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Heteroclinic and homoclinic solutions for a singular Hamiltonian system

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We consider an autonomous Hamiltonian system $\ddot{q} + V_q(q) = 0$ in $\mathbf{R}^2$, where the potential V has a global maximum at the origin and singularities at some points ξ_1, $\xi_2 \in \mathbf{R}^2 \setminus \{0\}$. Under some compactness conditions on V at infinity and assuming a strong force type condition at the singularities, we study, using variational arguments, the existence of various types of heteroclinic and homoclinic solutions of the system.

1 Introduction

The existence of various kinds of heteroclinic and homoclinic solutions of the autonomous second order Hamiltonian system

$$(HS) \qquad \ddot{q} + V_q(q) = 0$$

for $q \in \mathbb{R}^2$, will be studied. The potential V satisfies the following assumptions:

(V1) There exist distinct points ξ_1, $\xi_2 \in \mathbb{R}^2 \setminus \{0\}$ such that $V \in C^2(\mathbb{R}^2 \setminus \{\xi_1, \xi_2\})$;

(V2) $\lim_{x \to \xi_i} V(x) = -\infty$, for $i = 1, 2$;

(V3) There exist neighbourhoods, N_i, of ξ_i and functions $U_i \in C^1(N_i \setminus \{\xi_i\}, \mathbb{R})$ such that $|U_i(x)| \to +\infty$, as $x \to \xi_i$, and $|(U_i)_x(x)|^2 \leqslant -V(x)$ for $x \in N_i \setminus \{\xi_i\}$, $i = 1, 2$;

(V4) $V(x) \leqslant 0$, $V(x) = 0$ if and only if $x = 0$, and $V_{xx}(0)$ is negative definite.

(V5) There is a constant $V_0 < 0$ such that $\overline{\lim_{|x| \to \infty}} V(x) \leqslant V_0$.

Condition (V3) governs the rate at which $-V(x) \to \infty$ as $x \to \xi_i$, $i = 1, 2$. Conditions of this nature were introduced by Poincaré in [11], On the other hand, condition (V4) implies that the origin is a nondegenerate critical point, and (V5) guarantees that the critical point, 0, is a global maximum of V.

As an example of such potential, we can consider the function $V : \mathbb{R}^2 \setminus \{\xi_1, \xi_2\} \to \mathbb{R}$, defined by

$$V(\mathbf{x}) = \frac{-|\mathbf{x}|^4}{|\mathbf{x} - \xi_2|^2 |\mathbf{x} - \xi_1|^2},$$

where $\xi_1, \xi_2 \in \mathbb{R}^2 \setminus \{0\}$.

The Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$ will be employed. We will minimize the functional

$$I(q) = \int_{\mathbb{R}} \mathbf{L}(q)\, dt$$

where

$$\mathbf{L}(q) = \frac{1}{2}|\dot{q}|^2 - V(q)$$

is the associated Lagrangian, to prove the existence of three homoclinic solutions, winding around one or both of the singularities. Introducing a geometric condition, we prove that these solutions are geometrically distinct, and that they are minimal, energy 0 solutions of (HS) which do not intersect each other.

In § 3, similar arguments will be used to conclude that for each positive real number T, there are T-periodic solutions of (HS), winding around each of the singularities. This result is borrowed from Caldiroli & Jeanjean [4]: the singularities ξ_1 and ξ_2 have no interaction, since we have required that each class of periodic solutions remains in the region bounded by the respective homoclinic solution. Minimizing the Lagrangian over all these periodic functions, we will obtain at least two solutions of (HS) which we will call *minimal action periodics*. Since we may view the homoclinic solutions as periodic functions of infinite period, the danger in this approach is that the minimum action periodics may be precisely the homoclinic solutions. Hence, we will impose a new geometric condition in order to obtain a distinct class of solutions.

As a consequence, for each $i = 1, 2$, we may consider the region of $\mathbb{R}^2$ which is bounded by the homoclinic solution and the minimal periodic solution of the same homotopy type. In § 4, we will prove the existence of a solution of (HS) which asymptotes to the origin as $t \to -\infty$, and to the periodic solution as $t \to \infty$, i.e., is heteroclinic from 0 to the periodic solution, and that for all $t \in \mathbb{R}$ remains in that region. Since at those curves, the functional I will be infinite, we will need to define a renormalized functional, which we will denote by J. Although we give the proof for the particular case of a heteroclinic solution between the global maximum of V and the periodic that winds around ξ_1, the same methods can be applied to prove the existence of an heteroclinic solution between the origin and the periodic that winds around ξ_2. We will also consider the case of 2 heteroclinic solutions of (HS) which asymptote to the periodic solution as $t \to -\infty$ and approach the origin as $t \to \infty$. The direction of the winding plays a fundamental role in this case. By defining the appropriate direction for these curves, we exclude the possibility of obtaining the heteroclinic solutions we described earlier with time reversed. We will show the existence of 4 distinct solutions and their properties

The problem of studying Hamiltonian systems using variational methods, has been treated extensively in the literature [1, 3, 8, 9, 10, 16]. The case of singular Hamiltonian systems is rather important, due to the fact that potentials arising, e.g., from gravitational and electromagnetic force fields have infinitely deep wells. Gordon [7] studied a conservative dynamical system, which is associated with a potential, V, singular at a compact set S. He used condition (V3), the so called *strong force condition* introduced by Poincaré in [11], which is mainly used to verify compactness properties of the functional I. Furthermore, he showed the existence of periodic trajectories making loops around S for such systems through the application of standard variational and geometrical techniques.

As pointed out by Gordon [7], it is perhaps disappointing that the gravitational case is not included in the class of systems verifying the strong force condition. Nevertheless the failure of these methods in the Newtonian potential agree with experimental evidence. We were able to prove that basically every kind of heteroclinic and homoclinic solutions can be found in a system under the strong force condition. This kind of chaotic dynamic is clearly not happening in the gravitational case.

In the case of a potential having only one singularity, ξ, it was proved in Caldiroli & Jeanjean [4] and Rabinowitz [13], the existence of a wide class of solutions of (HS) winding around ξ: a minimal action periodic solution, p, and a homoclinic solution, q, winding once around ξ. The condition

$$\int_0^T \mathbf{L}(p)\, dt < \int_{\mathbb{R}} \mathbf{L}(q)\, dt,$$

is sufficient to prove the existence of a more general class of homoclinic solutions of (HS), winding a finite number of times around ξ. Caldiroli & Jeanjean [4] show the existence of a heteroclinic solution between 0 and the periodic solution, as a limit in C^1_{loc}-topology of the sequence of homoclinics solutions described above. Also in Rabinowitz [13], the effect of dropping the strong force condition is studied and the existence of a "generalized" homoclinic solution which may be a collision orbit, is proved.

In Caldiroli & Nolasco [5], the existence of infinitely many homoclinic solutions with an arbitrary winding number for a planar singular system, was proved under a geometrical property which is verified for example by potentials with some discrete rotational symmetry and by potentials given by the sum of a smooth radial term and a localized singular perturbation.

In the papers mentioned above, as in the present work, some topological and geometric properties specific to $\mathbb{R}^2$ are essential to obtain the multiple homoclinic and heteroclinic solutions. For the more general case, of a system in $\mathbb{R}^n$, $n > 2$, Bahri & Rabinowitz [2], using a minimax method, prove the existence of time periodic solutions of (HS), and their method applies to a singular potential as well. Tanaka [17] obtained a homoclinic solution of (HS) by solving an approximate system in the spirit of Bahri & Rabinowitz [2], and then using a limit process to obtain it.

2 Existence of homoclinic solutions

2.1 The variational setting

Consider

$$E = \{q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2) \mid \int_{\mathbb{R}} |\, \dot{q}\, |^2\, dt < \infty\},$$

endowed with the norm $\|q\|^2 = |q(0)|^2 + \int_{\mathbb{R}} |\, \dot{q}\, |^2\, dt$, and define the functional

$$I(q) = \int_{\mathbb{R}} \mathbf{L}(q)\, dt = \int_{\mathbb{R}} \left(\frac{1}{2}|\dot{q}|^2 - V(q)\right) dt.$$

As a direct consequence of (V4) we can conclude that I is bounded from below in E.

In this setting, the functional I satisfies some properties, proofs of which can be found, for example, in Rabinowitz [12].

Lemma 2.1 *Let $q \in E$. If $t_1 < t_2 \in \mathbb{R}$ are such that $q(t) \notin B_\epsilon(0)$ for $t \in [t_1, t_2]$, then*

$$I(q) \geqslant \sqrt{2\alpha_\epsilon}|q(t_2) - q(t_1)|$$

where $\alpha_\epsilon \equiv \inf_{x \notin B_\epsilon(0)}(-V(x))$.

Lemma 2.2 *If $q \in E$ and for some positive real number M, $I(q) < M$, then*

(a) $\lim_{t\to\pm\infty}|q(t)| = 0$;
(b) $q \in C(\mathbb{R}, \mathbb{R}^2)$;
(c) *If we further define*

$$\Lambda = \{q \in E : q(t) \neq \xi_i,\ \forall t \in \mathbb{R},\ i = 1, 2\},$$

there exists $\beta = \beta(M) > 0$ such that for all $t \in \mathbb{R}$ and $i = 1, 2$

$$|q(t) - \xi_i| > \beta.$$

In particular if $I(q) < \infty$, then $q \in \Lambda$.

The previous lemma implies that any function $\hat{q}$ in E with $I(\hat{q}) < \infty$, describes a closed curve in $\mathbb{R}^2$ which starts and ends at the origin, without ever entering the singularities. By the identification of $\mathbb{R}$ with S^1, we can view $\hat{q}$ as a function $q : S^1 \to \mathbb{R}^2$. If Q is any continuous extension of q to $B_1(0)$, it makes sense to consider the Brouwer degree of Q with respect to ξ_i, $d(Q, B_1(0), \xi_i)$, since $\xi_i \notin Q(S^1) = q(S^1)$. For any $q \in \Lambda$ we will associate with it an element of $\mathbb{Z}^2$,

$$D(q) = (d_{\xi_1}(q), d_{\xi_2}(q))$$

where $d_{\xi_i}(q)$ denotes the Brouwer degree of q with respect to ξ_i, $i = 1, 2$, in the sense explained above. For every $\gamma \in \mathbb{Z}^2$, set

$$\Lambda_\gamma = \{q \in \Lambda : D(q) = \gamma\}.$$

If $q \in \Lambda_\gamma$, q is said to be of homotopy class prescribed by γ.

We are interested in finding solutions of (HS), which are homoclinic to zero, and that wind around one or both of the singularities. For that purpose, we introduce the following subsets of Λ_γ:

$$\Lambda_1 = \{q \in \Lambda : D(q) = (\alpha, 0) \text{ for some } \alpha \in \mathbb{Z}^+\},$$

$$\Lambda_2 = \{q \in \Lambda : D(q) = (0, \beta) \text{ for some } \beta \in \mathbb{Z}^+\}$$

and

$$\Lambda_{1,2} = \{q \in \Lambda : D(q) = (\alpha, \beta) \text{ for some } \alpha, \beta \in \mathbb{Z}^+\}.$$

For each of $i = 1, 2$, define

$$c_i = \inf_{q\in\Lambda_i} I(q) \quad \text{and} \quad c_{1,2} = \inf_{q\in\Lambda_{1,2}} I(q).$$

At this point we will require that, for each $i = 1, 2$,

$$c_i < c_{1,2}. \tag{2.3}$$

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As an example of a potential V for which condition (2.3) is verified, consider $V : \mathbb{R}^2\backslash\{(a,b),(-a,b)\} \to \mathbb{R}$, given by

$$V(x,y) = \frac{-(x^2+y^2)^2}{((x-a)^2+(y-b)^2)((x+a)^2+(y-b)^2)},$$

where a and b are non negative real numbers. It can be proved that, letting (a,b) vary in the segment line joining $(a,0)$ to $(0,b)$, and using the continuity of the critical values c_i, $i = 1,2$, and $c_{1,2}$ with respect to (a,b), we can find (a_0,b_0) in that segment, so that

$$c_1(a_0,b_0) + c_2(a_0,b_0) = 2c_1(a_0,b_0) > c_{1,2}(a_0,b_0) > c_1(a_0,b_0) = c_2(a_0,b_0).$$

2.2 Existence of homoclinic solutions winding around each singularity

Theorem 2.4 *If* (V1)–(V5) *and* (2.3) *are satisfied, then for each* $i = 1,2$, *there exists* $q_i \in \Lambda_i$ *such that* $I(q_i) = c_i$. *Moreover,* q_i *is a homoclinic solution of (HS) winding around* ξ_i.

Proof As soon as we prove the existence of a solution of (HS) in the class Λ_i, it will have the desired asymptotic and winding properties as a consquence of Lemma 2.2 and the definition of Λ_i.

If $(Q_m)_{m\in\mathbb{N}}$ is a minimizing sequence for $c_1 = \inf_{q\in\Lambda_1} I(q)$, we can assume that there is $M > 0$, such that

$$I(Q_m) \leqslant M, \quad \forall m \in \mathbb{N}, \tag{2.5}$$

which implies that $(\dot{Q}_m)$ is a bounded sequence in $L^2(\mathbb{R},\mathbb{R}^2)$. For $\delta \in (0, \frac{1}{4}\min\{|\xi_1|,|\xi_2|\})$, and each $m \in \mathbb{N}$, there exists $T_m = T_m(\delta) \in \mathbb{R}$ for which

$$|Q_m(T_m)| = \delta \quad \text{and} \quad |Q_m(t)| < \delta, \quad \forall t < T_m.$$

By the invariance under change of time scale of the functional I, we can choose $T_m = 0$, for all $m \in \mathbb{N}$, and therefore

$$|Q_m(0)| = \delta \quad \text{and} \quad |Q_m(t)| < \delta, \quad \forall t < 0.$$

As a consequence $(Q_m)_{m\in\mathbb{N}}$ is a bounded sequence in E, and therefore, there exists $Q \in E$ such that Q_m converges weakly to Q in E and strongly in $L^\infty_{\text{loc}}(\mathbb{R},\mathbb{R}^2)$, possibly along a subsequence which we will denote again by (Q_m). The weak lower semicontinuity of I implies

$$I(Q) \leqslant c_1 \tag{2.6}$$

and by Lemma 2.2, $\lim_{t\to\pm\infty}|Q(t)| = 0$ and $Q \in \Lambda$. It remains to show that $d_{\xi_1}(Q) > 0$ and $d_{\xi_2}(Q) = 0$. For δ as above, since $|Q(t)| \to 0$ as $|t| \to \infty$, we can find a positive real number $T = T(\delta)$ for which

$$|Q(t)| \leqslant \frac{\delta}{2}, \quad \forall |t| \geqslant T-1. \tag{2.7}$$

If δ is arbitrarily small, for $|t| \geqslant T$, $Q(t)$ lies in a small neighbourhood of 0, hence it can't change much with respect to ξ_i, $i = 1,2$. Then we can find $\epsilon = \epsilon(\delta)$, with $\epsilon \to 0$ as $\delta \to 0$,

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so that

$$-\frac{\epsilon}{4} \leqslant \mathrm{WN}_{\xi_i}(Q|_{|t|\geqslant T}) \leqslant \frac{\epsilon}{4},$$

and therefore, denoting $D(Q) = (\alpha, \beta)$,

$$\alpha - \frac{\epsilon}{4} \leqslant \mathrm{WN}_{\xi_1}(Q|_{|t|\leqslant T}) \leqslant \alpha + \frac{\epsilon}{4} \quad \text{and} \quad \beta - \frac{\epsilon}{4} \leqslant \mathrm{WN}_{\xi_2}(Q|_{|t|\leqslant T}) \leqslant \beta + \frac{\epsilon}{4}. \tag{2.8}$$

Using similar arguments, the normalization made on the sequence (Q_m) implies

$$-\frac{\epsilon}{4} \leqslant \mathrm{WN}_{\xi_i}(Q_m|_{-\infty}^{-T}) \leqslant \frac{\epsilon}{4}, \tag{2.9}$$

and the strong convergence of Q_m to Q on bounded intervals and (2.8)

$$\alpha - \frac{\epsilon}{2} \leqslant \mathrm{WN}_{\xi_1}(Q_m|_{|t|\leqslant T}) \leqslant \alpha + \frac{\epsilon}{2} \quad \text{and} \quad \beta - \frac{\epsilon}{2} \leqslant \mathrm{WN}_{\xi_2}(Q_m|_{|t|\leqslant T}) \leqslant \beta + \frac{\epsilon}{2}, \tag{2.10}$$

for m sufficiently large. (2.9) and (2.10) imply

$$\alpha - \frac{3\epsilon}{4} \leqslant \mathrm{WN}_{\xi_1}(Q_m|_{-\infty}^{T}) \leqslant \alpha + \frac{3\epsilon}{4} \quad \text{and} \quad \beta - \frac{3\epsilon}{4} \leqslant \mathrm{WN}_{\xi_2}(Q_m|_{-\infty}^{T}) \leqslant \beta + \frac{3\epsilon}{4}.$$

Finally, for $D(Q_m) = (\alpha_m, 0)$, $m \in \mathbb{N}$, we can conclude

$$\alpha_m - \alpha - \frac{3\epsilon}{4} \leqslant \mathrm{WN}_{\xi_1}(Q_m|_{T}^{\infty}) \leqslant \alpha_m - \alpha + \frac{3\epsilon}{4} \tag{2.11}$$

and

$$-\beta - \frac{3\epsilon}{4} \leqslant \mathrm{WN}_{\xi_2}(Q_m|_{T}^{\infty}) \leqslant -\beta + \frac{3\epsilon}{4}. \tag{2.12}$$

If $\alpha \leqslant 0$ and $\beta = 0$, (2.11) implies

$$\mathrm{WN}_{\xi_1}(Q_m|_{T}^{\infty}) > 0,$$

while by (2.12)

$$-\frac{3\epsilon}{4} \leqslant \mathrm{WN}_{\xi_2}(Q_m|_{T}^{\infty}) \leqslant \frac{3\epsilon}{4}.$$

Consider the following sequence of functions:

$$v_m(t) = \begin{cases} 0 & \text{if } t \leqslant T-1, \\ (t-(T-1))Q_m(T) & \text{if } T-1 \leqslant t \leqslant T, \\ Q_m(t) & \text{if } t \geqslant T; \end{cases}$$

(2.12) implies $d_{\xi_1}(v_m) > 0$, and for δ as small as necessary $d_{\xi_2}(v_m) = 0$, and $v_m \in \Lambda_1$. On the other hand,

$$I(v_m) = I(Q_m) - \int_{-\infty}^{T} \mathbf{L}(Q_m)\, dt + \int_{T-1}^{T} \left(\frac{1}{2}|Q_m(T)|^2 - V(v_m)\right) dt.$$

Since $|Q(t)| \leqslant \delta/2$ for $t \in [T-1, T]$, the strong convergence of the (Q_m) implies that for m sufficiently large, $|Q_m(T)| \leqslant \delta$ and therefore

$$\int_{T-1}^{T} \frac{1}{2}|Q_m(T)|^2\, dt \leqslant \frac{\delta^2}{2}. \tag{2.13}$$

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Also, for $t \in [T-1, T]$

$$|v_m(t)| = |t-(T-1)||Q_m(T)| \leqslant |Q_m(T)| \leqslant \delta.$$

By condition (V4), for x in a sufficiently small neighbourhood of the origin there exists $\gamma > 0$ such that $-V(x) \leqslant \gamma\delta^2$, which implies that

$$\int_{T-1}^{T} -V(v_m)\, dt \leqslant \gamma\delta^2. \tag{2.14}$$

Finally, notice that since $|Q_m(t)| < \delta$, for $T-1 \leqslant t \leqslant T$ and $Q_m(-\infty) = 0$, there is at least one subinterval $(\underline{T}, \overline{T}) \subset (-\infty, T)$, for which $Q_m(t) \in B_\delta(0) \setminus B_{\delta/2}(0)$ for all $t \in (\underline{T}, \overline{T})$, and by Lemma 2.1,

$$\int_{-\infty}^{T} \mathbf{L}(Q_m)\, dt \geqslant \int_{\underline{T}}^{\overline{T}} \mathbf{L}(Q_m)\, dt \geqslant \sqrt{2\alpha_\delta}|Q_m(\underline{T}) - Q_m(\overline{T})|, \tag{2.15}$$

where $\alpha_\delta = \inf_{x \notin B_{\delta/2}(0)}(-V(x))$. (2.13)–(2.15) imply

$$I(v_m) \leqslant I(Q_m) - \sqrt{2\alpha_\delta}\delta + \frac{\delta^2}{2} + \gamma\delta^2,$$

and if we take $\delta < \frac{2\sqrt{2\alpha_\delta}\delta}{1+2\gamma}$

$$-\sqrt{2\alpha_\delta}\delta + \frac{\delta^2}{2} + \gamma\delta^2 < 0,$$

which leads to

$$I(v_m) < I(Q_m)$$

contradicting the definition of (Q_m).

Next, we consider the case of $\alpha \leqslant 0$ and $\beta \neq 0$. As a consequence of (2.11) and (2.12),

$$\mathrm{WN}_{\xi_1}(Q_m|_T^\infty) > 0 \quad \text{and} \quad \mathrm{WN}_{\xi_2}(Q_m|_T^\infty) \neq 0,$$

which implies that $v_m \in \Lambda_{1,2}$, for the sequence of functions v_m defined previously. Then, using the same arguments as above, we can choose δ so small so that

$$c_{1,2} \leqslant I(v_m) < I(Q_m),$$

which contradicts (2.3). Thus, $\alpha = d_{\xi_1}(Q) > 0$ has been verified. For the final case, suppose that $\beta \neq 0$. Once again by (2.12), we conclude that

$$\mathrm{WN}_{\xi_2}(Q_m|_T^\infty) \neq 0,$$

and by the additivity of Brouwer degree, we must necessarily have that

$$\mathrm{WN}_{\xi_2}(Q_m|_{-\infty}^T) \neq 0. \tag{2.16}$$

There are two distinct possibilities: if

$$\mathrm{WN}_{\xi_1}(Q_m|_T^\infty) \neq 0,$$

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then once again $v_m \in \Lambda_{1,2}$, and we are done as in the previous case. If, on the other hand

$$\mathrm{WN}_{\xi_1}(Q_m|_T^\infty) = 0,$$

then, and since $\alpha > 0$

$$\mathrm{WN}_{\xi_1}(Q_m|_{-\infty}^T) \neq 0,$$

and this fact, together with (2.16), imply that the functions

$$\hat{v}_m(t) = \begin{cases} Q_m(t) & \text{if } t \leqslant T-1 \\ (t-(T-1))Q_m(T-1) & \text{if } T-1 \leqslant t \leqslant T \\ 0 & \text{if } t \geqslant T \end{cases}$$

are in the class $\Lambda_{1,2}$, and we can choose δ to be so small so that

$$c_{1,2} \leqslant I(\hat{v}_m) < c_1,$$

which as before contradicts (2.3). Then $\beta = d_{\xi_2}(Q) = 0$, and we have proved that $Q \in \Lambda_1$.

Since $Q \in \Lambda_1$, $I(Q) \geqslant c_1$. This together with (2.6) gives us that

$$I(Q) = c_1.$$

Given that Q is the minimizer of I, it is known that Q is a classical solution of (HS) (see, for example, Rabinowitz [12]).

That $\lim_{t\to\pm\infty} |\dot{Q}(t)| = 0$ is a simple consequence of the fact that Q is a solution of (HS), and of the interpolation formula (for the proof see for example Friedman [6])

$$\|\dot{Q}\|_{L^\infty([a,b]} \leqslant \epsilon \|\ddot{Q}\|_{L^\infty([a,b]} + k(\epsilon)\|Q\|_{L^\infty([a,b]},$$

where $[a,b]$ is any bounded interval of $\mathbb{R}$, and $k(\epsilon)$ depends only $|b-a|$.

This finishes the proof for the existence of a homoclinic solution of (HS) winding around ξ_1. The existence of a homoclinic solution of (HS) winding around ξ_2 uses precisely the same arguments. □

The previous theorem, guarantees the existence of at least two homoclinic solutions of (HS), each one in the class Λ_i, $i = 1,2$. From now on, we will denote by q_1 and q_2 the homoclinic solutions of (HS) winding around ξ_1 and ξ_2, respectively, and by $c_i = I(q_i)$, $i = 1,2$. Since these functions are in the class Λ_i, $i = 1,2$, there will exist positive integers $\alpha = \alpha(q_1)$, and $\beta = \beta(q_2)$, such that

$$D(q_1) = (\alpha, 0) \quad \text{and} \quad D(q_2) = (0, \beta).$$

As a brief remark, notice that if $q_i \in \Lambda_i$, $i = 1,2$ are the solutions obtained in Theorem (2.4), then the functions $\hat{q}_i(t) = q_i(-t)$ are also solutions of (HS), with $I(\hat{q}_i) = I(q_i) = c_i$, as a consequence of the invariance of I under the transformation $t \mapsto -t$. Hence they must also be the minimizers of I over the classes of functions

$$\Lambda_1^- = \{q \in E : D(q) = (\alpha, 0) \quad \text{for } \alpha \in \mathbb{Z}^-\},$$

and

$$\Lambda_2^- = \{q \in E : D(q) = (0, \beta) \quad \text{for } \beta \in \mathbb{Z}^-\}.$$

2.3 Existence of homoclinic solution winding around both singularities

Our next goal is to prove the existence of a homoclinic solution of (HS) which winds around both the singularities. We can view the chain of solutions q_1, q_2 as an element of the weak closure of $\Lambda_{1,2}$. This means that, although the chain is not an element of $\Lambda_{1,2}$, we can find a function in $\Lambda_{1,2}$ as close as we want to that chain. Therefore

$$c_{1,2} \leqslant c_1 + c_2.$$

But, if equality holds, $\min_{\Lambda_{1,2}} I$ may not be achieved in $\Lambda_{1,2}$, since the limit of a minimizing sequence may be the chain q_1, q_2. In order to avoid such problems, we introduce the geometric condition

$$c_{1,2} < c_1 + c_2. \tag{2.17}$$

Theorem 2.18 *If* (V1)–(V5) *and condition* (2.17) *hold, there exists* $q \in \Lambda_{1,2}$ *such that* $I(q) = c_{1,2}$, *and* q *is a homoclinic solution of (HS) winding around both singularities.*

Proof Once again, as soon as we prove the existence of a minimizer of I over $\Lambda_{1,2}$, it will have the desired asymptotic properties by the definition of $\Lambda_{1,2}$. As in the proof of Theorem 2.4, it can be proved that any minimizing sequence, (Q_m), for I over $\Lambda_{1,2}$, is bounded in E, and therefore converges (possibly along a subsequence), weakly in $W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^2)$ and strongly in $L^{\infty}_{\text{loc}}(\mathbb{R}, \mathbb{R}^2)$ to some function Q in E. Lemma 2.2, implies that $\lim_{|t|\to\infty} |Q(t)| = 0$, and $Q \in \Lambda$. For $\delta \in (0, \frac{1}{4}\min\{|\xi_1|, |\xi_2|\})$, consider $T = T(\delta)$ so that

$$|Q(t)| < \delta/2, \quad \forall |t| \geqslant T - 1.$$

As in the proof of Theorem 2.4 we can find $\epsilon = \epsilon(\delta)$, with $\epsilon(\delta) \to 0$ as $\delta \to 0$, so that

$$\alpha_m - \alpha - \frac{\epsilon}{2} \leqslant \mathrm{WN}_{\xi_1}(Q_m|_T^{\infty}) \leqslant \alpha_m - \alpha + \frac{\epsilon}{2} \tag{2.19}$$

and

$$\beta_m - \beta - \frac{\epsilon}{2} \leqslant \mathrm{QWN}_{\xi_2}(Q_m|_T^{\infty}) \leqslant \beta_m - \beta + \frac{\epsilon}{2}, \tag{2.20}$$

where $(\alpha_m, \beta_m) = D(Q_m) \in \mathbb{N}^2$, $m \in \mathbb{N}$. If both α and β are non positive, (2.19) and (2.20) imply

$$\mathrm{WN}_{\xi_1}(Q_m|_T^{\infty}) > 0 \quad \text{and} \quad \mathrm{WN}_{\xi_2}(Q_m|_T^{\infty}) > 0,$$

and therefore, the sequence v_m, as in the proof of Theorem 2.4 is in the class $\Lambda_{1,2}$ and δ can be made arbitrarily small so that $I(v_m) < I(Q_m)$, contradicting the definition of (Q_m). Hence at least one of α or β must be positive. Without loss of generality suppose that

$$\alpha > 0 \quad \text{and} \quad \beta \leqslant 0$$

since the other case follows the same arguments. Using (2.20) we conclude

$$\mathrm{WN}_{\xi_2}(Q_m|_T^{\infty}) > 0.$$

If also $\mathrm{WN}_{\xi_1}(Q_m|_T^{\infty}) > 0$, we are done as above. If $\mathrm{WN}_{\xi_1}(Q_m|_T^{\infty}) = 0$, by the additivity of Brouwer degree $\mathrm{WN}_{\xi_1}(Q_m|_{-\infty}^{T}) > 0$, and we must consider two further possibilities: if $\mathrm{WN}_{\xi_2}(Q_m|_{-\infty}^{T}) \neq 0$, then the sequence $\hat{v}_m$, as defined in the proof of Theorem 2.4, is the

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class $\Lambda_{1,2}$ and δ can be chosen so small so that $I(\hat{v}_m) < I(Q_m)$ contradicting the definition of Q_m. On the other hand if $\mathrm{WN}_{\xi_1}(Q_m|_{-\infty}^{T}) = 0$, we can conclude that $\hat{v}_m \in \Lambda_1$, while $v_m \in \Lambda_2$, which implies that

$$I(\hat{v}_m) + I(v_m) \geqslant c_1 + c_2.$$

Letting $m \to \infty$,

$$c_{1,2} \geqslant c_1 + c_2,$$

contradicting condition (2.17).

Hence we have proved that $Q \in \Lambda_{1,2}$. The remainder of the proof follows exactly as in the proof of Theorem 2.4. □

We will denote the solution of (HS) obtained in Theorem 2.18 by $q_{1,2}$ and $c_{1,2} = I(q_{1,2})$.

2.4 Properties of the homoclinic solutions

Proposition 2.21 *If conditions* (V1)–(V5) *hold, the functions* q_i, $i = 1, 2$, *and* $q_{1,2}$ *are minimal solutions of (HS) in their homotopy class, i.e., they minimize the Lagrangian integrated between any pair of points in its homotopy type.*

Proof If the result is false for q_1, then,

$$\int_{\tau}^{\sigma} \mathbf{L}(q_1)\, dt \geqslant \int_{\alpha}^{\beta} \mathbf{L}(w)\, dt \tag{2.22}$$

for some $w \in W^{1,2}_{\mathrm{loc}}([\alpha, \beta], \mathbb{R}^2)$, with $w(\alpha) = q_1(\tau)$, $w(\beta) = q_1(\sigma)$ and $w|_{\alpha}^{\beta}$ of the same homotopy type as $q_1|_{\sigma}^{\tau}$. By construction, the function defined by

$$q(t) = \begin{cases} q_1(t - \alpha + \tau) & \text{if } t \leqslant \alpha \\ w(t) & \text{if } t \in [\alpha, \beta] \\ q_1(t - \beta + \sigma) & \text{if } t \geqslant \beta \end{cases}$$

$q \in \Lambda_1$, and as a consequence of (2.22)

$$I(q) \leqslant I(q_1) \tag{2.23}$$

For (2.23) not to contradict the fact that q_1 is the minimum value of I over Λ_1 equality must hold. But if this is the case, q is also a solution of (HS). Since it coincides with q_1 in an open interval of $\mathbb{R}$, by the uniqueness of solution of the IVP associated with (HS), we must have $q \equiv q_1$ over $\mathbb{R}$. Then, either we have the special case $q_1|_{\sigma}^{\tau} = w|_{\alpha}^{\beta}$ or we have a contradiction. And this proves the result. □

Proposition 2.24 *Under the conditions* (V1)–(V5), (2.3) *and* (2.17)*:*

(a) *All the functions obtained are energy zero solutions of (HS).*

(b) q_1, q_2 *and* $q_{1,2}$ *do not intersect each other other than at* $t = \pm\infty$.

(c) q_1, q_2*and* $q_{1,2}$ *are simple functions.*

Proof

(a) Since $\frac{d}{dt}E(t) = 0$,

$$E(t) = \frac{1}{2}|\dot{q}_i(t)|^2 + V(q_i) = C_i, \quad i = 1, 2$$

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for some constant C_i. Taking the limit as $t \to \infty$, we obtain $C_i = 0$ for $i = 1, 2$. The same reasoning applies to $q_{1,2}$.

(b) If q_1 and q_2 do intersect other than at $t = \pm\infty$, consider

$$t_1 \in \{t \in \mathbb{R} : q_1(t) \in q_2(\mathbb{R})\},$$

where $q_1(t) \cap q_2(\mathbb{R}) = \emptyset$ for all $t \geqslant t_1$. Then we can find $t_2 \in \mathbb{R}$ such that $q_2(t_2) = q_1(t_1)$ and $q_2(t) \cap q_1(\mathbb{R}) = \emptyset$ either for $t < t_2$ or for $t > t_2$. Without loss of generality suppose that the latter holds, and $t_1 = t_2 = 0$. By construction, the function

$$Q_1(t) = \begin{cases} q_1(t) & \text{if } t \leqslant 0 \\ q_2(-t) & \text{if } t \geqslant 0 \end{cases}$$

is in the class Λ_1, while the function

$$Q_2(t) = \begin{cases} q_1(-t) & \text{if } t \leqslant 0 \\ q_2(t) & \text{if } t \geqslant 0 \end{cases}$$

is in the class Λ_2. Moreover we must necessarily have that

$$\int_{-\infty}^{0} \mathbf{L}(q_2(t))\, dt = \int_{0}^{\infty} \mathbf{L}(q_1(t))\, dt,$$

since otherwise, we could produce a function either in Λ_1 or in Λ_2 (according to the chosen inequality), attaining a smaller value than $I(q_1)$ or $I(q_2)$, contradicting the definition of q_1 or q_2. But if that is the case, $I(Q_1) = I(q_1) = c_1$, which means that Q_1 is also a solution of (HS). By uniqueness of solution of the IVP for (HS), $Q_1 \equiv q_2$ in $\mathbb{R}$, which is false since they belong to classes with different homotopy types. In a similar way we can prove the other assertions.

(c) If q_1 has a self-intersection, then there are $-\infty < \sigma < \tau < \infty$ such that $q_1(\sigma) = q_1(\tau)$. If

$$D(q_1) = D(q_1|_\sigma^\tau) \tag{2.25}$$

we can define the function

$$P_1(t) = \begin{cases} q_1(t) & \text{if } t \leqslant \sigma \\ q_1(\tau - t + \sigma) & \text{if } \sigma \leqslant t \leqslant \tau \\ q_1(t) & \text{if } t \geqslant \tau \end{cases}$$

and as a consequence of (2.25), $D(P_1) = -D(q_1)$, which means that $P_1 \in \Lambda_1^-$. Also it is straightforward to check that

$$I(P_1) = I(q_1).$$

Hence P_1 is also a solution of (HS) and it coincides with q_1 in $\mathbb{R} \setminus (\sigma, \tau)$. By the uniqueness of the solution of the IVP for (HS), $P_1 \equiv q_1$ in $\mathbb{R}$. But $P_1 \in \Lambda_1^-$, while $q_1 \in \Lambda_1$, and since those sets are disjoint, (2.25) is not possible. Now, deleting the loop $q_1|_\sigma^\tau$ from q_1 produces a new function R_1 with $D(R_1) \neq (0, 0)$ and

$$I(R_1) < I(q_1). \tag{2.26}$$

To finish up the proof we consider two possibilities: if $d_{\xi_2}(q_1|_\sigma^\tau) = 0$, then $d_{\xi_2}(R_1) = 0$ which implies that $R_1 \in \Lambda_1$, or $R_1 \in \Lambda_1^-$ and (2.26) is a contradiction to the definition of

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q_1. If, on the other hand, $d_{\xi_2}(q_1|_\sigma^\tau) \neq 0$, then also $d_{\xi_2}(R_1) \neq 0$ which implies that $R_1 \in \Lambda_{1,2}$, and

$$c_{1,2} \leqslant I(R_1) < I(q_1) = c_1$$

contradicts (2.3). Thus q_1 has no self-intersections. To prove that q_2 and $q_{1,2}$ are also simple curves we use similar arguments. □

The last proposition implies that

$$D(q_1) = (1,0) \quad \text{and} \quad D(q_2) = (0,1).$$

3 Periodic solutions of (HS)

This section deals with periodic orbits of (HS). Due to the form of the potential V with global minima at ξ_1, ξ_2, it makes sense to look for such solutions winding around the singularities.

3.1 Preliminary results

For $T > 0$, define

$$E_T = \{q \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^2) : q \text{ is } T-\text{periodic}\},$$

endowed with the norm $\|q\|_T = \|q\|_{W^{1,2}([0,T])}$. In E_T we consider the functional

$$I_T(q) = \int_0^T \mathbf{L}(q(t))\,dt,$$

which verifies Lemma 2.1, with $\mathbb{R}$ replaced by $[0, T]$. Further define

$$\Lambda^T = \{q \in E_T : q(t) \neq \xi_i,\ i = 1, 2,\ t \in [0, T]\}.$$

Thereby, we have

Lemma 3.1 *If $q \in E_T$ and for some positive real number M, $I_T(q) < M$, then there exists $\beta = \beta(M) > 0$ such that for all $t \in [0, T]$ and $i = 1, 2$,*

$$|q(t) - \xi_i| > \beta.$$

In particular, if $I_T(q) < \infty$, then $q \in \Lambda_T$.

For $q \in E_T$, q is a closed curve in the interval $[0, T]$, so we can talk about its Brouwer degree with respect to each of the ξ_i. Furthermore consider $\mathbf{A}_i$, the bounded component of $\mathbb{R}^2 \backslash \{q_i(t) : t \in \mathbb{R}\}$, $i = 1, 2$, where q_1, q_2 are as in Section 2, and as a consequence of Proposition 2.24 (b), $\overline{\mathbf{A}}_1 \cap \overline{\mathbf{A}}_2 = \{0\}$. Define

$$\Lambda_1^T = \{q \in \Lambda^T : D(q) = (1,0) \quad \text{and} \quad q(t) \in \overline{\mathbf{A}}_1 \quad \forall t \in \mathbb{R}\}$$

and

$$\Lambda_2^T = \{q \in \Lambda^T : D(q) = (0,1) \quad \text{and} \quad q(t) \in \overline{\mathbf{A}}_2 \quad \forall t \in \mathbb{R}\}.$$

Also, for each $i = 1, 2$,

$$c_i(T) = \inf_{\Lambda_i^T} I_T(q).$$

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3.2 Existence of T-periodic solutions

Theorem 3.2 *If V satisfies* (V1)–(V5)*, for any positive T, there exists curves p_1 and p_2 in Λ_1^T and Λ_2^T, respectively, such that $I_T(p_1) = c_1(T)$ and $I_T(p_2) = c_2(T)$. Moreover, these are T-periodic solutions of (HS).*

Proof If $(q_m)_{m\in\mathbb{N}}$ is a minimizing sequence for I_T in Λ_1^T, there exist $M_1 \in \mathbb{R}^+$ such that $I_T(q_m) \leqslant M_1$, for all $m \in \mathbb{N}$, implying

$$\|\dot{q}_m\|^2_{L^2[0,T]} \leqslant \sqrt{2M_1}.$$

On the other hand, by construction $q_m([0,T]) \subset \mathbf{A}_1$, for all $m \in \mathbb{N}$, and the continuity of q_1 allow us to conclude

$$\|q_m\|_{L^2([0,T],\mathbb{R}^2)} \leqslant \sqrt{T}\|q_m\|_{L^\infty([0,T],\mathbb{R}^2)} \leqslant \sqrt{T}M_2,$$

for some positive constant M_2. We conclude that (q_m) is bounded in E_T, and this being the case, it converges (possibly along a subsequence) to a function $Q \in E_T$, weakly in E_T and strongly in $L^\infty([0,T],\mathbb{R}^2)$.

Since $I_T(q_m) \leqslant M_1$, $\forall m \in \mathbb{N}$, we can use Lemma 3.1 to conclude that, there exists $\beta = \beta(M_1)$ such that

$$|q_m(t) - \xi_1| \geqslant \beta(M_1) \quad \text{and} \quad |q_m(t) - \xi_2| \geqslant \beta(M_1).$$

Then, the strong convergence implies $|Q(t) - \xi_i| \geqslant \beta(M_1)$, $\forall t \in [0,T]$, and we have proved that $Q \in \Lambda^T$.

Since $Q \in E_T$, Q is T-periodic, and as a consequence of the strong convergence $Q \in \overline{\mathbf{A}}_1$. The continuity property of the Brouwer degree implies that $D(Q) = (1,0)$, and as a consequence $Q \in \Lambda_1^T$.

Finally, the weak lower semi-continuity of I_T implies

$$\int_0^T \mathbf{L}(Q)\,dt \leqslant \liminf_{m\to\infty} \int_0^T \mathbf{L}(q_m)\,dt = \liminf_{m\to\infty} I_T(q_m) = c_1(T),$$

and therefore $I_T(Q) = c_1(T)$.

Knowing that $I_T(Q) = \min_{\Lambda_1^T} I_T(q)$, it is a standard procedure to prove that Q is a classical solution of (HS).

The proof for the existence of a minimizer of I_T over Λ_2^T, follows exactly the same steps. □

We proved the existence of periodic solutions of (HS), with arbitrary period $T > 0$, winding around either ξ_1 or ξ_2. We will denote these functions as p_1^T and p_2^T respectively, and $c_i(T) = I_T(p_i^T)$, $i = 1,2$. Consider the sets

$$G_1 = \cup_{T>0}\{p_1^T\} \quad \text{and} \quad G_2 = \cup_{T>0}\{p_2^T\},$$

and let

$$\hat{c}_1 = \inf_{G_1} I_T(q) \quad \text{and} \quad \hat{c}_2 = \inf_{G_2} I_T(q),$$

As a consequence of Lemma 3.1, and for $i = 1, 2$,

$$|p_i^T - \xi_i| \geqslant \beta, \quad \forall t \in [0, T_i],$$

for some positive β, and all the functions p_i^T minimizing I_T over Λ_i^T. Hence

$$0 < 2\pi\beta \leqslant \int_0^{T_p} |\dot{p}_i^T|\, dt \leqslant (2T_p c_i(T))^{1/2},$$

implying

$$\hat{c}_i = \inf_{T>0} c_i(T) > 0.$$

On the other hand, heuristically we can view the homoclinic solutions q_1, q_2 obtained previously, as periodic solutions of (HS) with infinite period. In that sense,

$$\hat{c}_1 \leqslant c_1 \quad \text{and} \quad \hat{c}_2 \leqslant c_2.$$

But, if equality holds, $\inf_{G_i} I_T$ may not be achieved. To exclude such a possibility, we will require the geometric condition

$$\hat{c}_i < c_i \quad \text{for} \quad i = 1, 2 \tag{3.3}$$

i.e., the cost, as measured by I of winding around ξ_1 and ξ_2 in the class of all periodic solutions of (HS) is less than that for the homoclinics.

We will denote by p_i, $i = 1, 2$, the minimal action periodics, the periodic solutions of (HS) of period T_i, satisfying

$$I_{T_i}(p_i) = \hat{c}_i = \inf_{T>0} c_i(T).$$

3.3 Properties of the minimal action periodics

In Caldiroli & Jeanjean, [4], the following is proved.

Proposition 3.4 *Under conditions* (V1)–(V5) *and* (3.3)*, the functions p_i, $i = 1, 2$ are solutions of (HS) with energy level 0.*

The proof of the following Proposition, uses similar arguments to the ones used to prove analogous results in § 2.

Proposition 3.5 *If conditions* (V1)–(V5) *and* (3.3) *hold, for each $i = 1, 2$, the functions p_i, are simple functions and minimal solutions of (HS) in their homotopy class.*

Proposition 3.6 *If conditions* (V1)–(V5) *and* (3.3) *hold, then $p_1(t) \notin q_1(\mathbb{R})$ for all $t \in [0, T_1]$, and $p_2(t) \notin q_2(\mathbb{R})$ for all $t \in [0, T_2]$, i.e., in each homotopy class, the periodics do not intersect the homoclinics.*

Proof If the result is false for p_1, q_1, there exists t_0, $t_1 \in \mathbb{R}$ such that $q_1(t_0) = p_1(t_1)$. Without loss of generality, we can assume that $t_0 = t_1 = 0$. Since $p_1(t) \in \mathbf{A}_1$ for all $t \in \mathbb{R}$, at any point of intersection the two curves must be tangent, and as a consequence of

Proposition 3.4,

$$|\dot{q}_1(0)| = |\dot{p}_1(0)|.$$

This together with the fact that also $p_1(0) = q_1(0)$, will contradict the uniqueness of solution of the IVP for (HS) □

As an obvious consequence of Proposition 3.6, for each $i = 1, 2$, $0 \notin p_i([0, T_i])$. Furthermore, by our assumption (3.3), and as a consequence of Proposition 3.6, we will have in fact that

$$p_1([0, T_1]) \subset \mathbf{A}_1 \quad \text{and} \quad p_2([0, T_2]) \subset \mathbf{A}_2.$$

4 Existence of heteroclinic solutions from the equilibrium to the periodic solutions

4.1 Some preliminaries

We will prove the existence of a special class of heteroclinic solutions of (HS), i.e., solutions of (HS), with the following asymptotic behavior:

(i) $Q_i(t) \to 0$ as $t \to -\infty$;

(ii) $Q_i(t) - p_i(t) \to 0$ as $t \to \infty$;

i.e., solutions of (HS) that are heteroclinic from 0 to p_i, where p_i is the minimal periodic winding around ξ_i, obtained in § 3.

We will carry out the details for the case $i = 1$, but we must keep in mind that absolutely everything works for the case $i = 2$.

In order for the problem to be well defined, we must point out that the functions p_1 and q_1 may not be unique within their classes. Anyway, it is easy to see that if p_1 and $\hat{p}_1$ are two solutions of (HS) in the class G_1, with $I_T(p_1) = I_T(\hat{p}_1) = c_1^T$, by Proposition 3.6 they cannot intersect any homoclinic solution of (HS), and condition (3.3) guarantees that such functions can not accumulate to q_1. Also, using familiar arguments, one can show that they will not intersect each other. Henceforth, we denote by p_1 the outermost solution of (HS) in the class of periodic functions with $I_T(p_1) = \hat{c}_1$, and by q_1 the innermost solution of (HS) in the class Λ_1 with $I(q_1) = c_1$.

Consider $\mathbf{C}_1$ to be the subset of $\mathbb{R}^2$, defined by

$$\mathbf{C}_1 = \mathbf{A}_1 \cap \mathbf{B}_1$$

where $\mathbf{A}_1$ is the component of $\mathbb{R}^2 \setminus \{q_1(t)\}$ containing ξ_1, and $\mathbf{B}_1$ the component of $\mathbb{R}^2 \setminus \{p_1(t)\}$ containing the origin. We conclude that $\mathbf{C}_1$ is bounded, and that $\partial \mathbf{C}_1 = \{q_1(\mathbb{R})\} \cup \{p_1([0, T_1]\}$. Moreover, if we consider the problem of minimizing $I^+(q) \equiv \int_0^\infty \mathbf{L}(q)\, dt$, for q in the class of functions

$$\Lambda^+ = \{q \in W^{1,2}_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^2) : \int_0^\infty |\dot{q}|^2 dt < \infty \text{ and } q \text{ connects 0 to } p_1([0, T_1])\},$$

standard minimization arguments, as in [15], prove the existence of $z_1 \in \Lambda^+$ minimal solution of (HS), verifying the following:

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- $z_1(\infty) = \dot{z}_1(\infty) = 0$;
- $z_1(0) \in p_1([0, T_1])$;
- $z_1(t) \in \mathbf{C}_1$, for all $t \in \mathbb{R}^+$;
- $I^+(z_1) = c^+ \equiv \inf\limits_{q \in \Lambda^+} I^+(q)$.

4.2 Formulation of the variational problem

Define the class of functions

$$\Gamma = \{q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2) : (\Gamma\,1) - (\Gamma\,3) \text{ are verified}\}$$

where

(Γ 1) $q(-\infty) = 0$;
(Γ 2) $q(t) \in \overline{\mathbf{C}}_1$ for all $t \in \mathbb{R}$;
(Γ 3) $q(t)$ intersects $z_1(\mathbb{R}^+)$ an infinite number of times, along a strictly increasing sequence of real numbers $(t_i(q))_{i \in \mathbb{N}}$, with the following properties: for $(s_i(q))_{i \in \mathbb{N}_0}$ defined by

$s_0(q) = \infty$;
$z_1(s_i) = q(t_i(q))$, for $i \in \mathbb{N}$,

consider, for $i \in \mathbb{N}$, $\psi_i(t)$ as being the function obtained by gluing $q|_{t_i(q)}^{t_{i+1}(q)}$ to $z_1|_{s_{i+1}(q)}^{s_i(q)}$, and require

(i) $d_{\xi_1}(\psi_i) = 1$ and $d_{\xi_2}(\psi_i) = 0$, for all $i \in \mathbb{N}$;
(ii) the sequence $(s_i)_{i \in \mathbb{N}}$ is monotone non increasing, i.e., $s_{i+1}(q) \leqslant s_i(q)$, $\forall i \in \mathbb{N}$.

The class Γ is nonempty, since the function defined by

$$q(t) = \begin{cases} z_1(-t) & \text{if} \quad t \leqslant 0 \\ p_1(t + \alpha) & \text{if} \quad t \geqslant 0 \end{cases} \tag{4.1}$$

for $\alpha \in [0, T_1)$ such that $p_1(\alpha) = z_1(0)$, is clearly an element of Γ, with $t_i(q) = (i-1)T_1$, $i \in \mathbb{N}$. For $q \in \Gamma$, we can define

$$a_1(q) = \int_{-\infty}^{t_1(q)} \mathbf{L}(q)\, dt - \hat{c}_1$$

and for $i > 1$

$$a_i(q) = \int_{t_{i-1}(q)}^{t_i(q)} \mathbf{L}(q)\, dt - \hat{c}_1.$$

Finally, we consider the renormalized functional

$$J(q) = \sum_{i=1}^{\infty} a_i(q)$$

Lemma 4.2 *For each $q \in \Gamma$*

$$J(q) \geqslant -c^+ - \hat{c}_1.$$

Proof For each $i > 1$, we can extend ψ_i periodically to $\mathbb{R}$, and by (Γ 3) (i), $D(\psi_i) = (1,0)$. Hence $\psi_i \in \Lambda_1^T$, implying $I_{T_i}(\psi_i) \geqslant \hat{c}_1$, and as a consequence

$$a_{i+1}(q) \geqslant -\int_{s_{i+1}}^{s_i} \mathbf{L}(z_1)\, dt. \tag{4.3}$$

Summing over i, we obtain

$$J(q) \geqslant -\sum_{i=2}^{\infty} \int_{s_{i+1}}^{s_i} \mathbf{L}(z_1) + \int_{-\infty}^{t_1(q)} \mathbf{L}(q)\, dt - \hat{c}_1 \geqslant -\int_{s_1(q)}^{\infty} \mathbf{L}(z_1)\, dt - \hat{c}_1 \geqslant -c^+ - \hat{c}_1.$$

□

Finally, define

$$c = \inf_{\Gamma} J(q).$$

Using Lemma 4.2, and the fact that $J(q) = c^+ - \hat{c}_1$ for q defined by (4.1), we conclude that c is well defined and

$$-c^+ - \hat{c}_1 \leqslant c \leqslant c^+ - \hat{c}_1. \tag{4.4}$$

4.3 Existence and properties of the heteroclinic solutions asymptotic to the equilibrium and to the periodic p_1

Lemma 4.5 *If $q \in \Gamma$ and $J(q) \leqslant M$, then $\sum_{i=1}^{\infty} |a_i(q)| \leqslant M + 2(c^+ + \hat{c}_1)$.*

Proof If

$$N^-(q) = \{l \in \mathbb{N} : a_l(q) < 0\},$$

then, using (4.3) and the fact that $a_1(q) \geqslant -\hat{c}_1$

$$-\sum_{i\in N^-(q)} a_i(q) \leqslant \sum_{i\in N^-(q)} \int_{s_{i-1}(q)}^{s_i(q)} \mathbf{L}(z_1)\, dt + \hat{c}_1 \leqslant c^+ + \hat{c}_1,$$

and

$$J^+(q) \equiv J(q) - \sum_{i\in N^-(q)} a_i(q) \leqslant M + c^+ + \hat{c}_1.$$

Hence

$$\sum_{i=1}^{\infty} |a_i(q)| = -\sum_{i\in N^-(q)} a_i(q) + \sum_{i\in \mathbb{N}\setminus N^-(q)} a_i(q) \leqslant M + 2(c^+ + \hat{c}_1).$$

□

In the next results, we obtain some important properties for the functions ψ_i defined previously, which will help to obtain the desired asymptotic behavior of the minimizer of J over the class Γ.

Lemma 4.6 *If $q \in \Gamma$ is such that $J(q) \leqslant M$, there exists a strictly positive $\beta = \beta(M)$ for which $t_i(q) - t_{i-1}(q) \geqslant \beta$ for all $i \in \mathbb{N}$.*

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Proof By Lemma 4.5,

$$\sum_{i=1}^{\infty} |a_i(q)| \leqslant M + 2(c^+ + \hat{c}_1),$$

which implies

$$\int_{t_i(q)}^{t_{i+1}(q)} \mathbf{L}(q)\, dt = a_i(q) + \hat{c}_1 \leqslant |a_i(q)| + \hat{c}_1 \leqslant M + 2c^+ + 3\hat{c}_1 \equiv M_1,$$

and therefore, for each $i \in \mathbb{N}$, q is bounded in $L^2([t_i, t_{i+1}])$. If $l_{\varphi|_a^b}$ is the length of the curve corresponding to a function φ restricted to the interval $[a, b]$

$$l_i(q) \equiv l_{q|_{t_i}^{t_{i+1}}} \leqslant (t_{i+1}(q) - t_i(q))^{1/2}(2M_1)^{1/2},$$

implying

$$t_{i+1}(q) - t_i(q) \geqslant \frac{l_i(q)^2}{2M_1}.$$

By construction, for all $i \in \mathbb{N}$, $q|_{t_i(q)}^{t_{i+1}(q)}$ winds around the periodic p_1, and as a consequence $l_i(q) \geqslant l_{p_1}$. Thus

$$t_{i+1}(q) - t_i(q) \geqslant \frac{l_{p_1}^2}{2M_1}.$$

which implies that for any $\beta < \frac{l_{p_1}^2}{2M_1}$ the result is true. □

Proposition 4.7 *If $q \in \Gamma$ and $J(q) \leqslant M < \infty$, then as $i \to \infty$:*

(a) $t_{i+1}(q) - t_i(q) \to T_1$;

(b) $\|q - p_1\|_{L^\infty([t_i, t_{i+1}])} \to 0$.

Proof Since $J(q)$ is bounded

$$\int_{t_i}^{t_{i+1}} \mathbf{L}(q)\, dt - \hat{c}_1 \to 0 \text{ as } i \to \infty. \tag{4.8}$$

By property (Γ_3) (ii), the sequence $s_i(q)$ is monotone and bounded, thus there exists $s \geqslant 0$ such that $s_i(q) \to s$ as $i \to \infty$. As a consequence

$$\int_{s_{i+1}(q)}^{s_i(q)} \mathbf{L}(z_1) dt \to 0 \quad \text{as} \quad i \to \infty,$$

and together with (4.8) implies that

$$\lim_{i\to\infty} \int_0^{T_i} \mathbf{L}(\psi_i)\, dt = \hat{c}_1. \tag{4.9}$$

By Lemma 4.6, the sequence $(t_{i+1}(q) - t_i(q))_{i\in\mathbb{N}}$ is bounded by below. We claim, that for i sufficiently large, it is also bounded by above. If that was not the case, along a subsequence

$$t_{i+1}(q) - t_i(q) \to \infty \quad \text{as} \quad i \to \infty. \tag{4.10}$$

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This implies that, for some $t \in [0, T_i]$, and i large, the function ψ_i will have to be arbitrarily close to the origin, since otherwise, $|\psi_i(t)| > \delta$ for all $t \in [0, T_i]$ would imply, together with condition (V4)

$$I_{T_i}(\psi_i) \geqslant \int_0^{T_i} -V(\psi_i)\,dt \geqslant \beta_\delta(t_{i+1} - t_i) \to \infty \text{ as } i \to \infty,$$

where $\beta_\delta \equiv \min_{|x|\geqslant\delta}(-V(x))$, contradicting (4.8). We can consider $0 < \delta < \frac{1}{2}|\xi_1|$ such that $\psi_i(t) \cap B_\delta(0) \neq \emptyset$, and choose $\theta_i \in (0, T_i]$ such that $|\psi_i(\theta_i)| = \delta$. For each $i \in \mathbb{N}$, define

$$\phi_i(t) = \begin{cases} 0 & \text{if } t \leqslant \theta_i - 1 \\ \psi_i(\theta_i)(t + 1 - \theta_i) & \text{if } \theta_i - 1 \leqslant t \leqslant \theta_i \\ \psi_i(t) & \text{if } \theta_i \leqslant t \leqslant \theta_i + T_i \\ \psi_i(\theta_i)(\theta_i + T_i + 1 - t) & \text{if } \theta_i + T_i \leqslant t \leqslant \theta_i + T_i + 1 \\ 0 & \text{if } t \geqslant \theta_i + T_i + 1. \end{cases}$$

It is obvious that $\phi_i(\pm\infty) = \dot{\phi}_i(\pm\infty) = 0$, and as a consequence of $(\Gamma 3)$ (i), $D(\phi_i) = (1, 0)$. Hence $\phi_i \in \Lambda_1$, and

$$I(\phi_i) = \int_0^1 \mathbf{L}(\psi_i(\theta_i)t)\,dt + \int_{\theta_i}^{\theta_i+T_i} \mathbf{L}(\psi_i)\,dt \geqslant c_1, \tag{4.11}$$

for i sufficiently large. By (4.9), we can choose $\epsilon > 0$ such that for i sufficiently large,

$$\int_{\theta_i}^{\theta_i+T_i} \mathbf{L}(\psi_i)\,dt \leqslant \hat{c}_1 + \epsilon,$$

and we can choose δ as small as necessary, so that

$$\int_0^1 \mathbf{L}(\psi_i(\theta_i)t)\,dt \leqslant \epsilon/2.$$

Then

$$c_1 \leqslant I(\phi_i) \leqslant \epsilon + \hat{c}_1 + \epsilon.$$

and choosing $\epsilon < \frac{1}{4}(c_1 - \hat{c}_1)$, we obtain $c_1 - \hat{c}_1 \leqslant 0$, contradicting assumption (3.3). Thus $t_{i+1}(q) - t_i(q)$ is bounded away from 0 and ∞, and there will exist $\hat{T} > 0$ such that $t_{i+1} - t_i \to \hat{T}$ as $i \to \infty$, possibly along a subsequence. As usual, (4.9) give us an L^2 bound for $\dot{\psi}_i$ in $[0, T_i]$, for i sufficiently large, and, by construction, ψ_i is defined in the bounded region $\overline{\mathbf{C}}_1$, thus we can conclude that there exists $i_0 \in \mathbb{N}$ for which ψ_i is bounded in $W^{1,2}([0, T_i])$ for all $i \geqslant i_0$, implying the existence of $\psi \in W^{1,2}([0, \hat{T}])$ such that, as $i \to \infty$, $\psi_i \to \psi$ weakly in $W^{1,2}([0, \hat{T}])$ and strongly in $L^\infty([0, \hat{T}])$. The weak lower semicontinuity of I on compact sets, and the strong convergence of ψ_i to ψ, imply

$$\int_0^{\hat{T}} \mathbf{L}(\psi)\,dt \leqslant \liminf_{i\to\infty} \int_0^{T_i} \mathbf{L}(\psi_i)\,dt = \hat{c}_1.$$

Finally, $\psi(0) = \psi(\hat{T})$ implies $\psi \in \Lambda_1^T$ and therefore $I_{\hat{T}}(\psi) \geqslant \hat{c}_1$. Hence $I_{\hat{T}}(\psi) = \hat{c}_1$, and by construction $\psi \equiv p_1$ and $\hat{T} = T_1$, which will imply, in particular that for any $q \in \Gamma$, $s_i(q) \to 0$ as $i \to \infty$. We proved that $\psi_i \to p_1$ as $i \to \infty$ along a subsequence in $L^\infty([0, T])$, any other choice of subsequence would lead to the same result. □

Lemma 4.12 *For $q \in \Gamma$, if $J(q) < \infty$, $J(q)$ is independent of the choice of the sequence $(t_i(q))$ satisfying* (Γ 3).

Proof Let $(t_i(q))$ and $(\hat{t}_i(q))$ be two sequences satisfying (Γ 3). For any $l \in \mathbb{N}$, set

$$J_l(q) = \sum_{i=1}^{l} a_i(q) = \int_{-\infty}^{t_l(q)} \mathbf{L}(q)\, dt - (l-1)\hat{c}_1,$$

and

$$\hat{J}_l(q) = \sum_{i=1}^{l} \hat{a}_i(q) = \int_{-\infty}^{\hat{t}_l(q)} \mathbf{L}(q)\, dt - (l-1)\hat{c}_1.$$

Then

$$|J_l(q) - \hat{J}_l(q)| = \left| \int_{\hat{t}_l(q)}^{t_l(q)} \mathbf{L}(q)\, dt \right|.$$

By the definition of the sequences, and since we are taking the same index l for both, notice that $(\tilde{t}(q))_i$ defined by, e.g.,

$$\tilde{t}_i(q) = t_i(q) \quad \forall i \neq l, l+1,$$

and

$$\tilde{t}_l(q) = \min\{t_l, \hat{t}_l\}, \quad \tilde{t}_{l+1}(q) = \max\{t_{l+1}, \hat{t}_{l+1}\},$$

is also an admissible sequence for the problem, and by Proposition 4.7,

$$\tilde{t}_{l+1}(q) - \tilde{t}_l(q) \approx T_1,$$

for l large. Then, and assuming that $\min\{t_l, \hat{t}_l\} = t_l$ and $\max\{t_{l+1}, \hat{t}_{l+1}\} = \hat{t}_{l+1}$

$$\int_{t_l}^{\hat{t}_{l+1}} \mathbf{L}(q)\, dt + \int_{\hat{s}_{l+1}}^{s_l} \mathbf{L}(z_1)\, dt \approx \hat{c}_1,$$

which implies that

$$\int_{t_{l+1}}^{\hat{t}_{l+1}} \mathbf{L}(q)\, dt + \int_{\hat{s}_{l+1}}^{s_{l+1}} \mathbf{L}(z_1)\, dt \approx 0.$$

Since, both s_{l+1} and $\hat{s}_{l+1}$ are converging to 0 as $l \to \infty$, we can conclude

$$|J_l(q) - \hat{J}_l(q)| \approx 0,$$

and therefore

$$|J(q) - \hat{J}(q)| = \lim_{l \to \infty} |J_l(q) - \hat{J}_l(q)| = 0.$$

□

We now state the main Theorem of this section, establishing the existence of a solution of (HS), with the desired asymptotic behavior.

Theorem 4.13 *Under the assumptions* (V1)–(V5) *and* (3.3), *there exists $Q \in \Gamma$ such that $J(Q) = c$. Moreover, Q is a solution of (HS) with $Q(-\infty) = 0$ and $Q(t) - p_1(t) \to 0$ as $t \to \infty$.*

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Proof Let $(q_m)_{m\in\mathbb{N}}$, be a minimizing sequence for J over the class Γ, and choose $\delta > 0$, small enough so that

$$\overline{B_\delta(0)} \cap \{p_1(t) : t \in \mathbb{R}\} = \emptyset.$$

By $(\Gamma 1)$, there is a smallest value of $t_{0,m} = t_0(q_m)$ for which $|q_m(t_{0,m})| = \delta$ and $q_m(t) \in B_\delta(0)$ for $t \in (-\infty, t_{0,m})$, and the invariance of J under translations, allow us to assume

$$t_{0,m} = 0, \quad \forall m \in \mathbb{N}.$$

By construction, $|q_m(0)| = \delta$ for all $m \in \mathbb{N}$. For $L \in \mathbb{R}^+$, choose i_m so that

$$t_{i_m}(q_m) \leqslant L < t_{i_m+1}(q_m).$$

Then

$$\int_{-L}^{L} |\dot{q}_m|^2 \, dt \leqslant 2\left(\sum_{j=1}^{i_m+1} a_j(q_m) + (i_m+1)\hat{c}_1\right),$$

and using Lemma 4.5

$$\int_{-L}^{L} |\dot{q}_m|^2 \, dt \leqslant 2(M + 2(c^+ + \hat{c}_1) + (i_m+1)\hat{c}_1). \tag{4.14}$$

On the other hand, by Lemma 4.6 and our normalization

$$t_{i_m}(q_m) = t_{i_m}(q_m) - t_{0,m} = \sum_{i=1}^{i_m}(t_i(q_m) - t_{i-1}(q_m)) \geqslant \beta \sum_{i=1}^{i_m} 1 \geqslant \beta i_m,$$

and (4.14) becomes

$$\|\dot{q}_m\|^2_{L^2([-L,L])} \leqslant 2\left(M + 2(c^+ + \hat{c}_1) + \frac{L}{\beta}\hat{c}_1\right).$$

Finally,

$$\|q_m\|^2_{W^{1,2}([-L,L])} \leqslant 2\left(M + 2(c^+ + \hat{c}_1) + \frac{L}{\beta}\hat{c}_1 + \delta^2\right),$$

and $(q_m)_{m\in\mathbb{N}}$ is a bounded sequence in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$. As usual this means that q_m converges (possibly along a subsequence, that we will denote again by (q_m)) weakly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$ and strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$ to a function $Q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$.

As a direct consequence of the $L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2)$ convergence, we can conclude immediately that

$$Q(0) \in \partial B_\delta(0) \cap \mathbf{C}_1.$$

□

Next, it will be proved that $Q \in \Gamma$. By construction and Lemma 4.5

$$\int_{-\infty}^{0} \mathbf{L}(q_m) \leqslant M + 2c^+ + 3\hat{c}_1 \equiv C_2.$$

The weakly lower semi-continuity of $\int \mathbf{L}(q)dt$ implies

$$-\int_{-\infty}^{0} V(Q)\, dt \leqslant C_2,$$

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for some positive constant C_2, and, see for example [14], it is easy to conclude that

$$\lim_{t\to-\infty} Q(t) \in V^{-1}(0) = 0,$$

and (Γ_1) is verified. (Γ_2) is a direct consequence of the L^∞_{loc} convergence. To prove that Q verifies (Γ_3), we need the following result.

Lemma 4.15 *Under the same hypothesis of Theorem 4.13, for each $i \in \mathbb{N}$, there exists a positive real number A_i, such that*

$$|t_i(q_m)| \leqslant A_i, \quad \forall m \in \mathbb{N}. \tag{4.16}$$

Proof If for some $j \in \mathbb{N}$, condition (4.16) does not hold, for any positive A, we can find $m = m(A)$ sufficiently large, so that

$$t_j(q_m) \geqslant A,$$

which means that $t_j(q_m) \to \infty$ as $m \to \infty$, possibly along a subsequence. Then

$$\int_{-\infty}^{t_j(q_m)} \mathbf{L}(q_m)\, dt \leqslant \sum_{i\in\mathbb{N}} |a_i(q_m)| + j\hat{c}_1,$$

and by Lemma 4.5, we can find $C_j = C(j)$ independent of m, such that

$$\int_{-\infty}^{t_j(q_m)} \mathbf{L}(q_m)\, dt \leqslant C_j.$$

Since $t_j(q_m) \to \infty$ as $m \to \infty$, for m sufficiently large and all $r > 0$

$$\int_{-\infty}^{r} \mathbf{L}(q_m)\, dt \leqslant C_j,$$

and the lower semicontinuity of $\int \mathbf{L}(q)$, imply that, for all $r > 0$

$$\int_{-\infty}^{r} \mathbf{L}(Q)\, dt \leqslant C_j,$$

and therefore

$$\int_{\mathbb{R}} -V(Q)\, dt < \infty \tag{4.17}$$

By construction, $Q(t)$ avoids arbitrarily small neighbourhoods of ξ_1, ξ_2, thus (V1) and (4.17) imply

$$V(Q(t)) \to 0 \quad \text{as} \quad t \to \infty,$$

and as a consequence $Q(t) \to 0$ as $t \to \infty$. With our choice of δ, pick $0 < \rho < \delta/4$, and $\Theta = \Theta(\rho)$ such that $Q(t) \in B_\rho(0)$ for all $t > \Theta$. Then, for m sufficiently large, $q_m(\Theta) \in B_{2\rho}(0)$, and we can define the following sequence of functions:

$$q^*_m(t) = \begin{cases} \varphi_m(t) & \text{if } t \leqslant 0 \\ q_m(t+\Theta) & \text{if } t \geqslant 0 \end{cases}$$

where φ_m minimizes $\int_{-\infty}^{0} \mathbf{L}(y)dt$ over the class of functions

$$\{y_m \in W^{1,2}_{\text{loc}}(\mathbb{R}^+) : (A1) - (A2) \text{ are verified}\}$$

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where

$(A1)$ $y_m(-\infty) = 0$ and $y_m(0) = q_m(\Theta)$;

$(A2)$ For all $t \in \mathbb{R}^+$, $y_m(t) \in \overline{\mathbf{C}}_1 \cap B_{2\rho}(0)$.

It is easy to check that $q_m^* \in \Gamma$ and

$$J(q_m) - J(q_m^*) = \int_{-\infty}^{\Theta} \mathbf{L}(q_m)\, dt - \int_{-\infty}^{0} \mathbf{L}(\varphi_m)\, dt.$$

Since ρ can be choosen arbitrarily small

$$\int_{-\infty}^{0} \mathbf{L}(\varphi_m)\, dt \to 0 \quad \text{as} \quad \rho \to 0.$$

On the other hand, for $t \in (-\infty, \Theta]$, $q_m(t)$ starts at the equilibrium, intersects $\partial B_\delta(0)$ at $t = 0$, and returns to $\partial B_{2\rho}(0)$ at $t = \Theta$. Hence there exists $\gamma = \gamma(\delta) > 0$ such that

$$\int_{-\infty}^{\Theta} \mathbf{L}(q_m)\, dt \geqslant \gamma,$$

and we can choose ρ as small as necessary so that

$$\int_{-\infty}^{0} \mathbf{L}(\varphi_m)\, dt \leqslant \frac{\gamma}{2}.$$

Then

$$J(q_m) - J(q_m^*) \geqslant \frac{\gamma}{2} > 0,$$

contradicting the fact that q_m is a minimizing sequence for J over Γ, and the Lemma is proved. □

Lemma 4.15 implies that for each $i \in \mathbb{N}$, the sequence $t_i(q_m)$ is a bounded sequence in $\mathbb{R}$. We can then conclude that it has at least one convergent subsequence. Choose a convergent subsequence of $t_i(q_m)$, which we will denote again by $t_i(q_m)$ and define

$$t_i(Q) \equiv \lim_{m \to \infty} t_i(q_m).$$

The strong convergence of (q_m) imply that, for each $i \in \mathbb{N}$

$$Q(t_i(Q)) \in z_1(\mathbb{R}^+),$$

and we can define the sequence of nonnegative real numbers $(s_i(Q))$ by

$$z_1(s_i(Q)) = Q(t_i(Q)).$$

There is nothing we can say about the set

$$\{t \in \mathbb{R},\ Q(t) \in z_1(\mathbb{R}^+)\},$$

other than it contains the sequence $t_i(Q)$. We will prove $(\Gamma 3)$ for the sequence above defined, which, by Proposition 4.12, will not change the value of $J(Q)$. To verify (Γ_3), that Q intersects z_1 an infinite number of times is just a consequence of condition $(\Gamma 3)$ applied to the sequence (q_m) and the fact that for each $i \in \mathbb{N}$, we defined $t_i(Q)$ as the limit as $m \to \infty$ of $t_i(q_m)$, and by construction $t_i(Q) \in z_1(\mathbb{R}^+)$. We will prove next that $t_i(Q)$ verifies

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properties (i) and (ii). For property (i), we want to show that $D(\psi_i^Q) = (1, 0)$ for all $i \in \mathbb{N}$, where

$$\psi_i^Q(t) = \begin{cases} Q(t) & \text{if } t_i(Q) \leqslant t \leqslant t_{i+1}(Q) \\ z_1(t - t_{i+1}(Q) + s_{i+1}(Q)) & \text{if } t_{i+1}(Q) \leqslant t \leqslant t_{i+1}(Q) - s_{i+1}(Q) + s_i(Q) \end{cases}$$

It is immediate that

$$d_{\xi_2}(\psi_i^Q) = 0, \quad \forall i \in \mathbb{N}.$$

On the other hand, by Lemma 4.15 and the strong convergence, for each $i \in \mathbb{N}$

$$t_i(Q) \leqslant A_i \quad \forall i \in \mathbb{N}. \tag{4.18}$$

Then, for ψ_i^m defined by

$$\psi_i^m(t) = \begin{cases} q_m(t) & \text{if } t_i(q_m) \leqslant t \leqslant t_{i+1}(q_m) \\ z_1(t - t_{i+1}(q_m) + s_{i+1}(q_m)) & \text{if } t_{i+1}(q_m) \leqslant t \leqslant t_{i+1}(q_m) - s_{i+1}(q_m) + s_i(q_m) \end{cases}$$

we can state the following:

(a) By $(\Gamma 2)$, for all $t \in \mathbb{R}$, both $q_m(t)$ and $Q(t)$ are in the region $\overline{\mathbf{C}}_1$, and $\xi_1 \in \mathbb{R}^2 \setminus \overline{\mathbf{C}}_1$, implying

$$\psi_i^Q(t) \neq \xi_1, \text{ for } t \in [t_i(Q), t_{i+1}(Q) - s_{i+1}(Q) + s_i(Q)];$$
$$\psi_i^m(t) \neq \xi_1, \text{ for } t \in [t_i(q_m), t_{i+1}(q_m) - s_{i+1}(q_m) + s_i(q_m)];$$

(b) By construction, both ψ_i^Q and ψ_i^m are closed curves in their intervals of definition.

By (a) and the fact that Q is the L^∞_{loc} limit of q_m, we can choose $\epsilon > 0$ such that

$$|q_m(t) - \xi_1| > 2\epsilon, \quad \forall m \in \mathbb{N}$$

and for some m_0 sufficiently large,

$$\|Q - q_{m_0}\|_{L^\infty_{\text{loc}}(\mathbb{R}^2, \mathbb{R})} < \epsilon.$$

On the other hand, by (b), and for each $i \in \mathbb{N}$, we can extend both ψ_i^Q and ψ_i^m periodically to $\mathbb{R}$, and

$$\|\psi_i^Q - \psi_i^m\|_{L^\infty_{\text{loc}}(\mathbb{R}, \mathbb{R}^2)} < \epsilon.$$

In particular,

$$\|\psi_i^Q - \psi_i^m\|_{L^\infty([0, T_i(Q)], \mathbb{R}^2)} < \epsilon,$$

for $T_i(Q) \equiv t_{i+1}(Q) - t_i(Q) + s_i(Q) - s_{i+1}(Q)$, which is finite as a consequence of (4.18). The continuity property of Brouwer degree implies

$$d_{\xi_1}(\psi_i^Q) = d_{\xi_1}(\psi_i^m) = 1, \quad \forall i \in \mathbb{N}.$$

Finally, to prove that (ii) also holds, notice that, for each $i \in \mathbb{N}$

$$z_1(s_i(q_m)) = q_m(t_i(q_m)) \to Q(t_i(Q)) = z_1(s_i(Q))$$

as $m \to \infty$, via the definition of $s_i(Q)$. Hence we must also have that

$$s_i(q_m) \to s_i(Q), \quad m \to \infty.$$

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The fact that $s_{i+1}(q_m) \leqslant s_i(q_m)$, for all $i \in \mathbb{N}$, and the monotonicity property of limits, imply the result.

Thus, $Q \in \Gamma$. To finish the proof, we still have to establish some technical properties of Q.

Lemma 4.19 *Under the hypothesis of Theorem 4.13,*

$$J(Q) = c.$$

Proof For any $L \in \mathbb{R}^+$, such that $-L < t_1(Q)$, the definition of $(t_i(Q))$ implies that for any $\epsilon > 0$,

$$\liminf_{m\to\infty} t_i(q_m) \geqslant t_i(Q) - \epsilon. \tag{4.20}$$

By the weak lower semicontinuity of $\int \mathbf{L}(q)dt$ on compact sets and (4.20)

$$\begin{aligned}\int_{-L}^{t_i(Q)} \mathbf{L}(Q)\,dt &= \lim_{\epsilon\to 0}\int_{-L}^{t_i(Q)-\epsilon} \mathbf{L}(Q)\,dt \leqslant \lim_{\epsilon\to 0}\lim_{m\to\infty}\int_{-L}^{t_i(Q)-\epsilon} \mathbf{L}(q_m)\,dt\\ &\leqslant \lim_{\epsilon\to 0}\lim_{m\to\infty}\int_{-L}^{t_i(q_m)} \mathbf{L}(q_m)\,dt = \lim_{m\to\infty}\int_{-L}^{t_i(q_m)} \mathbf{L}(q_m)\,dt.\end{aligned}$$

Hence, for any $i \in \mathbb{N}$,

$$\begin{aligned}\sum_{j=1}^{i} a_j(Q) &= \int_{-\infty}^{t_i(Q)} \mathbf{L}(Q)\,dt - i\hat{c}_1 = \lim_{L\to\infty}\int_{-L}^{t_i(Q)} \mathbf{L}(Q)\,dt - i\hat{c}_1\\ &\leqslant \lim_{L\to\infty}\left(\liminf_{m\to\infty}\int_{-L}^{t_i(q_m)} \mathbf{L}(q_m)\,dt - i\hat{c}_1\right) = \liminf_{m\to\infty}\sum_{j=1}^{i} a_j(q_m)\\ &\leqslant \liminf_{m\to\infty}\sum_{j=1}^{i} |a_j(q_m)|.\end{aligned}$$

Since $J(q_m) \to c$ as $m \to \infty$, we may assume that for all m

$$J(q_m) \leqslant c + 1,$$

and by Lemma 4.5

$$\sum_{j=1}^{i} a_j(Q) \leqslant c + 1 + 2(c^+ + \hat{c}_1). \tag{4.21}$$

On the other hand, for $N^-(Q) = \{j \in \mathbb{N} : a_j(Q) < 0\}$ and $N^+(Q) = \mathbb{N} \setminus N^-(Q)$, once again by Lemma 4.5,

$$-\sum_{j<i, j\in N^-(Q)} a_j(Q) \leqslant -\sum_{j\in N^-(Q)} a_j(Q) \leqslant c^+ + \hat{c}_1.$$

Using this inequality and (4.21), for all $i \in \mathbb{N}$,

$$\sum_{j<i, j\in N^+(Q)}^{i} a_j(Q) = \sum_{j=1}^{i} a_j(Q) - \sum_{j<i, j\in N^-(Q)} a_j(Q) \leqslant c + 1 + 3(c^+ + \hat{c}_1).$$

Taking the limit as $i \to \infty$ in both series

$$\sum_{j \in N^+(Q)} a_j(Q) \leqslant c + 1 + 3(c^+ + \hat{c}_1) \quad \text{and} \quad -\sum_{j \in N^-(Q)} a_j(Q) \leqslant c^+ + \hat{c}_1.$$

Therefore,

$$J(Q) < \sum_{j \in \mathbb{N}} |a_j(Q)| < \infty.$$

Let ϵ be an arbitrary positive number. Since $J(Q)$ is finite, Proposition 4.7 implies $s_j(Q) \to 0$ as $j \to \infty$. Hence, we can find $N_1 = N_1(\epsilon)$, such that for all $j \geqslant N_1$

$$\sum_{i=j+1}^{\infty} a_i(Q) \leqslant \epsilon,$$

and

$$s_j(Q) \leqslant \epsilon.$$

This implies that for $j \geqslant N_1$

$$J(Q) = \sum_{i=1}^{j} a_i(Q) + \sum_{i=j+1}^{\infty} a_i(Q) \leqslant \sum_{i=1}^{j} a_i(Q) + \epsilon, \tag{4.22}$$

and

$$s_j(Q) \leqslant \epsilon. \tag{4.23}$$

On the other hand, by the strong convergence of q_m to Q and the weak lower semicontinuity of $\int \mathbf{L}(q)dt$, we can find $N_2 = N_2(\epsilon)$ such that, for all $m \geqslant N_2$

$$\sum_{i=1}^{N_1} a_i(Q) = \int_{-\infty}^{t_{N_1}(Q)} \mathbf{L}(Q)\, dt - N_1\hat{c}_1 \leqslant \sum_{i=1}^{N_1} a_i(q_m) + \epsilon. \tag{4.24}$$

By (4.22) and (4.24)

$$J(Q) \leqslant \sum_{i=1}^{N_1} a_1(q_m) + 2\epsilon, \quad \forall m \geqslant N_2.$$

Since (q_m) is a minimizing sequence, we can find $N_3 = N_3(\epsilon)$, so that

$$J(q_m) \leqslant c + \epsilon, \quad \forall m \geqslant N_3.$$

For $N_4 \geqslant \max\{N_2, N_3\}$

$$J(Q) \leqslant c + 3\epsilon - \sum_{i=N_1+1}^{\infty} a_i(q_m), \quad \forall m \geqslant N_4.$$

As in the proof of Lemma 4.2,

$$\sum_{i=N_1+1}^{\infty} a_i(q_m) \geqslant -\sum_{i=N_1+1}^{\infty} \int_{s_{i+1}(q_m)}^{s_i(q_m)} \mathbf{L}(z_1)\, dt = -\int_{0}^{s_{N_1+2}(q_m)} \mathbf{L}(z_1)\, dt.$$

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By its definition we can find $N_5 = N_5(\epsilon)$, such that

$$s_i(q_m) \leqslant s_i(Q) + \epsilon$$

for all $m \geqslant N_5$. Therefore, for $m \geqslant \max\{N_4, N_5\}$

$$\sum_{i=N_1+1}^{\infty} a_i(q_m) \geqslant -\int_0^{s_{N_1+2}(Q)+\epsilon} \mathbf{L}(z_1)\, dt.$$

Using (4.23)

$$J(Q) \leqslant c + 3\epsilon + \int_0^{2\epsilon} \mathbf{L}(z_1)\, dt.$$

Taking the limit as $\epsilon \to 0$, we will obtain

$$J(Q) \leqslant c.$$

Since $Q \in \Gamma$ and $c = \inf_{q\in\Gamma} J(q)$, we must have that $J(Q) = c$, as wanted. □

In all that follows, we will assume that the hypothesis of Theorem 4.13 are verified.

Lemma 4.25 *For* $(s_i(Q))$ *defined as in the proof of Theorem* 4.13, *we have that*

$$s_i(Q) > s_{i+1}(Q) \quad \forall i \in \mathbb{N},$$

unless that for some i_0, $s_{i_0}(Q) = 0$, *in which case* $Q(t) = p_1(t)$, *for all* $t \geqslant t_{i_0}(Q)$.

Proof If this is not the case, there is a smallest $j \in \mathbb{N}$, such that

$$0 \neq s_j(Q) = s_{j+1}(Q),$$

and we can consider the function

$$\hat{Q} = Q \backslash Q|_{t_j(Q)}^{t_{j+1}(Q)}.$$

It is immediate to see that $\hat{Q} \in \Gamma$, and

$$J(\hat{Q}) = J(Q) - a_j(Q).$$

Since $s_j(Q) = s_{j+1}(Q)$, $Q|_{t_j(Q)}^{t_{j+1}(Q)}$ is a closed curve, and as a consequence of $(\Gamma 3)$ (i), $D(Q|_{t_j(Q)}^{t_{j+1}(Q)}) = (1, 0)$, which will imply that $a_j(Q) > 0$ since neither $s_j(Q)$ nor $s_{j+1}(Q)$ are zero. Therefore,

$$J(\hat{Q}) < J(Q)$$

contradicting the fact that Q is the infimum of J over Γ. On the other hand, if there exists $j \in \mathbb{N}$ for which $s_j(Q) = 0$, notice that the function

$$\tilde{Q}(t) = \begin{cases} Q(t) & \text{if } t \leqslant t_j(Q) \\ p_1(t - t_j(Q)) & \text{if } t \geqslant t_j(Q) \end{cases}$$

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is also in Γ, and by the minimality property of p_1

$$J(\tilde{Q}) < J(Q).$$

Hence, unless Q coincides with $\tilde{Q}$, we have the usual contradiction. □

Proposition 4.26 *Q is a solution of (HS).*

Proof We will show that for all $\theta \in \mathbb{R}$, there exists an open neighborhood of θ, N_θ, such that $Q|_{N_\theta}$ is a solution of (HS). For $\theta \in \mathbb{R}$, choose $\epsilon > 0$ small, compared with $|Q(\theta)|$. Consider $\sigma = \sigma(\theta, \epsilon)$ and $s = s(\theta, \epsilon)$, sufficiently close to θ, so that we can find a function p minimizing $\int \mathbf{L}(y)dt$ over all $W^{1,2}$ functions joining $Q(\sigma)$ to $Q(s)$, i.e we consider the problem of finding an interior minimum of I over

$$\cup_{a>0}\{q \in W^{1,2}([0,a], \mathbb{R}^2) : q(0) = Q(\sigma),\ q(a) = Q(s) \text{and } \|Q - q\|_{L^\infty([0,a])} \leqslant \epsilon\}.$$

As remarked in §4.1, the existence of such minimizer is guaranteed, either defined in a bounded time interval, or as a chain of homoclinic solutions. Nevertheless, if we choose ϵ small enough so that $|Q(t)| > 4\epsilon$ for $t \in [\sigma, s]$, the latter possibility is excluded. If $Q|_\sigma^s$ is such minimizer, then we are done, since it is known that any minimizer of $I_{[a,b]}$ is a solution of (HS) on $[a, b]$. Otherwise, we can assume that there exists a distinct minimizer $p(t)$ defined for $t \in [0, s_p]$. By the minimality property of p, we can assume that

$$\int_0^{s_p} \mathbf{L}(p)\, dt < \int_\sigma^s \mathbf{L}(Q)\, dt. \tag{4.27}$$

We will show that (4.27) leads to a contradiction, and therefore $p|_0^{s_p} \equiv Q|_\sigma^s$, which will prove the result via the above remark.

First, note that if $p([0, s_p]) \cap (\partial\mathbf{C}_1 \cup z_1((0, \infty))) = \emptyset$ we will have an immediate contradiction since replacing $Q|_\sigma^s$ by $p|_0^{s_p}$, produces a function $\hat{Q} \in \Gamma$, with $J(\hat{Q}) < J(Q)$ as a consequence of (4.27). It remains to study the cases when

$$p([0, s_p]) \cap (\partial\mathbf{C}_1 \cup z_1((0, \infty))) \neq \emptyset.$$

We will proceed, by studying the several cases that may occur.

(i) $p([0, s_p]) \cap q_1(\mathbb{R}) \neq \emptyset$

Due to the properties of $q_1(\mathbb{R})$ and $p_1([0, T_1])$, we can choose σ and τ close enough to θ, so that $p([0, s_p]) \cap (p_1([0, T_1]) \cup z_1(\mathbb{R}^+)) = \emptyset$. If for all $t \in [0, s_p]$, $p(t) \in \overline{\mathbf{C}}_1$, then the function $\hat{Q}$ defined above is in Γ and once again by (4.27), $J(\hat{Q}) < J(Q)$. Hence, we may assume that for some $t \in [\sigma_p, s_p]$, $p(t)$ lies in the exterior of $\mathbf{C}_1$. By continuity, we can conclude that there exist at least one pair of points $0 < \sigma' < s' < s_p$ such that

$$p(\sigma'), p(s') \in q_1(\mathbb{R}) \quad \text{and} \quad p((\sigma', s')) \notin \overline{\mathbf{C}_1}.$$

Then there will exist α, $\beta \in \mathbb{R}$ such that

$$p(\sigma') = q_1(\alpha) \quad \text{and} \quad p(s') = q_1(\beta), \tag{4.28}$$

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and by the minimality of both p and q_1, we must necessarily have that

$$\int_{\sigma'}^{s'} \mathbf{L}(p)\,dt = \int_{\alpha}^{\beta} \mathbf{L}(q_1)\,dt,$$

in which case the function obtained by gluing $Q|_{\mathbb{R}\setminus(\sigma,s)}$ to $p|_{[0,s_p]\setminus(\sigma',s')}$ to $q_1|_{\sigma'}^{s'}$ is in Γ, and by (4.27) and (4.28), $J(\tilde{Q}) < J(Q)$.

(ii) $p([0,s_p]) \cap p_1((0,T_1)) \neq \emptyset$

This case follows as case (i), using the minimality of p_1. The case $p([0,s_p]) \cap p_1(0) \neq \emptyset$ will be studied in the proof of case (iii).

(iii) $p([0,s_p]) \cap z_1(\mathbb{R}^+) \neq \emptyset$

If this is the case, there exists an element of $(0,s_p)$, that without loss of generality we will assume to be θ, and $\alpha \geqslant 0$, for which

$$p(\theta) = z_1(\alpha).$$

If $\alpha = 0$, we can consider

$$\tilde{Q}(t) = \begin{cases} Q(t) & \text{if } t \leqslant \sigma \\ p(t-\sigma+\sigma_p) & \text{if } \sigma \leqslant t \leqslant \theta \\ p_1(t-\theta) & \text{if } t \geqslant \theta, \end{cases}$$

and it is easy to check that $\tilde{Q} \in \Gamma$, and once again by (4.27) and the minimality of p_1, $J(\tilde{Q}) < J(Q)$. On the other hand, if $\alpha \neq 0$, consider $i_0 \in \mathbb{N}$ such that $t_{i_0}(Q) \in [\sigma,s]$. Note that if, $\cup_{i\in\mathbb{N}}\{t_i(Q)\} \cap [\sigma,s] = \emptyset$, then the function $\hat{Q}$ above defined is in Γ and we are done. By the previous lemma, we can choose ϵ so small, and shrink $[\sigma,s]$ if necessary, so that $s_{i_0-1}(Q) \notin (\alpha, s_{i_0}(Q))$ if $\alpha \leqslant s_{i_0}(Q)$ or that $s_{i_0+1}(Q) \notin (s_{i_0}(Q),\alpha)$ if $\alpha \geqslant s_{i_0}(Q)$. Then, the function $\hat{Q}$ above defined is in Γ, with $t_i(\hat{Q}) = t_i(Q)$ for $i \neq i_0$ and $t_{i_0}(\hat{Q}) = \alpha$, and as usual $J(\hat{Q}) < J(Q)$. □

The proof of Theorem 4.13 is now complete.

Corollary 4.29 *If Q is the solution of (HS) obtained in the previous results, then*

$$\frac{1}{2}|\dot{Q}|^2 + V(Q(t)) = 0, \quad \forall t \in \mathbb{R} \tag{4.30}$$

i.e., Q is a solution of (HS) of energy level 0.

Proof Since Q is a solution of (HS)

$$\exists \alpha_Q \in \mathbb{R} : \frac{1}{2}|\dot{Q}|^2 + V(Q(t)) = \alpha_Q \quad \forall t \in \mathbb{R}. \tag{4.31}$$

Moreover, by (Γ_1), $Q(-\infty) = 0$, and by (V1), $V(0) = 0$. As in the proof of analogous result in Theorem 2.4, one can prove that $\dot{Q}(-\infty) = 0$. Taking the limit as $t \to -\infty$ in (4.31), we obtain that $\alpha_Q = 0$. □

Corollary 4.32 *For any $t \in \mathbb{R}$, $Q(t) \cap \partial\mathbf{C}_1 = \emptyset$.*

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Proof We will prove that $Q(t) \cap q_1(\mathbb{R}) = \emptyset$. That $Q(t) \cap p_1([0, T_1]) = \emptyset$ follows the same arguments.

If the intersection is not empty, there are $\alpha, \beta \in \mathbb{R}$ such that $Q(\alpha) = q_1(\beta)$. By (4.30), at $t = \alpha$,

$$\frac{1}{2}|\dot{Q}(\alpha)|^2 + V(Q(\alpha)) = 0,$$

and by Proposition 3.4, applied at $t = \beta$

$$\frac{1}{2}|\dot{q}_1(\beta)|^2 + V(q_1(\beta)) = 0.$$

But since $V(Q(\alpha)) = V(q_1(\beta))$, we must also have that $|\dot{Q}(\alpha)| = |\dot{q}_1(\beta)|$. By the fact that $q(t) \in \overline{\mathbf{C}}_1$ for all $t \in \mathbb{R}$, either $\dot{Q}(\alpha) = \dot{q}_1(\beta)$ and the following reasoning yields the contradiction, or $\dot{Q}(\alpha) = -\dot{q}_1(\beta)$ and we use the same arguments with one of the functions with time reversed. In any case, the equalities will violate the uniqueness of solution of the (IVP) for (HS)

$$\begin{aligned} \ddot{q} + V'(q) &= 0 \\ q(0) = Q(\alpha) &= q_1(\beta) \\ \dot{q}(0) = \dot{Q}(\alpha) &= \dot{q}_1(\beta)\ . \end{aligned}$$

□

An immediate consequence of this corollary, and of Lemma 4.25 is that the sequence $(s_i(Q))$ is strictly monotone.

Corollary 4.33

$$\bar{t}(Q) \equiv \{t \in \mathbb{R} : Q(t) \cap z_1(\mathbb{R}^+) \neq \emptyset\} = \cup_{i \in \mathbb{N}} t_i(Q).$$

Proof If this is not the case, there exists $\theta \in \bar{t}(Q)$ and $\theta \neq t_i(Q)$ for all $i \in \mathbb{N}$. Then we can find $i_0 \in \mathbb{N}$ such that

$$t_{i_0-1}(Q) < \theta < t_i(Q).$$

For $\theta_s > 0$ defined by $Q(\theta) = z_1(\theta_s)$, the function $\hat{Q}$ obtained by replacing $Q|^{\theta}_{t_{i_0-1}(Q)}$ by $z_1|^{\theta_s}_{s_{i_0-1}}$ is in the class Γ and by the minimality of z_1 verifies $J(\hat{Q}) < J(Q)$ which is a contradiction to the fact that Q is the minimizer of J over Γ. □

4.4 Some more heteroclinic solutions

In the previous section we obtained a solution of (HS), which is heteroclinic to 0 and p_1, winding positvely around ξ_1. As we remarked earlier, in a symmetric way we can obtain another solution of (HS) which asymptotes to 0 as $t \to -\infty$, and to the periodic p_2 as $t \to \infty$, winding positvely around ξ_2. We will denote by h_j^+, $j = 1, 2$, these solutions of (HS).

As a consequence of the invariance of $\int \mathbf{L}(q)dt$, under the transformation $t \mapsto -t$, the function $\tilde{h}_j^+(t) = h_j^+(-t)$ is also a solution of (HS), and is heteroclinic from p_j to 0, winding negatively around ξ_j. Motivated by this fact, and choosing the direction of

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winding carefully, we can obtain two solutions of (HS) asymptotic to 0 as $t \to +\infty$ and to p_j as $t \to -\infty$ winding positively around ξ_j. As before, we will obtain them as the minimizer of a functional over an appropriate class of functions, and the proof of their existence parallels the proof of Theorem 4.13, and we will denote them by h_j^-, $j = 1, 2$.

Proposition 4.34 *Under the conditions of Theorem 4.13, the functions $h_j^\pm$, $j = 1, 2$, are all minimal solutions of (HS), in the sense that, for all $q \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^2)$, $\sigma < \tau$ in $\mathbb{R}$ and $\alpha < \beta$ in $\mathbb{R}$, such that $q(\tau) = h_j^\pm(\beta)$ and $q(\sigma) = h_j^\pm(\alpha)$, $q([\sigma, \tau]) \subset \overline{\mathbf{C}}_1$ and $q|_\sigma^\tau$ has the same homotopy type as $h_j^\pm|_\alpha^\beta$, we have that*

$$\int_\alpha^\beta \mathbf{L}(h_j^\pm)\, dt < \int_\sigma^\tau \mathbf{L}(q)\, dt$$

unless $\tau - \sigma = \beta - \alpha$ and $q = h_j^\pm$ (whenever the equality makes sense).

Proof If that was not the case, we should have

$$\int_\sigma^\tau \mathbf{L}(q)\, dt \leqslant \int_\alpha^\beta \mathbf{L}(h_j^\pm)\, dt, \tag{4.35}$$

for a function q and real numbers $\sigma < \tau$ as above. But then, we can consider the function

$$H_j^\pm(t) = \begin{cases} h_j^\pm(t - \sigma + \alpha) & \text{if } t \leqslant \sigma \\ q(t) & \text{if } \sigma < t < \tau. \\ h_j^\pm(t - \tau + \beta) & \text{if } t \geqslant \tau \end{cases}$$

By the assumptions that $q(\sigma, \tau) \subset \overline{\mathbf{C}}_1$ and that $q|_\sigma^\tau$ has the same homotopy type as $h_j^\pm|_\alpha^\beta$, we conclude that $H_j^\pm \in \Gamma_j^\pm$, for $j = 1, 2$, and by (4.35)

$$J_j^\pm(H_j^\pm) - J_j^\pm(h_j^\pm) \leqslant 0,$$

and this contradicts the fact that $h_j^\pm$ is a minimizer of $J_j^\pm$ over $\Gamma_j^\pm$, for $j = 1, 2$, unless equality holds. But then $J_j^\pm(H_j^\pm) = b_j^\pm$ and therefore is also a solution of (HS) which agrees with $h_j^\pm$ in all $\mathbb{R}$ except in the interval $[\sigma, \tau]$. This contradicts the uniqueness of solution of the IVP for (HS), unless $q \equiv h_j^\pm$ and $\beta - \alpha = \tau - \sigma$. □

5 Conclusion

In this paper we have studied the asymptotic behavior of the solutions of an Hamiltonian system with a potential having a double well of infinite depth, satisfying a compactness condition.

Using variational methods, we proved the existence of periodic soltions and of homoclinic solutions winding any prescribed number of times around each or both the singularities. We also showed the existence of heteroclinic solutions asymptotic to the critical point as $t \to -\infty$ and to the periodic solutions as $t \to \infty$.

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