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106.27 An interesting application of Ptolemy's inequality

Martin Lukarevski proposed the following inequalities in the *Monthly* [1] and the *Gazette* [2]

$$\frac{a}{l_a} + \frac{b}{l_b} + \frac{c}{l_c} \geqslant 2\sqrt{3}, \tag{1}$$

$$\frac{a^2}{l_a^2} + \frac{b^2}{l_b^2} + \frac{c^2}{l_c^2} \geqslant 4, \tag{2}$$

where a, b, c are the side lengths and l_a, l_b, l_c are the lengths of the corresponding angle bisectors of a given triangle. The goal of this note is to give a geometric proof of the following inequality of which (1) and (2) are consequences:

$$\frac{ab}{m_a m_b} + \frac{bc}{m_b m_c} + \frac{ca}{m_c m_a} \geqslant 4. \tag{3}$$

I found it interesting that I can only prove (3) via Ptolemy's inequality [3]. Consider triangle ABC. Let M, N, P be midpoints of BC, CA, AB respectively. Let $BC = a$, $CA = b$, $AB = c$, $AM = m_a$, $BN = m_b$, $CP = m_c$. Let G be the centroid of triangle ABC.

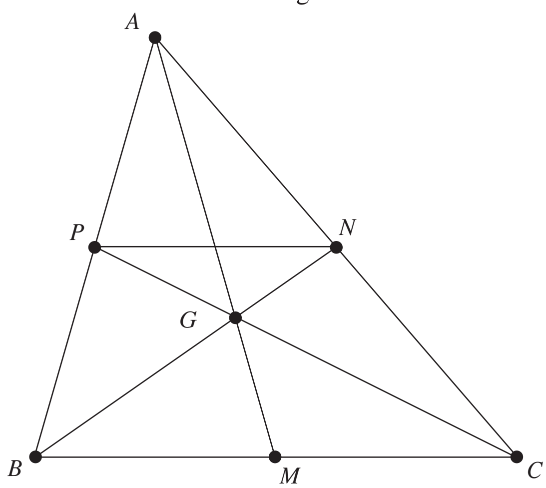

FIGURE 1

Quadrilateral $APGN$ has $AG = \frac{2m_a}{3}$, $GN = \frac{m_b}{3}$, $GP = \frac{m_c}{3}$, $NP = \frac{a}{2}$, $AN = \frac{b}{2}$ and $AP = \frac{c}{2}$. After applying Ptolemy's inequality to quadrilateral $APGN$ we have

$$bm_c + cm_b \geqslant 2am_a.$$

Therefore

$$\begin{aligned} abm_c + bcm_a + cam_b &= a(bm_c + cm_b) + bcm_a \\ &\geqslant 2a^2m_a + bcm_a \qquad (4) \\ &= m_a(2a^2 + bc). \end{aligned}$$

Quadrilateral $BCNP$ has $BC = a$, $NP = \frac{1}{2}a$, $CN = \frac{1}{2}b$, $PB = \frac{1}{2}c$, $BN = m_b$ and $CP = m_c$. After applying Ptolemy.s inequality to quadrilateral $BCNP$ we have

$$2a^2 + bc \geqslant 4m_bm_c. \qquad (5)$$

From (4) and (5) we have

$$abm_c + bcm_a + cam_b \geqslant 4m_am_bm_c. \qquad (6)$$

Dividing both sides of (6) by $m_am_bm_c$ we have (3).

If the equality holds in (3), then $APGN$ and $BCNP$ are cyclic quadrilaterals. Since $NP \parallel BC$, we have $NC = PB$. Hence $AC = AB$. Therefore AM is the perpendicular bisector of PN and BC. Quadrilateral $APGN$ is cyclic, so $\angle GPN = \angle GAN = \angle PAG$. Because $PN \perp AG$, we have $\angle APG = 90°$. Hence CP is the perpendicular bisector of AB. So $AC = BC$. Therefore $AB = AC = BC$. Conversely, if triangle ABC is equilateral, then it is easy to see that the equality in (3) holds.

We now show that (1) and (2) are consequences of (3). Let AD be the angle bisector of $\angle BAC$. Without loss of generality, we can assume $AB \leqslant AC$.

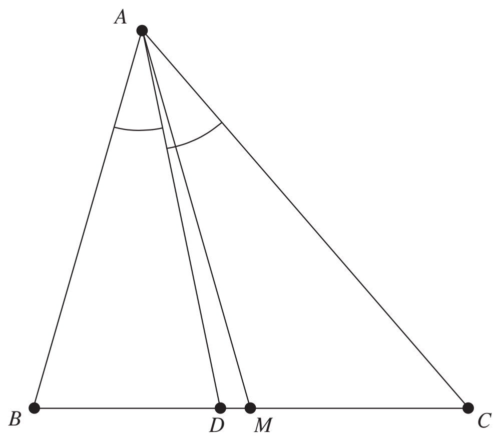

FIGURE 2

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Then

$$\frac{DB}{DC} = \frac{AB}{AC} \leqslant 1.$$

Hence $DC \geqslant \frac{1}{2}BC = MC$. Hence M lies in between D and C. Therefore

$$\begin{aligned}\angle DMA &= \angle MAC + \angle ACB \\ &\leqslant \angle DAC + \angle ACB \\ &= \angle DAB + \angle ACB \\ &\leqslant \angle DAB + \angle ABC \\ &= \angle ADM.\end{aligned}$$

Hence $l_a = AD \leqslant AM = m_a$. Similarly, $l_b \leqslant m_b$ and $l_c \leqslant m_c$. Therefore

$$\begin{aligned}\frac{a}{l_a} + \frac{b}{l_b} + \frac{c}{l_c} &\geqslant \frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \\ &\geqslant \sqrt{3\left(\frac{ab}{m_a m_b} + \frac{bc}{m_b m_c} + \frac{ca}{m_c m_a}\right)} \qquad (7) \\ &\geqslant 2\sqrt{3},\end{aligned}$$

and

$$\begin{aligned}\frac{a^2}{l_a^2} + \frac{b^2}{l_b^2} + \frac{c^2}{l_c^2} &\geqslant \frac{1}{3}\left(\frac{a}{l_a} + \frac{b}{l_b} + \frac{c}{l_c}\right)^2 \\ &\geqslant 4. \qquad (8)\end{aligned}$$

Here, in (7) and (8), we have used (3), $l_a \leqslant m_a$, $l_b \leqslant m_b$, $l_c \leqslant m_c$ and the inequality $x^2 + y^2 + z^2 \geqslant \frac{1}{3}(x + y + z)^2 \geqslant xy + yz + zx$ for all real numbers x, y, z. The equality in any of (1) or (2) or (3) holds if, and only if, the triangle is equilateral. To end this Note, the author is not able to find an algebraic proof of (3). Is there such a proof of (3)?

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