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DEFORMING A STARSHAPED CURVE INTO A CIRCLE BY AN AREA-PRESERVING FLOW

JIANBO FANG

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Abstract

We show that a class of area-preserving flows can deform every starshaped curve into a circle.

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1. Introduction

The curve-shortening flow has received much attention since the 1980s. Gage in [8] and Gage and Hamilton in [10] showed that a convex initial curve remains so and becomes more and more circular during the evolution process until it shrinks to a round point in a finite time. More generally, Grayson in [12] used complex arguments to show that any embedded curve will become convex and so shrinks to a point under the curve-shortening flow. His argument is called *Grayson's theorem*.

Finding analogues of Grayson's theorem for expansion curve flow is also an interesting problem. In 1996, Tsai [17] showed that a starshaped curve eventually becomes convex under the nonhomogeneous expanding curve flow,

$$\frac{\partial X}{\partial t} = F(\kappa)N_{\text{out}},$$

where $F(z) : \mathbb{R} \to \mathbb{R}^+$ is an arbitrary positive smooth decreasing function which satisfies $\lim_{z\to\infty} F(z) = \infty$ and $dF(z)/dz < 0$ and κ is the curvature and N_{out} the unit outward normal vector of the evolving curve X. Chow and Tsai [6] showed that the rescaling convex curve is convergent to the unit circle. Subsequently, the convexity results have been generalised to embedded plane curves with turning angle greater than $-\pi$ (see [5]).

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Based on these results, it is of interest to study the *convexity theorem* under general nonlocal curve flow. In 1984, Gage first introduced and studied the area-preserving flow,

$$\frac{\partial X}{\partial t} = \Big(\kappa - \frac{2\pi}{L}\Big)N, \tag{1.1}$$

where X, L, κ and N are the position function, the length, the curvature and the inner unit normal vector of the evolving curve, respectively. In [9], Gage proved that a convex curve evolving according to (1.1) remains convex and converges to a circle. For recent work on the flow (1.1), see [3]. A natural question is whether, as in the case of the curve-shortening flow, the initial curve can be extended to a general embedded curve. Mayer [15] carried out a numerical experiment and found that the flow (1.1) can develop singularities when the initial embedded curve bends violently. For recent work associated with the convexity theorem for curve flows with a global forcing term, see Dittberner [7].

If the initial curve is starshaped, whether or not the flow (1.1) has long-term existence is still an open problem. Very recently, Gao and Pan [11] further restricted the starshaped condition on the initial embedded curve to centrosymmetric and starshaped, and proved that the flow (1.1) has long-term behaviour. In higher dimensions, Huisken [14] considered the volume-preserving mean curvature flow. In 2015, Guan and Li [13] dealt with the mean curvature flow in space forms for hypersurfaces.

Let γ_0 be an arbitrary starshaped (with respect to the origin of $\mathbb{R}^2$) embedded closed curve parameterised by a smooth embedding $X_0(u)$. We consider the evolving flow,

$$\frac{\partial X}{\partial t} = (p\kappa - 1)N, \tag{1.2}$$

where X, κ and N are the position vector, the curvature and the inner unit normal vector and $p = -\langle X, N\rangle$ is the support function of the evolving curve. The flow (1.2) is different from the famous flows studied by Angenent [1, 2] and Oaks [16], where the speed depends on κ and T (or equivalently N). Here p also depends on X.

Since (1.2) is never parabolic when viewed as a system, we need to fix a parametrisation and express (1.2) as a single parabolic equation. By introducing polar coordinates (r, θ) in $\mathbb{R}^2$, (1.2) for starshaped curves is equivalent to

$$\frac{\partial r}{\partial t} = \frac{r^2 r_{\theta\theta}}{g^3} + \frac{r_\theta^4}{rg^3}. \tag{1.3}$$

The proof of the equivalence is given in Section 2. We obtain the following main result.

THEOREM 1.1. *A curve X_0 starshaped with respect to the origin in the plane evolving under the flow (1.2) remains so, decreases its length and keeps the area which it encloses constant. As time goes to infinity, the curve converges to a circle.*

2. Preparation

Assume that K is an open and starshaped domain with respect to the origin and that the boundary of K is of class C^2. In this situation, $X = \partial K$ is called a *starshaped*

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curve and the support function is defined by $p = -\langle X, N\rangle$. If we express the curve by $X(s) = (x(s), y(s))$, then $T(s) = (\dot{x}(s), \dot{y}(s))$ and $N(s) = (-\dot{y}(s), \dot{x}(s))$. Notice that

$$p(s) = x(s)\dot{y}(s) - \dot{x}(s)y(s) = \det(X(s), T(s)).$$

Therefore, X is starshaped if and only if $p(s) > 0$ for all s. Using the polar coordinate system (r, θ) for the plane, we can express any closed curve as $X(s) = r(s)P(\theta(s))$, where $\theta \in [0, 2\pi]$, r denotes the radial function of X and $P(\theta) = (\cos(\theta), \sin(\theta))$. With $Q(\theta) = (-\sin(\theta), \cos(\theta))$, the unit tangential vector is $T = r_s P + r\theta_s Q$ and the curvature is

$$\kappa = \frac{r^2 + 2r_\theta^2 - rr_{\theta\theta}}{(r^2 + r_\theta^2)^{3/2}}. \tag{2.1}$$

First, by adding a tangential component, we show that the flow remains equivalent but θ can be made constant. Second, we reduce the flow (1.2) to a Cauchy problem of a single equation for the radial function $r = r(\theta, t)$ and we use this to explore some simple properties of the flow (1.2).

Let $g := \sqrt{\langle \partial X/\partial\varphi, \partial X/\partial\varphi\rangle}$ be the metric of the evolving curve. Set $\beta = p\kappa - 1$. Under the flow (1.2), g evolves according to

$$\frac{\partial g}{\partial t} = \frac{1}{g}\Big\langle \frac{\partial}{\partial t}\frac{\partial X}{\partial\varphi}, \frac{\partial X}{\partial\varphi}\Big\rangle = g\Big\langle \frac{\partial}{\partial s}(\alpha T + \beta N), T\Big\rangle = \Big(\frac{\partial\alpha}{\partial s} - \beta\kappa\Big)g.$$

The interchange of the operators $\partial/\partial s$ and $\partial/\partial t$ is given by

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial t}\Big(\frac{1}{g}\frac{\partial}{\partial\varphi}\Big) = \frac{\partial}{\partial s}\frac{\partial}{\partial t} - \Big(\frac{\partial\alpha}{\partial s} - \beta\kappa\Big)\frac{\partial}{\partial s}$$

and we have the evolution equations of T and N,

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\frac{\partial X}{\partial s} = \frac{\partial}{\partial s}\frac{\partial X}{\partial t} - \Big(\frac{\partial\alpha}{\partial s} - \beta\kappa\Big)T = \Big(\alpha\kappa + \frac{\partial\beta}{\partial s}\Big)N,$$
$$\frac{\partial N}{\partial t} = \Big\langle\frac{\partial N}{\partial t}, T\Big\rangle T + \Big\langle\frac{\partial N}{\partial t}, N\Big\rangle N = -\Big(\alpha\kappa + \frac{\partial\beta}{\partial s}\Big)T.$$

If there is a family of starshaped curves evolving under the flow (1.2), then we can describe the evolving curve by

$$X(\theta(t), t) = r(\theta(t), t)P(\theta(t)). \tag{2.2}$$

Equation (2.2) is valid at least for a short time and in Corollary 2.5 we show that the starshaped property is preserved and hence (2.2) is valid for the life of the flow.

Since $X_\theta = r_\theta P + rQ$,

$$g = \Big\|\frac{\partial X}{\partial\theta}\Big\| = \Big(r^2 + \Big(\frac{\partial r}{\partial\theta}\Big)^2\Big)^{1/2}, \quad T = \frac{\partial r}{\partial s}P + \frac{r}{g}Q, \quad N = -\frac{r}{g}P + \frac{\partial r}{\partial s}Q. \tag{2.3}$$

Using (1.2), (2.2) and (2.3),

$$\frac{\partial r}{\partial t}P + r\frac{\partial\theta}{\partial t}Q = \alpha T + \beta N = \Big(\alpha\frac{\partial r}{\partial s} - \frac{r\beta}{g}\Big)P + \Big(\frac{\alpha r}{g} + \beta\frac{\partial r}{\partial s}\Big)Q.$$

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Comparing the coefficients on both sides leads to the evolution equations

$$\frac{\partial r}{\partial t} = \alpha \frac{\partial r}{\partial s} - \frac{r\beta}{g}, \quad \frac{\partial \theta}{\partial t} = \frac{\alpha}{g} + \frac{\beta}{r}\frac{\partial r}{\partial s}.$$

From now on, we choose $\alpha = -g\beta r_s/r = -\beta r_\theta/r$, making the polar angle θ independent of t, that is,

$$\frac{\partial \theta}{\partial t} \equiv 0.$$

From (2.1), (2.2) and (2.3) and the formula

$$p = -\langle X, N\rangle = -\Big\langle rP, -\frac{r}{g}P + \frac{r_\theta}{g}Q\Big\rangle = \frac{r^2}{g},$$

we immediately obtain

$$\frac{\partial r}{\partial t} = \frac{r^2 r_{\theta\theta}}{g^3} + \frac{r_\theta^4}{rg^3}. \tag{2.4}$$

On the other hand, if $r = r(\theta, t) > 0$ is defined on $[0, 2\pi] \times [0, \omega)$ and satisfies (2.4), then the family of curves $\{X = rP \mid t \in [0, \omega)\}$ satisfies the flow (1.2). So, we can reduce the flow (1.2) to (1.3) with initial value $r_0(\theta) > 0$.

Lemma 2.1. *Suppose that X_0 is starshaped with respect to the origin. The flow (1.2) is equivalent to (1.3) with a positive initial value $r_0(\theta)$ in some interval $[0, \omega)$.*

Since $L(T) = \int_0^{2\pi} g(\theta, t)\, d\theta = \int_0^{2\pi} \sqrt{r^2 + (r_\theta)^2}\, d\theta$, one can define an operator F from the space $C^{2,\alpha}([0, 2\pi] \times [0, \omega))$ to $C^\beta([0, 2\pi] \times [0, \omega))$, for $0 < \beta < \alpha \le 1$, by

$$F(r) = \frac{\partial r}{\partial t} - \frac{r^2 r_{\theta\theta}}{g^3} - \frac{r_\theta^2}{rg^3}.$$

The Frechet derivative of F at some point $r_0 > 0$ is

$$DF(r_0)u = F(r) = \frac{\partial u}{\partial t} - \frac{1}{\sqrt{r_0^2 + (\partial r_0/\partial\theta)^2}}\frac{\partial^2 u}{\partial \theta^2} + \text{lower derivatives of } u,$$

so (2.4) is uniformly parabolic near its initial value r_0. The implicit function theorem for Banach spaces implies that the Cauchy problem (2.4) has a unique solution in some small time interval. So, Lemma 2.1 gives the short-time existence.

Lemma 2.2. *The flow (1.2) has a unique solution in some time interval $[0, \omega)$ for $\omega > 0$.*

Next, we use the maximum principle to derive C^0 and C^1 estimates for the radial function r.

Lemma 2.3. *Given a starshaped curve X_0 with respect to the origin, under the flow (1.2), the radial function r is uniformly bounded from above and below.*

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Proof. From (2.4), at critical points,

$$\frac{\partial r}{\partial t} = \frac{r_{\theta\theta}}{r}.$$

Thus, the maximum principle yields the desired result. □

As mentioned at the beginning of Section 2, X is starshaped if and only if $p = r^2/\sqrt{r^2 + r_\theta^2} > 0$. Thus, we need to bound r_θ.

Lemma 2.4. *Let X_0 be a starshaped curve with respect to the origin. If $r(\theta, t)$ is the solution of (2.4) in a time interval $[0, T]$, then*

$$|r_\theta| \le C,$$

where C is a constant which depends only on the initial curve.

Proof. Tedious computation yields the evolution equation of r_θ,

$$(r_\theta)_t = \frac{r^2}{g^3} r_{\theta\theta\theta} + \frac{(6r^2 r_\theta^3 + r_\theta^5 - r^4 r_\theta - 3r^3 r_\theta r_{\theta\theta}) r_{\theta\theta}}{r g^5} - \frac{4r^2 r_\theta^5 + r_\theta^7}{r^2 g^5}.$$

With $h = r_\theta^2/2$, one obtains $h_\theta = r_\theta r_{\theta\theta}$, $h_{\theta\theta} = r_{\theta\theta}^2 + r_\theta r_{\theta\theta\theta}$ and

$$\begin{aligned} h_t &= r_\theta (r_\theta)_t = r_\theta (r_t)_\theta \\ &= \frac{r^2 h_{\theta\theta}}{g^3} - \frac{r^2 r_{\theta\theta}^2}{g^3} + \frac{(6r^2 r_\theta^3 + r_\theta^5 - r^4 r_\theta - 3r^3 r_\theta r_{\theta\theta}) h_\theta}{r g^5} - \frac{32 r^2 h^3 + 16 h^4}{r^2 g^5}. \end{aligned} \tag{2.5}$$

Suppose that h attains a maximum value at the point (θ^*, t^*). At the critical point,

$$h_t \le -\frac{32 r^2 h^3 + 16 h^4}{r^2 g^5} \le 0.$$

From the parabolic maximal principle, $r_\theta^2 \le C^2 = \max r_\theta^2(\theta, 0)$, which completes the proof. □

Lemmas 2.3 and 2.4 yield the following result.

Corollary 2.5. *The evolving curve $X(\theta, t)$ remains starshaped with respect to the origin of $\mathbb{R}^2$.*

3. Long-term existence

In this section, we show that the flow (1.2) exists in the time interval $[0, \infty)$ if the initial curve is starshaped. In order to prove the long-term existence, we need to estimate the second derivative $r_{\theta\theta}$, which depends only on the initial curve. Although it is hard to estimate $r_{\theta\theta}$ from the evolution equation directly, we can do so from the curvature estimate.

Lemma 3.1. *Under the flow (1.2), the function $r_{\theta\theta}$ has uniform upper and lower bounds.*

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PROOF. The curvature evolution equation under the flow (1.2) is given by

$$\kappa_t = p_{ss}\kappa + 2p_s\kappa_s + p\kappa_{ss} + p\kappa^3 - \kappa^2. \tag{3.1}$$

From $p = -\langle X, N\rangle$,

$$p_s = \kappa\langle X, T\rangle, \quad p_{ss} = \kappa + \kappa_s\langle X, T\rangle - p\kappa^2.$$

Then (3.1) can be written as

$$\kappa_t = p\kappa_{ss} + 3\kappa\kappa_s\langle X, T\rangle.$$

Since p is uniformly bounded above and below, the parabolic maximum principle tells us that the curvature κ has uniform upper and lower bounds. The lemma follows from (2.1) and Lemmas 2.3 and 2.4. □

By the classical theory of parabolic equations, the higher regularity of the solution associated to (2.4) comes from the uniform gradient estimates in Lemmas 2.4 and 3.1. Moreover, the solution for the radial function $r(\cdot, t)$ exists for all times $t \in [0, \infty)$.

4. The final shape and convergence

In this section, we show that the area-preserving flow (1.2) can deform every smooth, closed and starshaped curve into a circle.

From the equations of [4],

$$\frac{\partial A}{\partial t} = -\int_0^L (p\kappa - 1)\, ds = -L + L = 0.$$

Since

$$L^2 \le \int p\, ds \int p\kappa^2\, ds = 2A \int p\kappa^2\, ds,$$

it is easy to see that

$$\frac{\partial L}{\partial t} = -\int_0^L p\kappa^2\, ds + 2\pi \le -\frac{L^2 - 4\pi A}{2A} \le 0.$$

Thus,

$$\frac{d}{dt}(L^2 - 4\pi A) = -\frac{L}{A}(L^2 - 4\pi A) \le 0.$$

If a family of starshaped curves evolves under the flow (1.2), then the area enclosed by X is invariant and the length L is decreasing. The classical isoperimetric inequality gives the following lemma.

LEMMA 4.1. *Under the flow (1.2), the area of the evolving curve $A(t) \equiv A_0$ and the length L satisfies $4\pi A_0 \le L^2(t) \le L_0^2$. Furthermore, the isoperimetric deficit $L^2 - 4\pi A$ is decreasing, which means that the evolving curve becomes more and more circular.*

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Next, we use the comparison principle to obtain the C^∞ convergence.

THEOREM 4.2. *The flow (1.2) evolves any starshaped curve into a circle in the sense of C^∞ topology.*

PROOF. From Lemmas 2.3 and 2.4, (2.5) can be written as

$$h_t = c_1 h_{\theta\theta} - c_1 r_{\theta\theta}^2 + (c_2 - c_3 h_\theta) h_\theta - c_4 h^2,$$

where the c_i $(i = 1, 2, 3, 4)$ are positive constants.

Consider $\tilde{h}(t) = 1/c_4(t - \tilde{c}_4)$, which satisfies $\tilde{h}_t = -c_4\tilde{h}^2$ and $\lim_{t\to\infty} \tilde{h}(t) = 0$. By a comparison of ordinary differential equations, $h(t) \leq \tilde{h}(t)$. Since $h(t) = r_\theta^2/2$,

$$\lim_{t\to\infty} \frac{r_\theta^2(\theta, t)}{2} = 0.$$

This means that as time t goes to infinity,

$$\lim_{t\to\infty} r(\theta, t) = \text{constant},$$

which is the required result. □

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JIANBO FANG, School of Mathematics and Statistics,
Guizhou University of Finance and Economics,
Guiyang 550025, PR China
e-mail: fjbwcj@126.com