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A new approach to singularity-free inverse kinematics using dual-quaternionic error chains in the Davies method

Andre Schneider de Oliveira†*, Edson Roberto De Pieri‡, Ubirajara Franco Moreno‡ and Daniel Martins§

†*Department of Informatics, Federal University of Technology – Parana, Curitiba, PR, Brazil*
‡*Department of Automation and Systems, Federal University of Santa Catarina, Florianopolis, SC, Brazil*
E-mails: edson@das.ufsc.br, moreno@das.ufsc.br
§*Department of Mechanical Engineering, Federal University of Santa Catarina, Florianopolis, SC, Brazil*
E-mail: daniel@emc.ufsc.br

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SUMMARY
The manipulation in singular regions promotes an instantaneous reduction in mechanism mobility, which can result in some disturbances in the trajectory tracking. The application of the quaternionic elements for motion representation not only guarantees an orthonormal transformation but also results in the smallest variance and minimizes the acceleration peaks. The use of a unit quaternion avoids these phenomena, but there are dimensional limitations that make it impossible to translate the representation. This work presents a methodology for using dual quaternions in the analysis of robot kinematics using the Davies method, which avoids kinematic singularities and ensures the optimal torque profiles.

KEYWORDS: Mechanism mobility; Kinematic singularities; Davies method; Dual quaternions; Optimal torque profiles.

1. Introduction

In robotics, the kinematics perform the conversion between the joint and operational spaces. Some restrictions, which are directly related to the mechanism structure, define a nonlinear mapping between these spaces. A temporary phenomenon (i.e., a limited period of time) exists that is directly related to mechanism pose and introduces nonlinearities in this transformation; this phenomenon is called the singularity.

Singularities represent configurations where the structure mobility is reduced, i.e., as in ref. [1], and it is not possible to impose an arbitrary motion to an end effector. The singularities can be divided into two types: boundary singularities (also called as geometric singularities), which occur near the extension or contraction limits, and internal singularities (also called kinematic singularities), which occur within the reachable space of the manipulator and are usually caused by the alignment of two or more axes or by particular configurations of the end effector. The consequence of this effect is that in an internal singularity, there are infinite solutions for the inverse kinematics. In proximity to the singularities, small velocities of the end effectors generate high speeds in the joints due to the gradual reduction of the mobility.

The occurrence of singularities frequently happens on mobile base systems where it is possible to accomplish the reorientation of the whole structure. In manipulator-vehicle systems where a

* Corresponding author. E-mail: andreoliveira@utfpr.edu.br

serial kinematic chain is fixed on a vehicle, this vehicle may have three degrees of re-orientation, e.g., spacecraft and underwater. In this case, the quaternion is frequently used to avoid kinematic singularity and the phenomenon of rotational axis alignment, i.e., the "gimbal-lock," as seen in ref. [2].

The kinematic singularity is one focus of current robotic research where different methodologies are applied. Several studies employ quaternions to represent the orientation due to its uniqueness in orientation angle extraction and to avoid this phenomenon. For example, Xu *et al.*[3] present a path planning system for a spatial vehicle manipulator where the use of unit quaternions guarantees that singularities of orientation will not occur. Bai *et al.*[4] discuss a method of workspace modeling for spherical parallel manipulators where the Euler angles are expressed by unit quaternions.

Other approaches employ the quaternions to correct the orientation error, e.g., Erdong and Zhaowei[5] discuss an application of unit quaternions for orientation error stabilization in spacecraft pose control. Castillo-Cruces and Wahrburg[6] present a control strategy for surgical interventions applied to a human–robot cooperative system that uses unit quaternions to determine the orientation error. Tabandeh *et al.*[7] present a modified genetic algorithm to solve the inverse kinematic of a serial manipulator using unit quaternion feedback.

In transformation representation through homogeneous transformation matrices (HTMs), the error accumulated from successive multiplications leads to a loss of orthonormality and to singular matrices. The application of quaternionic elements for movement representation is not only to guarantee an orthonormal transformation but also to ensure the smallest variation and minimize the acceleration peaks. These phenomena can be seen in ref. [8] in which the authors present the potential of this element for a reduction in the discrete movement representation errors.

The application of unit quaternions is restricted to the orientation representation. The application of these elements for translational motion is only possible by applying complex, nonlinear mathematical relations; e.g., Sahu *et al.*[9] apply a methodology that combines the use of unit quaternions and dual quaternions to perform a linear motion. However, this approach becomes confusing because using elements of different algebras led the authors to equivocate the algebraic definition, and consequently, the wrong use of the elements, i.e., the double number definition was employed in the dual number relations. Qiao *et al.*[10] introduce the inverse kinematic solution of a serial 6R chain that employed the dual quaternions using a correlation with the transformation matrix and not an algebraic definition, which led to an incorrect definition of the algebraic properties and, consequently, increased the solution complexity. Sariyildiz and Temeltas[11] present an inverse kinematic method using dual quaternions, but the approach is purely geometrical and dependent on the mechanism topology, i.e., the application in other mechanisms is not trivial.

This paper presents a methodology for the use of dual quaternions in kinematic mapping using the Davies method, which avoids the occurrence of kinematic singularities and inherits the algebraic properties of the quaternions. This work is organized in the following manner. Section 2 discusses the problem of singularity occurrence in the classic method. Section 3 is an overview of the Clifford algebra for the correct definition of the elements and the algebraic relationships. Sections 4 and 5 define the quaternions and biquaternions. Section 6 presents some methodologies for the dual-quaternionic approach. Section 7 describes and discusses the computational performance analysis. Section 8 presents a singularity-free trajectory tracking. Finally, Section 9 presents the conclusions.

2. The Problem of Singularity Occurrence

To explain the problem of singularity occurrence, it is necessary to analyze the method of inverse kinematic determination. The inverse kinematics for a system of screws can be determined using the screws through the Davies method. This method is an adaptation of the Kirchhoff's law of mechanisms. According to Kirchhoff's law, the algebraic sum of the potential difference in an electrical circuit is zero. By analogy, in mechanisms, the sum of the relative kinematic velocity pairs along a kinematic chain is zero; in this case

$$\sum_{i=1}^{n} \hat{\$}_i \dot{q}_i = 0, \tag{1}$$

where a screw axis $\$_i$ can be expressed by the corresponding normalized axis $\hat{\$}_i$ and its magnitude $\dot{q}_i$.

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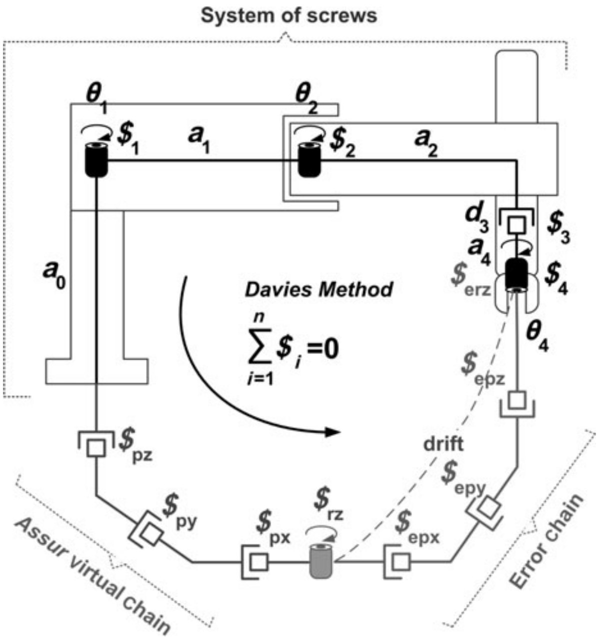

Fig. 1. System of screws.

The normalized screw axis is defined as

$$\hat{\$} = \begin{bmatrix} \hat{s} \\ s_0 \times \hat{s} + h_s \end{bmatrix}, \tag{2}$$

where s_0 is the position vector of the screw in relation to reference system, $\hat{s}$ is the direction of screw axis, and h_s is the screw step.

Joint topology defines the screw step as

$$\hat{\$} = \begin{cases} \begin{bmatrix} \hat{s} \\ s_0 \times \hat{s} \end{bmatrix} & \begin{cases} \text{for a} \\ \text{revolute joint} \\ h_s = 0 \end{cases} \\ \begin{bmatrix} 0 \\ \hat{s} \end{bmatrix} & \begin{cases} \text{for a} \\ \text{prismatic joint} \\ h_s = \infty \end{cases} \end{cases}. \tag{3}$$

However, robot manipulators are usually open kinematic chains, i.e., it is not possible to apply the Davies method. To close the chain, Campos *et al.*[12] presented the concept of virtual chains of Assur, i.e., the inclusion of virtual kinematic chains. As a result, it is possible to control and to impose movements to the kinematic chain. However, during implementation of the interactive procedure, error can occur that causes the chain to open, and consequently, the solution degenerates. A similar phenomenon occurs in the algorithm presented by Siciliano *et al.*[1] and is called a "drift." As a result and due to the integration procedure, the locations of the joint angle end effectors are different than the desired pose. Simas *et al.*[13] employ virtual error chains, which prevent the chain from opening and solve this issue.

The initial aim of this paper is to clarify the problem of chain opening using a trajectory-tracking example and to present the solution using the dual-quaternionic approach. This method is employed using a classic SCARA manipulator (shown in Fig. 1), which is composed of three rotary joints and one prismatic joint. The manipulator has the following physical characteristics: the height of the base (d_0), link length 1 (a_1), link length 2 (a_2), and the length of the wrist (d_4). The joint variables are the

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angular displacement 1 (θ_1), angular displacement 2 (θ_2), prismatic joint displacement (d_3), and the wrist joint displacement (θ_4).

The relative speed formulation, according to Eq. (1), can be subdivided into small plots of the primary joints, i.e., the virtual chain of Assur; the secondary joints, i.e., the manipulator; and the error chain, which results in

$$\overbrace{[\$_1 \ \ \$_2 \ \ \$_3 \ \ \$_4][\dot{q}_1 \ \ \dot{q}_2 \ \ \dot{q}_3 \ \ \dot{q}_4]^T}^{N_s(\overline{q}_s)\dot{q}_s} + \overbrace{[-\$_{e_{rz}} \ -\$_{e_{pz}} \ \ -\$_{e_{py}} \ \ -\$_{e_{px}}][\dot{q}_{e_{rz}} \ \ \dot{q}_{e_{pz}} \ \ \dot{q}_{e_{py}} \ \ \dot{q}_{e_{px}}]^T}^{N_e(q_e)\dot{q}_e} \\ + \overbrace{[-\$_{r_z} \ -\$_{p_x} \ \ -\$_{p_y} \ \ -\$_{p_z}][\dot{q}_{r_z} \ \ \dot{q}_{p_x} \ \ \dot{q}_{p_y} \ \ \dot{q}_{p_z}]^T}^{N_p(q_p)\dot{q}_p} = 0, \tag{4}$$

or simply

$$N_p(q_p)\dot{q}_p + N_s(\overline{q}_s)\dot{q}_s + N_e(q_e)\dot{q}_e = 0, \tag{5}$$

where N_p is the sub-matrix that represents the screws of primary joints on the virtual chain of *Assur*, N_s is the sub-matrix that represents the screws of the secondary joints, i.e., the manipulator, N_e corresponds to the screws of error chain, and $\dot{q}_p$, $\dot{q}_s$, and $\dot{q}_e$ are the magnitude vectors of the primary joints velocity, the secondary joint velocity, and the error, respectively.

Therefore, as in ref. [13], the secondary joint velocities may be obtained from Eq. (5) through the rearrangement and integration of the solution

$$q_s(t_{i+1}) = q_s(t_i) - \Big(N_s^{-1}\big(q_s(t_i)\big)N_e\big(q_e(t_i)\big)K_e q_e(t_i)\Big)\Delta t, \tag{6}$$

where K_e is a positive-definite gain matrix.

This methodology is based on the use of an interactive procedure that corrects for the chain opening error. Furthermore, this methodology employs a feedback loop for the operational pose in space, which requires an orientation representation. The error chain $q_e(t_i)$ is a direct association of the minimal representation error, which is based on the position and orientation in the operational space that is represented by six independent variables. This representation expresses the movement by the degrees of freedom of the operational space, is called the minimal representation, and is defined as

$$k(q) = x_{ee} = \begin{bmatrix} \phi_e \\ p_e \end{bmatrix} = \left[\phi_x; \phi_y; \phi_z; p_x; p_y; p_z\right]^T, \tag{7}$$

where x_{ee} is the end effector description in the operational space, p_e is the position vector in the operational space, and ϕ_e is the orientation vector in the operational space.

The error chain is obtained from the difference between the trajectory in the operational space and the mapping in this space of the inverse kinematics result or simply

$$\tilde{x}_{ee} = x_{ee_d} - x_{ee}. \tag{8}$$

The minimal representation of the mechanism pose is obtained through the kinematic transformation. The position is obtained directly from the position vector that represents the transformed object. The orientation angles need to be extracted from the composite transformation that represents the kinematics of the mechanism. However, the extraction of the Euler angles from the HTMs commonly introduces singularities (see refs. [1], [14] and [15] for more information). The occurrence of kinematic singularities degenerates the mobility of the system and may compromise the primary objective.

The object of this study is a SCARA manipulator that must follow a triangular, planar trajectory, and the transition between the points is performed using a trapezoidal speed profile, as described in Fig. 2. The trajectory is specified in the regions of kinematic singularities. A more detailed discussion about singularity analysis of this manipulator can be seen in refs. [16] and [17]. In the second sector of the trajectory, the manipulator approaches a singular configuration, and peaks are generated in the

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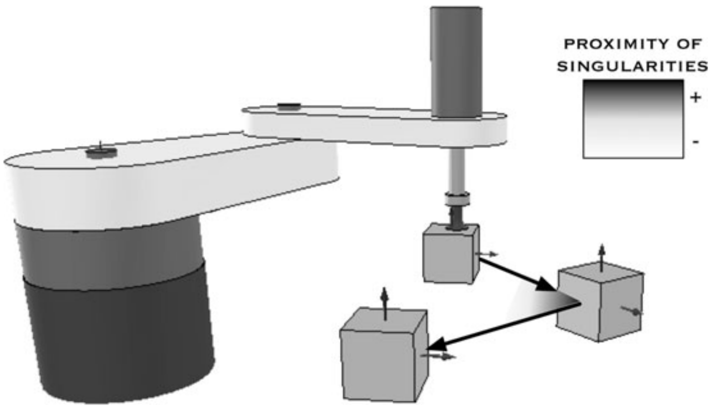

Fig. 2. Triangular trajectory in the operational space.

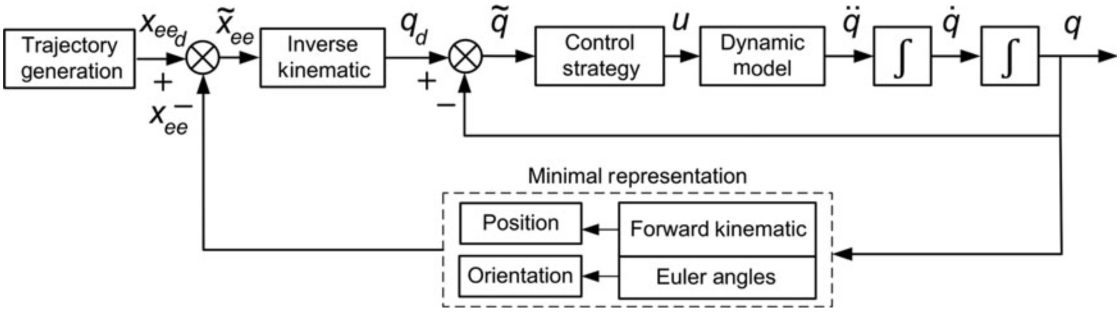

Fig. 3. Dynamic simulation environment.

joint space to ensure the stability of the movement. When the manipulator enters these regions, a loss of mechanism mobility occurs.

The control strategy employed is a proportional derivative (PD) in the joint space, but the trajectory is specified in terms of the operational space. Therefore, the inverse kinematics convert the points in the trajectory reference signal to the controller, and the forward kinematics convert the end effector position measurements in the feedback signals from the controller. The simulation structure is described in Fig. 3.

The forward kinematics using the Denavit–Hartenberg convention for this robot are

$$H_4^0 = \begin{bmatrix} R_a & R_b & 0 & L_x \\ R_c & R_d & 0 & L_y \\ 0 & 0 & -1 & L_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{where} \quad \begin{array}{l} R_a = c_{124} + s_{124} \\ R_b = -c_{12}s_4 + s_{12}c_4 \\ R_c = s_{12}c_4 + c_{12}s_4 \\ R_d = -s_{124} - c_{124} \end{array} \quad \text{and} \quad \begin{array}{l} L_x = a_1c_1 + a_2c_{12} \\ L_y = a_1s_1 + a_2s_{12} \\ L_z = a_0 - d_3 - a_4 \end{array}, \tag{9}$$

where $c_{124} = \cos(\theta_1 + \theta_2 + \theta_4)$ and $s_{124} = \sin(\theta_1 + \theta_2 + \theta_4)$.

The control strategy requires the rearrangement of the dynamic model to isolate the joint accelerations, which introduces the inversion of the inertia matrix $B(q)$ and results in

$$\ddot{q} = B(q)^{-1}\Big(\tau - \big(C(q, \dot{q})\dot{q} + F_v\dot{q} + g(q)\big)\Big), \tag{10}$$

where $B(q)$ is the inertia matrix, $C(q, \dot{q})$ are the contributions of centrifugal and Coriolis forces, F_v: is the diagonal matrix of viscous friction coefficients, $g(q)$ are the gravitational terms, and τ is a generalized forces vector.

The classic dynamic model of a manipulator in the joint space is dependent on the joint variables and its derivatives. In the singular regions, the inertia matrix loses its full rank and becomes non-invertible with a null determinant through its direct influence of the angle of joint 2 in the main

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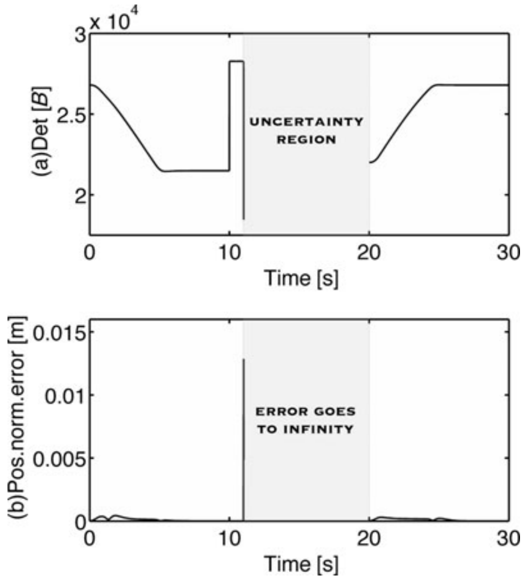

Fig. 4. Analysis of the singularity occurrence in trajectory tracking using the HTM method; (a) $Det[B]$ profile, and (b) normalized error in end effector pose.

diagonal. It is possible to identify the occurrence of kinematic singularities by the behaviour of inertia matrix.

The dynamic analysis of the manipulator and their physical parameters were based on the specifications given by the authors of ref. [18]. The extraction of the orientation angles for the dynamic determination of the manipulator is achieved from the HTM. Using this method, is it possible to monitor the inertia matrix $B(q)$, which needs to be inverted depending on the pose (position and RPY angles). This matrix has singularities, i.e., it becomes a singular matrix and is non-invertible; consequently, the determinant is zero. Figure 4 shows the trajectory tracking. The occurrence of singularities is demonstrated through the use of the HTMs and leads to an indeterminate region, which prevents a full execution of the trajectory using this method.

A new method for movement representation, which avoids the singularity occurrence, is presented in this work. However, the dual-quaternionic elements are defined using a Clifford algebra, which is discussed below.

3. Clifford Algebra

One of the initial studies in the development of Clifford algebra was published by Sir William R. Hamilton and presents the generalization of the complex numbers in a system called quaternions. The quaternions are appropriate objects to describe the three-dimensional space transformations, e.g., the rotations.

Later, the term biquaternion was introduced by Hamilton to designate the quaternion with seven complex terms. These entities have the form $q + \omega r$, where q and r are the usual quaternions. The function of the operator ω has become "unclear," but the correct definition is essential for structuring the algebra of this element. Nevertheless, in ref. [19], a paper titled "On the three types of complex number and planar transformations" was presented, and three types of complex numbers are defined in the geometric and the kinematic context. The difference between the elements is the behavior of the square operator ω.

From an algebraic point of view, each type of complex number is an ordered pair (a, b) of real numbers with different rules for their multiplication. By expanding this definition for the ordered pairs (r, s), the classical Hamilton quaternions ($q = a_1 + a_2 i + a_3 j + a_4 k$) have the following sets:

- *Hypercomplex number or Octonion* $(r + ls)$, where $l^2 = -1$;
- *Dual quaternion* $(r + \varepsilon s)$, where $\varepsilon^2 = 0$;
- *Double quaternion* $(r + ds)$, where $d^2 = +1$.

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The Clifford algebra allows for the generalization of the Hamilton quaternions and biquaternions. In this context, it is possible to examine the consequences of the multiplication rule's influence on operator ω using an algebraic operation. This operation is an associative algebra generated by a number of base elements $e_1, e_2, \ldots, e_n$.

The definition of the algebric properties is based on the multiplication rule of the elements. The Clifford algebra is defined as $Cl(p, q, r)$, where p is the number of generator bases whose square is equal to 1, q is related to the number of equal squares that equal -1, and r is the number of squares equal to 0.

After discussing the Clifford algebras and the main arithmetic operations, we will explore the algebras that define the Hamilton quaternions to expand the dimensions and obtain the Clifford biquaternions.

4. Biquaternions and Dual Quaternions

The Clifford biquaternions are the result of coupling between two traditional Hamilton quaternions. These elements are defined according to their characteristic operator ω, which defines the algebra. Therefore, a biquaternion can be classified as having a hypercomplex number, or an octonion, as a double quaternion or as a dual quaternion, as discussed above. A biquaternion is considered a dual quaternion when the square of operator ω is zero, which results in the operator being represented by ε. A dual quaternion (DQ) is defined as

$$z = q_1 + q_2\varepsilon, \tag{11}$$

where q_1 and q_2 are two classic Hamilton quaternions.

These elements can also be represented by the rigid body movements in space. However, those transformations can be applied in different geometric elements, as seen in ref. [20]. The transformation of a dual quaternion is given by

$$\xi' = h\xi h^*, \tag{12}$$

where ξ is the geometric element, h is the transformation that respects the condition $hh^* = 1$, and $h^* = g^* + \frac{1}{2}tg^*e$ is the conjugated of h.

The basic geometric transformations of a spatial rigid body can be represented by biquaternions, which include the linear movements. In this way, the biquaternion that represents the rigid body motion in space is defined as

$$h = g + \frac{1}{2}tg\varepsilon, \tag{13}$$

where g represents the rotational portions using a unit quaternion and t represents the linear movement through a pure quaternion.

From a quaternionic viewpoint, the rotation is defined by a unit quaternion, e.g., $g = a_0 + a_1 i + a_2 y + a_3 k$, and the linear movement is represented as a pure quaternion, i.e., $t = b_1 i + b_2 j + b_3 k$. The biquaternion transformation (Eq. (13)) can be expanded and generates

$$\begin{aligned} h = {} & (a_0 + a_1 i + a_2 y + a_3 k) + \frac{1}{2}(a_0 b_1 - a_2 b_3 + a_3 b_2) i\varepsilon + \frac{1}{2}(a_0 b_2 + a_1 b_3 - a_3 b_1) j\varepsilon \\ & + \frac{1}{2}(a_0 b_3 - a_1 b_2 + a_2 b_1) k\varepsilon - \frac{1}{2}(a_1 b_1 + a_2 b_2 + a_3 b_3)\varepsilon. \end{aligned} \tag{14}$$

Just like the quaternion, the biquaternion may be represented using Clifford algebra. One algebraic expression that defines the dual quaternions is $Cl^+(0, 3, 1)$. This algebra is employed using four generators: three generators $\{e_1, e_2, e_3\}$ whose square is -1 and a generator $\{e\}$ whose square is zero. In this even algebra (Cl^+), the bases are associated to generate elements of only an even grade, which results in seven complex elements: six elements of grade 2, $\{e_1e_2, e_2e_3, e_3e_1, e_1e, e_2e, e_3e\}$, and one element of grade 4, $\{e_1e_2e_3e\}$.

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The element of this algebra that represents the rigid body transformation in space is similar to that defined in Eq. (13), but the operator is replaced by generator e, which results in

$$h = g + \frac{1}{2}tge. \tag{15}$$

Therefore, the rotation transformation (R_ϕ) is a unit quaternion and is expressed as

$$g = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)[e_2e_3 + e_3e_1 + e_1e_2], \tag{16}$$

where transformation g is also equal to $-g$, i.e., $(-g)\xi(-g^*) = g\ \xi\ g^*$.

The translational component is a pure quaternion and is defined in the following algebra:

$$t = xe_1 + ye_2 + ze_3. \tag{17}$$

The conjugation of the transformation elements in this algebra is solely given by the conjugation of the rotational components. For example, the conjugate of the composed transformation element (defined in Eq. (15)) is given by

$$h^* = g^* + \frac{1}{2}tg^*e. \tag{18}$$

The rotational component g is a unit quaternion and is defined in this algebra as $g = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2$, and the translational component t is a pure quaternion, which is defined in this algebra by $t = b_1e_2e_3 + b_2e_3e_1 + b_3e_1e_2$. The transformation defined in Eq. (15) can be expanded and it generates

$$h = (a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2) + \frac{1}{2}(a_0b_1 - a_2b_3 + a_3b_2)e_1e + \frac{1}{2}(a_0b_2 + a_1b_3 - a_3b_1)e_2e$$
$$+ \frac{1}{2}(a_0b_3 - a_1b_2 + a_2b_1)e_3e + \frac{1}{2}(a_1b_1 + a_2b_2 + a_3b_3)e_1e_2e_3e. \tag{19}$$

The equivalence between the elements of the Clifford algebra $Cl^+(0, 3, 1)$ and the dual quaternions can be verified by comparing Eqs. (14)–(19), which results in the following relations:

$$\begin{array}{ccc} i = e_2e_3, & j = e_3e_1, & k = e_1e_2, \\ i\varepsilon = e_1e, & j\varepsilon = e_2e, & k\varepsilon = e_3e, \\ \text{and} & \varepsilon = -e_1e_2e_3e. & \end{array} \tag{20}$$

The expansion of the algebra provides a new representation point with an element of grade 3, which permits the application of the relational operations between the geometric elements (ξ). A new point definition, in algebra $Cl(0, 3, 1)$, also respects the quadratic condition $pp^* = 1$ and is defined as

$$p = e_1e_2e_3 + (xe_2e_3e + ye_3e_1e + ze_1e_2e). \tag{21}$$

In a geometric transformation composed and represented by a dual quaternion according to Eq. (15), the spatial position of the element is obtained by applying this transformation on a base element, as defined in Eq. (12). The minimal representation of a rigid body orientation in space is performed in a standardized way regardless of the manipulator topology. Unlike extraction of the angles from the HTM, some mathematical articles based on the function $Atan_2^{-1}$ are applied to determine these angles, e.g., refs. [21] and [22].

However, the rigid body orientation is extracted from the biquaternion directly. The Z-Y-X Euler angles, also called the Roll(φ)–Pitch(ϑ)–Yaw(ψ) of the dual quaternion of transformation

$h = a_0 + a_1e_2e_3 + a_2e_3e_1 + a_3e_1e_2 + a_4e_1e_2e_3e + a_5e_1e + a_6e_2e + a_7e_3e$, are obtained as

$$\psi = 2\sin^{-1}(A_1), \quad \vartheta = 2\sin^{-1}(A_2), \quad \text{and} \quad \varphi = 2\sin^{-1}(A_3), \tag{22}$$

where A_1, A_2, and A_3 represent the angles a_1, a_2, and a_3 in degrees.

A more detailed discussion of the movement representation using dual quaternions is presented in refs. [23] and [24].

After demonstrating the Clifford algebra, which defines the dual quaternions and its properties, a way to represent the spatial transformation of a rigid body will be discussed using the elementary transformation applied in robot kinematics.

5. Dual-quaternionic Approach

The classic direct kinematic calculation is based on the assignment of some intermediate coordinate system, which represents the joint position of a kinematic chain. The transformation of the variables attached to the end effector position and the orientation of the joint variables is achieved through the homogeneous transformation between each coordinate system.

However, the determined position and orientation of the intermediate coordinate systems is not univocal but allows for different solutions for the direct kinematics. There are some methods for direct kinematic systematization. The Denavit–Hartenberg convention is a systematic methodology for the classical determination of the transformations between the rigid bodies. This method determines the forward kinematics of the mechanism, i.e., defines the transformation sequences and not how to accomplish them. These transformations can be represented through the dual quaternions, which are defined by the algebra previously presented.

5.1. D-H convention

The Denavit–Hartenberg (D-H) convention is based on the transformation representation between two links and are defined by four parameters. These parameters are used to mount the composite transformation of the element; each parameter is associated with a particular transformation, i.e., $H_n^{n-1} = R_z(\theta_i)T_z(d_i)T_x(a_i)R_x(\alpha_i)$; and all parameters are associated with a HTM, see ref. [1] for more details. However, the same methodology can be applied to a transformation representation that is associated with these parameters through a dual quaternion

$$h_n^{n-1} = \left(g_z(\theta_\mathbf{n}) + \frac{1}{2}\big(t_z(\mathbf{d_n}) + t_x(\mathbf{a_n})\big)g_z(\theta_\mathbf{n})e\right)g_x(\alpha_\mathbf{n}). \tag{23}$$

In an expanded form, this equation becomes

$$\begin{aligned} h_n^{n-1} = &\Bigg(\overbrace{\left(\cos\left(\frac{\theta_\mathbf{n}}{2}\right)+\sin\left(\frac{\theta_\mathbf{n}}{2}\right)e_1e_2\right)}^{g_z(\theta_\mathbf{n})} + \left(\tfrac{1}{2}\Big(\overbrace{\mathbf{d_n}e_3}^{t_z(\mathbf{d_n})} + \overbrace{\mathbf{a_n}e_1}^{t_x(\mathbf{a_n})}\Big)\right. \\ &\left.\overbrace{\left(\cos\left(\frac{\theta_\mathbf{n}}{2}\right)+\sin\left(\frac{\theta_\mathbf{n}}{2}\right)e_1e_2\right)}^{g_z(\theta_\mathbf{n})} e\right)\Bigg)\overbrace{\left(\cos\left(\frac{\alpha_\mathbf{n}}{2}\right)+\sin\left(\frac{\alpha_\mathbf{n}}{2}\right)e_2e_3\right)}^{g_x(\alpha_\mathbf{n})}, \end{aligned} \tag{24}$$

where $\theta_\mathbf{n}$, $\mathbf{d_n}$, $\mathbf{a_n}$, and $\alpha_\mathbf{n}$ are the D-H parameters and $\mathbf{n}$ is the link number.

The D-H dual-quaternionic methodology was applied to the same SCARA manipulator from the previous example. The transformations between the intermediate coordinate systems of the mechanism have the same rotational influence and are defined as

$$g_{n=1}^{3} = \cos\left(\frac{\theta_n}{2}\right)+\sin\left(\frac{\theta_n}{2}\right)e_2e_3. \tag{25}$$

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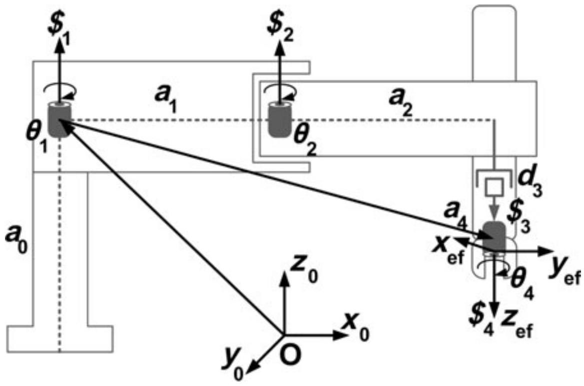

Fig. 5. Method of successive screw displacements applied on a SCARA manipulator.

The translation influences are

$$\begin{aligned} t_1 &= a_1\cos(\theta_1)e_1 + a_1\sin(\theta_1)e_2 + (a_0 - d_3 - a_4)e_3, \\ t_2 &= a_2\cos(\theta_2)e_1 + a_2\sin(\theta_2)e_2, \\ t_3 &= 0. \end{aligned} \tag{26}$$

5.2. Method of successive screw displacements

An alternative to the D-H convention is a method of successive screw displacements and is presented in ref. [14]. This methodology is based on the use of a differential geometry element: the screw. This element consists of a directed line called the axis and a scalar parameter with a unit length, i.e., the screw step. Both the axis and the step are employed to represent the composition of the translational and rotational motions simultaneously, which results in a helical movement.

The method of successive screw displacements is a methodology for the kinematic analysis of open chains through the use of a screw. One particularity of this method is the necessity to use only a fixed reference coordinate system and a coordinate system of the end effector, which is unlike the D-H convention that employs a coordinate system for each joint, as described in Fig. 5.

The initial procedure in this method is to determine the reference manipulator position. This position can be selected arbitrarily. However, it is advised to choose a known configuration where the joint angles can be easily determined, e.g., the setting where all joint angles are zero. The position and the orientation of joints axes also need to be determined, which is accomplished by placing a screw in each joint axis to determine the direction of screw axis (s_i), i.e., the joint axis, and screw axis position (s_0), i.e., the joint location. It is possible to mount the transformations associated with each screw with the desired manipulator position expressed in terms of end effector position. The coupling of these influences generates the direct kinematics defined as

$$h_n = \left(g_{\mathbf{s_i}}(\theta_{\mathbf{n}}) + \frac{1}{2}\big(t_x(\mathbf{q_x}) + t_y(\mathbf{q_y}) + t_z(\mathbf{q_z})\big)g_{\mathbf{s_i}}(\theta_{\mathbf{n}})e\right), \tag{27}$$

where g_{s_i} is the rotational component of the screw displacement, and q_x, q_y, and q_z are the translational components of screw displacement.

This equation can also be expressed in an expanded form:

$$\begin{aligned} h_n = &\overbrace{\left(\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)[\mathbf{s_x}e_2e_3 + \mathbf{s_y}e_3e_1 + \mathbf{s_z}e_1e_2]\right)}^{g_{\mathbf{si}}(\theta_{\mathbf{n}})} + \frac{1}{2}\Big(\overbrace{q_xe_1}^{t_x(\mathbf{q_x})} + \overbrace{q_ye_2}^{t_y(\mathbf{q_y})} + \overbrace{q_ze_3}^{t_z(\mathbf{q_z})}\Big) \\ &\overbrace{\left(\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)[\mathbf{s_x}e_2e_3 + \mathbf{s_y}e_3e_1 + \mathbf{s_z}e_1e_2]\right)}^{g_{\mathbf{si}}(\theta_{\mathbf{n}})}e, \end{aligned} \tag{28}$$

where $s_i = (s_x, s_y, s_z)$ is screw direction vector.

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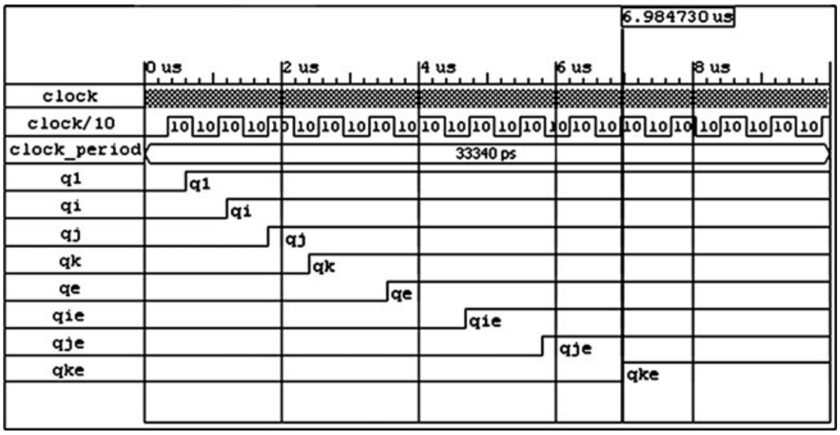

Fig. 6. Analysis of the computational performance of the multiplication of two DQs.

To illustrate the dual quaternions application and to determine the transformation between the screw systems, this methodology is applied for the previous example, as described in Fig. 5. The initial pose of the system is defined as $p = 1 + (a1 + a2)e_1e$.

The transformations between the screws results in the same rotation when using the D-H methodology, defined in Eq. (25). However, the translational components are given by

$$\begin{aligned} t_1 &= (a_0 - d_3 - a_4)e_3 \\ t_2 &= a_1(1 - \cos(\phi_2))e_1 - a_1\sin(\phi_2)e_2 \\ t_3 &= (a_1 + a_2)(1 - \cos(\phi_4))e_1 \\ &\quad -(a_1 + a_2)\sin(\phi_4)e_2 \end{aligned} \quad . \tag{29}$$

Transformations can also be analyzed in terms of the computational performance, which will be discussed below.

6. Computational Performance

The analysis of computational performance methods is achieved by implementing the methodologies in a dedicated digital signal processor. With the objective to evaluate the execution time in both transformation methods, the authors of ref. [25] present the computational analysis of the screw transformation using different methods (including the dual quaternions). However, the actual computational resources allow scaling and parallel processing and require a different analysis.

For this purpose, a high-performance DSP with a RISC CPU and 16-bit resolution that was able to execute 30 MIPS was used. This processor performs multiply-accumulate operations in one clock cycle due to a dedicated peripheral unit. The first test conducted was the multiplication of two dual quaternions to determine the execution time of this operation, as described in Fig. 6, which resulted in a time range of 6.98 μs, or 209 clock cycles.

To compare the performance of both methods, a classic anthropomorphic (6 degrees of freedom) arm for forward kinematics determination with the assignment of the coordinate systems using the D-H convention was used. In the first case, HTMs were used to perform transformations, as shown in Fig. 7, and in the second case, the dual quaternions were used, as seen in Fig. 8. The result is that the kinematics using the dual quaternions were determined in 1.343 clock cycles, i.e., 47.78 μs. Processing the HTM was performed in 6.270 clock cycles, i.e., 209.02 μs. Consequently, the dual quaternions have a lower computational cost than the HTMs.

The direct kinematics for both transformation methods can be expressed in terms of the mathematical operations, e.g., addition, subtraction, and multiplication. The computational performance is also influenced by the memory access operations, and this influence is primarily the writing process that can take up to 8 clock cycles. The comparison between the operations carried out by dual quaternions and the HTMs are described in Table I. The dual quaternions require fewer mathematical operations and memory accesses to perform the transformations. Performance is

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Table I. Comparison of the mathematical operations between the DQs and the HTM.

	Addition/ Subtraction	Multiplication	Write memory	Read memory
Multiplication of HTM	180	216	36	432
Multiplication of DQ	40	48	8	96
Kinematic by HTM	930	1116	186	2664
Kinematic by DQ	280	336	56	672

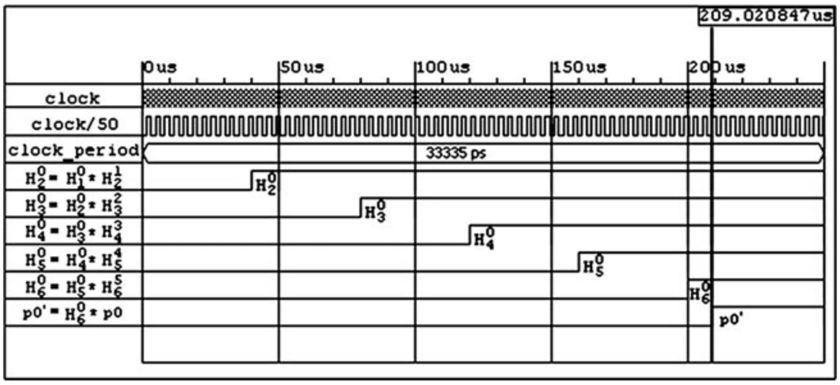

Fig. 7. Analysis of the computational performance of the transformation using the HTMs.

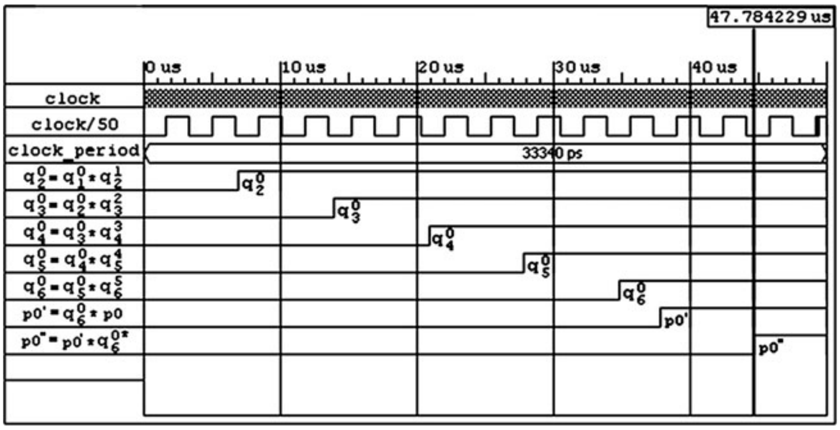

Fig. 8. Analysis of the computational performance of the transformation using the DQs.

improved with the dual quaternions because of the characteristic of the dual element ($\varepsilon^2 = 0$) which simplifies the multiplication operation.

7. Singularity-free Trajectory Tracking

To prove the potential of the dual quaternions in kinematics and control, the same environment for the previous example for trajectory tracking is used. Furthermore, the movement representation is performed using the dual-quaternionic elements.

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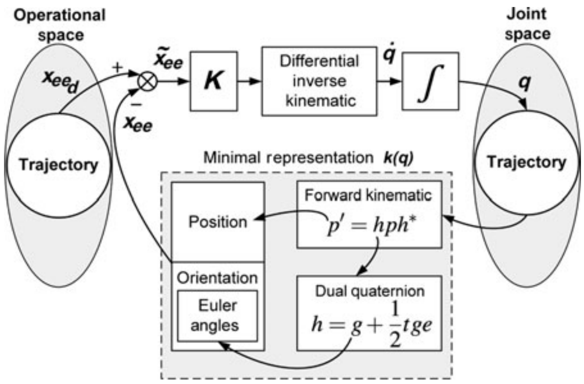

Fig. 9. Dual-quaternionic feedback.

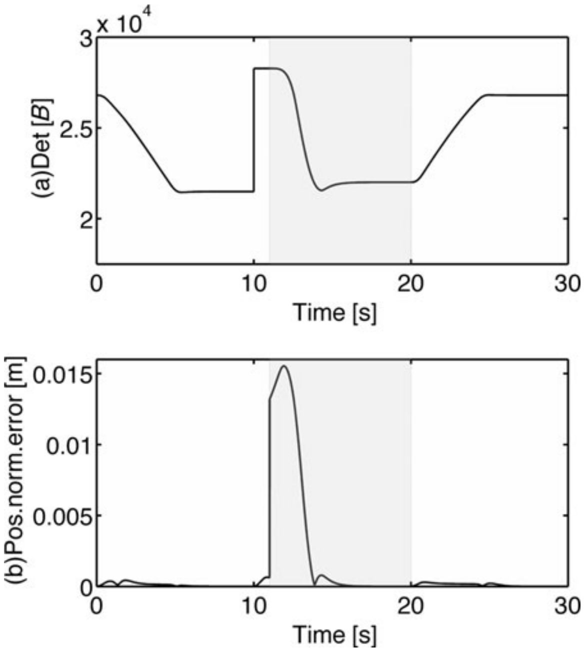

Fig. 10. Analysis of the singularity occurrence in trajectory tracking using the DQ method; (a) $Det[B]$ profile, and (b) normalized error in end effector pose.

In this method, the operational space feedback is performed by the forward kinematic and by the dual quaternions, as shown in Fig. 9. The extraction of the orientation angles is determined without the kinematic singularities, which is unlike the method using the HTMs. The phenomenon of kinematic singularities is not present because this element avoids the occurrence of kinematic singularities but in these regions the motion is kinematically disturbed. The use of dual quaternion allows for full trajectory tracking, as presented in Fig. 10.

The graphic of the controller output for the trajectory execution without the occurrence of singularities is presented in Fig. 11, and the correct trajectory tracking is demonstrated. n some transitions, the manipulator approximates of kinematic singularities, which results in an acceleration peak for the quickly adaptation of pose and stabilization the trajectory tracking. For example, a saturation and velocity profile to promote a realistic dynamic simulation is imposed.

In this case study, the dual quaternions allow for manipulation in regions of kinematic singularities unlike the HTMs. However, errors are added to the trajectory tracking to maintain the movement stability. These effects are a consequence of the imposition of torque peaks for a quick change of pose.

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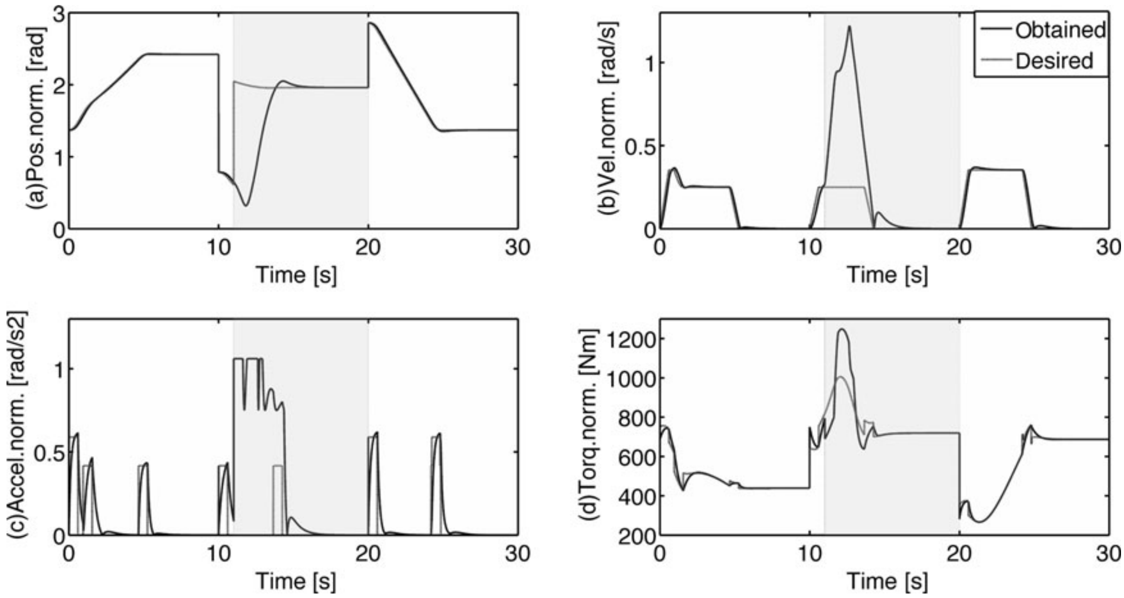

Fig. 11. (a) Normalized joint position profiles, (b) normalized joint velocity profiles, (c) normalized joint acceleration profiles, and (d) normalized joint torque profiles.

8. Conclusion

This paper presented a methodology for using quaternionic elements in a kinematic analysis. The use of quaternionic transformations is related to a non-occurrence of kinematic singularities, but the restrictions require the expansion of the dimensions and, hence, the employment of dual quaternions. The use of this element is based on the definition of the Clifford algebra, which expands its operations to translations. This feature increases the possibilities of kinematic analysis and explores the potential of these elements.

The methodology is evaluated in different contexts. In terms of forward kinematics, the dual quaternions were used in the classic D-H methodology and successive screw displacement methods. It is demonstrated that the use of this element requires less computational performance than the HTMs. In the inverse kinematics, the dual quaternions are applied through the method of kinematic restrictions, which is based on the screw displacement. The consequence of this approach is that kinematic singularities do not occur, but solutions for this method in the indeterminate regions of the HTM method do exist. In the redundant systems, these elements minimize the acceleration peaks that are imposed by the kinematic singularities.

The dual quaternions are promising elements for kinematic analysis because they prevent the occurrence of singularities and minimize the acceleration peaks. The spatial transformation is demonstrated with only eight variables, and the orientation is expressed in three independent operational spaces variables. Finally, these elements also require less computational load for the transformation determination.

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