0$, a convex combination $\sum_{i\in F} c_i e_i$, of the unit vector basis $(e_i)_i$ of $c_{00}(\mathbb{N})$ is called an (n,ε)-basic special convex combination (or an (n,ε)-basic s.c.c.) if

(i) $F \in \mathcal{S}_n$ and
(ii) for any $G \subset F$ with $G \in S_{n-1}$, we have that $\sum_{i\in G} c_i < \varepsilon$.

We will also call $\sum_{i\in F} c_i^{1/2} e_i$ a $(2,n,\varepsilon)$-basic special convex combination.

As follows from Proposition 2.4, every n-average $a(n,L)$ is an $(n, 3/\min L)$-basic s.c.c., and this yields the following.

Proposition 2.6. *Let $M \in [\mathbb{N}]^\infty$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Then there is a $k \in \mathbb{N}$, such that for any $F \subset M$, such that F is maximal in $\mathcal{S}_n$ and $k \le \min F$, there exists an (n,ε)-basic s.c.c. $x \in c_{00}(\mathbb{N})$ with supp$(x) = F$.*

Clearly, this also implies the existence of $(2,n,\varepsilon)$-basic special convex combinations by taking the square roots of the coefficients of an (n,ε)-b.s.c.c.

Definition 2.7. Let $x_1 < \ldots < x_d$ be vectors in $c_{00}(\mathbb{N})$, and define $t_i = \min \operatorname{supp}(x_i)$, $i = 1,\ldots,d$. We say that the vector $\sum_{i=1}^d c_i x_i$ is an (n,ε)-special convex combination (or an (n,ε)-s.c.c.) for some $n \in \mathbb{N}$ and $\varepsilon > 0$ if $\sum_{i=1}^d c_i e_{t_i}$ is an (n,ε)-basic s.c.c. and a $(2,n,\varepsilon)$-special convex combination if $\sum_{i=1}^d c_i e_{t_i}$ is a $(2,n,\varepsilon)$-basic s.c.c.

3. Asymptotic structures

Let us recall the definitions of the asymptotic notions that appear in the results of this paper and were mentioned in the Introduction. Namely, asymptotic models, joint spreading models and the notions of Asymptotic ℓ_p and Asymptotic c_0 spaces. For a more thorough discussion, including several open problems and known results, we refer the reader to [9, Section 3].

Definition 3.1 [20]. An infinite array of sequences $(x_j^i)_j$, $i \in \mathbb{N}$, in a Banach space X, is said to generate a sequence $(e_i)_i$, in a seminormed space E, as an asymptotic model if for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a $k_0 \in \mathbb{N}$, such that for any natural numbers $k_0 \le k_1 < \cdots < k_n$ and any scalars $a_1,\ldots,a_n$ in $[-1,1]$, we have

$$\Big| \Big\| \sum_{i=1}^n a_i x_{k_i}^i \Big\| - \Big\| \sum_{i=1}^n a_i e_i \Big\| \Big| < \varepsilon.$$

A Banach space X is said to admit a unique asymptotic model with respect to a family $\mathscr{F}$ of normalised sequences in X if whenever two infinite arrays, consisting of sequences from $\mathscr{F}$, generate asymptotic models, then those must be equivalent. Typical families under consideration are those of normalised weakly null sequences, denoted $\mathscr{F}_0(X)$, normalised Schauder basis sequences, denoted $\mathscr{F}(X)$, or the family of all normalised block sequences of a fixed basis of X, if it has one, denoted $\mathscr{F}_b(X)$.

Definition 3.2 [6]. Let $M \in [\mathbb{N}]^\infty$ and $k \in \mathbb{N}$. A plegma (respectively, strict plegma) family in $[M]^k$ is a finite sequence $(s_i)_{i=1}^l$ in $[M]^k$ satisfying the following.

(i) $s_{i_1}(j_1) < s_{i_2}(j_2)$ for every $1 \le j_1 < j_2 \le k$ and $1 \le i_1, i_2 \le l$.
(ii) $s_{i_1}(j) \le s_{i_2}(j)$ (respectively, $s_{i_1}(j) < s_{i_2}(j)$) for all $1 \le i_1 < i_2 \le l$ and $1 \le j \le k$.

For each $l \in \mathbb{N}$, the set of all sequences $(s_i)_{i=1}^l$ which are plegma families in $[M]^k$ will be denoted by $Plm_l([M]^k)$ and that of the strict plegma ones by S- $Plm_l([M]^k)$.

Definition 3.3 [6]. A finite array of sequences $(x_j^i)_j$, $1 \le i \le l$, in a Banach space X, is said to generate another array of sequences $(e_j^i)_j$, $1 \le i \le l$, in a seminormed space E, as a joint spreading model if for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a $k_0 \in \mathbb{N}$, such that for any $(s_i)_{i=1}^l \in S\text{-}Plm_l([\mathbb{N}]^n)$ with $k_0 \leq s_1(1)$ and for any $l \times n$ matrix $A = (a_{ij})$ with entries in $[-1, 1]$, we have that

$$\Big| \Big\| \sum_{i=1}^{l} \sum_{j=1}^{n} a_{ij} x^i_{s_i(j)} \Big\| - \Big\| \sum_{i=1}^{l} \sum_{j=1}^{n} a_{ij} e^i_j \Big\| \Big| < \varepsilon.$$

A Banach space X is said to admit a uniformly unique joint spreading model with respect to a family of normalised sequences $\mathscr{F}$ in X if there exists a constant C, such that whenever two arrays $(x^i_j)_j$ and $(y^i_j)_j$, $1 \leq i \leq l$, of sequences from $\mathscr{F}$ generate joint spreading models, then those must be C-equivalent. Moreover, a Banach space admits a uniformly unique joint spreading model with respect to a family $\mathscr{F}$ if and only if it admits a unique asymptotic model with respect to $\mathscr{F}$ (see, e.g. [6, Remark 4.21] or [9, Proposition 3.12]). In particular, if a space admits a uniformly unique joint spreading model with respect to some family $\mathscr{F}$ satisfying certain conditions described in [6, Proposition 4.9], then this is equivalent to some ℓ_p. In order to show that a space admits some ℓ_p as a uniformly unique joint spreading model, it may be more convenient to prove (ii) of the following lemma, thereby avoiding the use of plegma families.

Lemma 3.4. *Let X be a Banach space and $\mathscr{F}$ be a family of normalised sequences in X. Let also $1 \leq p < \infty$. The following are equivalent.*

(i) *X admits ℓ_p as a uniformly unique joint spreading model with respect to the family $\mathscr{F}$.*
(ii) *There exist constants $c, C > 0$, such that for every array $(x^i_j)_j$, $1 \leq i \leq l$, of sequences from $\mathscr{F}$, there is $M = \{m_1 < m_2 < \ldots\}$, an infinite subset of the naturals, such that for any choice of $1 \leq i_j \leq l$, $j \in M$, every $F \subset M$ with $m_k \leq F$ and $|F| \leq k$ and any choice of scalars a_j, $j \in F$,*

$$c\|(a_j)_{j\in F}\|_p \leq \Big\| \sum_{j\in F} a_j x^{i_j}_j \Big\| \leq C\|(a_j)_{j\in F}\|_p.$$

Proof. Note that (i) implies that there are constants $c, C > 0$, such that for every array $(x^i_j)_j$, $1 \leq i \leq l$, of sequences from $\mathscr{F}$, there is $N = \{n_1 < n_2 < \ldots\}$, an infinite subset of the naturals, such that for any k, any strict plegma family $(s_i)_{i=1}^l \in S\text{-}Plm_l([N]^k)$ with $n_k \leq s_1(1)$ and any $l \times k$ matrix $A = (a_{ij})$ of scalars, we have that

$$c\|(a_{ij})_{i=1,j=1}^{l,k}\|_p \leq \Big\| \sum_{i=1}^{l} \sum_{j=1}^{k} a_{ij} x^i_{s_i(j)} \Big\| \leq C\|(a_{ij})_{i=1,j=1}^{l,k}\|_p.$$

Let $N' = \{n_{2kl} : k \in \mathbb{N}\}$ and observe that for $k_1, \ldots, k_d \in \mathbb{N}$, there is a strict plegma family $(s_i)_{i=1}^l \in S\text{-}Plm_l([N]^d)$, such that $n_{2k_jl} \in \{s_i(j) : i = 1, \ldots, l\}$ for all $j = 1, \ldots, d$. Hence, we may find $M \subset N'$ satisfying (ii) with constants c, C. Finally, by repeating the sequences in the array, it follows easily that (ii) yields (i). □

We recall the main result from [6], stating that whenever a Banach space admits a uniformly unique joint spreading model with respect to some family satisfying certain stability conditions, then it satisfies a property concerning its bounded linear operators called the Uniform Approximation on Large Subspaces property (see [6, Theorem 5.17] and [6, Theorem 5.23]).

Definition 3.5 [24]. A Banach space X is called Asymptotic ℓ_p, $1 \leq p < \infty$, (respectively, Asymptotic c_0) if there exists a constant C, such that in a two-player n-turn game $G(n, p, C)$, where in each turn $k = 1, \ldots, n$, player (S) picks a finite codimensional subspace Y_k of X, and then player (V) picks a normalised vector $x_k \in Y_k$, player (S) has a winning strategy to force player (V) to pick a sequence $(x_k)_{k=1}^n$ that is C-equivalent to the unit vector basis of ℓ_p^n (respectively, ℓ_∞^n).

Although this is not the initial formulation, it is equivalent and follows from [24, Subsection 1.5]. The typical example of a nonclassical Asymptotic ℓ_p space is the Tsirelson space from [17]. This is a reflexive Asymptotic ℓ_1 space, and it is the dual of Tsirelson's original space from [27] which is Asymptotic c_0. Finally, whenever a Banach space is Asymptotic ℓ_p or Asymptotic c_0, it admits a uniformly unique joint spreading model with respect to $\mathscr{F}_0(X)$ (see, e.g. [6, Corollary 4.12]).

The above definition is the coordinate-free version of the notion of an asymptotic ℓ_p Banach space with a basis introduced by Milman and Tomczak-Jaegermann in [25].

Definition 3.6 [25]. Let X be a Banach space with a Schauder basis $(e_i)_i$ and $1 \le p < \infty$. We say that the Schauder basis $(e_i)_i$ of X is asymptotic ℓ_p if there exist positive constants D_1 and D_2, such that for all $n \in \mathbb{N}$, there exists $N(n) \in \mathbb{N}$ with the property that whenever $N(n) \le x_1 < \cdots < x_n$ are vectors in X, then

$$\frac{1}{D_1}\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{\frac{1}{p}} \le \Big\|\sum_{i=1}^{n} x_i\Big\| \le D_2\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{\frac{1}{p}}.$$

Specifically, we say that $(e_i)_i$ is D-asymptotic ℓ_p for $D = D_1 D_2$. The definition of an asymptotic c_0 space is given similarly.

It is easy to show that if X has a Schauder basis that is asymptotic ℓ_p, then X is Asymptotic ℓ_p. Moreover, if X is Asymptotic ℓ_p, then it contains an asymptotic ℓ_p sequence. In particular, note that if X has a Schauder basis and Y is an Asymptotic ℓ_p subspace of X, then Y contains a further subspace that is isomorphic to an asymptotic ℓ_p block subspace.

A noteworthy remark is that sequential asymptotic properties, array asymptotic properties and global asymptotic properties of a Banach space X can alternatively be interpreted as properties of special weakly null trees. A collection $\{x_A : A \in [\mathbb{N}]^{\le n}\}$ in X is said to be a normalised weakly null tree of height n, if for every $A \in [\mathbb{N}]^{\le n-1}$, $(x_{A\cup\{j\}})_{j>\max(A)}$ is a normalised weakly null sequence. Such a tree is said to originate from a sequence $(x_j)_j$ if for all $A = \{a_1, \dots, a_i\}$, we have $x_A = x_{a_i}$. Similarly, a tree $\{x_A : A \in [\mathbb{N}]^{\le n}\}$ is said to originate from an array of sequences $(x_j^{(i)})_j$, $1 \le i \le n$ if for all $A = \{a_1, \dots, a_i\}$, we have $x_A = x_{a_i}^{(i)}$. Then, X has a uniformly unique ℓ_p spreading model if and only if there exists $C > 0$, so that every tree $\{x_A : A \in [\mathbb{N}]^{\le n}\}$ originating from a normalised weakly null sequence $(x_j)_j$ in X has a maximal branch that is C-equivalent to the unit vector basis of ℓ_p^n. Similarly, X has a unique ℓ_p asymptotic model if the same can be said about all trees originating from normalised weakly null arrays in X. Finally, a Banach space X is an Asymptotic ℓ_p space (or an Asymptotic c_0 space if $p = \infty$) if there exists $C > 0$, so that every normalised weakly null tree of height n has a maximal branch $x_{\{a_1\}}, x_{\{a_1,a_2\}}, \dots, x_{\{a_1,a_2,\dots,a_n\}}$ that is C-equivalent to the unit vector basis of ℓ_p^n. For more details, see [14, Remark 3.11].

4. Measures on countably branching well-founded trees

In this section, we recall certain results from [8] concerning measures on countably branching well-founded trees. These will be used to prove that for all $1 \le p < \infty$, the space $\mathfrak{X}_{\mathrm{awi}}^{(p)}$ admits ℓ_p as a unique asymptotic model. In particular, Proposition 4.1 and Lemma 4.6 will be used to prove Lemma 7.2, which is one of the key ingredients in the proof of the main result, Theorem 1.4.

Let $\mathcal{T} = (A, <_{\mathcal{T}})$, where A is a countably infinite set equipped with a partial order $<_{\mathcal{T}}$. In the sequel, we use $t \in \mathcal{T}$ instead of $t \in A$. We assume that $<_{\mathcal{T}}$ is such that there is a unique minimal element in $\mathcal{T}$, and for each $t \in \mathcal{T}$, the set $S_t = \{s \in \mathcal{T} : s \le_{\mathcal{T}} t\}$ is finite and totally ordered, that is, $\mathcal{T}$ is a rooted tree. We also assume that $\mathcal{T}$ is well founded, that is, it contains no infinite totally ordered sets, and countably branching, that is, every nonmaximal node has countably infinite immediate successors.

Observe that $\widetilde{\mathcal{T}} = (\{S_t : t \in \mathcal{T}\}, <_{\widetilde{\mathcal{T}}})$, where $<_{\widetilde{\mathcal{T}}}$ denotes inclusion, is also a tree, and that $\mathcal{T}$ is in fact isomorphic to $\widetilde{\mathcal{T}}$ via the mapping $t \mapsto S_t$. Given $t \in \mathcal{T}$, we will denote S_t by $\tilde{t}$, identifying it as

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an element of $\widetilde{\mathcal{T}}$. For each $\tilde{t} \in \widetilde{\mathcal{T}}$, we denote by $S(\tilde{t})$ the set of immediate successors of $\tilde{t}$ in $\widetilde{\mathcal{T}}$. In particular, if $\tilde{t}$ is maximal, then $S(\tilde{t})$ is empty. Moreover, for $\tilde{t} \in \widetilde{\mathcal{T}}$, we denote $V_{\tilde{t}} = \{\tilde{s} \in \widetilde{\mathcal{T}} : \tilde{t} \leq_{\widetilde{\mathcal{T}}} \tilde{s}\}$ and view $\widetilde{\mathcal{T}}$ as a topological space with the topology generated by the sets $V_{\tilde{t}}$ and $\widetilde{\mathcal{T}} \setminus V_{\tilde{t}}$, $\tilde{t} \in \widetilde{\mathcal{T}}$, that is, the pointwise convergence topology. This is a compact metric topology, such that for each $\tilde{t} \in \widetilde{\mathcal{T}}$, the sets of the form $V_{\tilde{t}} \setminus (\cup_{\tilde{s} \in F} V_{\tilde{s}})$, $F \subset S(\tilde{t})$ finite, form a neighbourhood base of clopen sets for $\tilde{t}$. We denote by $\mathcal{M}_+(\widetilde{\mathcal{T}})$ the cone of all bounded positive measures $\mu : \mathcal{P}(\widetilde{\mathcal{T}}) \to [0, +\infty)$. For $\mu \in \mathcal{M}_+(\widetilde{\mathcal{T}})$, we define the support of μ to be the set $\operatorname{supp}(\mu) = \{\tilde{t} \in \widetilde{\mathcal{T}} : \mu(\{\tilde{t}\}) > 0\}$. Finally, we say that a subset $\mathcal{A}$ of $\mathcal{M}_+(\widetilde{\mathcal{T}})$ is bounded if $\sup_{\mu \in \mathcal{A}} \mu(\widetilde{\mathcal{T}}) < \infty$.

Proposition 4.1. *Let $(\mu_i)_i$ be a bounded and disjointly supported sequence in $\mathcal{M}_+(\widetilde{\mathcal{T}})$. Then for every $\varepsilon > 0$, there is an $L \in [\mathbb{N}]^\infty$ and a subset G_i of supp(μ_i) for each $i \in L$, satisfying the following.*

(i) *$\mu_i(\widetilde{\mathcal{T}} \setminus G_i) \leq \varepsilon$ for every $i \in L$.*
(ii) *The sets G_i, $i \in L$, are pairwise incomparable.*

For the proof, we refer the reader to [8, Proposition 3.1].

Definition 4.2. Let $(\mu_i)_i$ be a sequence in $\mathcal{M}_+(\widetilde{\mathcal{T}})$ and $\nu \in \mathcal{M}_+(\widetilde{\mathcal{T}})$. We say that ν is the successor-determined limit of $(\mu_i)_i$ if for all $\tilde{t} \in \widetilde{\mathcal{T}}$, we have $\nu(\{\tilde{t}\}) = \lim_i \mu_i(S(\tilde{t}))$. In this case, we write $\nu = \text{succ-}\lim_i \mu_i$.

Recall that a bounded sequence $(\mu_i)_i$ in $\mathcal{M}_+(\widetilde{\mathcal{T}})$ converges in the w^*-topology to a $\mu \in \mathcal{M}_+(\widetilde{\mathcal{T}})$ if and only if for all clopen sets $V \subset \widetilde{\mathcal{T}}$, we have $\lim_i \mu_i(V) = \mu(V)$ if and only if for all $\tilde{t} \in \widetilde{\mathcal{T}}$, we have $\lim_i \mu_i(V_{\tilde{t}}) = \mu(V_{\tilde{t}})$. In this case, we write $\mu = w^*\text{-}\lim_i \mu_i$.

Lemma 4.3. *Let $(\mu_i)_i$ be a bounded sequence in $\mathcal{M}_+(\widetilde{\mathcal{T}})$. There exist a subsequence $(\mu_{i_n})_n$ of $(\mu_i)_i$ and $\nu \in \mathcal{M}_+(\widetilde{\mathcal{T}})$ with $\nu = \text{succ-}\lim_n \mu_{i_n}$.*

Remark 4.4. It is possible for a bounded sequence $(\mu_i)_i$ in $\mathcal{M}_+(\widetilde{\mathcal{T}})$ to satisfy $w^*\text{-}\lim_i \mu_i \neq \text{succ-}\lim_i \mu_i$. Take, for example, $\widetilde{\mathcal{T}} = [\mathbb{N}]^{\leq 2}$ (all subsets of $\mathbb{N}$ with at most two elements with the partial order of initial segments), and define $\mu_i = \delta_{\{i,i+1\}}$, $i \in \mathbb{N}$. Then $w^*\text{-}\lim_i \mu_i = \delta_\emptyset$, whereas $\text{succ-}\lim_i \mu_i = 0$.

Although these limits are not necessarily the same, there is an explicit formula relating $\text{succ-}\lim_i \mu_i$ to $w^*\text{-}\lim_i \mu_i$.

Lemma 4.5. *Let $(\mu_i)_i$ be a bounded and disjointly supported sequence in $\mathcal{M}_+(\widetilde{\mathcal{T}})$, such that $w^*\text{-}\lim_i \mu_i = \mu$ exists, and for all $\tilde{t} \in \widetilde{\mathcal{T}}$, the limit $\nu(\{\tilde{t}\}) = \lim_i \mu_i(S(\tilde{t}))$ exists as well. Then for every $\tilde{t} \in \widetilde{\mathcal{T}}$ and enumeration $(\tilde{t}_j)_j$ of $S(\tilde{t})$, we have*

$$\mu(\{\tilde{t}\}) = \nu(\{\tilde{t}\}) + \lim_j \lim_i \mu_i\Big(\cup_{k \geq j} (V_{\tilde{t}_k} \setminus \{\tilde{t}_k\})\Big). \tag{4.1}$$

In particular, $\mu(\{\tilde{t}\}) = \nu(\{\tilde{t}\})$ if and only if the double limit in (4.1) is zero.

Lemma 4.6. *Let $(\mu_i)_i$ be a bounded and disjointly supported sequence in $\mathcal{M}_+(\widetilde{\mathcal{T}})$, such that $\text{succ-}\lim_i \mu_i = \nu$ exists. Then there exist an infinite $L \subset \mathbb{N}$ and partitions A_i, B_i of supp(μ_i), $i \in L$, such that the following are satisfied.*

(i) *If for all $i \in L$, we define the measure μ_i^1 by $\mu_i^1(C) = \mu_i(C \cap A_i)$, then $\nu = w^*\text{-}\lim_{i\in L} \mu_i^1 = \text{succ-}\lim_{i \in L} \mu_i^1$.*
(ii) *If for all $i \in L$, we define the measure μ_i^2 by $\mu_i^2(C) = \mu_i(C \cap B_i)$, then for all $\tilde{t} \in \widetilde{\mathcal{T}}$, the sequence $(\mu_i^2(S(\tilde{t})))_i$ is eventually zero. In particular, $\text{succ-}\lim_{i\in L} \mu_i^2 = 0$.*

For the proofs, we refer the reader to [8, Lemma 4.10] and [8, Lemma 4.12].

Remark 4.7. Although the results from [8] were formulated for trees $\mathcal{T}$ defined on infinite subsets of $\mathbb{N}$, this is not a necessary restriction, and they can be naturally extended to the more general setting of countably branching well-founded trees.

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PART I. The case of ℓ_1

5. Definition of the space $\mathfrak{X}^{(1)}_{\mathrm{awi}}$

The method of saturation with asymptotically weakly incomparable constraints, that is used in the construction of both spaces presented in this paper, was introduced in [8], where it was shown that (b)$_1$ $\not\Rightarrow$(a)$_1$. There, it was also used to prove an even stronger result, namely, the existence of a Banach space with a basis admitting ℓ_1 as a unique asymptotic model, and in which any infinite subsequence of the basis generates a non-Asymptotic ℓ_1 subspace. This method requires the existence of a well-founded tree defined either on the basis of the space or on a family of functionals of its norming set. In this section, we define the space $\mathfrak{X}^{(1)}_{\mathrm{awi}}$ by introducing its norm via a norming set, which is a subset of the norming set of a Mixed Tsirelson space $\mathcal{T}[(m_j, \mathcal{S}_{n_j})_j]$ for an appropriate choice of $(m_j)_j$ and $(n_j)_j$ described below. The key ingredient in the definition of this norming set is the notion of asymptotically weakly incomparable sequences of functionals, which is also introduced in this section. This notion will allow the space $\mathfrak{X}^{(1)}_{\mathrm{awi}}$ to admit ℓ_1 as a unique asymptotic model, while at the same time, it will force the norm to be small on the branches of a tree, in every subspace of $\mathfrak{X}^{(1)}_{\mathrm{awi}}$, showing that the space does not contain Asymptotic ℓ_1 subspaces.

5.1. *Definition of the space $\mathfrak{X}^{(1)}_{awi}$*

Define a pair of strictly increasing sequences of natural numbers $(m_j)_j$, $(n_j)_j$ as follows:

$$m_1 = 2 \qquad\qquad n_1 = 1$$
$$m_{j+1} = m_j^{m_j} \qquad\qquad n_{j+1} = 2^{2m_{j+1}} n_j.$$

Definition 5.1. Let $V_{(1)}$ denote the minimal subset of $c_{00}(\mathbb{N})$ that

(i) contains 0 and all $\pm e_j^*$, $j \in \mathbb{N}$ and
(ii) is closed under the operations $(m_j, \mathcal{S}_{n_j})_j$, that is, if $j \in \mathbb{N}$ and $f_1 < \ldots < f_n$ is an $\mathcal{S}_{n_j}$-admissible sequence (see Section 2.1) in $V_{(1)} \setminus \{0\}$, then $m_j^{-1} \sum_{i=1}^n f_i$ is also in $V_{(1)}$.

Remark 5.2.

(i) If $f \in V_{(1)} \setminus \{0\}$, then either $f \in \{\pm e_j^* : j \in \mathbb{N}\}$, or it is of the form $f = m_j^{-1} \sum_{i=1}^n f_i$ with $f_1 < \ldots < f_n$ an $\mathcal{S}_{n_j}$-admissible sequence in $V_{(1)}$ for some $j \in \mathbb{N}$.
(ii) As usual, we view the elements of $V_{(1)}$ as functionals acting on $c_{00}(\mathbb{N})$, inducing a norm $\|\cdot\|_{V_{(1)}}$. The completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{V_{(1)}})$ is the Mixed Tsirelson space $\mathcal{T}[(m_j, \mathcal{S}_{n_j})_j]$ introduced for the first time in [5]. The first space with a saturated norm defined by a countable family of operations is the Schlumprecht space [26], which is a fundamental discovery and was used by Gowers and Maurey [19] to define the first hereditarily indecomposable (HI) space.

We now recall the notion of tree analysis which appeared for the first time in [4]. This has become a standard tool in proving upper bounds for the estimations of functionals on certain vectors in Mixed Tsirelson spaces. However, it is the first time where the tree analysis has a significant role in the definition of the norming set $W_{(1)}$. Additionally, it is also a key ingredient in the proof that $\mathfrak{X}^{(1)}_{\mathrm{awi}}$ contains no Asymptotic ℓ_1 subspaces.

Let $\mathcal{A}$ be a rooted tree. For a node $\alpha \in \mathcal{A}$, we denote by $S(\alpha)$ the set of all immediate successors of α, by $|\alpha|$ the height of α, that is, $|\alpha| = \#\{\beta \in \mathcal{A} : \beta <_{\mathcal{A}} \alpha\}$, and finally, we denote by $h(\mathcal{A})$ the height of $\mathcal{A}$, that is, the maximum height over its nodes.

Definition 5.3. Let $f \in V_{(1)} \setminus \{0\}$. For a finite tree $\mathcal{A}$, a family $(f_\alpha)_{\alpha\in\mathcal{A}}$ is called a tree analysis of f if the following are satisfied.

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(i) $\mathcal{A}$ has a unique root denoted by 0 and $f_0 = f$.
(ii) Each f_α is in $V_{(1)}$, and if $\beta < \alpha$ in $\mathcal{A}$, then $\mathrm{range}(f_\alpha) \subset \mathrm{range}(f_\beta)$.
(iii) For every maximal node $\alpha \in \mathcal{A}$, we have that $|\alpha| = h(\mathcal{A})$.
(iv) For every nonmaximal node $\alpha \in \mathcal{A}$, either f_α is the result of some $(m_j, \mathcal{S}_{n_j})$ operation of $(f_\beta)_{\beta\in S(\alpha)}$, i.e., $f_\alpha = m_j^{-1} \sum_{\beta\in S(\alpha)} f_\beta$, or $f_\alpha \in \{\pm e_j^* : j \in \mathbb{N}\}$ and $S(\alpha) = \{\beta\}$ with $f_\beta = f_\alpha$.
(v) For every maximal node $\alpha \in \mathcal{A}$, $f_\alpha \in \{\pm e_j^* : j \in \mathbb{N}\}$.

Remark 5.4.

(i) It follows by minimality that every f in $V_{(1)} \setminus \{0\}$ admits a tree analysis, but it may not be unique. For example, $f = (m_1 m_2)^{-1} e_1^*$ admits two distinct tree analyses.
(ii) The standard definition of a tree analysis does not include 5.3 (iii). This property is included for technical reasons and is used below in the equality of Remark 5.8 (i).

Definition 5.5. Let $f \in V_{(1)}$.

(i) If $f = 0$ or $f \in \{\pm e_j^* : j \in \mathbb{N}\}$, then we define the weight $w(f)$ of f as $w(f) = 0$ and $w(f) = 1$, respectively.
(ii) If f is the result of an (m_j, S_{n_j})-operation for some $j \in \mathbb{N}$, then $w(f) = m_j$.

Remark 5.6. It is not difficult to see that $w(f)$, for $f \in V_{(1)}$, is not uniquely determined, that is, f could be the result of more than one distinct $(m_j, \mathcal{S}_{n_j})$-operation. However, if we fix a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ of f, then for $\alpha \in \mathcal{A}$ with $f_\alpha = (m_{j_\alpha})^{-1} \sum_{\beta\in S(\alpha)} f_\beta$, the tree analysis determines the weight $w(f_\alpha)$, being equal to m_{j_α}. Thus, for $f \in V_{(1)}$ and a fixed tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ of f, with $w(f_\alpha)$, we will denote the weight m_{j_α} determined by $(f_\alpha)_{\alpha\in\mathcal{A}}$, for every $\alpha \in \mathcal{A}$. In addition, we will denote by $\bar{f}_\alpha$ the pair (f_α, m_{j_α}).

Definition 5.7. Let $f \in V_{(1)}$ and $(f_\alpha)_{\alpha\in\mathcal{A}}$ be a tree analysis of f. Then for $\alpha \in \mathcal{A}$, we define the relative weight $w_f(f_\alpha)$ of f_α as

$$w_f(f_\alpha) = \begin{cases} \prod_{\beta<\alpha} w(f_\beta) & \text{if } \alpha \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Remark 5.8. Let $f \in V_{(1)}$ and $(f_\alpha)_{\alpha\in\mathcal{A}}$ be a tree analysis of f.

(i) For every $k = 1, \dots, h(\mathcal{A})$

$$f = \sum_{|a|=k} w_f(f_\alpha)^{-1} f_\alpha.$$

This can be proved by induction and essentially relies on the fact that $(f_\alpha)_{\alpha\in\mathcal{A}}$ satisfies 5.3 (iii).
(ii) If $\mathcal{B}$ is a maximal pairwise incomparable subset of $\mathcal{A}$, then

$$f = \sum_{\beta\in\mathcal{B}} w_f(f_\beta)^{-1} f_\beta.$$

(iii) For every $\alpha \in \mathcal{A}$, whose immediate predecessor β in $\mathcal{A}$ (if one exists) satisfies $f_\beta \notin \{\pm e_j^* : j \in \mathbb{N}\}$, we have $w_f(f_\alpha) \geq 2^{|\alpha|}$.

Fix an injection σ that maps any pair $(f, w(f))$, for $f \in V_{(1)}$ and $w(f)$ a weight of f, to some m_j with $m_j > \max\mathrm{supp}(f)\, w(f)$ whenever $f \neq 0$.

Definition 5.9. Define a partial order $<_\mathcal{T}$ on the set of all pairs $(f, w(f))$ for $f \in V_{(1)}$ and $w(f)$ a weight of f, as follows: $(f, w(f)) <_\mathcal{T} (g, w(g))$ either if $f = 0$ or if there exist $f_1 < \dots < f_n \in V_{(1)}$ and weights $w(f_1), \dots, w(f_n)$, such that

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(i) $(f_i)_{i=1}^n$ is $\mathcal{S}_1$-admissible,
(ii) $w(f_1) = \sigma(0,0)$ and $w(f_i) = \sigma(f_{i-1}, w(f_{i-1}))$ for every $i = 2, \dots, n$,
(iii) there are $1 \leq i_1 < i_2 \leq n$, such that $f = f_{i_1}$ and $g = f_{i_2}$.

It is easy to see that $<_{\mathcal{T}}$ induces a tree structure rooted at $\bar{0} = (0,0)$. Let us denote this tree by $\mathcal{T}$, and observe that this is a countably branching well-founded tree, due to 5.9(i). For $t = (f, w(f)) \in \mathcal{T}$, we set $f_t = f$ and $w(t) = w(f)$.

It is clear that unlike the case where the tree is defined on the basis of the space, here, incomparable segments need not necessarily have disjoint supports. This forces us to introduce the notion of essentially incomparable nodes, which was first defined in [8]. To this end, we first need to define an additional tree structure that is readily implied by $\mathcal{T}$ via the projection $(f, w(f)) \mapsto w(f)$.

Definition 5.10. Define a partial order $<_{\mathcal{W}}$ on $\{m_j : j \in \mathbb{N}\}$ as follows: $m_i <_{\mathcal{W}} m_j$ if there exist $t_1, t_2 \in \mathcal{T}$, such that $t_1 <_{\mathcal{T}} t_2$, $w(t_1) = m_i$ and $w(t_2) = m_j$.

As an immediate consequence of the fact that $\mathcal{T}$ is a countably branching well-founded tree, we have that $<_{\mathcal{W}}$ also defines a tree structure. Let us denote this tree by $\mathcal{W}$ and note that it is also countably branching and well founded.

Remark 5.11. The above definition implies that if $t_1, t_2 \in \mathcal{T}$ are such that $w(t_1) <_{\mathcal{W}} w(t_2)$, then there exist $t_3, t_4 \in \mathcal{T}$, such that $t_3 <_{\mathcal{T}} t_4$, $w(t_3) = w(t_1)$ and $w(t_4) = w(t_2)$. The tree structure of $\mathcal{T}$ implies that t_3 is uniquely defined, and we will say that t_3 generates $w(t_2)$. This is not the case, however, for t_4, and, moreover, it is not necessary that $t_3 <_{\mathcal{T}} t_2$.

Definition 5.12.

(i) A subset A of $\mathcal{T} \setminus \{\bar{0}\}$ is called essentially incomparable if whenever $t_1, t_2 \in A$ are such that $w(t_1) <_{\mathcal{W}} w(t_2)$, then for the unique $t_3 \in \mathcal{T}$ with $w(t_3) = w(t_1)$ that generates $w(t_2)$, we have that $f_{t_3} < f_{t_1}$.
(ii) A subset A of $\mathcal{T}$ is called weight incomparable if for any $t_1 \neq t_2$ in A, $w(t_1) \neq w(t_2)$ and the weights $w(t_1)$ and $w(t_2)$ are incomparable in $\mathcal{W}$.
(iii) A sequence $(A_j)_j$ of subsets of $\mathcal{T}$ is called pairwise weight incomparable if for every $j_1 \neq j_2$ in $\mathbb{N}$, $t_1 \in A_{j_1}$ and $t_2 \in A_{j_2}$, $w(t_1) \neq w(t_2)$ and the weights $w(t_1)$ and $w(t_2)$ are incomparable in $\mathcal{W}$.

Remark 5.13.

(i) If A is an essentially (respectively, weight) incomparable subset of $\mathcal{T}$, then every $B \subset A$ is also essentially (respectively, weight) incomparable.
(ii) Any subsequence of a pairwise weight incomparable sequence in $\mathcal{T}$ is also pairwise weight incomparable.
(iii) Any weight incomparable subset of $\mathcal{T}$ is essentially incomparable.
(iv) Let $A = \{(f, 1) : f \in \{\pm e_j^* : j \in \mathbb{N}\}\}$. Then A is essentialy incomparable, and, additionally, if $B \subset \mathcal{T}$ is essentially incomparable, then the same holds for $A \cup B$.

We can finally describe the rule used to define the norming set $W_{(1)}$ of $\mathfrak{X}_{\mathrm{awi}}^{(1)}$, namely, asymptotically weakly incomparable constraints.

Definition 5.14. Let J be an initial segment of $\mathbb{N}$ or $J = \mathbb{N}$. Then a sequence $(f_j)_{j \in J}$ of functionals with successive supports in $V_{(1)} \setminus \{0\}$ is called asymptotically weakly incomparable (AWI) if each f_j admits a tree analysis $(f_{j,\alpha})_{\alpha \in \mathcal{A}_j}$, $j \in J$, such that the following are satisfied.

(i) There is a partition $\{\bar{f}_j : j \in J\} = C_1^0 \cup C_2^0$, such that C_1^0 is essentially incomparable and C_2^0 is weight incomparable.
(ii) For every $k, j \in J$ with $j \geq k+1$, there exists a partition

$$\{\bar{f}_{j,\alpha} : \alpha \in \mathcal{A}_j \text{ and } |\alpha| = k\} = C_{1,j}^k \cup C_{2,j}^k,$$

such that $\cup_{j=k+1}^{\infty} C_{1,j}^k$ is essentially incomparable and $(C_{2,j}^k)_{j=k+1}^{\infty}$ is pairwise weight incomparable.

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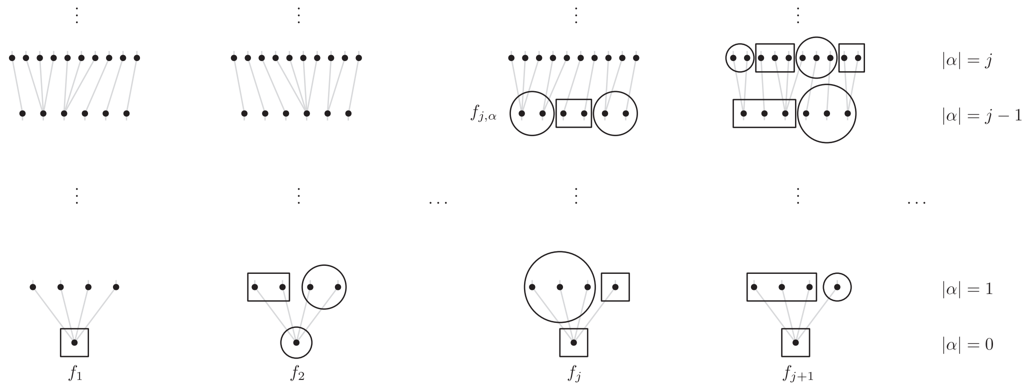

Figure 1. *The collection of nodes of a fixed level in rectangles across all tree analyses forms an essentially incomparable subset, while circles across a fixed level form a family of pairwise weight incomparable subsets.*

Before defining the space $\mathfrak{X}^{(1)}_{\mathrm{awi}}$, we prove that AWI sequences are stable under taking subsequences and under taking restrictions of functionals to subsets. This fact will imply the unconditionality of the basis of $\mathfrak{X}^{(1)}_{\mathrm{awi}}$.

Remark 5.15. Let $f \in V_{(1)}$ and $(f_\alpha)_{\alpha\in\mathcal{A}}$ be a tree analysis of f. Let Δ be a nonempty subset of $\mathrm{supp}(f)$, and set $g = f|_\Delta$. First, note that $g \in V_{(1)}$. Moreover, $(f_\alpha)_{\alpha\in\mathcal{A}}$ naturally induces a tree analysis $(g_\alpha)_{\alpha\in\mathcal{B}}$ for g as follows: $\mathcal{B} = \{\alpha \in \mathcal{A} : \mathrm{supp}(f_\alpha) \cap \Delta \neq \emptyset\}$ and $g_\alpha = f_\alpha|_\Delta$, $\alpha \in \mathcal{B}$. Finally, it is easy to see that $w(g) = w(f)$.

Proposition 5.16. *Let J be an initial segment of $\mathbb{N}$ or $J = \mathbb{N}$ and $(f_j)_{j\in J}$ be an AWI sequence in $V_{(1)}$.*

(i) *Every subsequence of $(f_j)_{j\in J}$ is also an AWI sequence in $V_{(1)}$.*
(ii) *If Δ_j is a nonempty subset of supp(f_j) and $g_j = f_j|_{\Delta_j}$, $j \in J$, then $(g_j)_{j\in J}$ is an AWI sequence in $V_{(1)}$.*
(iii) *If $(g_j)_{j\in J}$ is a sequence in $V_{(1)}$, such that $|g_j| = |f_j|$ for all $j \in J$, then $(g_j)_{j\in J}$ is also AWI.*

Proof. Let for every $j \in J$, $(f_{j,\alpha})_{\alpha\in\mathcal{A}_j}$ be a tree analysis of f_j with

$$\{\bar{f}_j : j \in J\} = C^0_1 \cup C^0_2$$

and for every $k, j \in J$ with $j > k$

$$\{\bar{f}_{j,\alpha} : \alpha \in \mathcal{A}_j \text{ and } |\alpha| = k\} = C^k_{1,j} \cup C^k_{2,j},$$

witnessing that $(f_j)_{j\in J}$ is AWI. We will define the desired partitions proving the cases (i)–(iii).

To prove (i), let N be a subset of J and define

$$F^0_i = \{\bar{f}_j : j \in N\} \cap C^0_i, \quad i = 1, 2.$$

Then $\{\bar{f}_j : j \in N\} = F^0_1 \cup F^0_2$, where F^0_1 is essentially incomparable and F^0_2 is weight incomparable. For the remaining part, let $k \in N$, and note that for $N_k = \{j \in N : j \geq k\}$, $\cup_{j\in N_k} C^k_{1,j}$ is essentially incomparable and $(C^k_{2,j})_{j\in N_k}$ is pairwise weight incomparable.

To prove (ii), Remark 5.15 implies that $g_j \in V_{(1)}$, $w(g_j) = w(f_j)$, and we let $(g_{j,\alpha})_{\alpha\in\mathcal{B}_j}$ be the tree analysis of g induced by $(f_{j,\alpha})_{\alpha\in\mathcal{A}_j}$, $j \in J$. Define

$$F^0_i = \{\bar{g}_j : j \in J \text{ and } g_j = f_j|_{\Delta_j} \text{ with } \bar{f}_j \in C^0_i\}, \quad i = 1, 2,$$

and observe that $\{\bar{g}_j : j \in J\} = F_1^0 \cup F_2^0$. Moreover, for $j \in J$, $\operatorname{supp}(g_j) \subset \operatorname{supp}(f_j)$ and $w(g_j) = w(f_j)$, and, hence, whenever $g_i \neq g_j$ are in F_1^0 with $w(g_i) <_{\mathcal{W}} w(g_j)$, we have $w(f_i) <_{\mathcal{W}} w(f_j)$, implying that the generator $t_3 \in \mathcal{T}$ of $w(f_j) = w(g_j)$ with $w(t_3) = w(f_i) = w(g_i)$ is such that $f_{t_3} < f_i$, and thus $f_{t_3} < g_i$. This yields that F_1^0 is essentially incomparable. Clearly, F_2^0 is weight incomparable. Next, for $k, j \in J$ with $j > k$, define

$$F_{i,j}^k = \{\bar{g}_{j,\alpha} : g_{j,\alpha} = f_{j,\alpha}|_{\Delta_j} \text{ with } \bar{f}_{j,\alpha} \in C_{i,j}^k\}, \quad i = 1, 2.$$

Note that for each $k \in J$, $(F_{2,j}^k)_{j=k+1}^\infty$ is pairwise weight incomparable, and the proof that $\cup_{j=k+1}^\infty F_{1,j}^k$ is essentially incomparable is identical to that for F_0^1. Finally, the proof of (iii) is similar that of (ii). □

Definition 5.17. Let $W_{(1)}$ be the smallest subset of $V_{(1)}$ that is symmetric, contains the singletons and for every $j \in \mathbb{N}$ and every $\mathcal{S}_{n_j}$-admissible AWI sequence $(f_i)_{i=1}^n$ in $W_{(1)}$, we have that $m_j^{-1} \sum_{i=1}^n f_i$ is in $W_{(1)}$. Moreover, let $\mathfrak{X}_{\text{awi}}^{(1)}$ denote the completion of $c_{00}(\mathbb{N})$ with respect to the norm induced by $W_{(1)}$.

Remark 5.18.

(i) The norming set $W_{(1)}$ can be defined as the increasing union of a sequence $(W_{(1)}^n)_{n=0}^\infty$, where $W_{(1)}^0 = \{\pm e_k^* : k \in \mathbb{N}\} \cup \{0\}$ and

$$W_{(1)}^{n+1} = W_{(1)}^n \cup \left\{\frac{1}{m_j} \sum_{l=1}^d f_l : j, d \in \mathbb{N} \text{ and } (f_l)_{l=1}^d \text{ is an } \mathcal{S}_{n_j}\text{-admissible AWI sequence in } W_{(1)}^n\right\}.$$

(ii) Note that Remark 5.13 (iv) implies that any sequence of singletons is AWI. Hence, we have that

$$W_{(1)}^1 = W_{(1)}^0 \cup \left\{\frac{1}{m_j} \sum_{k \in E} \epsilon_k e_k^* : \ j \in \mathbb{N}, \ E \in \mathcal{S}_{n_j} \text{ and } \epsilon_k \in \{-1, 1\} \text{ for } k \in E\right\}.$$

(iii) Proposition 5.16 yields that the standard unit vector basis of $c_{00}(\mathbb{N})$ forms an 1-unconditional Schauder basis for $\mathfrak{X}_{\text{awi}}^{(1)}$.

6. Outline of proof

Although unconditionality of the basis of $\mathfrak{X}_{\text{awi}}^{(1)}$ is almost immediate, it is not, however, straightforward to show that $\mathfrak{X}_{\text{awi}}^{(1)}$ admits ℓ_1 as an asymptotic model. Indeed, this requires Lemma 7.2, which is based on the combinatorial results concerning measures on well-founded trees of Section 4, which first appeared in [8]. This lemma yields that for any choice of successive families $(F_j)_j$ of normalised blocks in $\mathfrak{X}_{\text{awi}}^{(1)}$ and for any $\varepsilon > 0$, we may pass to a subsequence $(F_j)_{j \in M}$ and find a family $(G_j)_{j \in M}$ of subsets of $W_{(1)}$, such that for any choice of $x_j \in F_j$, $j \in M$, there is a $g_j \in G_j$ with $g_j(x_j) > 1 - \varepsilon$ so that $(g_j)_{j \in M}$ is AWI. Thus, we are able to prove, employing Lemma 3.4, the aforementioned result.

To prove the nonexistence of Asymptotic ℓ_1 subspaces in $\mathfrak{X}_{\text{awi}}^{(1)}$, we start with the notion of exact pairs. This is a key ingredient in the study of Mixed Tsirelson spaces, used for the first time by Schlumprecht [26].

Definition 6.1. We call a pair (x, f), where $x \in \mathfrak{X}_{\text{awi}}^{(1)}$ and $f \in W_{(1)}$, an m_j-exact pair if the following hold.

(i) $\|x\| \leq 3$, $f(x) = 1$ and $w(f) = m_j$.
(ii) If $g \in W_{(1)}$ with $w(g) < w(f)$, then $|g(x)| \leq 18 w(g)^{-1}$.
(iii) If $g \in W_{(1)}$ with $w(g) > w(f)$, then $|g(x)| \leq 6(m_j^{-1} + m_j w(g)^{-1})$.

If, additionally, for every $g \in W_{(1)}$ that has a tree analysis $(g_\alpha)_{\alpha \in \mathcal{A}}$, such that $w(g_\alpha) \neq m_j$ for all $\alpha \in \mathcal{A}$, we have $|g(x)| \leq 18 m_j^{-1}$, then we call (x, f) a strong exact pair.

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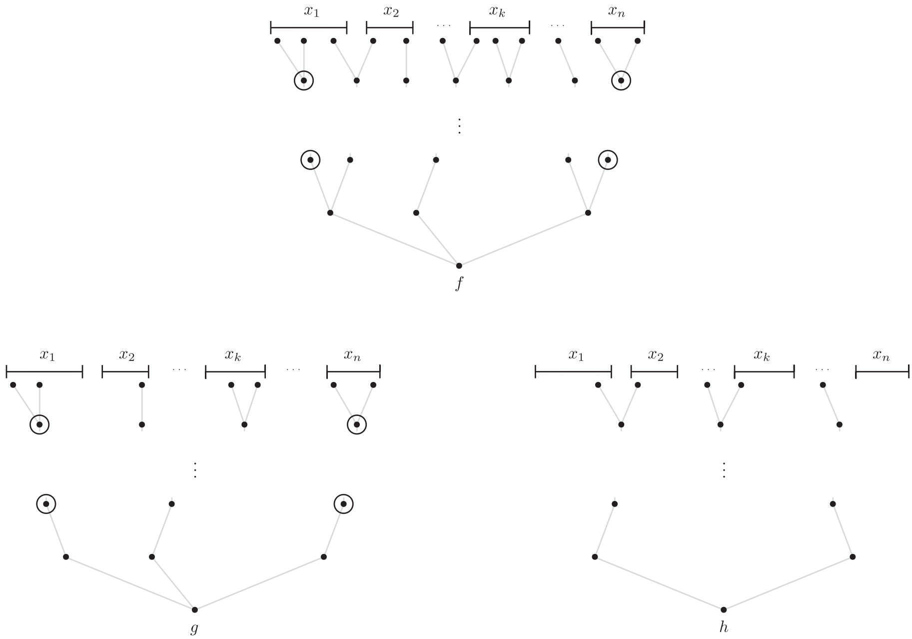

Figure 2. *The tree analysis of f and the induced tree analyses of g and h. The circled nodes α are such that $w(f_\alpha) = w(f_k)$ and $supp(x_k) \cap supp(f_\alpha) \neq \emptyset$ for some $k \in \{1, \dots, n\}$.*

That is, roughly speaking, for an exact pair (x, f), the evaluation of a functional g in $W_{(1)}$, on x, admits an upper bound depending only on the weight of g. In the case of an m_j-strong exact pair (x, f), any g in $W_{(1)}$ with a tree analysis $(g_\alpha)_{\alpha\in\mathcal{A}}$, such that $w(g_\alpha) \neq m_j$, has negligible evaluation on x. We will consider certain exact pairs which we call standard exact pairs (SEP) (see Definition 8.7) and which we prove to be strong exact pairs. It is the case that such pairs can be found in any block subspace of $\mathfrak{X}^{(1)}_{\text{awi}}$, and this is used to prove the reflexivity of $\mathfrak{X}^{(1)}_{\text{awi}}$ as well as the following proposition which yields the nonexistence of Asymptotic ℓ_1 subspaces.

Proposition 6.2. *Given $0 < c < 1$, there is $n \in \mathbb{N}$ so that in any block subspace Y there is a sequence $(x_1, f_1), \dots, (x_n, f_n)$ of SEPs, where $x_i \in Y$, $i = 1, \dots, n$, with $\bar{f}_1 <_{\mathcal{T}} \dots <_{\mathcal{T}} \bar{f}_n$, such that $\|x_1 + \cdots + x_n\| < c\, n$.*

To this end, we first employ the following lemma that highlights the importance of the asymptotically weakly incomparable constraints.

Lemma 6.3. *Let $(x_1, f_1), \dots, (x_n, f_n)$ be SEPs with $\bar{f}_1 <_{\mathcal{T}} \dots <_{\mathcal{T}} \bar{f}_n$. Then, for any $f \in W_{(1)}$ with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ and $k \in \mathbb{N}$, the number of f_i's, $i = 1, \dots, n$, such that there exists $\alpha \in \mathcal{A}$ with $|\alpha| = k$, $w(f_i) = w(f_\alpha)$ and $supp(x_i) \cap supp(f_\alpha) \neq \emptyset$, is at most $ek!$, where e denotes Euler's number.*

Then, we consider a sequence of standard exact pairs $(x_1, f_1), \dots, (x_n, f_n)$ with $\bar{f}_1 <_{\mathcal{T}} \dots <_{\mathcal{T}} \bar{f}_n$ and fix $0 < c < 1$. Pick an $m \in \mathbb{N}$, such that $3/2^m < c$. For $f \in W_{(1)}$, with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$, we consider partitions $f = g + h$ and $g = g_1 + g_2$ as follows: First, set

$$G = \cup\{\text{range}(x_k) \cap \text{range}(f_\alpha) : k \in \{1, \dots, n\} \text{ and } \alpha \in \mathcal{A} \text{ with } w(f_\alpha) = w(f_k)\},$$

and define $g = f|_G$ and $h = f|_{\mathbb{N}\setminus G}$ (see Figure 2).

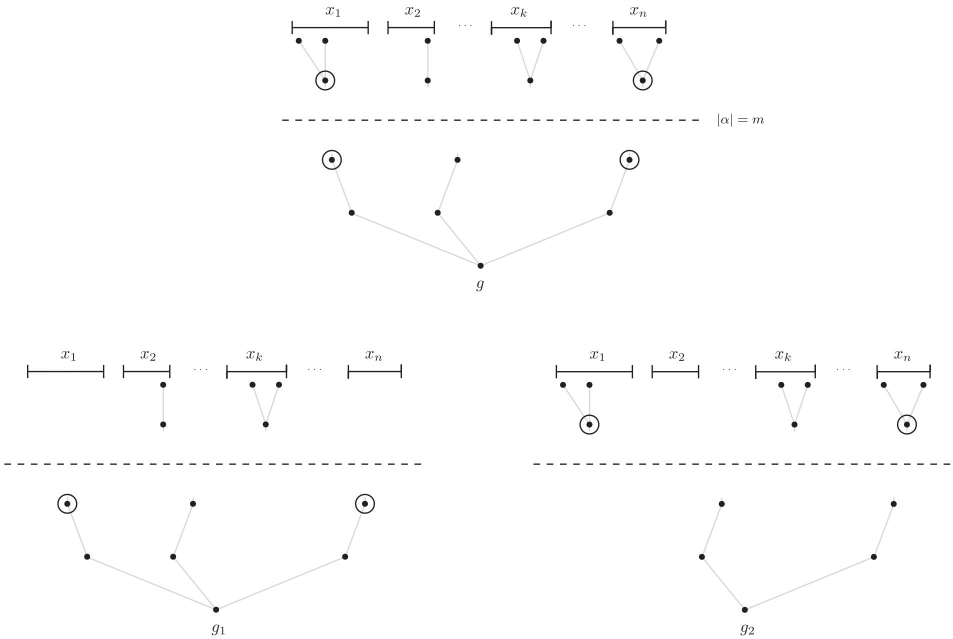

Figure 3. *We consider the m-th level of the induced tree analysis of g. Nodes α with $w(f_\alpha) = w(f_k)$ and $|\alpha| \le m$ are used to define g_1, while such nodes of height greater than m define g_2, restricted on each x_k for $k = 1, \dots, n$.*

To define g_1, consider the tree analysis $(g_\alpha)_{\alpha \in \mathcal{A}_g}$ of g that is induced by $(f_\alpha)_{\alpha \in \mathcal{A}}$, that is, $g_\alpha = f_\alpha|_G$ for $\alpha \in \mathcal{A}_g$ and $\mathcal{A}_g = \{\alpha \in \mathcal{A} : \operatorname{supp}(f_\alpha) \cap G \neq \emptyset\}$. Then, we define

$$\mathcal{B}_k^1 = \{\alpha \in \mathcal{A}_g : |a| \le m,\ w(f_\alpha) = w(f_k) \text{ and } w(f_\beta) \neq w(f_k) \text{ for all } \beta < \alpha \text{ in } \mathcal{A}_g\}$$

for $k = 1, \dots, n$,

$$G_1 = \cup_{k=1}^n \cup \left\{\operatorname{supp}(g_\alpha) \cap \operatorname{supp}(x_k) : \alpha \in \mathcal{B}_k^1\right\},$$

and finally $g_1 = g|_{G_1}$ (see Figure 3). Observe that Lemma 6.3 implies that

$$\#\{k \in \{1, \dots, n\} : g_1(x_k) \neq 0\} \le \ell = e \sum_{k=1}^{m} k!. \tag{6.1}$$

Moreover, the induced tree analysis $(h_\alpha)_{\alpha \in \mathcal{A}_h}$ of h is such that $w(h_\alpha) \neq w(f_k)$ for all $k = 1, \dots, n$, and, therefore, the fact that (x_k, f_k) are strong exact pairs yields

$$|h(x_k)| \le \frac{18}{w(f_k)}, \qquad k = 1, \dots, n. \tag{6.2}$$

Considering a further partition of $g_2|_{\operatorname{supp}(x_k)}$, we show that

$$|g_2(x_k)| \le \frac{18}{w(f_k)} + \frac{3}{2^m}, \qquad k = 1, \dots, n. \tag{6.3}$$

Hence, (6.1), (6.2) and (6.3) imply

$$\left|f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right| \leq \frac{36+\ell}{n} + \frac{3}{2^m}.$$

Then, our choice of m yields Proposition 6.2 for sufficiently large n and $w(f_1)$, where $w(f_1)$ is chosen appropriately to deal with the case where $f \in \{\pm e_j^* : j \in \mathbb{N}\}$.

Assuming that Y is a C-asymptotic ℓ_1 block subspace of $\mathfrak{X}^{(1)}_{\text{awi}}$, we pick a sequence of standard exact pairs $(x_1, f_1), \ldots, (x_n, f_n)$ in $Y \times W_{(1)}$, satisfying the conclusion of Proposition 6.2 and derive a contradiction.

The remainder of this part of the paper is organised as follows. In Section 7, we prove that $\mathfrak{X}^{(1)}_{\text{awi}}$ admits ℓ_1 as a unique asymptotic model. Next, in Section 8, we prove existence and properties of standard exact pairs. The final section of this part contains the results leading up to the proof that $\mathfrak{X}^{(1)}_{\text{awi}}$ does not contain Asymptotic ℓ_1 subspaces.

7. Asymptotic models generated by block sequences of $\mathfrak{X}^{(1)}_{\text{awi}}$

We show that the space $\mathfrak{X}^{(1)}_{\text{awi}}$ admits a unique asymptotic model, or equivalently, a uniformly unique joint spreading model with respect to $\mathscr{F}_b(\mathfrak{X}^{(1)}_{\text{awi}})$ that is equivalent to the unit vector basis of ℓ_1. The key ingredient in the proof is the following lemma concerning bounded positive measures on the tree of initial segments of $\mathcal{T}$.

Remark 7.1. Let us first recall some notation from Section 4. We denote by $\widetilde{\mathcal{T}}$ the tree of initial segments of $\mathcal{T}$ equipped with the partial order induced by inclusion and consider the isomorphism $t \mapsto \tilde{t} = \{s \in \mathcal{T} : s \leq_{\mathcal{T}} t\}$, between $\mathcal{T}$ and $\widetilde{\mathcal{T}}$. Similarly, by $\widetilde{\mathcal{W}}$, we denote the tree of initial segments of $\mathcal{W}$ and consider the isomorphism $w \mapsto \tilde{w} = \{v \in \mathcal{W} : v \leq_{\mathcal{W}} w\}$ between $\mathcal{W}$ and $\widetilde{\mathcal{W}}$. Finally, for $t \in \mathcal{T}$, we set $\tilde{w}(t) = \{\tilde{w} \in \widetilde{\mathcal{W}} : w \leq_{\mathcal{W}} w(t)\}$.

Lemma 7.2. *Let $(\mu_i)_i$ be a bounded finitely and disjointly supported sequence in $\mathcal{M}_+(\widetilde{\mathcal{T}})$. Assume that the sets $\cup\{supp(f_t) : \tilde{t} \in supp(\mu_i)\}$, $i \in \mathbb{N}$, are disjoint. Then, for every $\varepsilon > 0$, there exists an infinite subset of the natural numbers L and for each $i \in L$ subsets G_i^1, G_i^2 of $\widetilde{\mathcal{T}}$, such that*

(i) *G_i^1, G_i^2 are disjoint subsets of $supp(\mu_i)$ for every $i \in L$,*
(ii) *$\mu_i(\widetilde{\mathcal{T}} \setminus G_i^1 \cup G_i^2) < \varepsilon$ for every $i \in L$,*
(iii) *$\{t \in \mathcal{T} : \tilde{t} \in \cup_{i\in L} G_i^1\}$ is essentially incomparable and*
(iv) *if $F_i^2 = \{t \in \mathcal{T} : \tilde{t} \in G_i^2\}$, $i \in L$, then the sequence $(F_i^2)_{i\in L}$ is pairwise weight incomparable.*

Proof. Passing to a subsequence if necessary, we may assume that the (unique) root of $\widetilde{\mathcal{T}}$ is not in the support of any μ_i, $i \in \mathbb{N}$, succ-lim$_i\, \mu_i$ exists and that there exist partitions $\text{supp}(\mu_i) = A_i \cup B_i$, $i \in \mathbb{N}$, satisfying the conclusion of Lemma 4.6. Define for each $i \in \mathbb{N}$, the measures $\mu_i^1, \mu_i^2 \in \mathcal{M}_+(\widetilde{\mathcal{T}})$ given by $\mu_i^1(C) = \mu_i(A_i \cap C)$ and $\mu_i^2(C) = \mu_i(B_i \cap C)$, and let $\nu = w^*\text{-}\lim_i \mu_i^1 = \text{succ-lim}_i\, \mu_i^1$. Pick a finite subset F of $\widetilde{\mathcal{T}}$, such that $\nu(\widetilde{\mathcal{T}} \setminus F) < \varepsilon/2$. Then, $\nu = w^*\text{-}\lim_i \mu_i^1$ implies that $\lim_i \mu_i^1(\widetilde{\mathcal{T}}) = \nu(\widetilde{\mathcal{T}})$, and, thus, since $\nu = \text{succ-lim}_i\, \mu_i^1$, we have

$$\lim_i \left|\mu_i^1(\widetilde{\mathcal{T}}) - \mu_i^1(\cup_{\tilde{t}\in F} S(\tilde{t}))\right| = \left|\nu(\widetilde{\mathcal{T}}) - \lim_i \sum_{\tilde{t}\in F} \mu_i^1(S(\tilde{t}))\right| = \nu(\widetilde{\mathcal{T}} \setminus F) < \frac{\varepsilon}{2}.$$

Hence, we can find $i_0 \in \mathbb{N}$, such that for all $i \geq i_0$, we have

$$\left|\mu_i(A_i) - \mu_i\Big(A_i \cap \big(\cup_{\tilde{t}\in F} S(\tilde{t})\big)\Big)\right| = \left|\mu_i^1(\widetilde{\mathcal{T}}) - \mu_i^1(\cup_{\tilde{t}\in F} S(\tilde{t}))\right| < \frac{\varepsilon}{2}. \tag{7.1}$$

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We set $\Sigma = \sigma(\{t \in \mathcal{T} : \tilde{t} \in F\})$ and

$$R = \{r \in \mathcal{T} : w(r) \in \Sigma, \text{ and there is } s \in \mathcal{T} \text{ with } w(s) \in \Sigma, \text{ such that } r <_{\mathcal{T}} s\}.$$

Note that Σ and R are finite, since F is finite. Thus, using the fact that the sets $\cup\{\operatorname{supp}(f_t) : \tilde{t} \in \operatorname{supp}(\mu_i)\}$ for $i \in \mathbb{N}$ are disjoint, find $i_1 \in \mathbb{N}$ with $i_1 \geq i_0$ so that

$$\cup_{r \in R} \operatorname{supp}(f_r) < \operatorname{supp}(f_t), \text{ for all } \tilde{t} \in \cup_{i \geq i_1} \operatorname{supp}(\mu_i^1). \tag{7.2}$$

For $G_i^1 = A_i \cap (\cup_{\tilde{t} \in F} S(\tilde{t}))$, (7.1) implies that $|\mu_i(A_i) - \mu_i(G_i^1)| < \varepsilon/2$, $i \geq i_1$. We will show that $\{t \in \mathcal{T} : \tilde{t} \in \cup_{i \geq i_1} G_i^1\}$ is essentially incomparable, that is, that (iii) is satisfied. To this end, first observe that if $\tilde{t} \in \cup_{i \geq i_1} G_i^1$, then $w(t) \in \Sigma$. Let $\tilde{t}_1, \tilde{t}_2 \in \cup_{i \geq i_1} G_i^1$ with $w(t_1) <_{\mathcal{W}} w(t_2)$. It is immediate that if $t_3 \in \mathcal{T}$ is the generator of $w(t_2)$ with $w(t_3) = w(t_1)$, then $t_3 \in R$, and, hence, (7.2) implies that $f_{t_3} < f_{t_1}$, proving the desired result.

For the remaining part of the proof, recall the root of $\widetilde{\mathcal{T}}$ avoids the supports of all μ_i^2, $i \geq i_1$. This implies that every $\tilde{t} \in \cup_{i \geq i_1} \operatorname{supp}(\mu_i^2)$ is the successor of some node in $\widetilde{\mathcal{T}}$. Then, since for all $i \geq i_1$, the set $B_i = \operatorname{supp}(\mu_i^2)$ is finite (as a subset of the finite support of μ_i), and for each $\tilde{t} \in \widetilde{\mathcal{T}}$, the sequence $(\mu_i^2(S(\tilde{t})))_{i \geq i_1}$ is eventually zero, we may pass to a subsequence so that for all $i_1 \leq i < j$, we have $\{w(t) : \tilde{t} \in \operatorname{supp}(\mu_i^2)\} \cap \{w(t) : \tilde{t} \in \operatorname{supp}(\mu_j^2)\} = \emptyset$. We can, therefore, define the bounded sequence of disjointly supported measures $(\nu_i)_{i \geq i_1}$ on $\widetilde{\mathcal{W}}$ given by $\nu_i(\{\tilde{w}\}) = \mu_i^2(\{\tilde{t} \in \widetilde{\mathcal{T}} : \tilde{w}(t) = \tilde{w}\})$. Hence, applying Proposition 4.1 and passing to a subsequence, we obtain a subset E_i of $\operatorname{supp}(\nu_i)$, such that $\nu_i(\widetilde{\mathcal{W}} \setminus E_i) < \varepsilon/2$ and the sets E_i, $i \geq i_1$, are pairwise incomparable. It is easy to verify that if $G_i^2 = \{\tilde{t} \in B_i : \tilde{w}(t) \in E_i\}$ and $F_i^2 = \{t \in \mathcal{T} : \tilde{t} \in G_i^2\}$, $i \geq i_1$, then $(F_i^2)_{i \geq i_1}$ is pairwise weight incomparable and $|\mu_i(B_i) - \mu_i(G_i^2)| = \mu_i^2(\widetilde{\mathcal{T}} \setminus G_i^2) < \varepsilon/2$ for every $i \geq i_1$. □

Lemma 7.3. *Let $x \in \mathfrak{X}_{awi}^{(1)}$, $f \in W_{(1)}$ and a tree analysis $(f_\alpha)_{\alpha \in \mathcal{A}}$ of f, such that $f_\alpha(x) \geq 0$ for every $\alpha \in \mathcal{A}$. Let $\varepsilon_1, \ldots, \varepsilon_{h(\mathcal{A})}$ be positive reals and G_i be a subset of $\{\alpha \in \mathcal{A} : |\alpha| = i\}$, such that $\sum_{\alpha \in G_i} w_f(f_\alpha)^{-1} f_\alpha(x) > f(x) - \varepsilon_i$ for every $1 \leq i \leq h(\mathcal{A})$, and $f(x) > \sum_{i=1}^{h(\mathcal{A})} \varepsilon_i$. Then, there exists a $g \in W_{(1)}$ satisfying the following conditions.*

- (i) *$\operatorname{supp}(g) \subset \operatorname{supp}(f)$ and $w(g) = w(f)$.*
- (ii) *$g(x) > f(x) - \sum_{i=1}^{h(\mathcal{A})} \varepsilon_i$.*
- (iii) *g has a tree analysis $(g_\alpha)_{\alpha \in \mathcal{A}_g}$, such that for every $\alpha \in \mathcal{A}_g$, there is a unique $\beta \in G_{|\alpha|}$ with $\operatorname{supp}(g_\alpha) \subset \operatorname{supp}(f_\beta)$ and $w(g_\alpha) = w(f_\beta)$.*

Proof. Let $\mathcal{A}_k$ denote the set of all nodes in $\mathcal{A}$, such that $|\alpha| = k$, $1 \leq k \leq h(\mathcal{A})$. We define g by constructing the tree analysis $(g_\alpha)_{\alpha \in \mathcal{A}_g}$. First, define by induction $B_1 = \mathcal{A}_1 \setminus G_1$ and for $2 \leq k \leq h(\mathcal{A})$:

$$B_k = \{\alpha \in \mathcal{A}_k : \alpha \notin G_k \text{ or there is a } \beta \in B_{k-1}, \text{ such that } \alpha \in S(\beta)\}.$$

It follows easily that $\alpha \in B_k$ if and only if there exists $\beta \leq \alpha$, such that $\beta \notin G_{|\beta|}$. Let $\mathcal{C}_g = \mathcal{A}_{h(\mathcal{A})} \setminus B_{h(\mathcal{A})}$. Note that $f_\alpha \in \{\pm e_j^* : j \in \mathbb{N}\}$ for every $\alpha \in \mathcal{C}_g$, and let $\Delta_g = \cup\{\operatorname{supp}(f_\alpha) : \alpha \in \mathcal{C}_g\}$. Then $g = f|_{\Delta_g}$ and $(g_\alpha)_{\alpha \in \mathcal{A}_g}$ is the tree analysis induced by $(f_\alpha)_{\alpha \in \mathcal{A}}$.

Observe that, by construction, g satisfies (i) and (iii). To see that it also satisfies (ii), we show by induction that for every $1 \leq k \leq h(\mathcal{A})$

$$\sum_{\alpha \in \mathcal{A}_k \setminus B_k} \frac{f_\alpha(x)}{w_f(f_\alpha)} > f(x) - \sum_{i=1}^{k} \varepsilon_i. \tag{7.3}$$

This indeed proves (ii), since the left-hand side of (7.3) for $k = h(\mathcal{A})$ is equal to $g(x)$. We now prove (7.3) by induction. Assume that the inequality holds for some $1 \leq k < h(\mathcal{A})$. Then, for every $\alpha \in \mathcal{A}_k \setminus B_k$, we have

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$$f_\alpha(x) = \sum_{\beta\in S(\alpha)\cap G_{k+1}} \frac{f_\beta(x)}{w(f_\alpha)} + \sum_{\beta\in S(\alpha)\setminus G_{k+1}} \frac{f_\beta(x)}{w(f_\alpha)}$$

and

$$\sum_{\alpha\in\mathcal{A}_k\setminus B_k}\sum_{\beta\in S(\alpha)\setminus G_{k+1}} \frac{f_\beta(x)}{w_f(f_\alpha)w(f_\alpha)} = \sum_{\alpha\in\mathcal{A}_k\setminus B_k}\sum_{\beta\in S(\alpha)\setminus G_{k+1}} \frac{f_\beta(x)}{w_f(f_\beta)} < \varepsilon_{k+1}.$$

Hence

$$\begin{aligned}\sum_{\alpha\in\mathcal{A}_k\setminus B_k}\sum_{\beta\in S(\alpha)\cap G_{k+1}} \frac{f_\beta(x)}{w_f(f_\alpha)w(f_\alpha)} &= \sum_{\alpha\in\mathcal{A}_k\setminus B_k}\frac{f_\alpha(x)}{w_f(f_\alpha)} - \sum_{\alpha\in\mathcal{A}_k\setminus B_k}\sum_{\beta\in S(\alpha)\setminus G_{k+1}} \frac{f_\beta(x)}{w_f(f_\beta)}\\ &> \Big(f(x) - \sum_{i=1}^{k}\varepsilon_i\Big) - \varepsilon_{k+1}\end{aligned}$$

which, along with the previous inequality, proves the desired result since

$$\{\beta\in\mathcal{A} : \beta\in S(a)\setminus G_{k+1} \text{ for some } \alpha\in\mathcal{A}_k\setminus B_k\} = \mathcal{A}_{k+1}\setminus B_{k+1}.$$

□

Lemma 7.4. *Let $(x_j^1)_j,\dots,(x_j^l)_j$ be normalised block sequences in $\mathfrak{X}^{(1)}_{awi}$. For every $\varepsilon>0$, there exists an $L\in[\mathbb{N}]^\infty$ and a $g_j^i\in W_{(1)}$ with $g_j^i(x_j^i)>1-\varepsilon$, $1\le i\le l$ and $j\in L$, such that for any choice of $1\le i_j\le l$, the sequence $(g_j^{i_j})_{j\in L}$ is AWI.*

Proof. Let $(\varepsilon_k)_{k=0}^\infty$ be a sequence of positive reals, such that $\sum_{k=0}^\infty \varepsilon_k<\varepsilon/2$. For every $1\le i\le l$ and $j\in\mathbb{N}$, pick an $f_j^i\in W_{(1)}$ and a tree analysis $(f_{j,\alpha}^i)_{\alpha\in\mathcal{A}_j^i}$ of f_j^i, such that $f_j^i(x_j^i)>1-\varepsilon/2$ and $f_{j,\alpha}^i(x_j^i)>0$ for every $\alpha\in\mathcal{A}_j^i$. For $1\le i\le l$ and $j\in\mathbb{N}$, we set $t_j^i=\bar f_j^i$ and $t_{j,\alpha}^i=\bar f_{j,\alpha}^i$, $\alpha\in\mathcal{A}_j^i$. We will choose, by induction, an $L\in[\mathbb{N}]^\infty$ and, for every $1\le i\le l$, $j\in L$ and $k\in\mathbb{N}$, a subset $G_j^{k,i}$ of $\{\alpha\in\mathcal{A}_j^i : |\alpha|=k\}$ satisfying the following conditions. For $k\in\mathbb{N}$, we set $L_{>k}=\{j\in L : j>k\}$.

(i) For every $j\in L$, there is a partition $\{t_j^i : i=1,\dots,l\}=C_{1,j}^0\cup C_{2,j}^0$, such that $\cup_{j\in L}C_{1,j}^0$ is essentially incomparable and $(C_{2,j}^0)_{j\in L}$ is pairwise weight incomparable.

(ii) For every $1\le i\le l$, $k\in\mathbb{N}$ and $j\in L_{>k}$, there is a partition $G_j^{k,i}=G_{1,j}^{k,i}\cup G_{2,j}^{k,i}$, such that for any choice of $1\le i_j\le l$, $\cup_{j\in L_{>k}}\{t_{j,\alpha}^{i_j} : \alpha\in G_{1,j}^{k,i_j}\}$ is essentially incomparable and $(\{t_{j,\alpha}^{i_j} : \alpha\in G_{2,j}^{k,i_j}\})_{j\in L_{>k}}$ is pairwise weight incomparable.

(iii) For every $i=1,\dots,l$, $j\in L$ and $k\in\mathbb{N}$ with $k\le h(\mathcal{A}_j^i)$

$$\sum_{\alpha\in G_j^{k,i}} w_{f_j^i}(f_{j,\alpha}^i)^{-1} f_{j,\alpha}^i(x_j^i) > f_j^i(x_j^i)-\varepsilon_k.$$

Observe then that (iii) and an application of the previous lemma yield, for every $1\le i\le l$ and $j\in L$, a functional $g_j^i\in W_{(1)}$, such that

$$g_j^i(x_j^i) > f_j^i(x_j^i) - \sum_{k=1}^\infty \varepsilon_k > 1-\varepsilon.$$

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Fix a choice of $1 \leq i_j \leq l$, $j \in L$. Then, (i) implies that $\{t_j^{i_j} : j \in L\} \cap (\cup_{j \in L} C^0_{1,j})$ is essentially incomparable, and that $\{t_j^{i_j} : j \in L\} \cap (\cup_{j \in L} C^0_{2,j})$ is weight incomparable. Finally, (ii) and Lemma 7.3 (iii) yield that, for every $1 \leq i \leq l$, $k \in \mathbb{N}$ and $j \in L_{>k}$, there is a partition

$$\{\bar{g}^{i_j}_{j,\alpha} : a \in \mathcal{A}^{i_j}_j \text{ and } |\alpha| = k\} = C^{k,i_j}_{1,j} \cup C^{k,i_j}_{2,j},$$

such that $\cup_{j \in L_{>k}} C^{k,i_j}_{1,j}$ is essentially incomparable and $(C^{k,i_j}_{2,j})_{j \in L_{>k}}$ is pairwise weight incomparable. Hence, g^i_j, $1 \leq i \leq l$ and $j \in L$ satisfy the desired conditions.

To obtain L, let us first assume that $\sup_{i,j} h(\mathcal{A}^i_j) = +\infty$ (if $\sup_{i,j} h(\mathcal{A}^i_j) < +\infty$, then a finite version of the same proof works). Moreover, passing to a subsequence, we may further assume that $\max_i h(\mathcal{A}^i_j) > k$ whenever $j > k$, for $j, k \in \mathbb{N}$. Define, for each $j \in \mathbb{N}$, the measure μ^0_j on $\widetilde{\mathcal{T}}$ given by

$$\mu^0_j = \sum_{i=1}^{l} f^i_j(x^i_j)\delta_{\tilde{t}^i_j}.$$

Applying Lemma 7.2, we obtain an $L_0 \in [\mathbb{N}]^\infty$, such that, for every $j \in L_0$, there exist disjoint subsets $G^0_{1,j}$ and $G^0_{2,j}$ of $\operatorname{supp}(\mu^0_j)$ so that the following hold.

(α_0) $\mu^0_j(\widetilde{\mathcal{T}} \setminus G^0_{1,j} \cup G^0_{2,j}) < \varepsilon_0$ for every $j \in L_0$.
(β_0) Define $C^0_{1,j} = \{t \in \mathcal{T} : \tilde{t} \in G^0_{1,j}\}$ for $j \in L_0$. Then $\cup_{j \in L_0} C^0_{1,j}$ is essentially incomparable.
(γ_0) Define $C^0_{2,j} = \{t \in \mathcal{T} : \tilde{t} \in G^0_{2,j}\}$ for $j \in L_0$. Then the sequence $(C^0_{2,j})_{j \in L_0}$ is pairwise weight incomparable.

Note that (α_0) implies that $\operatorname{supp}(\mu^0_j) = G^0_{1,j} \cup G^0_{2,j}$ since $f^i_j(x^i_j) > 1 - \varepsilon/2$, that is, $\{\tilde{t}^i_j : i = 1, \dots, l\} = C^0_{1,j} \cup C^0_{2,j}$. We proceed by induction on $\mathbb{N}$. Suppose we have chosen $L_0, \dots, L_{k-1}$ and $G^0_{1,j_0}, G^0_{2,j_0}, \dots, G^{k-1}_{1,j_{k-1}}, G^{k-1}_{2,j_{k-1}}$ for some $k \in \mathbb{N}$ and every $j_i \in L_i$, for $i = 0, \dots, k-1$. Set $L^0_k = \{j \in L_{k-1} : h(\mathcal{A}^i_j) < k \text{ for all } 1 \leq i \leq l\}$. Then, for each $j \in L_{k-1} \setminus L^0_k$, define the following measure on $\widetilde{\mathcal{T}}$

$$\mu^k_j = \sum_{i=1}^{l} \sum_{\substack{\alpha \in \mathcal{A}^i_j \\ |\alpha| = k}} \frac{f^i_{j,\alpha}(x^i_j)}{w_{f^i_j}(f^i_{j,\alpha})} \delta_{\tilde{t}^i_{j,\alpha}}.$$

Again, applying Lemma 7.2 yields an $L^1_k \in [L_{k-1} \setminus L^0_k]^\infty$ and disjoint subsets $G^k_{1,j}$ and $G^k_{2,j}$ of $\operatorname{supp}(\mu^k_j)$, $j \in L^1_k$, such that

(α_k) $\mu^k_j(\widetilde{\mathcal{T}} \setminus G^k_{1,j} \cup G^k_{2,j}) < \varepsilon_k$ for every $j \in L^1_k$,
(β_k) $\{t \in \mathcal{T} : \tilde{t} \in \cup_{j \in L^1_k} G^k_{1,j}\}$ is essentially incomparable and
(γ_k) the sequence $(\{t : \tilde{t} \in G^k_{2,j}\})_{j \in L^1_k}$ is pairwise weight incomparable.

Then, set $L_k = L^0_k \cup L^1_k$ and $G^k_{i,j} = \{\alpha \in \mathcal{A}^i_j : |\alpha| = k\}$, for $1 \leq i \leq l$ and $j \in L^0_k$. Finally, choose L to be a diagonalisation of $(L_k)_k$, that is, $L(k) \in L_k$ for $k \in \mathbb{N}$. Observe that (β_k) and (γ_k) imply (ii), while (α_k) implies (iii). □

Proposition 7.5. *The space $\mathfrak{X}^{(1)}_{awi}$ admits a unique asymptotic model, with respect to $\mathscr{F}_b(\mathfrak{X}^{(1)}_{awi})$, equivalent to the unit vector basis of ℓ_1.*

Proof. Equivalently, we will show that $\mathfrak{X}^{(1)}_{\text{awi}}$ admits ℓ_1 as a uniformly unique joint spreading model with respect to $\mathscr{F}_b(\mathfrak{X}^{(1)}_{\text{awi}})$. To this end, let $(x^1_j)_j, \dots, (x^l_j)_j$ be normalised block sequences in $\mathfrak{X}^{(1)}_{\text{awi}}$. Passing to

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a subsequence, we may assume that $\mathrm{supp}(x_j^{i_1}) < \mathrm{supp}(x_{j+1}^{i_2})$ for every $i_1, i_2 = 1, \dots, l$ and $j \in \mathbb{N}$. Fix $\varepsilon > 0$ and apply Lemma 7.4 to obtain an $L \in [\mathbb{N}]^\infty$ and a functional $g_j^i \in W_{(1)}$, for each $1 \le i \le l$ and $j \in L$, such that

(i) $\mathrm{supp}(g_j^i) \subset \mathrm{supp}(x_j^i)$ and $g_j^i(x_j^i) > 1 - \varepsilon$, for all $1 \le i \le l$ and $j \in L$ and
(ii) the sequence $(g_j^{i_j})_{j \in L}$ is AWI for any choice of $1 \le i_j \le l$, $j \in L$.

Fix a choice of $1 \le i_j \le l$, $j \in L$, and let $k \in \mathbb{N}$ and $F \subset L$ with $L(k) \le F$ and $|F| \le k$. Note that $(g_j^{i_j})_{j \in F}$ is an $\mathcal{S}_1$-admissible sequence in $W_{(1)}$ and is in fact AWI, as implied by (ii). Hence, $g = 1/2 \sum_{j \in F} g_j^{i_j}$ is in $W_{(1)}$, and, thus, for any choice of scalars $(a_j)_{j \in F}$, we calculate

$$\Big\| \sum_{j \in F} a_j x_j^{i_j} \Big\| \ge \Big\| \sum_{j \in F} |a_j| x_j^{i_j} \Big\| \ge g\Big(\sum_{j \in F} |a_j| x_j^{i_j} \Big) \ge \frac{1 - \varepsilon}{2} \sum_{j \in F} |a_j|.$$

Then, Lemma 3.4 yields the desired result. □

8. Standard exact pairs

We pass to the study of certain basic properties of Mixed Tsirelson spaces which have appeared in several previous papers (see [5] and [9]). The goal of this section is to define the standard exact pairs in $\mathfrak{X}_{\mathrm{awi}}^{(1)}$ and present their basic properties. In the next section, we will use the existence of sequences of such pairs in any block subspace of $\mathfrak{X}_{\mathrm{awi}}^{(1)}$ to show that it is not Asymptotic ℓ_1. The proof of the properties of the standard exact pairs are based on the definition of an auxiliary space and the basic inequality which are given in Appendix A.

8.1. Special convex combinations

We return our attention to special convex combinations, defined in Section 2.3. These types of vectors are used to prove the presence of standard exact pairs in every block subspace of $\mathfrak{X}_{\mathrm{awi}}^{(1)}$.

Remark 8.1. Let $(x_k)_k$ be a block sequence in $\mathfrak{X}_{\mathrm{awi}}^{(1)}$. Then Proposition 2.4 implies that, for every $\varepsilon > 0$, $n, m \in \mathbb{N}$ and $M \in [\mathbb{N}]^\infty$, there exist $F \subset M$ with $m \le F$ and scalars $(a_k)_{k \in F}$, such that $\sum_{k \in F} a_k x_k$ is a (n, ε)-s.c.c.

Lemma 8.2. *Let $(x_k)_k$ be a normalised block sequence in $\mathfrak{X}_{awi}^{(1)}$. For every $\varepsilon > 0$, there exists $M \in [\mathbb{N}]^\infty$, such that for every $j \in \mathbb{N}$, every $\mathcal{S}_{n_j}$-admissible sequence $(x_k)_{k \in F}$ with $F \subset M$ and any choice of scalars $(a_k)_{k \in F}$, we have*

$$\Big\| \sum_{k \in F} a_k x_k \Big\| \ge \frac{1 - \varepsilon}{m_j} \sum_{k \in F} |a_k|.$$

Proof. Apply Lemma 7.4 to obtain $M \in [\mathbb{N}]^\infty$ and an $f_k \in W_{(1)}$ with $f_k(x_k) > 1 - \varepsilon$, for each $k \in M$, such that $(f_k)_{k \in M}$ is AWI. We may also assume that $\mathrm{supp}(f_k) \subset \mathrm{supp}(x_k)$, $k \in M$. Pick an $F \subset M$, such that $(x_k)_{k \in F}$ is S_{n_j}-admissible. Then, $(f_k)_{k \in F}$ is $\mathcal{S}_{n_j}$-admissible and clearly $(f_k)_{k \in F}$ is AWI. Hence, $f = {m_j}^{-1} \sum_{k \in F} f_k$ is in $W_{(1)}$, and we calculate

$$\| \sum_{k \in F} a_k x_k \| = \| \sum_{k \in F} |a_k| x_k \| \ge f(\sum_{k \in F} |a_k| x_k) \ge \frac{1 - \varepsilon}{m_j} \sum_{k \in F} |a_k|.$$

□

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Proposition 8.3. *Let Y be a block subspace of $\mathfrak{X}^{(1)}_{awi}$. Then, for every $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists a (n, ε)-s.c.c. $x = \sum_{k=1}^m c_k x_k$ with $\|x\| > 1/2$, where $x_1, \dots, x_m$ are in the unit ball of Y.*

Proof. Towards a contradiction, assume that the conclusion is false. That is, for any $\mathcal{S}_n$-admissible sequence $(x_k)_{k=1}^m$ in the unit ball of Y, such that the vector $x = \sum_{k=1}^m c_k x_k$ is a (n, ε)-s.c.c., we have that $\|x\| \le 1/2$.

Start with a normalised block sequence $(x_k^0)_k$ in Y and pass to a subsequence satisfying the conclusion of Lemma 8.2 for $\varepsilon = 1/2$. Using the choice of the sequence $(m_k)_k$, we may find $j \in \mathbb{N}$, such that

$$2^{n_j/n} \ge 4m_j. \tag{8.1}$$

Set $d = \lfloor n_j/n \rfloor$ and, using Remark 8.1, define inductively block sequences $(x_k^i)_k$, $i = 1, \dots, d$, such that for each $i = 1, \dots, d$ and $k \in \mathbb{N}$, there is an $\mathcal{S}_n$-admissible sequence $(x_m^{i-1})_{m \in F_k^i}$ and coefficients $(c_m^i)_{m \in F_k^i}$, such that $\tilde{x}_k^i = \sum_{m \in F_k^i} c_m^i x_m^{i-1}$ is a (n, ε)-s.c.c. and $x_k^i = 2\tilde{x}_k^i$.

Using the negation of the desired conclusion, it is straightforward to check by induction that $\|x_k^i\| \le 1$ for every $i = 1, \dots, d$ and $k \in \mathbb{N}$. Moreover, note that each vector x_k^i can be written in the form

$$x_k^i = 2^i \sum_{m \in G_k^i} d_m^i x_m^0$$

for some subset G_k^i of $\mathbb{N}$, such that $(x_m^0)_{m \in G_k^i}$ is $\mathcal{S}_{ni}$-admissible and $\sum_{m \in G_k^i} d_m^i = 1$. As the sequence $(x_k^0)_k$ satisfies the conclusion of Lemma 8.2, we deduce that

$$1 \ge \|x_1^d\| \ge \frac{2^d}{2m_j} > \frac{2^{n_j/n}}{4m_j},$$

since $n_j - n < dn$, and this contradicts (8.1). □

Proposition 8.4. *Let $x = \sum_{i=1}^m c_i x_i$ be a (n, ε)-s.c.c. in $\mathfrak{X}^{(1)}_{awi}$ with $\|x_i\| \le 1$, $i = 1, \dots, m$, and $f \in \mathcal{W}_{(1)}$ with $w(f) = m_j$, such that $n_j < n$. Then we have*

$$|f(x)| \le \frac{1 + 2\varepsilon w(f)}{w(f)}.$$

Proof. Let $f = m_j^{-1} \sum_{l=1}^d f_l$, where $(f_l)_{l=1}^d$ is an $\mathcal{S}_{n_j}$-admissible AWI sequence in $W_{(1)}$, and define

$$A = \Big\{ i \in \{1, \dots, m\} : \text{there is at most one } 1 \le l \le d, \\ \text{such that } \operatorname{range}(x_i) \cap \operatorname{range}(f_l) \ne \emptyset \Big\}.$$

Note that $|f(x_i)| \le 1/m_j$, for each $i \in A$, and, hence

$$\left| f\left(\sum_{i=1}^m c_i x_i \right) \right| \le \frac{1}{m_j} \sum_{i \in A} c_i + \sum_{i \notin A} c_i. \tag{8.2}$$

Set $B = \{1, \dots, m\} \setminus A$. The spreading property of the Schreier families implies that the vectors $(x_i)_{i \in B \setminus \{\min(B)\}}$ are $\mathcal{S}_{n_j}$-admissible. Moreover, the singleton $\{x_{\min B}\}$ is $\mathcal{S}_0$-admissible. Thus, $\sum_{i \in B \setminus \min B} c_i < \varepsilon$ and $c_{\min B} < \varepsilon$. Applying this to (8.2) immediately yields the desired conclusion. □

8.2. Rapidly increasing sequences

These sequences are a standard tool in the study of HI and related constructions. They are the building blocks of standard exact pairs.

Definition 8.5. Let $C \geq 1$, I be an interval of $\mathbb{N}$ and $(j_k)_{k\in I}$ be a strictly increasing sequence of naturals. A block sequence $(x_k)_{k\in I}$ in $\mathfrak{X}^{(1)}_{\text{awi}}$ is called a $(C, (j_k)_{k\in I})$-rapidly increasing sequence (RIS) if

(i) $\|x_k\| \leq C$ for every $k \in I$,
(ii) $\max \operatorname{supp}(x_{k-1}) \leq \sqrt{m_{j_k}}$ for every $k \in I \setminus \{\min I\}$ and
(iii) $|f(x_k)| \leq C/w(f)$ for every $k \in I$ and $f \in W_{(1)}$ with $w(f) < m_{j_k}$.

Proposition 8.6. *Let Y be a block subspace of $\mathfrak{X}^{(1)}_{awi}$ and $C > 2$. Then there exists a strictly increasing sequence $(j_k)_{k\in\mathbb{N}}$ of naturals and a $(C, (j_k)_{k\in\mathbb{N}})$-RIS $(x_k)_{k\in\mathbb{N}}$ in Y, such that $1/2 < \|x_k\| \leq 1$, for all $k \in \mathbb{N}$.*

Proof. We define the sequences $(j_k)_k$ and $(x_k)_k$ inductively as follows. First, choose x_1, using Proposition 8.3, to be a (n_1, m_1^{-2})-s.c.c. x_1 in Y with $1/2 < \|x_1\| \leq 1$, and set $j_1 = 1$. Suppose that we have chosen $j_1, \dots, j_{k-1}$ and $x_1, \dots, x_{k-1}$ for some $k \in \mathbb{N}$. Then, choose $j_k \in \mathbb{N}$ with $j_k > j_{k-1}$ and $\sqrt{m_{j_k}} > \max \operatorname{supp}(x_{k-1})$, and use Proposition 8.3 to find an $(n_{j_k}, m_{j_k}^{-2})$-s.c.c. x_k in Y with $\min \operatorname{supp}(x_k) > \max \operatorname{supp}(x_{k-1})$ and $1/2 < \|x_k\| \leq 1$. Proposition 8.4 then yields that x_k satisfies (iii) of Definition 8.5, and, hence, we conclude that the sequences $(j_k)_{k\in\mathbb{N}}$ and $(x_k)_{k\in\mathbb{N}}$ satisfy the desired conclusion. □

8.3. *Standard exact pairs*

We are ready to define standard exact pairs and prove their existence in every block subspace of $\mathfrak{X}^{(1)}_{\text{awi}}$.

Definition 8.7. Let $C \geq 1$ and $j_0 \in \mathbb{N}$. We call a pair (x, f), for $x \in \mathfrak{X}^{(1)}_{\text{awi}}$ and $f \in W_{(1)}$, a (C, m_{j_0})-SEP if there exists a $(C, (j_k)_{k=1}^n)$-RIS $(x_k)_{k=1}^n$ with $j_0 < j_1$, such that

(i) $x = m_{j_0} \sum_{k=1}^n a_k x_k$ and $\sum_{k=1}^n a_k x_k$ is a $(n_{j_0}, m_{j_0}^{-2})$-s.c.c.,
(ii) x_k is a $(n_{j_k}, m_{j_k}^{-2})$-s.c.c. and $1/2 < \|x_k\| \leq 1$ for every $k = 1, \dots, n$ and
(iii) $f = m_{j_0}^{-1} \sum_{k=1}^n f_k$, where $(f_k)_{k=1}^n$ is an $S_{n_{j_0}}$-admissible AWI sequence in $W_{(1)}$ with $f_k(x_k) > 1/4$, for every $k = 1, \dots, n$.

The following proposition is an immediate consequence of the definition of standard exact pairs, the existence of seminormalised rapidly increasing sequences in every block subspace of $\mathfrak{X}^{(1)}_{\text{awi}}$, as follows from Proposition 8.6 and Lemma 7.4 applied to a sequence.

Proposition 8.8. *Let Y be a block subspace of $\mathfrak{X}^{(1)}_{awi}$. Then, for every $C > 2$ and $j_0, m \in \mathbb{N}$, there exists a (C, m_{j_0})-SEP (x, f) with $x \in Y$ and $m \leq \min supp(x)$.*

Proof. Applying Proposition 8.6, we obtain a $(C, (j_k)_{k\in\mathbb{N}})$-RIS $(x_k)_{k\in\mathbb{N}}$ in Y, such that $m \leq \operatorname{minsupp}(x_1)$ and $1/2 < \|x_k\| \leq 1$, $k \in \mathbb{N}$, with $j_0 < j_1$. Then, applying Lemma 7.4 for $\varepsilon = 1/2$ and passing to a subsequence, we obtain an AWI sequence $(f_k)_{k\in\mathbb{N}}$ in $W_{(1)}$ so that $f_k(x_k) > (1-\varepsilon)/2 = 1/4$, $k \in \mathbb{N}$. We may assume that $\operatorname{supp}(f_k) \subset \operatorname{supp}(x_k)$, $k \in \mathbb{N}$. Remark 8.1 then yields the desired SEP. □

Definition 8.9. Let I be an interval of $\mathbb{N}$ and $(x_k)_{k\in I}$ be a block sequence in $\mathfrak{X}^{(1)}_{\text{awi}}$. For every $f \in W_{(1)}$, we define the sets $I_f = \{k \in I : \operatorname{supp}(x_k) \subset \operatorname{range}(f)\}$, $J_f = I_f \cap \{k \in I : \operatorname{supp}(x_k) \cap \operatorname{supp}(f) \neq \emptyset\}$ and $I'_f = \{k \in I : \operatorname{supp}(x_k) \cap \operatorname{range}(f) \neq \emptyset\}$.

If (x, f) is a (C, m_{j_0})-SEP and $g \in W_{(1)}$, then when we write I_g^x or J_g^x we mean I_g or J_g, respectively, with respect to the sequence $(x_k)_{k=1}^n$ as in Definition 8.7.

Remark 8.10. Let I be an interval of $\mathbb{N}$ and $(x_k)_{k\in I}$ be a block sequence in $\mathfrak{X}^{(1)}_{\text{awi}}$. Then, for every $f \in W_{(1)}$, the following hold.

(i) I_f is a finite subset of I and $\#\{k \in I : \operatorname{supp}(x_k) \cap \operatorname{range}(f) \neq \emptyset\} \leq \# I_f + 2$.
(ii) If $f = m_j^{-1} \sum_{l=1}^d f_l$, then $\cup_{l=1}^d I_{f_l} \subset I_f$.
(iii) If there exists $k \in I$, such that $\operatorname{range}(f) \subsetneq \operatorname{range}(x_k)$, then $I_f = \emptyset$.

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Proposition 8.11. *For every (C, m_{j_0})-SEP (x, f), the following hold.*

(i) *For every $g \in W_{(1)}$*

$$\left|g(x)\right| \leq \begin{cases} \frac{2C}{m_{j_0}}, & g = \pm e_i^* \text{ for some } i \in \mathbb{N} \\ 2C\left[\frac{1}{m_{j_0}} + \frac{m_{j_0}}{w(g)}\right], & w(g) \geq m_{j_0} \\ \frac{6C}{w(g)}, & w(g) < m_{j_0} \end{cases}$$

(ii) *If $g \in W_{(1)}$ with a tree analysis $(g_\alpha)_{\alpha \in \mathcal{A}}$, such that $I^x_{g_\alpha} = \emptyset$ for all $\alpha \in \mathcal{A}$ with $w(g_\alpha) = m_{j_0}$, then*

$$|g(x)| \leq \frac{6C}{m_{j_0}}.$$

For the proof, we refer the reader to Appendix A.

Remark 8.12. Proposition 8.11 (ii) remains valid if we replace $I^x_{g_\alpha}$ with $J^x_{g_\alpha}$.

Corollary 8.13. *The space $\mathfrak{X}^{(1)}_{awi}$ is reflexive.*

Proof. The unit vector basis of $c_{00}(\mathbb{N})$ forms an unconditional Schauder basis for $\mathfrak{X}^{(1)}_{\text{awi}}$, and it is also boundedly complete since the space admits a unique ℓ_1 asymptotic model. Hence, it suffices to show that $\mathfrak{X}^{(1)}_{\text{awi}}$ does not contain ℓ_1. To this end, suppose that $\mathfrak{X}^{(1)}_{\text{awi}}$ contains ℓ_1 and, in particular, from James's ℓ_1 distortion theorem [22], there is a normalised block sequence $(x_k)_k$ in $\mathfrak{X}^{(1)}_{\text{awi}}$, such that for $0 < \varepsilon < 1/2$

$$\left\|\sum_{k=1}^n a_k x_k\right\| \geq (1-\varepsilon) \sum_{k=1}^n |a_k|$$

for all $n \in \mathbb{N}$ and any choice of scalars $a_1, \ldots, a_n$. Choose $j_0 \in \mathbb{N}$, such that $12/m_{j_0} < 1 - \varepsilon$. Let also $y_1 < \ldots < y_n$, where each y_i is a special convex combination of $(x_k)_k$ for all $i = 1, \ldots, n$, such that $x = m_{j_0} \sum_{i=1}^n a_k y_k$ is a $(3, m_{j_0})$-SEP (note that $\|y_i\| \geq 1 - \varepsilon > 1/2$ for all $i = 1, \ldots, n$). Then, Proposition 8.11 yields that $\|x\| \leq 12$ and, since $\|x\| = m_{j_0}\|\sum_{i=1}^n a_k y_k\| \geq m_{j_0}(1-\varepsilon)$, we derive a contradiction. $\square$

9. The space $\mathfrak{X}^{(1)}_{\text{awi}}$ does not contain asymptotic ℓ_1 subspaces

In this last section of the first part of the paper, we show that $\mathfrak{X}^{(1)}_{\text{awi}}$ does not contain Asymptotic ℓ_1 subspaces. It is worth pointing out that unlike the constructions in [8], we are not able to prove the existence of a block tree which is either c_0 or ℓ_p, for some $1 < p < \infty$, of height greater or equal to ω, in any subspace of $\mathfrak{X}^{(1)}_{\text{awi}}$.

Definition 9.1. We say that a sequence $(x_1, f_1), \ldots, (x_n, f_n)$, with $x_i \in \mathfrak{X}^{(1)}_{\text{awi}}$ and $f_i \in W_{(1)}$ for $i = 1, \ldots, n$, is a dependent sequence if each pair (x_i, f_i) is a $(3, m_{j_i})$-SEP and $\bar{f}_1 <_{\mathcal{T}} \ldots <_{\mathcal{T}} \bar{f}_n$.

Definition 9.2. Given a dependent sequence $(x_1, f_1), \ldots, (x_n, f_n)$, for $f \in W_{(1)}$ with a tree analysis $(f_\alpha)_{\alpha \in \mathcal{A}}$ and each $1 \leq k \leq h(\mathcal{A})$ define

$$D^k_f = \big\{\alpha \in \mathcal{A} : |\alpha| = k \text{ and there exists } 1 \leq i \leq n, \text{ such that}$$
$$w(f_\alpha) = w(f_i) \text{ and } \operatorname{supp}(f_\alpha) \cap \operatorname{range}(f_i) \neq \emptyset\big\}$$

and

$$E^k_f = \big\{i \in \{1, \ldots, n\} : w(f_\alpha) = w(f_i) \text{ for some } \alpha \in D^k_f\big\}.$$

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Remark 9.3. Let $f_1, \dots, f_n, f$ be as in the above definition and fix $k \in \mathbb{N}$. If f_α and f_β are such that $\alpha, \beta \in D_f^k$ and $w(f_\alpha) < w(f_\beta)$, then $w(f_\alpha) <_{\mathcal{W}} w(f_\beta)$ since $w(f_\alpha) = w(f_{i_1})$ and $w(f_\beta) = w(f_{i_2})$ for some $1 \leq i_1 < i_2 \leq n$. This implies that $\{\bar{f}_\alpha, \bar{f}_\beta\}$ is not essentially incomparable. Indeed, if it were essentially incomparable, then $f_{i_1} < f_\alpha$, and this contradicts the fact that $\operatorname{supp}(f_\alpha) \cap \operatorname{range}(f_{i_1}) \neq \emptyset$ in the definition of D_f^k.

Proposition 9.4. *Let $(x_1, f_1), \dots, (x_n, f_n)$ be a dependent sequence and $f \in W_{(1)}$. Then $\# E_f^k \leq ek!$ for every $k \in \mathbb{N}$ (where e denotes Euler's number).*

Proof. Denote by $(a_k)_k$ the sequence satisfying the recurrence relation $a_1 = 2$ and $a_k = ka_{k-1} + 1$, $k \geq 2$. We will show that $\# E_f^k \leq a_k$ for every $k \in \mathbb{N}$. Note that this yields the desired result since $a_k = \sum_{j=0}^k k!/j! \leq ek!$.

Let $(f_\alpha)_{\alpha \in \mathcal{A}}$ be a tree analysis of f. We proceed by induction. For $k = 1$, the definition of $W_{(1)}$, and in particular, that of AWI sequences, yields a partition

$$\{\bar{f}_\alpha : \alpha \in \mathcal{A} \text{ and } |\alpha| = 1\} = C_1^0 \cup C_2^0,$$

such that C_1^0 is essentially incomparable and C_2^0 is weight incomparable. Then, note that Remark 9.3 implies that

$$\#\{w(f_\alpha) : \alpha \in D_f^1 \text{ and } \bar{f}_\alpha \in C_1^0\} \leq 1. \tag{9.1}$$

Moreover, since C_2^0 is weight incomparable, we have that

$$\#\{w(f_\alpha) : \alpha \in D_f^1 \text{ and } \bar{f}_\alpha \in C_2^0\} \leq 1, \tag{9.2}$$

and, hence, (9.1) and (9.2) imply that $\# E_f^1 \leq 2$.

Assume that for some $k \in \mathbb{N}$, we have $\# E_g^k \leq a_k$ for all functionals g in $W_{(1)}$, with respect to the dependent sequence $(x_1, f_1), \dots, (x_n, f_n)$. We will show that $\# E_f^{k+1} \leq a_{k+1}$. Let $\{\alpha \in \mathcal{A} : |\alpha| = 1\} = \{\alpha_1, \dots, \alpha_d\}$, where $f_{\alpha_1} < \dots < f_{\alpha_d}$ and consider the tree analyses $(f_\alpha)_{\alpha \in \mathcal{A}_i}$, where $\mathcal{A}_i = \{\alpha \in \mathcal{A} : \alpha_i \leq \alpha\}$ for $1 \leq i \leq d$. The fact that f is in $W_{(1)}$, that is, $(f_{\alpha_i})_{i=1}^d$ is AWI, implies that there exist partitions

$$\{\bar{f}_\alpha : \alpha \in \mathcal{A}_i \text{ with } |\alpha| = k\} = C_{1,i}^k \cup C_{2,i}^k, \quad i \geq k+1,$$

such that $\cup_{i=k+1}^d C_{1,i}^k$ is essentially incomparable and $(C_{2,i}^k)_{i=k+1}^d$ is pairwise weight incomparable. Here, $|\alpha|$ is the height of α in the tree $\mathcal{A}_i$. Then, using Remark 9.3 and arguing as in the previous paragraph, we have

$$\#\{w(f_\alpha) : \alpha \in D_f^{k+1} \text{ and } \alpha \in \cup_{i=k+1}^d C_{1,i}^k\} \leq 1. \tag{9.3}$$

Moreover, it follows easily that $D_f^{k+1} \cap C_{2,i_0}^k \neq \emptyset$ for at most one $k < i_0 \leq d$, and, thus, if such an i_0 exists, we have

$$\# \cup_{i=k+1, i \neq i_0}^d E_{f_{\alpha_i}}^k \leq 1.$$

If no such i_0 exists, we have

$$\# \cup_{i=k+1}^d E_{f_{\alpha_i}}^k \leq 1.$$

In any case, since the inductive hypothesis yields that $\#E^k_{f_{\alpha_{i_0}}} \le a_k$, we have

$$\# \cup_{i=k+1}^d E^k_{f_{\alpha_i}} \le a_k + 1.$$

Note that the inductive hypothesis also implies that

$$\#E^k_{f_{\alpha_i}} \le a_k, \quad 1 \le i \le k,$$

and, hence, since $E^{k+1}_f = \cup_{i=1}^d E^k_{f_{\alpha_i}}$, we conclude that

$$\#E^{k+1}_f \le \sum_{i=1}^k \#E^k_{f_{\alpha_i}} + \# \cup_{i=k+1}^d E^k_{f_{\alpha_i}} \le k a_k + a_k + 1.$$

This completes the inductive step and the proof. □

Lemma 9.5. *Let (x, f_0) be a $(3, m_{j_0})$-SEP. If f is a functional in $W_{(1)}$ with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ and*

$$\mathcal{B} = \{\alpha \in \mathcal{A} : w(f_\alpha) = m_{j_0} \text{ and } w(f_\beta) \ne m_{j_0} \text{ for every } \beta < \alpha\},$$

then there exists a partition range$(f) = G \cup D$, *such that*

(i) $|f|_D(x)| \le 18/m_{j_0}$ *and*
(ii) $|\sum_{\alpha\in\mathcal{B}} f_\alpha|_G(x)| \le 3$.

Proof. Let $x = m_{j_0} \sum_{k=1}^n a_k x_k$ for some $(3, (j_k)_{k=1}^n)$-RIS $(x_k)_{k=1}^n$, and set

$$I^x_{f_\alpha} = \{k \in \{1, \dots, n\} : \text{supp}(x_k) \subset \text{range}(f_\alpha)\}, \quad \alpha \in \mathcal{A}.$$

For every $\alpha \in \mathcal{B}$ and every $k \in I^x_{f_\alpha}$, Definition 8.5 (iii) implies that

$$|f_\alpha(a_k x_k)| \le \frac{3a_k}{m_{j_0}} \tag{9.4}$$

since $j_0 < j_k$. Set $G = \cup\{\text{range}(x_k) : k \in \cup_{\alpha\in\mathcal{B}} I^x_{f_\alpha}\}$ and $D = \text{range}(f) \setminus G$. Then, (9.4) immediately yields that G satisfies (ii). To see that D satisfies (i), note that if $\alpha \in \mathcal{A}$ with $w(f_\alpha) = m_{j_0}$ and $\text{supp}(f_\alpha) \cap D = \emptyset$, there exists $\beta \in \mathcal{B}$, such that $\beta \le \alpha$ and $J^x_{f_\alpha|D} \subset J^x_{f_\beta|D}$. However, it is easy to see that $J^x_{f_\beta|D} = \emptyset$, and thus $J^x_{f_\alpha|D} = \emptyset$. Hence, (i) follows from Proposition 8.11 (ii) and Remark 8.12. □

Proposition 9.6. *For every $0 < c < 1$, there exists $d \in \mathbb{N}$, such that for any dependent sequence $(x_1, f_1), \dots, (x_n, f_n)$ where $d \le n$, and any $f \in W_{(1)}$, we have*

$$\left| f(\frac{1}{n}\sum_{i=1}^n x_i) \right| < c.$$

Proof. First, pick an $m \in \mathbb{N}$, such that $3/2^m < c$, and fix a dependent sequence $(x_1, f_1), \dots, (x_n, f_n)$. Let $f \in W_{(1)}$ with $(f_\alpha)_{\alpha\in\mathcal{A}}$ be a tree analysis of f, and set

$$G = \cup\{\text{range}(x_k) \cap \text{range}(f_\alpha) : k \in \{1, \dots, n\} \text{ and } \alpha \in \mathcal{A} \text{ with } w(f_\alpha) = w(f_k)\}$$

and $H = \mathbb{N} \setminus G$. Let $g = f|_G$ and $h = f|_H$. Then, consider the tree analysis $(g_\alpha)_{\alpha\in\mathcal{A}_g}$ for g, induced by $(f_\alpha)_{\alpha\in\mathcal{A}}$, and define

$$\mathcal{B}^1_k = \{\alpha \in \mathcal{A}_g : |a| \le m,\ w(f_\alpha) = w(f_k) \text{ and } w(f_\beta) \ne w(f_k) \text{ for all } \beta < \alpha \text{ in } \mathcal{A}_g\}$$

for $k = 1, \dots, n$ and

$$G_1 = \cup_{k=1}^{n} \cup \{\mathrm{supp}(g_\alpha) \cap \mathrm{supp}(x_k) : \alpha \in \mathcal{B}_k^1\}.$$

Let $g_1 = g|_{G_1}$ and $g_2 = g|_{\mathbb{N}\setminus G_1}$. Recall Remark 5.8 (ii), and observe that for $k = 1, \dots, n$,

$$g_1(x_k) = \sum_{\alpha \in \mathcal{B}_k^1} \frac{1}{w_g(g_\alpha)} g_\alpha(x_k) \quad \text{and} \quad g_2(x_k) = \sum_{\alpha \in \mathcal{B}_k^2} \frac{1}{w_g(g_\alpha)} g_\alpha(x_k),$$

where

$$\mathcal{B}_k^2 = \{\alpha \in \mathcal{A}_g : |a| > m,\ w(f_\alpha) = w(f_k) \text{ and } w(f_\beta) \neq w(f_k) \text{ for all } \beta < \alpha \text{ in } \mathcal{A}_g\}. \tag{9.5}$$

Consider the tree analysis $(h_\alpha)_{\alpha\in\mathcal{A}_h}$ of h, induced by $(f_\alpha)_{\alpha\in\mathcal{A}}$. Note that, for every α in $\mathcal{A}_h$ and $k = 1, \dots, n$, such that $w(h_\alpha) = w(f_k)$, we have $\mathrm{range}(h_\alpha) \cap \mathrm{range}(x_k) = \emptyset$, and, hence, $k \notin I_{h_\alpha}$. Proposition 8.11 (ii) then implies that for every $k = 1, \dots, n$

$$|h(x_k)| \leq \frac{18}{w(f_k)}.$$

Thus, we obtain

$$|h(\frac{1}{n}\sum_{k=1}^{n} x_k)| \leq \frac{18}{n}. \tag{9.6}$$

Next, we apply Lemma 9.5 for g_2 and each (x_k, f_k), $k = 1, \dots, n$, to obtain partitions $\mathrm{supp}(g_2) \cap \mathrm{supp}(x_k) = G_k^2 \cup D_k^2$, such that

(a) $|g|_{D_k^2}(x_k)| \leq 18/w(f_k)$ and
(b) $|\sum_{\beta\in\mathcal{B}_k^2} g_\beta|_{G_k^2}(x_k)| \leq 3$.

Then, (b) and Remark 5.8 (iii) yield that

$$|g_2|_{G_k^2}(x_k)| = |\sum_{\beta\in\mathcal{B}_k^2} w_g(g_\beta)^{-1} g_\beta|_{G_k^2}(x_k)| \leq \sum_{\beta\in\mathcal{B}_k^2} 2^{-m} |g_\beta|_{G_k^2}(x_k)| \leq \frac{3}{2^m},$$

and, hence, using (a) we obtain

$$|g_2(\frac{1}{n}\sum_{k=1}^{n} x_k)| \leq \frac{1}{n}\sum_{k=1}^{n} \frac{18}{w(f_k)} + \frac{3}{2^m} \leq \frac{18}{n} + \frac{3}{2^m}. \tag{9.7}$$

Finally, observe that it follows immediately from Proposition 9.4 that

$$\{k \in \{1, \dots, n\} : g_1(x_k) \neq 0\} \leq \ell = e\sum_{k=1}^{m} k!,$$

and, thus, by Proposition 8.11 (i),

$$|g_1(\frac{1}{n}\sum_{k=1}^{n} x_k)| \leq \frac{\ell}{n} 6. \tag{9.8}$$

Then for d, such that

$$\frac{36+6\ell}{d}+\frac{3}{2^m}<c,$$

(9.6), (9.7) and (9.8) yield the desired result. □

Proposition 9.7. *The space $\mathfrak{X}^{(1)}_{awi}$ does not contain Asymptotic ℓ_1 subspaces.*

Proof. Suppose that $\mathfrak{X}^{(1)}_{\mathrm{awi}}$ contains a C'-Asymptotic ℓ_1 subspace Y. By standard arguments, for every $\varepsilon>0$, there exists a block subspace of Y which is $C'+\varepsilon$ Asymptotic ℓ_1. Passing to a further block subspace, we may assume that Y is block and C-asymptotic ℓ_1 in the sense of [25], that is, Y admits a Schauder basis $(y_i)_i$, which is a block subsequence of $(e_i)_i$, such that for every $n\in\mathbb{N}$, there exists $N(n)\in\mathbb{N}$ with the property that whenever $N(n)\le x_1\le\ldots\le x_n$ are blocks of $(y_i)_i$ then

$$\frac{1}{C}\sum_{k=1}^n\|x_k\|\le\left\|\sum_{k=1}^n x_k\right\|. \tag{9.9}$$

Applying Proposition 9.6 for $c=1/2C$, we obtain $n\in\mathbb{N}$, such that for any dependent sequence $(x_1,f_1),\ldots,(x_n,f_n)$, we have

$$\left\|\frac{x_1+\cdots+x_n}{n}\right\|<\frac{1}{2C}.$$

We apply Proposition 8.8 iteratively to construct a dependent sequence in Y as follows: We find $x_1\in Y$ with $N(n)\le\mathrm{supp}(x_1)\cup\mathrm{supp}(f_1)$, $w(f_1)=\sigma(\bar{0})$, and set $\bar{f_1}=(f_1,\sigma(\bar{0}))$, and for $1<k\le n$, we find $x_k\in Y$ with $w(f_k)=\sigma(\bar{f}_{k-1})$, and set $\bar{f_k}=(f_k,\sigma(\bar{f}_{k-1}))$. Note that the sequence $(f_k)_{k=1}^n$ is $\mathcal{S}_1$-admissible since $n\le N(n)$. Then, (9.9) implies that

$$\left\|\frac{x_1+\cdots+x_n}{n}\right\|\ge\frac{1}{2C},$$

since $\|x_k\|>1/2$ for each $k=1,\ldots,n$ as follows from Definition 8.7, which is a contradiction. □

Question 9.8. Let $\xi<\omega_1$ and $1<p\le\infty$. Does there exist a Banach space X with a Schauder basis admitting a unique ℓ_1 asymptotic model, such that any block subspace of X contains an ℓ_p (or c_0 if $p=\infty$) block tree of height greater or equal to ω^ξ?

PART II. The case of ℓ_p for $1<p<\infty$

10. Introduction

In this second part, we treat the case of $1<p<\infty$ and, in particular, that of $p=2$. The cases where $p\neq 2$ follow as an easy modification. The definition of $\mathfrak{X}^{(2)}_{\mathrm{awi}}$ and the the proofs of its properties are for the most part almost identical to those of $\mathfrak{X}^{(1)}_{\mathrm{awi}}$. We start with the 2-convexification of a Mixed Tsirelson space and define a countably branching well-founded tree on its norming set. Then, employing the notion of asymptotically weakly incomparable constraints, we define the norming set $W_{(2)}$ of $\mathfrak{X}^{(2)}_{\mathrm{awi}}$. To prove that the space admits ℓ_2 as a unique asymptotic model, we use Lemma 3.4 by first applying the combinatorial results of Section 4, in a manner similar to that of Section 7, and prove lower ℓ_2 estimates for arrays of block sequences of $\mathfrak{X}^{(2)}_{\mathrm{awi}}$ by passing to a subsequence. Then, a result similar to [16, Proposition 2.9] shows that any block sequence of $\mathfrak{X}^{(2)}_{\mathrm{awi}}$ also has an upper ℓ_2 estimate. Finally, to prove that $\mathfrak{X}^{(2)}_{\mathrm{awi}}$ does not contain Asymptotic ℓ_2 subspaces, just like in Part I, we show that any block subspace contains a vector, that is an ℓ_2-average of standard exact pairs, with arbitrarily small norm. The existence of standard exact pairs follows again from similar arguments, while the proof that these

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are strong exact pairs requires a variant of the basic inequality, which we include in Appendix B. In particular, for a block subspace Y and $0 < c < 1$, we show that there is a sequence of standard exact pairs $(x_1, f_1), \dots, (x_n, f_n)$ in Y, such that $\bar{f}_1 \leq_{\mathcal{T}} \dots \leq_{\mathcal{T}} \bar{f}_n$ and $\|x_1 + \cdots + x_n\| < c\sqrt{n}$. To prove this, we consider the evaluation of an f in $W_{(2)}$ on such a sequence and partition f into $g + h$ and then g into $g_1 + g_2$ as in the proof of Proposition 9.6. An upper bound for h follows from the fact that standard exact pairs are strong exact pairs, while that of g_1 is, again, an immediate consequence of Lemma 9.4. Finally, for g_2, unlike the case of Part I, we cannot estimate its action on each x_k, $k = 1, \dots, n$ using similar arguments. Instead, we need to carefully apply the Cauchy-Schwarz inequality to provide an upper estimate for its action on $x_1 + \cdots + x_k$. We demonstrate this in Lemma 14.2.

11. The space $\mathfrak{X}^{(2)}_{\mathrm{awi}}$

Define a pair of strictly increasing sequences of natural numbers $(m_j)_j$, $(n_j)_j$ as follows:

$$m_1 = 4 \qquad\qquad n_1 = 1$$
$$m_{j+1} = m_j^{m_j} \qquad\qquad n_{j+1} = 2^{2m_{j+1}} n_j.$$

Definition 11.1. Let $V_{(2)}$ denote the minimal subset of $c_{00}(\mathbb{N})$ that

(i) contains 0 and all $\pm e_j^*$, $j \in \mathbb{N}$ and
(ii) whenever $f_1 < \dots < f_n$ is an $\mathcal{S}_{n_j}$-admissible sequence in $V_{(2)} \setminus \{0\}$ for some $j \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{Q}$ with $\sum_{i=1}^n \lambda_i^2 \leq 1$, then $m_j^{-1} \sum_{i=1}^n \lambda_i f_i$ is in $V_{(2)}$.

The notion of the weight $w(f)$ of a functional f in $V_{(2)}$ is identical to that in Section 5. We also define, in a similar manner, the notion of tree analysis of a functional in $V_{(2)}$, taking into account the ℓ_2 version of the $(m_j, \mathcal{S}_{n_j})$-operations, in the definition of $V_{(2)}$. Again, it follows from minimality that every f in $V_{(2)} \setminus \{0\}$ admits a tree analysis and finally, for a functional f in $V_{(2)} \setminus \{0\}$ admitting a tree analysis $(f_\alpha)_{\alpha \in \mathcal{A}}$, we define $w_f(f_\alpha)$ as in Definition 5.5.

Definition 11.2. Let $f \in V_{(2)}$ with a tree analysis $(f_\alpha)_{\alpha \in \mathcal{A}}$.

(i) Let $\beta \in \mathcal{A}$ with $\beta \neq \emptyset$. Then, if $\alpha \in \mathcal{A}$ is the immediate predecessor of β, we will denote by λ_β the coefficient of f_β in the normal form of f_α, that is,

$$f_\alpha = m_j^{-1} \sum_{\beta \in S(\alpha)} \lambda_\beta f_\beta,$$

where $S(\alpha)$ denotes the set of immediate successors of α and $w(f_\alpha) = m_j$.
(ii) For each $\beta \in \mathcal{A}$, we define

$$\lambda_{f,\beta} = \begin{cases} \prod_{\alpha < \beta} \lambda_\alpha, & \beta \neq \emptyset \\ 1. \end{cases}$$

Remark 11.3. Let $f \in V_{(2)}$ with a tree analysis $(f_\alpha)_{\alpha \in \mathcal{A}}$.

(i) For every $k = 1, \dots, h(\mathcal{A})$

$$f = \sum_{|a|=k} \frac{\lambda_{f,\alpha}}{w_f(f_\alpha)} f_\alpha.$$

(ii) If $\mathcal{B}$ is a maximal pairwise incomparable subset of $\mathcal{A}$, then

$$f = \sum_{\alpha \in \mathcal{B}} \frac{\lambda_{f,\alpha}}{w_f(f_\alpha)} f_\alpha.$$

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(iii) For every $\alpha \in \mathcal{A}$, whose immediate predecessor β in $\mathcal{A}$ (if one exists) satisfies $f_\beta \notin \{\pm e_j^* : j \in \mathbb{N}\}$, we have $w_f(f_\alpha) \geq 4^{|\alpha|}$.

(iv) If $\mathcal{B}$ is a pairwise incomparable subset of $\mathcal{A}$, then

$$\sum_{\alpha \in \mathcal{B}} \lambda_{f,\alpha}^2 \leq 1.$$

Next, as in Section 5, we define a tree $\mathcal{T}$ on the set of all pairs $(f, w(f))$, for $f \in V_{(2)}$ and $w(f)$ a weight of f and consider the trees $\widetilde{\mathcal{T}}$, $\mathcal{W}$ and $\widetilde{\mathcal{W}}$, which are induced by $\mathcal{T}$ and defined identically to those in Section 5. These are countably branching well-founded trees. Finally, let us recall all three incomparability notions of Definition 5.12, as well as the notion of asymptotically weakly incomparable sequences in Definition 5.14.

Definition 11.4. Let $W_{(2)}$ be the smallest subset of $V_{(2)}$ that is symmetric, contains the singletons and whenever $j \in \mathbb{N}$, $f_1 < \ldots < f_n$ is an $\mathcal{S}_{n_j}$-admissible AWI sequence in $V_{(2)}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$ with $\sum_{i=1}^n \lambda_i^2 \leq 1$, then $m_j^{-1} \sum_{i=1}^n \lambda_i f_i \in W_{(2)}$. Denote by $\mathfrak{X}_{\mathrm{awi}}^{(2)}$ completion of $c_{00}(\mathbb{N})$ with respect to the norm induced by $W_{(2)}$.

Remark 11.5.

(i) The norming set $W_{(2)}$ can be defined as an increasing union of a sequence $(W_{(2)}^n)_{n=0}^\infty$, where $W_{(2)}^0 = \{\pm e_i^* : i \in \mathbb{N}\}$ and

$$W_{(2)}^{n+1} = W_{(2)}^n \cup \Big\{ \frac{1}{m_j} \sum_{i=1}^m \lambda_i f_i : j, m \in \mathbb{N},\ (\lambda_i)_{i=1}^m \subset \mathbb{Q} \text{ with } \sum_{i=1}^m \lambda_i^2 \leq 1 \text{ and } (f_i)_{i=1}^m \text{ is an } \mathcal{S}_{n_j}\text{-admissible AWI sequence in } W_{(2)}^n \Big\}.$$

(ii) Proposition 5.16 yields that the standard unit vector basis of $c_{00}(\mathbb{N})$ forms an 1-unconditional Schauder basis for $\mathfrak{X}_{\mathrm{awi}}^{(2)}$.

The following lemma is a result similar to [16, Proposition 2.9], in which we prove upper ℓ_2 estimates for block sequences of $\mathfrak{X}_{\mathrm{awi}}^{(2)}$.

Proposition 11.6. *For any block sequence $(x_k)_k$ in $\mathfrak{X}_{awi}^{(2)}$, any finite subset F of the naturals and $f \in W_{(2)}$, we have*

$$|f(\sum_{k \in F} x_k)| \leq 2\sqrt{2}(\sum_{k \in F} \|x_k\|^2)^{\frac{1}{2}}.$$

Proof. Recall from Remark 11.5 that $W_{(2)} = \cup_{n=0}^\infty W_{(2)}^n$. We will show by induction that for every $n \in \mathbb{N}$, every $f \in W_{(2)}^n$ and any finite subset F of $\mathbb{N}$, we have

$$|f(\sum_{k \in F} x_k)| \leq 2\sqrt{2}(\sum_{k \in F} \|x_k\|^2)^{\frac{1}{2}}.$$

Clearly, this holds for all $f \in W_{(2)}^0$. Hence, let us assume that it also holds for all functionals in $W_{(2)}^n$ for some $n \geq 0$ and fix $f \in W_{(2)}^{n+1}$. Then $f = m_j^{-1} \sum_{=1}^m \lambda_i f_i$, where $(f_i)_{i=1}^m$ is an S_{n_j}-admissible AWI sequence in $W_{(2)}^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{Q}$ with $\sum_{i=1}^m \lambda_i^2 \leq 1$. Define

$$I_k = \{i \in \{1, \ldots, m\} : \mathrm{supp}(x_k) \cap \mathrm{range}(f_i) \neq \emptyset\}, \quad k \in F,$$

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$F_1 = \{k \in F : \#I_k \leq 1\}$ and $F_2 = F \setminus F_1$. We also define

$$K_i = \{k \in F_1 : \operatorname{supp}(x_k) \cap \operatorname{range}(f_i) \neq \emptyset\}, \quad i = 1, \ldots, m.$$

Note that if $k \in F_1$, then $k \in K_i$ for at most one $i \in \{1, \ldots, m\}$. Thus, using the inductive hypothesis and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |f(\sum_{k\in F_1} x_k)| &= m_j^{-1}|\sum_{i=1}^m \lambda_i f_i(\sum_{k\in K_i} x_k)| \\ &\leq \frac{2\sqrt{2}}{m_j}\sum_{i=1}^m |\lambda_i|(\sum_{k\in K_i}\|x_k\|^2)^{\frac{1}{2}} \\ &\leq \sqrt{2}(\sum_{i=1}^m \lambda_i^2)^{\frac{1}{2}}(\sum_{i=1}^m\sum_{k\in K_i}\|x_k\|^2)^{\frac{1}{2}} \\ &\leq \sqrt{2}(\sum_{k\in F_1}\|x_k\|^2)^{\frac{1}{2}}. \end{aligned} \tag{11.1}$$

Moreover, for each $k \in F_2$, it is easy to see that

$$|m_j^{-1}\sum_{i\in I_k}\lambda_i f_i(x_k)| \leq (\sum_{i\in I_k}\lambda_i^2)^{\frac{1}{2}}\|x_k\|. \tag{11.2}$$

Observe that for each $i \in \{1, \ldots, m\}$ there are at most two k's in F_2, such that $\operatorname{supp}(x_k) \cap \operatorname{range}(f_i) \neq \emptyset$ and, thus, applying the Cauchy-Schwarz inequality and (11.2), we have

$$\begin{aligned} |f(\sum_{k\in F_2} x_k)| = m_j^{-1}|\sum_{i=1}^m \lambda_i f_i(\sum_{k\in F_2} x_k)| &= m_j^{-1}|\sum_{k\in F_2}\sum_{i\in I_k}\lambda_i f_i(x_k)| \\ &\leq \sum_{k\in F_2}(\sum_{i\in I_k}\lambda_i^2)^{\frac{1}{2}}\|x_k\| \\ &\leq (\sum_{k\in F_2}\sum_{i\in I_k}\lambda_i^2)^{\frac{1}{2}}(\sum_{k\in F_2}\|x_k\|^2)^{\frac{1}{2}} \\ &\leq \sqrt{2}(\sum_{k\in F_2}\|x_k\|^2)^{\frac{1}{2}}. \end{aligned} \tag{11.3}$$

Finally, (11.1) and (11.3) yield the desired result. □

12. Asymptotic models generated by block sequences of $\mathfrak{X}^{(2)}_{\mathrm{awi}}$

In this section, we prove that $\mathfrak{X}^{(2)}_{\mathrm{awi}}$ admits ℓ_2 as a unique asymptotic model. This follows as an easy modification of the results of Section 7, which yield lower ℓ_2 estimates, combined with the upper ℓ_2 estimates of Proposition 11.6. Let us first recall Proposition 7.2, and note that this in fact holds for the trees defined in the previous section. Applying this, we obtain the following variant of Lemma 7.3, using similar arguments.

Lemma 12.1. *Let $x \in \mathfrak{X}^{(2)}_{awi}$, $f \in W_{(2)}$ and a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ of f, such that $f_\alpha(x) > 0$ for every $\alpha \in \mathcal{A}$. Let $\varepsilon_1, \ldots, \varepsilon_{h(\mathcal{A})}$ be positive reals and G_i be a subset of $\{\alpha \in \mathcal{A} : |\alpha| = i\}$, such that*

$$\sum_{\alpha\in G_i}\frac{\lambda_{f,\alpha}}{w_f(f_\alpha)}f_\alpha(x) > f(x) - \varepsilon_i$$

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for $i = 1, \ldots, h(\mathcal{A})$, *and* $f(x) > \sum_{i=1}^{h(\mathcal{A})} \varepsilon_i$. *Then, there exists a* $g \in W_{(2)}$, *such that*

(i) $supp(g) \subset supp(f)$ *and* $w(g) = w(f)$.
(ii) $g(x) > f(x) - \sum_{i=1}^{h(\mathcal{A})} \varepsilon_i$.
(iii) g *has a tree analysis* $(g_\alpha)_{\alpha\in\mathcal{A}_g}$, *such that, for every* $\alpha \in \mathcal{A}_g$ *with* $|\alpha| = i$, *there is a unique* $\beta \in G_i$, *such that* $supp(g_\alpha) \subset supp(f_\beta)$ *and* $w(g_\alpha) = w(f_\beta)$.

Lemma 12.2. *Let* $(x_j^1)_j, \ldots, (x_j^l)_j$ *be normalised block sequences in* $\mathfrak{X}_{awi}^{(2)}$. *For every* $\varepsilon > 0$, *there exists an* $L \in [\mathbb{N}]^\infty$ *and a* $g_j^i \in W_{(2)}$ *with* $g_j^i(x_j^i) > 1 - \varepsilon$ *for* $i = 1, \ldots, l$ *and* $j \in L$, *such that for any choice of* $i_j \in \{1, \ldots, l\}$, *the sequence* $(g_j^{i_j})_{j\in L}$ *is AWI.*

Proof. The proof is similar to that of Proposition 7.4 with μ_j^k defined as

$$\mu_j^k = \sum_{i=1}^{l} \sum_{\substack{\alpha\in\mathcal{A}_j^i \\ |\alpha|=k}} \frac{\lambda_{f_j^i,\alpha} f_{j,\alpha}^i(x_j^i)}{w_{f_j^i}(f_{j,\alpha}^i)} \delta_{\bar{f}_{j,\alpha}^i}$$

and applying Lemma 12.1 instead of 7.3. □

Proposition 12.3. *The space* $\mathfrak{X}_{awi}^{(2)}$ *admits a unique asymptotic model, with respect to* $\mathscr{F}_b(\mathfrak{X}_{awi}^{(2)})$, *equivalent to the unit vector basis of* ℓ_2.

Proof. Let $(x_j^1)_j, \ldots, (x_j^l)_j$ be normalised block sequences in $\mathfrak{X}_{\mathrm{awi}}^{(2)}$. Working as in the proof of Proposition 7.5 applying Lemma 12.2, we have that, passing to a subsequence, for any choice of $1 \le i_j \le l$ for $j \in \mathbb{N}$, any $F \in \mathcal{S}_1$ and any choice of scalars $(a_j)_{j\in F}$, there is a functional $g \in W_{(2)}$ with

$$g = \frac{1}{4} \sum_{j\in F} \frac{a_j}{(\sum_{j\in F} a_j^2)^{\frac{1}{2}}} g_j^{i_j},$$

such that $g_j^{i_j}(x_j^{i_j}) \ge 1 - \varepsilon$ and $\mathrm{supp}(g_j^{i_j}) \subset \mathrm{supp}(x_j^{i_j})$ for $j \in F$. Hence, we calculate

$$\Big\| \sum_{j\in F} a_j x_{m_j}^{i_j} \Big\| \ge g\Big(\sum_{j\in F} a_j x_{m_j}^{i_j} \Big) \ge \frac{1-\varepsilon}{4} (\sum_{j\in F} a_j^2)^{\frac{1}{2}}. \tag{12.1}$$

Moreover, Lemma 11.6 implies that

$$\Big\| \sum_{j\in F} a_j x_j^{i_j} \Big\| \le 2\sqrt{2} (\sum_{j\in F} a_j^2)^{\frac{1}{2}}. \tag{12.2}$$

Thus, (12.1), (12.2) and Lemma 3.4 yield the desired result. □

By the above proposition, $\mathfrak{X}_{\mathrm{awi}}^{(2)}$ cannot contain an isomorphic copy of c_0 or ℓ_1. Therefore, by James's theorem [21] for spaces with an unconditional basis, we obtain the following.

Proposition 12.4. *The space* $\mathfrak{X}_{awi}^{(2)}$ *is reflexive.*

13. Standard exact pairs

The definitions of rapidly increasing sequences and standard exact pairs in $\mathfrak{X}_{\mathrm{awi}}^{(2)}$ are almost identical to these in Part I. We show that standard exact pairs are in fact strong exact pairs. This requires a variant of the basic inequality that we prove in Appendix B.

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Definition 13.1. Let $C \geq 1$, $I \subset \mathbb{N}$ be an interval and $(j_k)_{k\in I}$ be a strictly increasing sequence of naturals. A block sequence $(x_k)_{k\in I}$ in $\mathfrak{X}^{(2)}_{\mathrm{awi}}$ is called a $(C,(j_k)_{k\in I})$-rapidly increasing sequence (RIS) if

(i) $\|x_k\| \leq C$ for every $k \in I$,
(ii) $\max \operatorname{supp}(x_{k-1}) \leq \sqrt{m_{j_k}}$ for every $k \in I \setminus \{\min I\}$ and
(iii) $|f(x_k)| \leq C/w(f)$ for every $k \in I$ and $f \in W_{(2)}$ with $w(f) < m_{j_k}$.

Definition 13.2. Let $C \geq 1$ and $j_0 \in \mathbb{N}$. We call a pair (x,f) where $x \in \mathfrak{X}^{(2)}_{\mathrm{awi}}$ and $f \in W_{(2)}$, a $(2,C,m_{j_0})$-standard exact pair if there exists a $(C,(j_k)_{k=1}^n)$-RIS $(x_k)_{k=1}^n$ with $j_0 < j_1$, such that

(i) $x = m_{j_0}\sum_{k=1}^n a_k x_k$ and $\sum_{k=1}^n a_k x_k$ is a $(2,n_{j_0},m_{j_0}^{-4})$-s.c.c.,
(ii) x_k is a $(2,n_{j_k},m_{j_k}^{-4})$-s.c.c. and $1/2 < \|x_k\| \leq 1$ for every $k = 1,\dots,n$,
(iii) $f = m_{j_0}^{-1}\sum_{k=1}^n f_k$, where $f_k \in W_{(2)}$ with $f_k(x_k) > 1/4$ for every $k = 1,\dots,n$
(iv) and $48m_{j_0}^2 \leq \min \operatorname{supp}(x)$.

The proof of the following proposition, which demonstrates the existence of SEPs in any subspace of $\mathfrak{X}^{(2)}_{\mathrm{awi}}$, is similar to that of Proposition 8.8 and is omitted.

Proposition 13.3. *Let Y be a block subspace of $\mathfrak{X}^{(2)}_{awi}$. Then, for every $C > 2$ and $j_0, m \in \mathbb{N}$, there exists a $(2,C,m_{j_0})$-SEP (x,f) with $x \in Y$ and $m \leq \min supp(x)$.*

Proposition 13.4. *For every $(2,C,m_{j_0})$-SEP (x,f), the following hold.*

(i) *For every $g \in W_{(2)}$*

$$\left|g(x)\right| \leq \begin{cases} 4C\left[\frac{1}{m_{j_0}} + \frac{m_{j_0}}{w(g)}\right], & w(g) \geq m_{j_0} \\ \frac{12C}{w(g)}, & w(g) < m_{j_0}. \end{cases}$$

(ii) *If $g \in W_{(2)}$ with a tree analysis $(g_\alpha)_{\alpha\in\mathcal{A}}$, such that $I^x_{g_\alpha} = \emptyset$ for all $\alpha \in \mathcal{A}$ with $w(g_\alpha) = m_{j_0}$, then*

$$|g(x)| \leq \frac{6C}{m_{j_0}}.$$

Proof. We refer the reader to Appendix B. □

14. The space $\mathfrak{X}^{(2)}_{\mathrm{awi}}$ does not contain asymptotic ℓ_2 subspaces

To prove that $\mathfrak{X}^{(2)}_{\mathrm{awi}}$ contains no asymptotic ℓ_2 subspaces, we use almost identical arguments as in the case of $\mathfrak{X}^{(1)}_{\mathrm{awi}}$. In particular, we show that any block subspace contains a vector, that is an ℓ_2-average of standard exacts pairs, with arbitrarily small norm. Again, this requires Lemma 9.4. However, in this case, we employ Lemma 14.2 to carefully calculate certain upper bounds, using the Cauchy-Schwarz inequality.

Definition 14.1. We say that a sequence $(x_1,f_1),\dots,(x_n,f_n)$, where $x_i \in \mathfrak{X}^{(2)}_{\mathrm{awi}}$ and $f_i \in W_{(2)}$ for $i = 1,\dots,n$, is a dependent sequence if each (x_i,f_i) is a $(2,3,m_{j_i})$-SEP and $\bar{f}_1 <_{\mathcal{T}} \dots <_{\mathcal{T}} \bar{f}_n$.

Lemma 14.2. *Let (x,f) be a $(2,3,m_j)$-SEP, where $x = m_j\sum_{k=1}^n a_k x_k$, and let $g_1 < \dots < g_m \in W_{(2)}$ with $w(g_i) = m_j$ and $I^x_{g_i} \neq \emptyset$ for all $i = 1,\dots,m$. Then, for any choice of scalars $\lambda_1,\dots,\lambda_m$, we have*

$$|\sum_{i=1}^m \lambda_i g_i(x)| \leq (4\sqrt{2}+1)(\sum_{i=1}^m \lambda_i^2)^{\frac{1}{2}}.$$

Proof. For each $i = 1,\dots,m$, let

$$g_i = \frac{1}{m_j}\sum_{\ell\in L_i}\lambda_{i\ell}g^i_\ell, \quad \sum_{\ell\in L_i}\lambda_{i\ell}^2 \leq 1$$

and define

$$K_1 = \{k \in \{1,\ldots,n\} : k \in \cup_{i=1}^m \cup_{\ell\in L_i} I^x_{g^i_\ell}\}, \quad K_2 = \{1,\ldots,n\} \setminus K_1.$$

Then, Lemma 11.6 and the Cauchy-Schwarz inequality imply that

$$\begin{aligned}
|\sum_{i=1}^m \lambda_i g_i(m_j \sum_{k\in F_1} a_k x_k)| &= |\sum_{i=1}^m \lambda_i m_j^{-1} \sum_{\ell\in L_i} \lambda_{i\ell} g^i_\ell (m_j \sum_{k\in I^x_{g^i_\ell}} a_k x_k)| \\
&\leq 2\sqrt{2} |\sum_{i=1}^m \lambda_i \sum_{\ell\in L_i} \lambda_{i\ell} (\sum_{k\in I^x_{g^i_\ell}} a_k^2)^{\frac{1}{2}}| \\
&\leq 2\sqrt{2} |\sum_{i=1}^m \lambda_i (\sum_{\ell\in L_i} \lambda_{i\ell}^2)^{\frac{1}{2}} (\sum_{k\in \cup_{\ell\in L_i} I^x_{g^i_\ell}} a_k^2)^{\frac{1}{2}}| \\
&\leq 2\sqrt{2} (\sum_{i=1}^m \lambda_i^2)^{\frac{1}{2}} (\sum_{k\in K_1} a_k^2)^{\frac{1}{2}}. \qquad (14.1)
\end{aligned}$$

For each $k = 1,\ldots,n$, let

$$x_k = \sum_{q\in Q_k} b_{kq} y^k_q, \quad \sum_{q\in Q_k} b_{kq}^2 \leq 1.$$

Define for each $i = 1,\ldots,m$ and $\ell \in L_i$

$$\begin{aligned}
M_i^\ell &= \{k \in K_2 : \text{ there is } q \in Q_k \text{ with } \operatorname{supp}(y^k_q) \subset \operatorname{range}(g^i_\ell)\} \text{ and for } k \in M_i \\
N^k_{i\ell} &= \{q \in Q_k : \operatorname{supp}(y^k_q) \subset \operatorname{range}(g^i_\ell)\}.
\end{aligned}$$

Also, for $k \in K_2$, define

$$O_k = \{q \in Q_k : \text{ there are } i \in \{1,\ldots,m\} \text{ and } \ell \in L_i \text{ with } q \in N^k_{i\ell}\}.$$

Finally, also define

$$F_1 = \cup_{i=1}^m \cup_{\ell\in L_i} \cup_{k\in M_i^\ell} \cup_{q\in N^k_{i\ell}} \operatorname{supp}(y^k_q), \quad F_2 = \mathbb{N} \setminus F_1.$$

Note that the sets $N^k_{i\ell}$, $i \in \{1,\ldots,m\}$, $\ell \in L_i$, $k \in M_i^\ell$, are pairwise disjoint with union $\cup_{k\in K_2} O_k$. Applying Lemma 11.6 and the Cauchy-Schwarz inequality once again, we have

$$\begin{aligned}
|\sum_{i=1}^m \lambda_i g_i|_{F_1} (m_j \sum_{k\in K_2} a_k x_k)| &= |\sum_{i=1}^m \lambda_i \sum_{\ell\in L_i} \lambda_{i\ell} g^i_\ell (\sum_{k\in M_i^\ell} \sum_{q\in N^k_{i\ell}} a_k b_{kq} y^k_q)| \\
&\leq 2\sqrt{2} \sum_{i=1}^m \lambda_i \sum_{\ell\in L_i} \lambda_{i\ell} (\sum_{k\in M_i^\ell} \sum_{q\in N^k_{i\ell}} a_k^2 b_{kq}^2)^{1/2} \\
&\leq 2\sqrt{2} \sum_{i=1}^m \lambda_i (\sum_{\ell\in L_i} \lambda_{i\ell}^2)^{1/2} (\sum_{\ell\in L_i} \sum_{k\in M_i^\ell} \sum_{q\in N^k_{i\ell}} a_k^2 b_{kq}^2)^{1/2}
\end{aligned}$$

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$$
\begin{aligned}
&\leq 2\sqrt{2}\sum_{i=1}^{m}\lambda_i(\sum_{\ell\in L_i}\sum_{k\in M_i^\ell}\sum_{q\in N_{i\ell}^k}a_k^2b_{kq}^2)^{1/2}\\
&\leq 2\sqrt{2}(\sum_{i=1}^{m}\lambda_i^2)^{1/2}(\sum_{i=1}^{m}\sum_{\ell\in L_i}\sum_{k\in M_i^\ell}\sum_{q\in N_{i\ell}^k}a_k^2b_{kq}^2)^{1/2}\\
&= 2\sqrt{2}(\sum_{i=1}^{m}\lambda_i^2)^{1/2}(\sum_{k\in K_2}a_k^2\sum_{q\in O_k}b_{kq}^2)^{1/2}\\
&\leq 2\sqrt{2}(\sum_{i=1}^{m}\lambda_i^2)^{1/2}(\sum_{k\in K_2}a_k^2)^{1/2}. \qquad (14.2)
\end{aligned}
$$

For each $i = 1, \dots, m$ and $k \in K_2$, define

$$Q_k^i = \{q \in Q_k : \text{there is an } \ell \in L_i, \text{ such that } \operatorname{supp}(y_q^k) \cap \operatorname{supp}(g_\ell^i) \neq \emptyset \text{ and } \operatorname{supp}(y_q^k) \not\subset \operatorname{supp}(g_\ell^i)\}.$$

Observe that, since $(g_\ell^i)_{\ell\in L_i}$ is $\mathcal{S}_{n_j}$-admissible, $(y_q^k)_{Q_k^i}$ is $\mathcal{S}_{n_j+1}$-admissible for all $i = 1, \dots, m$, and Proposition 2.4 thus implies that

$$\sum_{q\in Q_k^i} b_{kq}^2 < \frac{3}{\min\operatorname{supp}(x_k)}.$$

For $i \in \{1, \dots, m\}$, put $K_2^i = \{k \in K_2 : \operatorname{range}(g_i) \cap \operatorname{range} x_k \neq \emptyset\}$. The condition $i \in \{1, \dots, m\}$, $I_{g_i}^x \neq \emptyset$, for $i \in \{1, \dots, m\}$ implies that each $k \in K_2$ is in at most two sets K_2^i. We then calculate

$$
\begin{aligned}
|\sum_{i=1}^{m}\lambda_i g_i|_{F_2}(m_j\sum_{k\in K_2}a_kx_k)| &= |\sum_{i=1}^{m}\lambda_i g_i(m_j\sum_{k\in K_2\cap I_{g_i}}a_k\sum_{q\in Q_k^i}b_{kq}y_q^k)|\\
&\leq 2\sqrt{2}m_j\sum_{i=1}^{m}\lambda_i(\sum_{k\in K_2\cap I_{g_i}}a_k^2\sum_{q\in Q_k^i}b_{kq}^2)^{1/2}\\
&\leq 2\sqrt{2}m_j(\sum_{i=1}^{m}\lambda_i^2)^{1/2}(\sum_{i=1}^{m}\sum_{k\in K_2\cap I_{g_i}}a_k^2\frac{3}{\min\operatorname{supp}(x_k)})^{1/2}\\
&\leq 4\sqrt{3}m_j(\sum_{i=1}^{m}\lambda_i^2)^{1/2}(\sum_{k\in K_2}a_k^2\frac{1}{\min\operatorname{supp}(x_k)})^{1/2}\\
&\leq (\sum_{i=1}^{m}\lambda_i^2)^{1/2}\frac{4\sqrt{3}m_j}{\min\operatorname{supp}(x)^{1/2}}\\
&\leq (\sum_{i=1}^{m}\lambda_i^2)^{1/2} \text{ (by Definition 13.2 (iv)).} \qquad (14.3)
\end{aligned}
$$

Hence, (14.1), (14.2) and (14.3) yield the desired result. □

Proposition 14.3. *For every $0 < c < 1$, there exists $d \in \mathbb{N}$, such that whenever $d \leq n$ and $(x_1, f_1), \dots, (x_n, f_n)$ is a dependent sequence, then*

$$\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}x_i\| < c.$$

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Proof. Pick an $m \in \mathbb{N}$, such that

$$2^{-m+3} < c \tag{14.4}$$

and fix a dependent sequence $(x_1, f_1), \ldots, (x_n, f_n)$. Let $f \in W \setminus W_0$ and consider the partitions $f = h+g$ and $g = g_1 + g_2$ as in the proof of Proposition 9.6. Then, the same arguments and Proposition 13.4 yield that

$$|h(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} x_k)| \leq \frac{18}{\sqrt{n}}. \tag{14.5}$$

Moreover, Proposition 9.4, again, implies that

$$\#\{k \in \{1, \ldots, n\} : g_1(x_k) \neq 0\} \leq \ell = e \sum_{k=1}^{m} k!$$

and, thus, by Propositions 11.6 and 13.4 (i),

$$\left| g_1\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} x_k \right) \right| \leq 2\sqrt{2} \frac{\sqrt{\ell}}{\sqrt{n}} 24 = 48 \sqrt{\frac{2\ell}{n}}. \tag{14.6}$$

Finally, we treat g_2 differently from Proposition 9.6. Recall that for $k = 1, \ldots, n$,

$$\mathcal{B}_k^2 = \{\alpha \in \mathcal{A}_f : |\alpha| > m, w(f_\alpha) = w(f_k), \text{ and } w(f_\beta) \neq w(f_k) \text{ for } \beta < \alpha \text{ in } \mathcal{A}_f\}.$$

Define

$$G_2 = \cup_{k=1}^{n} \cup \{\mathrm{range}(x_k) \cap \mathrm{range}(f_\alpha) : \alpha \in \mathcal{B}_k^2\},$$

so that $g_2 = g|_{G_2}$. We further split G_2 as follows

$$G_2^1 = \cup_{k=1}^{n} \cup \{\mathrm{supp}(x_k) \cap \mathrm{supp}(f_\alpha) : \alpha \in \mathcal{B}_k^2 \text{ and } I_{f_\alpha}^{x_k} = \emptyset\} \text{ and } G_2^2 = G_2 \setminus G_2^1.$$

Proposition 13.4 (ii) implies that for $k \in \{1, \ldots, n\}$,

$$|g_2|_{G_2^1}(x_k)| \leq \frac{18}{w(f_k)},$$

and, thus,

$$|g_2|_{G_2^1}(\frac{1}{\sqrt{n}} \sum_{k \in K_1} x_k)| \leq \frac{18}{\sqrt{n}}. \tag{14.7}$$

To complete the computation, we need to evaluate the action of $g_2|_{G_2^2}$. To that end, for $s = m+1, m+2, \ldots$ and $k \in \{1, \ldots, n\}$, put

$$\mathcal{B}_{k,s}^2 = \{\alpha \in \mathcal{B}_k^2 : |\alpha| = s\},$$

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so that for each $s > m$, the sets $\mathcal{B}^2_{k,s}$, $k \in \{1, \dots, n\}$ are pairwise disjoint and the set $\cup_{k=1}^n \mathcal{B}^2_{k,s}$ is pairwise incomparable. We use Lemma 14.2 and the definition of G_2^2 to calculate

$$\begin{aligned}
|g_2|_{G_2^2}(\frac{1}{\sqrt{n}}\sum_{k=1}^n x_k)| &= |\frac{1}{\sqrt{n}}\sum_{k=1}^n \sum_{\alpha\in\mathcal{B}_k^2} \frac{\lambda_{f_\alpha}}{w_f(f_\alpha)} f_\alpha|_{G_2^2}(x_k)| \\
&\leq \frac{4\sqrt{2}+1}{\sqrt{n}} \sum_{k=1}^n (\sum_{\alpha\in\mathcal{B}_k^2} \frac{\lambda_{f_\alpha}^2}{w_f(f_\alpha)^2})^{1/2} \\
&\leq (4\sqrt{2}+1)(\sum_{k=1}^n \sum_{\alpha\in\mathcal{B}_k^2} \frac{\lambda_{f_\alpha}^2}{w_f(f_\alpha)^2})^{1/2} \\
&\leq (4\sqrt{2}+1)(\sum_{s=m+1}^\infty \frac{1}{4^s} \sum_{\alpha\in\cup_{k=1}^n \mathcal{B}^2_{k,s}} \lambda_{f_\alpha}^2)^{1/2} \\
&\leq (4\sqrt{2}+1)(\sum_{s=m+1}^\infty \frac{1}{4^s})^{1/2} = \frac{4\sqrt{2}+1}{2^m\sqrt{3}} \leq \frac{4}{2^m}. \qquad (14.8)
\end{aligned}$$

Then, (14.5), (14.6), (14.7) and (14.8) yield that

$$|f(x)| \leq \frac{36+48\sqrt{2}\ell}{\sqrt{n}} + \frac{4}{2^m} \leq \frac{36+48\sqrt{2}\ell}{\sqrt{n}} + \frac{c}{2},$$

and, thus, for d, such that

$$\frac{36+48\sqrt{2}\ell}{\sqrt{d}} < \frac{c}{2},$$

we have the desired result. □

Proposition 14.4. *The space $\mathfrak{X}^{(2)}_{awi}$ does not contain Asymptotic ℓ_2 subspaces.*

Proof. It is an immediate consequence of Proposition 14.3, using similar arguments as in Proposition 9.7. □

Remark 14.5. Unlike the case of ℓ_1, for every $1 < p < \infty$, it is in fact possible to define a reflexive Banach space with a Schauder basis, admitting a unique ℓ_p asymptotic model with respect to the family of normalised block sequences, whose any block subspace contains an ℓ_1 block tree of height ω^ξ. Such a space can be defined using the attractors method, which was first introduced in [3] and later used in [10].

15. Appendix A

In this section, we prove the properties of standard exact pairs in $\mathfrak{X}^{(1)}_{\mathrm{awi}}$, given in Proposition 8.11. This requires three steps. First, we need to define an auxiliary space which is also a Mixed Tsirelson space. Then, on the special convex combinations of its basis, we give upper bounds on the evaluations of the functionals in its norming set $W^{(1)}_{\mathrm{aux}}$. Finally, for a standard exact pair (x, f), via the basic inequality, we reduce the upper bounds of the evaluations of functionals in $W_{(1)}$ acting on x, to the corresponding one of a functional g in $W^{(1)}_{\mathrm{aux}}$ on a normalised special convex combination of the basis of the auxiliary space.

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15.1. The auxiliary space

Definition 15.1. Let $W_{\mathrm{aux}}^{(1)}$ be the minimal subset of $c_{00}(\mathbb{N})$, such that

(i) $\pm e_i$ is in $W_{\mathrm{aux}}^{(1)}$ for all $i \in \mathbb{N}$ and
(ii) for every $j \in \mathbb{N}$ and every $\mathcal{S}_{n_j+1}$-admissible sequence of functionals $(f_i)_{i=1}^d$ in $W_{\mathrm{aux}}^{(1)}$, we have that $f = m_j^{-1}\sum_{i=1}^d f_i$ is in $W_{\mathrm{aux}}^{(1)}$.

The purpose of the following two lemmas is to provide upper bounds for the norms of linear combinations of certain vectors in the auxiliary space.

Lemma 15.2. *Let $j \in \mathbb{N}$ and $\varepsilon > 0$ with $\varepsilon \le m_j^{-1}$. For every (n_j, ε)-basic s.c.c. $x = \sum_{k\in F} c_k e_k$, the following hold.*

(i) *For every $f \in W_{aux}^{(1)}$*

$$\left|f(x)\right| \le \begin{cases} \varepsilon, & f = \pm e_i^* \text{ for some } i \in \mathbb{N} \\ \frac{1}{w(f)}, & w(f) \ge m_j \\ \frac{2}{w(f)m_j}, & w(f) < m_j. \end{cases}$$

(ii) *If $f \in W_{aux}^{(1)}$ with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$, such that $w(f_\alpha) \ne m_j$ for all $\alpha \in \mathcal{A}$ and $\varepsilon < m_j^{-2}$, then $|f(x)| < 2m_j^{-2}$.*

Proof. We may assume that $\mathrm{supp}(f) \subset F$ and $f(e_i) \ge 0$ for every $i \in \mathbb{N}$. If $f = \pm e_i^*$ for some $i \in F$, then $|f(x)| = c_i < \varepsilon$, since x is an (n_j, ε)-basic s.c.c. and $\{i\} \in \mathcal{S}_0$.

Suppose that $m_j \le w(f)$. Then $\|f\|_\infty \le 1/w(f)$, and, hence

$$\left|f(x)\right| \le \|f\|_\infty \|x\|_1 \le \frac{1}{w(f)}.$$

In the case where $w(f) = m_i < m_j$, let $f = m_i^{-1}\sum_{l=1}^d f_l$ with $(f_l)_{l=1}^d$ an $\mathcal{S}_{n_i+1}$-admissible sequence in $W_{\mathrm{aux}}^{(1)}$. For $l = 1, \ldots, d$, define $D_l = \{k \in F : f_l(e_k) > m_j^{-1}\}$ and $D = \cup_{l=1}^d D_l$. Then, [13, Lemma 3.16] implies that $D_l \in \mathcal{S}_{(\log_2(m_j)-1)(n_{j-1}+1)}$ for each $l = 1, \ldots, d$ and, hence, since $(f_l)_{l=1}^d$ is $S_{n_{j-1}+1}$-admissible (recall that $i < j$ since $m_i < m_j$) and $D_l \subset \mathrm{supp}(f_l)$, $l = 1, \ldots, d$, we conclude that the sequence $(D_l)_{l=1}^d$ is $S_{n_{j-1}+1}$-admissible and

$$D = \cup_{l=1}^d D_l \in S_{n_{j-1}+1} * \mathcal{S}_{(\log_2(m_j)-1)(n_{j-1}+1)} = \mathcal{S}_{\log_2(m_j)(n_{j-1}+1)}.$$

Since x is an (n_j, ε)-basic s.c.c. and $\log_2(m_j)(n_{j-1}+1) < n_j$, the above implies that $\sum_{k\in D} c_k < \varepsilon$, and, thus

$$\begin{aligned} f(x) &= \frac{1}{m_i}\sum_{l=1}^d f_l\Big(\sum_{k\in F} c_k e_k\Big) = \frac{1}{m_i}\Big(\sum_{l=1}^d f_l|_D\Big(\sum_{k\in F} c_k e_k\Big) + \sum_{l=1}^d f_l|_{\mathbb{N}\setminus D}\Big(\sum_{k\in F} c_k e_k\Big)\Big) \\ &\le \frac{1}{m_i}\Big(\sum_{k\in D} c_k + \frac{1}{m_j}\Big) \le \frac{1}{m_i}\Big(\varepsilon + \frac{1}{m_j}\Big) \le \frac{2}{m_i m_j}. \end{aligned}$$

Finally, if there is a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ of f with $w(f_\alpha) \ne m_j$ for every $\alpha \in \mathcal{A}$, [13, Lemma 3.16] implies that $D = \{k \in F : f(e_k) > m_j^{-2}\} \in \mathcal{S}_{(2\log_2(m_j)-1)(n_{j-1}-1)}$, and since $(2\log_2(m_j)-1)(n_{j-1}-1) < n_j$, we have that $\sum_{k\in D} c_i < \varepsilon$. Hence, we conclude that

$$f(x) = \sum_{k\in D} c_k f(x_k) + \sum_{k\in F\setminus D} c_k f(x_k) \le \varepsilon + \frac{1}{m_j^2} < \frac{2}{m_j^2}.$$

□

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15.2. The basic inequality

Proposition 15.3 (basic inequality). *Let $(x_k)_{k\in I}$ be a $(C,(j_k)_{k\in I})$-RIS in $\mathfrak{X}^{(1)}_{awi}$ with $4 \leq \min supp(x_{\min I})$, $(a_k)_{k\in I}$ be a sequence of nonzero scalars and $f \in W_{(1)}$ with $I_f \neq \emptyset$. Define $t_k = \max supp(x_k)$, $k \in I$. Then there exist*

(i) $g \in W^{(1)}_{aux} \cup \{0\}$ *with* $w(g) = w(f)$ *if* $g \neq 0$ *and* $\{k : t_k \in supp(g)\} \subset I_f$,
(ii) $h \in \{sign(a_k)e^*_{t_k} : k \in I_f\} \cup \{0\}$ *with* $k_0 \in I_f$ *and* $k_0 < \min supp(g)$ *if* $h = sign(a_{k_0})e^*_{t_{k_0}}$ *and*
(iii) $j_0 \geq \min\{j_k : k \in I_f\}$,

such that

$$|f(\sum_{k\in I_f} a_k x_k)| \leq C(1+\frac{1}{\sqrt{m_{j_0}}})[h + g(\sum_{k\in I_f} a_k e_{t_k})]. \tag{15.1}$$

Proof. Recall that $W_{(1)}$ is the increasing union of the sequence $(W^n_{(1)})_{n=0}^\infty$ defined in Remark 5.18. We prove the statement by induction on $n = 0, 1, \ldots$ for every $f \in W^n_{(1)}$ and every RIS.

For $n = 0$ and $f \in W^0_{(1)}$, the fact that $I_f \neq \emptyset$ implies that $I_f = \{k_0\}$, that is, $f = e^*_{t_{k_0}}$ or $f = -e^*_{t_{k_0}}$ for some $k_0 \in I$. In either case, it is immediate to check that $h = \text{sign}(a_{k_0})e^*_{t_{k_0}}$, $g = 0$ and $j_0 = j_{k_0}$ are as desired.

Fix $n \in \mathbb{N}$, and assume that the conclusion holds for every $f \in W^n_{(1)}$ and every RIS. Pick an $f \in W^{n+1}_{(1)}$ with $f = m_i^{-1}\sum_{l=1}^d f_l$, where $(f_l)_{l=1}^d$ is an $\mathcal{S}_{n_i}$-admissible sequence in $W^n_{(1)}$. We will first treat the two extreme cases, namely, the cases where $i \geq \max\{j_k : k \in I_f\}$ and $i < \min\{j_k : k \in I_f\}$.

For the first case, set $k_0 = \max I_f$ and $j_0 = j_{k_0}$ and choose $k_1 \in I_f$ that maximises the quantity $|a_k|$ for $k \in I_f$. Then, since $(x_k)_{k\in I}$ is a RIS, items (i) and (ii) of Definition 8.5 yield that

$$\begin{aligned}|f(\sum_{k\in I_f\setminus\{k_0\}} a_k x_k)| &\leq \frac{1}{m_i}\max \text{supp}(x_{k_0-1})\|\sum_{k\in I_f\setminus\{k_0\}} a_k x_k\|_\infty \\ &\leq \frac{\max \text{supp}(x_{k_0-1})}{m_{j_{k_0}}}C|a_{k_1}| \leq \frac{C}{\sqrt{m_{j_{k_0}}}}|a_{k_1}|,\end{aligned}$$

and, thus

$$\begin{aligned}|f(\sum_{k\in I_f} a_k x_k)| &\leq \frac{C}{\sqrt{m_{j_{k_0}}}}|a_{k_1}| + |f(a_{k_0}x_{k_0})| \\ &\leq \frac{C}{\sqrt{m_{j_{k_0}}}}|a_{k_1}| + C|a_{k_1}| = C(1+\frac{1}{\sqrt{m_{j_{k_0}}}})|a_{k_1}| \\ &= C(1+\frac{1}{\sqrt{m_{j_{k_0}}}})\text{sign}(a_{k_1})e^*_{t_{k_1}}(\sum_{k\in I_f} a_k e_{t_k}).\end{aligned} \tag{15.2}$$

That is, $h = \text{sign}(a_{k_1})e^*_{t_{k_1}}$, $g = 0$ and j_{k_0} yield the conclusion.

For the second case, the inductive hypothesis implies that, for every $l = 1, \ldots, d$ with $I_{f_l} \neq \emptyset$, there are g_l, h_l and $j_{0,l}$ as in (i)–(iii) of the statement, that satisfy the conclusion for the functional f_l. Define $J_f = \{k \in I_f : f(x_k) \neq 0\} \setminus \cup_{l=1}^d I_{f_l}$. Then, for every $k \in J_f$, Definition 8.5 (iii) yields that

$$|f(a_k x_k)| \leq \frac{C}{m_i}|a_k| = \frac{C}{m_i}\text{sign}(a_k)e^*_{t_k}\left(\sum_{k\in I_f} a_k e_{t_k}\right),$$

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and, hence, we calculate

$$
\begin{aligned}
|f(\sum_{k\in I_f} a_k x_k)| &\leq |f(\sum_{k\in \cup_{l=1}^d I_{f_l}} a_k x_k)| + |f(\sum_{k\in J_f} a_k x_k)| \\
&\leq \frac{C}{m_i}\sum_{k\in J_f}\mathrm{sign}(a_k)e^*_{t_k}(\sum_{k\in I_f} a_k e_{t_k}) + \frac{C}{m_i}\sum_{l=1}^d [(1+\frac{1}{\sqrt{m_{j_{0,l}}}})(h_l+g_l)](\sum_{k\in I_{f_l}} a_k e_{t_k}) \\
&\leq C(1+\frac{1}{\sqrt{m_{j_{\min I_f}}}})[\frac{1}{m_i}(\sum_{k\in J_f}\mathrm{sign}(a_k)e^*_{t_k} + \sum_{l=1}^d h_l + g_l)](\sum_{k\in I_f} a_k e_{t_k}).
\end{aligned}
\tag{15.3}
$$

Define

$$
g = \frac{1}{m_i}(\sum_{k\in J_f}\mathrm{sign}(a_k)e^*_{t_k} + \sum_{l=1}^d h_l + g_l).
$$

Moreover, for each $l = 1,\dots,d$, define

$$
K_l = \left\{k \in J_f : \min\{l' = 1,\dots,d : \mathrm{supp}(x_k)\cap \mathrm{range}(f_{l'}) \neq \emptyset\} = l\right\}
$$

and

$$
I_l = \{t_k : k\in K_l\} \cup \{\mathrm{supp}(h_l)\} \cup \{\min \mathrm{supp}(g_l)\}.
$$

Let us make the following remarks. First, observe that $\#K_l \leq 2$. In particular, consider the case where $K_l = \{k_1, k_2\}$ for some $l = 1,\dots,d$. Then, $k_1 < \min I_{f_l} \leq \max I_{f_l} < k_2$, and since $\mathrm{supp}(h_l)\cup\mathrm{supp}(g_l)$ is a subset of $\{t_k : k \in I_{f_l}\}$, we have $t_{k_1} < \mathrm{supp}(h_l) < \mathrm{supp}(g_l) < t_{k_2}$. Moreover, if $l < d$ and $\mathrm{range}(f_{l+1})\cap\mathrm{supp}(x_{k_2}) \neq \emptyset$, then $k_2 \notin K_{l+1}$ and clearly $k_2 < I_{l+1}$. In the case where K_l is a singleton for some $l = 1,\dots,d$, then either $\mathrm{supp}(h_l) < \mathrm{supp}(g_l) < k$ or $k < \mathrm{supp}(h_l) < \mathrm{supp}(g_l)$ holds for $K_l = \{k\}$. Hence, we conclude that $I_1 < \cdots < I_d$. Moreover, let us finally note that $\min\mathrm{supp}(f_l) \leq I_l$ and $\#I_l \leq 4$ for every $l = 1,\dots,d$. For each $l = 1,\dots,d$, let $K_l = \{k_1^l, k_2^l\}$, where k_2^l or k_2^l can be ommited if necessary. Then,

$$
g = \frac{1}{m_i}\sum_{l=1}^d(\mathrm{sign}(a_{k_1^l})e^*_{t_{k_1^l}} + h_l + g_l + \mathrm{sign}(a_{k_2^l})e^*_{t_{k_2^l}}).
\tag{15.4}
$$

We will show that the sequence $(e^*_{t_k})_{k\in J_f}{}^\frown(h_l)_{l=1}^d{}^\frown(g_l)_{l=1}^d$ is $\mathcal{S}_{n_i+1}$-admissible, when the functionals ordered as implied by (15.4), that is, according to the minimum of their supports. This yields that $g \in W^{(1)}_{\mathrm{aux}}$, and thus $h = 0$, g and $j_0 = j_{\min I_f}$ satisfy the conclusion, as follows from (15.3). More specifically, we will show that $\cup_{l=1}^d I_l \in \mathcal{S}_{n_i+1}$. To this end, note that $(I_l)_{l=1}^d$ is $\mathcal{S}_{n_i}$-admissible, since $(f_l)_{l=1}^d$ is $\mathcal{S}_{n_i}$-admissible, $I_1 < \cdots < I_d$ and $\min\mathrm{supp}(f_l) \leq I_l$ for every $l = 1,\dots,d$. Thus, $\cup_{l=1}^d I_l \in \mathcal{S}_{n_i} * \mathcal{A}_4$, since $\#I_l \leq 4$ for all $l = 1,\dots,d$. Using item (ii) of Lemma 2.1 and the fact that $4 \leq \min\mathrm{supp}(x_{\min I})$, we conclude that $\cup_{l=1}^d I_l \in \mathcal{S}_{n_i+1}$.

Finally, in the remaining case where $\min\{j_k : k\in I_f\} \leq i < \max\{j_k : k \in I_f\}$, define $I_f^1 = \{k \in I_f : j_k \leq i\}$ and $I_f^2 = \{k \in I_f : j_k > i\}$, and observe that $I_f = I_f^1 \cup I_f^2$, $\max\{j_k : k\in I_f^1\} \leq i$ and $i < \min\{j_k : k \in I_f^2\}$. Applying the result of the first case for $(x_k)_{k\in I_f^1}$ and that of the second for

$(x_k)_{k\in I_f^2}$, we have

$$
\begin{aligned}
|f(\sum_{k\in I_f} a_k x_k)| &\leq |f(\sum_{k\in I_f^1} a_k x_k)| + |f(\sum_{k\in I_f^2} a_k x_k)| \\
&\leq C(1+\frac{1}{\sqrt{m_{j_{\max I_f^1}}}})h(\sum_{k\in I_f} a_k e_{t_k}) + |f(\sum_{k\in I_f^2} a_k x_k)| \\
&\leq C(1+\frac{1}{\sqrt{m_{j_{\max I_f^1}}}})h(\sum_{k\in I_f} a_k e_{t_k}) + C(1+\frac{1}{\sqrt{m_{j_{\min I_f^2}}}})g(\sum_{k\in I_f} a_k e_{t_k}) \\
&\leq C(1+\frac{1}{\sqrt{m_{j_{\max I_f^1}}}})[h+g(\sum_{k\in I_f} a_k e_{t_k})],
\end{aligned}
\tag{15.5}
$$

where $h = \mathrm{sign}(a_{k_1})e^*_{t_{k_1}}$, $k_1 \in I_f^1$ maximises the quantity $|a_k|$ for $k \in I_f^1$ and $g \in W_{\mathrm{aux}}^{(1)}$ with $w(g) = w(f)$. □

Remark 15.4. Let $(x_k)_{k\in I}$ and f be as in the statement of Proposition 15.3.

(i) Define $E = \mathrm{range}(f)$, and note that the sequence $(x'_k)_{k\in I'_f}$, where $x'_k = x_k|_E$, $k \in I'_f$, is also a $(C, (j_k)_{k\in I'_f})$-RIS. Then,

$$f(\sum_{k\in I} a_k x_k) = f(\sum_{k\in I'_f} a_k x'_k)$$

and $\{k \in I'_f : \mathrm{supp}(x'_k) \subset \mathrm{range}(f)\} = I'_f$. Hence, the basic inequality yields h, g and j_0 as in items (i)–(iii), such that

$$|f(\sum_{k\in I} a_k x_k)| \leq C(1+\frac{1}{\sqrt{m_{j_0}}})[h+g(\sum_{k\in I'_f} a_k e_{z_k})],$$

where $z_k = \max \mathrm{supp}(x'_k)$, $k \in I'_f$.

(ii) Let $j \in \mathbb{N}$. It follows from the proof of Proposition 15.3 that if f has a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$, such that $I_{f_\alpha} = \emptyset$ for every $\alpha \in \mathcal{A}$ with $w(f_\alpha) = m_j$, then the functional $g \in W_{\mathrm{aux}}^{(1)} \cup \{0\}$ that the basic inequality yields for $(x_k)_{k\in I}$ and f has a tree analysis $(g_\beta)_{\beta\in\mathcal{B}}$ with $w(g_\beta) \neq m_j$ for every $\beta \in \mathcal{B}$, whenever $g \neq 0$.

15.3. Evaluations on standard exact pairs

We prove the following lemma, which yields Proposition 8.11 as an immediate corollary.

Lemma 15.5. *For every (C, m_{j_0})-SEP (x, f), the following hold.*

(i) *For every $f' \in W_{(1)}$*

$$\left|f'(x)\right| \leq \begin{cases} \frac{C}{m_{j_0}}(1+\frac{1}{\sqrt{m_{j_0}}}), & f' = \pm e_i^* \text{ for some } i \in \mathbb{N} \\ C(1+\frac{1}{\sqrt{m_{j_0}}})[\frac{1}{m_{j_0}} + \frac{m_{j_0}}{w(f')}], & w(f') \geq m_{j_0} \\ C(1+\frac{1}{\sqrt{m_{j_0}}})[\frac{1}{m_{j_0}} + \frac{2}{w(f')}], & w(f') < m_{j_0}. \end{cases}$$

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(ii) *If $f' \in W_{(1)}$ with a tree analysis $(f'_\alpha)_{\alpha\in\mathcal{A}}$, such that $I_{f'_\alpha} = \emptyset$ for every $\alpha \in \mathcal{A}$ with $w(f'_\alpha) = m_{j_0}$, then*

$$|f'(x)| \leq \frac{3C}{m_{j_0}}(1 + \frac{1}{\sqrt{m_{j_0}}}).$$

Proof. Let $(x_k)_{k=1}^n$ be a $(C, (j_k)_{k=1}^n)$-RIS witnessing that (x, f) is a (C, m_{j_0})-SEP. Applying Proposition 15.3, we obtain h and g as in items (i) and (ii), respectively, that satisfy (15.1) for x and f', namely,

$$|f'(x)| \leq Cm_{j_0}(1 + \frac{1}{\sqrt{m_{j_0}}})[h(\tilde{x}) + g(\tilde{x})],$$

where $\tilde{x} = \sum_{k\in I} a_k e_{z_k}$, $z_k = \max \operatorname{supp}(x_k|_{\operatorname{range}(f')})$ and $I = \{k = 1, \ldots, n : \operatorname{supp}(x_k) \cap \operatorname{range}(f') \neq \emptyset\}$. Note that $\tilde{x}$ is a $(n_{j_0}, m_{j_0}^{-2})$-b.s.c.c. and, hence, since $\operatorname{supp}(h) \in \mathcal{S}_0$, we have $h(\tilde{x}) < m_{j_0}^{-2}$.

To prove (i), first observe that if $g = 0$, which is the case, for example, when $f' = \pm e_i^*$ for some $i \in \mathbb{N}$, then we already have established a valid upper bound for $|f'(x)|$. Hence, suppose that $g \neq 0$. Then, using Lemma 15.2 and the fact that $w(g) = w(f')$, we obtain the following upper bounds for $g(\tilde{x})$

$$g(\tilde{x}) \leq \begin{cases} \frac{1}{w(f')}, & w(f') \geq m_{j_0} \\ \frac{2}{w(f')m_{j_0}}, & w(f') < m_{j_0}, \end{cases}$$

which yield the desired upper bounds for $|f'(x)|$.

Finally, item (ii) of Remark 15.4 implies that g admits a tree analysis $(g_\beta)_{\beta\in\mathcal{B}}$, such that $w(g_\beta) \neq m_{j_0}$ for every $\beta \in \mathcal{B}$. We derive the desired upper bound using item (ii) of Lemma 15.2, which yields that $|g(\tilde{x})| \leq 2m_{j_0}^{-2}$. □

16. Appendix B

We prove another version of the basic inequality that reduces evaluations on standard exact pairs of $\mathfrak{X}^{(2)}_{\text{awi}}$ to evaluations on the basis of an auxiliary space. The results are almost identical to those of Appendix A, and we include them for completeness.

16.1. The auxiliary space

Definition 16.1. Let $W^{(2)}_{\text{aux}}$ be the minimal subset of $c_{00}(\mathbb{N})$, such that

(i) $\pm e_i^*$ is in $W^{(2)}_{\text{aux}}$ for all $i \in \mathbb{N}$ and
(ii) whenever $j \in \mathbb{N}$, $(f_i)_{i=1}^d$ is an $\mathcal{S}_{n_j+1}$-admissible sequence in $W^{(2)}_{\text{aux}}$ and $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}$ with $\sum_{i=1}^d \lambda_i^2 \leq 1$, then $f = 2m_j^{-1}\sum_{i=1}^d \lambda_i f_i$ is in $W^{(2)}_{\text{aux}}$.

Remark 16.2. For each $f \in W^{(2)}_{\text{aux}}$, the weight of f is defined as $w(f) = 0$ if $f = \pm e_i^*$ for some $i \in \mathbb{N}$ and $w(f) = m_j/2$ in the case where $f = 2m_j^{-1}\sum_{i=1}^d \lambda_i f_i$.

The following lemma is a slightly modified version of [13, Lemma 3.16]. We use it to prove Lemma 16.4.

Lemma 16.3. *Let $f \in W^{(2)}_{aux}$ with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$.*

(i) *For all $j \in \mathbb{N}$, we have*

$$\{k \in supp(f) : w_f(e_k^*) < m_j\} \in \mathcal{S}_{(\log_2(m_j)-1)(n_{j-1}+1)}.$$

(ii) *If $j \in \mathbb{N}$ is such that $w(f_\alpha) \neq m_j$ for each $\alpha \in \mathcal{A}$, then*

$$\{k \in supp(f) : w_f(e_k^*) < m_j^2\} \in \mathcal{S}_{(2\log_2(m_j)-1)(n_{j-1}+1)}.$$

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Proof. The proof is similar to [13, Lemma 3.16]. □

Next, we prove a lemma similar to 15.2, for the evaluations of functionals in $W^{(2)}_{\text{aux}}$ on the ℓ_2 version of basic special convex combinations.

Lemma 16.4. *Let $j \in \mathbb{N}$ and $\varepsilon > 0$ with $\varepsilon \le m_j^{-2}$. For every $(2, n_j, \varepsilon)$-basic s.c.c. $x = \sum_{k\in F} c_k e_k$, the following hold.*

(i) *For every $f \in W^{(2)}_{aux}$*

$$|f(x)| \le \begin{cases} \frac{1}{w(f)}, & w(f) \ge m_j/2 \\ \frac{2}{w(f)m_j}, & w(f) < m_j/2. \end{cases}$$

(ii) *If $f \in W^{(2)}_{aux}$ with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$, such that $w(f_\alpha) \neq m_j$ for all $\alpha \in \mathcal{A}$ and $\varepsilon < m_j^{-4}$, then $|f(x)| < 2m_j^{-2}$.*

Proof. Without loss of generality, we may assume that $\text{supp}(f) \subset F$ and $f(e_k) \ge 0$ for every $k \in F$. If $m_j/2 \le w(f)$, then $\|f\|_2 \le 1/w(f)$, and, hence

$$|f(x)| \le \|f\|_2 \|x\|_2 \le \frac{1}{w(f)}.$$

Suppose now that $m_i < m_j$, and let $f = 2m_i^{-1} \sum_{l=1}^d \lambda_l f_l$, where $(f_l)_{l=1}^d$ is an $\mathcal{S}_{n_i+1}$-admissible sequence in $W^{(2)}_{\text{aux}}$. For $l = 1, \dots, d$, define

$$D_l = \{k \in \text{supp}(f_l) : w_{f_l}(e_k^*) < m_j\}, \quad F_l = \text{supp}(f_l) \setminus D_l.$$

Then, Lemma 16.3 (i) implies that $D_l \in \mathcal{S}_{(2\log_2(m_j)-1)(n_{j-1}+1)}$ for each $l = 1, \dots, d$, and, hence, since $(f_l)_{l=1}^d$ is $S_{n_{j-1}+1}$-admissible (recall that $i < j$ since $m_i < m_j$) and $D_l \subset \text{supp}(f_l)$, $l = 1, \dots, d$, we have

$$D = \cup_{l=1}^d D_l \in S_{n_{j-1}+1} * \mathcal{S}_{(2\log_2(m_j)-1)(n_{j-1}+1)} = \mathcal{S}_{2\log_2(m_j)(n_{j-1}+1)}.$$

Therefore, since x is an $(2, n_j, \varepsilon)$-basic s.c.c. and $2\log_2(m_j)(n_{j-1}+1) < n_j$, we have $\sum_{k\in D} c_k^2 < \varepsilon$. Moreover, observe that for $l = 1, \dots, d$ and $k \in F_l$

$$f_l(e_k) = \frac{\lambda_{f_l,\alpha_k}}{w_{f_l}(e_k^*)} \le \frac{\lambda_{f_l,\alpha_k}}{m_j},$$

where a_k is the node in the induced tree analysis of f_l with $f_{l,\alpha_k} = e_k^*$, and

$$\sum_{l=1}^d \lambda_l^2 \sum_{k\in F_l} \lambda_{f_l,\alpha_k}^2 \le 1.$$

We then calculate, using the Cauchy-Schwarz inequality

$$\begin{aligned} f(x) &= \frac{2}{m_i}\Big(\sum_{l=1}^d \lambda_l f_l|_D\Big(\sum_{k\in F} c_k e_k\Big) + \sum_{l=1}^d \lambda_l f_l|_{\mathbb{N}\setminus D}\Big(\sum_{k\in F} c_k e_k\Big)\Big) \\ &= \frac{2}{m_i}\Big(\sum_{l=1}^d \lambda_l \sum_{k\in D_l} \frac{c_k \lambda_{f_l,\alpha_k}}{w_{f_l}(e_k^*)} + \sum_{l=1}^d \lambda_l \sum_{k\in F_l} \frac{c_k \lambda_{f_l,\alpha_k}}{w_{f_l}(e_k^*)}\Big) \end{aligned}$$

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$$
\begin{aligned}
&\le \frac{2}{m_i}\Big(\sum_{l=1}^{d}\lambda_l \sum_{k\in D_l} c_k \lambda_{f_l,\alpha_k} + \frac{1}{m_j}\sum_{l=1}^{d}\lambda_l \sum_{k\in F_l} c_k\lambda_{f_l,\alpha_k}\Big)\\
&\le \frac{2}{m_i}\Big(\sum_{l=1}^{d}\lambda_l\big(\sum_{k\in D_l} c_k^2\big)^{\frac12}\big(\sum_{k\in D_l}\lambda^2_{f_l,\alpha_k}\big)^{\frac12} + \frac{1}{m_j}\sum_{l=1}^{d}\lambda_l\big(\sum_{k\in F_l} c_k^2\big)^{\frac12}\big(\sum_{k\in F_l}\lambda^2_{f_l,\alpha_k}\big)^{\frac12}\Big)\\
&\le \frac{2}{m_i}\Big(\big(\sum_{l=1}^{d}\lambda_l^2\big)^{\frac12}\big(\sum_{k\in D} c_k^2\big)^{\frac12} + \frac{1}{m_j}\big(\sum_{l=1}^{d}\sum_{k\in F_l} c_k^2\big)^{\frac12}\big(\sum_{l=1}^{d}\lambda_l^2\sum_{k\in F_l}\lambda^2_{f_l,\alpha_k}\big)^{\frac12}\Big)\\
&\le \frac{2}{m_i}\Big(\sqrt{\varepsilon}+\frac{1}{m_j}\Big) \le \frac{4}{m_i m_j}.
\end{aligned}
$$

Finally, if there is a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ of f, such that $w(f_\alpha) \neq m_j$ for every $\alpha \in \mathcal{A}$, Lemma 16.3 (ii) implies that

$$
D = \{k \in \operatorname{supp}(f) : w_f(e_k^*) < m_j^2\} \in \mathcal{S}_{(2\log_2(m_j)-1)(n_{j-1}-1)},
$$

and since $(2\log_2(m_j)-1)(n_{j-1}-1) < n_j$, we have that $\sum_{k\in D} c_k^2 < \varepsilon$. Hence, using similar arguments as above, we conclude that

$$
f(x) \le \sqrt{\varepsilon} + \frac{1}{m_j^2} < \frac{2}{m_j^2}.
$$

□

16.2. The basic inequality

Proposition 16.5 (basic inequality). *Let $(x_k)_{k\in I}$ be a $(C,(j_k)_{k\in I})$-RIS in $\mathfrak{X}^{(2)}_{awi}$ with $4 \le \min supp(x_{\min I})$, $(a_k)_{k\in I}$ be a sequence of nonzero scalars and $f \in W_{(2)}$ with $I_f \neq \emptyset$. Define $t_k = \max supp(x_k)$, $k \in I$. Then there exist*

(i) *$g \in W^{(2)}_{aux} \cup \{0\}$ with $w(g) = w(f)/2$ if $g \neq 0$ and $\{k : t_k \in supp(g)\} \subset I_f$,*
(ii) *$h \in \{sign(a_k)e^*_{t_k} : k \in I_f\} \cup \{0\}$ with $k_0 \in I_f$ and $k_0 < \min supp(g)$ if $h = sign(a_{k_0})e^*_{t_{k_0}}$ and*
(iii) *$j_0 \ge \min\{j_k : k \in I_f\}$,*

such that

$$
|f(\sum_{k\in I_f} a_k x_k)| \le C\Big(1+\frac{1}{\sqrt{m_{j_0}}}\Big)\Big[h + g\Big(\sum_{k\in I_f} a_k e_{t_k}\Big)\Big].
$$

Proof. As in Proposition 15.3, we prove the statement by induction on $n = 0, 1, \ldots$ for every $f \in W^n_{(2)}$ and every RIS. The case of $n = 0$ follows easily.

Fix $n \in \mathbb{N}$, and assume that the conclusion holds for every $f \in W^n_{(2)}$ and every RIS. Pick an $f \in W^{n+1}_{(2)}$ with $f = m_i^{-1}\sum_{l=1}^{d}\lambda_l f_l$, where $(f_l)_{l=1}^d$ is an $\mathcal{S}_{n_i}$-admissible AWI sequence in $W^n_{(2)}$ and $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}$ with $\sum_{l=1}^{d}\lambda_l^2 \le 1$. The proof of the case where $i \ge \max\{j_k : k \in I_f\}$ is identical to that of Proposition 15.3.

Suppose then that $i < \min\{j_k : k \in I_f\}$. The inductive hypothesis implies that, for every $l = 1, \ldots, d$ with $I_{f_l} \neq \emptyset$, there are g_l, h_l and $j_{0,l}$ as in (i)–(iii) of the statement, that satisfy the conclusion for the functional f_l. Define $J_f = \{k \in I_f : f(x_k) \neq 0\} \setminus \cup_{l=1}^{d} I_{f_l}$. For every $k \in J_f$, since $i < j_k$, Definition 8.5 (iii) yields that

$$
|f(a_k x_k)| \le \Big(\sum_{l\in L_k}\lambda_l^2\Big)^{\frac12}\frac{C}{m_i}|a_k| = \Big(\sum_{l\in L_k}\lambda_l^2\Big)^{\frac12}\frac{C}{m_i}\operatorname{sign}(a_k)e^*_{t_k}\Big(\sum_{k\in I_f} a_k e_{t_k}\Big),
$$

where

$$L_k = \{l \in \{1, \ldots, d\} : \mathrm{supp}(x_k) \cap \mathrm{supp}(f_l) \neq \emptyset\}.$$

Hence, we calculate

$$\begin{aligned}
&|f(\sum_{k\in I_f} a_k x_k)| \leq |f(\sum_{k\in J_f} a_k x_k)| + |f(\sum_{k\in \cup_{l=1}^d I_{f_l}} a_k x_k)| \\
&\leq \frac{C}{m_i}\sum_{k\in J_f}(\sum_{l\in L_k}\lambda_l^2)^{\frac{1}{2}}\mathrm{sign}(a_k)e^*_{t_k}(\sum_{k\in I_f} a_k e_{t_k}) + \frac{C}{m_i}\sum_{l=1}^{d}[(1+\frac{1}{\sqrt{m_{j_{0,l}}}})\lambda_l(h_l+g_l)](\sum_{k\in I_{f_l}} a_k e_{t_k}) \\
&\leq C(1+\frac{1}{\sqrt{m_{j_{\min I_f}}}})[\frac{1}{m_i}(\sum_{k\in J_f}(\sum_{l\in L_k}\lambda_l^2)^{\frac{1}{2}}\mathrm{sign}(a_k)e^*_{t_k} + \sum_{l=1}^{d}\lambda_l h_l + \lambda_l g_l)](\sum_{k\in I_f} a_k e_{t_k}).
\end{aligned}$$

Define

$$g = \frac{2}{m_i}(\sum_{k\in J_f}\frac{1}{2}(\sum_{l\in L_k}\lambda_l^2)^{\frac{1}{2}}\mathrm{sign}(a_k)e^*_{t_k} + \sum_{l=1}^{d}\frac{\lambda_l}{2}h_l + \frac{\lambda_l}{2}g_l).$$

Then, observe that each $l = 1, \ldots, d$, belongs to L_k for at most two $k \in J_f$, and thus, using the same arguments as in Proposition 15.3, we have that $g \in W^{(2)}_{\mathrm{aux}}$, and this completes the proof for cases where $i < j_k$ for all $k \in I_f$.

Finally, the proof of the remaining case is the same as in Proposition 15.3. □

16.3. *Evaluations on standard exact pairs*

Finally, we prove the following lemma which shows that standard exact pairs are in fact strong exact pairs.

Lemma 16.6. *For every $(2, C, m_{j_0})$-SEP (x, f), the following hold.*

(i) *For every $g \in W$*

$$|g(x)| \leq \begin{cases} 2C(1+\frac{1}{\sqrt{m_{j_0}}})[\frac{1}{m_{j_0}} + \frac{m_{j_0}}{w(g)}], & w(g) \geq m_{j_0} \\ 2C(1+\frac{1}{\sqrt{m_{j_0}}})[\frac{1}{m_{j_0}} + \frac{2}{w(g)}], & w(g) < m_{j_0}. \end{cases}$$

(ii) *If $g \in W$ with a tree analysis $(g_\alpha)_{\alpha\in\mathcal{A}}$, such that $I_{g_\alpha} = \emptyset$ for every $\alpha \in \mathcal{A}$ with $w(g_\alpha) = m_{j_0}$, then*

$$|g(x)| \leq \frac{3C}{m_{j_0}}(1+\frac{1}{\sqrt{m_{j_0}}}).$$

Proof. Apply the basic inequality and the evaluations of functionals in $W^{(2)}_{\mathrm{aux}}$ on 2-b.s.c.c. from Lemma 16.4. The proof is identical to that of Lemma 15.5. □

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