On the Routh-Steiner theorem and some generalisations

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1. *Introduction*

Coxeter in his book [1, p. 211], considered the following theorem of affine geometry;

Theorem 1: If the sides BC, CA, AB of a triangle ABC are divided at L, M, N in the respective ratios $\lambda : 1$, $\mu : 1$, $\nu : 1$, the cevians AL, BM, CN form a triangle whose area is

$$\frac{(\lambda\mu\nu - 1)^2}{(\lambda\mu + \lambda + 1)(\mu\nu + \mu + 1)(\nu\lambda + \nu + 1)}$$

times that of ABC.

He emphasised that this result was discovered by Steiner, but simultaneously cited two references: the first was Steiner's work [2, pp. 163-168] and the second was Routh's work [3, p. 82]. Later in his book he referred to the result as 'Routh's theorem' [1, p. 219], admitting to the contribution of both scientists in revealing the theorem.

Coxeter gave a general proof of this result using barycentric coordinates attributed to Möbius. These are homogeneous coordinates (t_1, t_2, t_3), where t_1, t_2, t_3 are masses at the vertices of a triangle of reference $A_1A_2A_3$. In particular (1,0,0) is A_1, (0,1,0) is A_2, (0,0,1) is A_3 and (t_1, t_2, t_3) corresponds to a point P such that the areas of the triangles PA_2A_3, PA_3A_1, PA_1A_2 are proportional to the barycentric coordinates t_1, t_2, t_3 of P, respectively (see Figure 1). If $t_1 + t_2 + t_3 = 1$ then the normalised barycentric coordinates (t_1, t_2, t_3) are called *areal* coordinates. In this case the areas of the triangles PA_2A_3, PA_3A_1, PA_1A_2 are t_1, t_2, t_3 times the area of the whole triangle $A_1A_2A_3$, respectively.

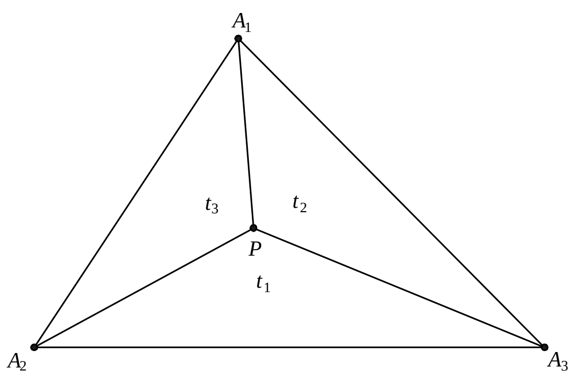

FIGURE 1: The areas of the triangles PA_2A_3, PA_3A_1, PA_1A_2 are proportional to the barycentric coordinates t_1, t_2, t_3 respectively.

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Similarly to what Coxeter did in his book [1, p. 220], A. Bényi and B. Ćurgus [4], recently proved, a version of Theorem 1 and obtained a unification of the theorems of Ceva and Menelaus. In fact they unified two expressions of Routh in his treatise on area ratios [3, p. 82], from which they derived theorems of both Ceva and Menelaus as special cases.

Theorem 1 implies the following general property:

Theorem 2: If the sides of a triangle $A_2A_3A_1$ are divided at $A_{i,1}, A_{i,2}, \ldots, A_{i,n-1}$, $1 \leqslant i \leqslant 3$ in the respective ratios $\lambda_{i,1} : \lambda_{i,2} : \ldots : \lambda_{i,n-1}$, $1 \leqslant i \leqslant 3$, then the ratio of the area of any sub-polygon to the area of the whole triangle depends only on $\{\lambda_{i,j}\}$, where $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant n - 1$ (see Figure 2 for the case $n = 5$).

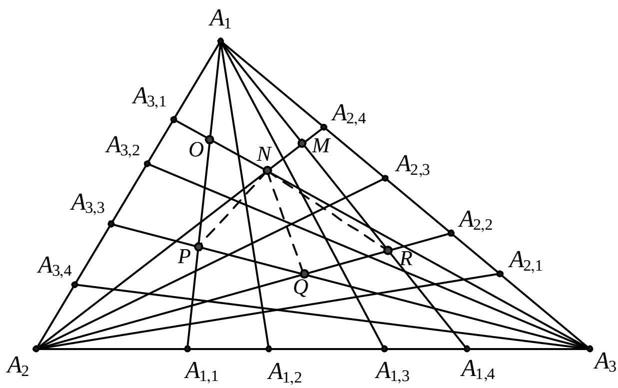

FIGURE 2: The ratio of the area of any sub-polygon to the area of the whole triangle depends only on $\{\lambda_{i,j}\}$.

A sub-polygon of the triangle is defined as a polygon whose vertices are points of intersections of the cevians $\{A_iA_{i,j}\}$, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant n - 1$.

To prove the theorem we compute first the barycentric coordinates for each vertex of the sub-polygon; these are points of intersections of the barycentric equations of the cevians $\{A_iA_{i,j}\}$, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant n - 1$. Next we divide the sub-polygon into non-overlapping triangles (in Figure 2 the polygon *MNOPQR* is divided into 4 triangles). Then we use the method which Coxeter gave in his book [1, p. 219] for proving Theorem 1 (which will be illustrated in the next section) to conclude that the ratio of the area of each triangle to the area of the whole triangle depends only on $\{\lambda_{i,j}\}$ and the result follows.

2. *Patterns*

When there are 'symmetric' divisions of the sides of a triangle, some simple expressions for the ratios of areas can arise. In particular we have the following theorem:

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Theorem 3: Suppose the sides of a triangle $A_2A_3A_1$ are divided at $A_{i,1}$, $A_{i,2}$, $1 \leqslant i \leqslant 3$ in the respective ratios $1 : \lambda : 1$ (see Figure 3). Let I, J, K, L, M, N be the points of intersection of the corresponding cevians as shown in the table:

point	*intersection of cevians*
I	$A_1A_{1,1} \cap A_3A_{3,2}$
J	$A_2A_{2,1} \cap A_3A_{3,2}$
K	$A_2A_{2,1} \cap A_1A_{1,2}$
L	$A_3A_{3,1} \cap A_1A_{1,2}$
M	$A_2A_{2,2} \cap A_3A_{3,1}$
N	$A_1A_{1,1} \cap A_2A_{2,2}$

Then the ratio of the area of the hexagon $IJKLMN$ to the area of the triangle $A_2A_3A_1$ is

$$\frac{2\lambda^2}{(3 + \lambda)(2\lambda + 3)}.$$

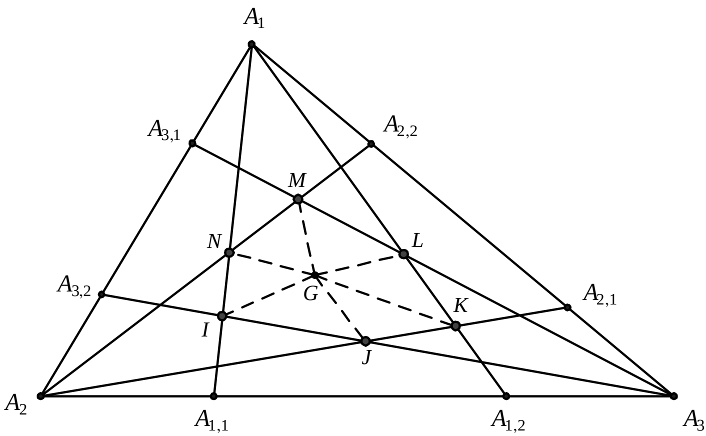

FIGURE 3: Dividing each side in the ratios $1 : \lambda : 1$

Proof: First note that such a hexagon exists in any triangle since it exists in an equilateral triangle, by symmetry of the division ratios, and every other triangle is affine equivalent to an equilateral triangle. Let G be the centre of gravity of $A_1A_2A_3$ (see Figure 3). Since the barycentric coordinates of A_1, A_2, A_3 are (1,0,0), (0,1,0), (0,0,1) respectively, the barycentric coordinates of G are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. By symmetry, the area of the hexagon $IJKLMN$ is 6 times the area of the triangle GMN. Therefore, it is sufficient to compute the barycentric coordinates of M and N.

In order to find the barycentric equations of the corresponding cevians, we exhibit the barycentric coordinates of some of the division points. Now $A_{1,1}$ divides A_2A_3 in the ratio $1 : \lambda + 1$, $A_{2,2}$ divides A_3A_1 in the ratio $\lambda + 1 : 1$ and $A_{3,1}$ divides A_1A_2 in the ratio $1 : \lambda + 1$. Thus the barycentric coordinates are given in the table:

point	*barycentric coordinates*
$A_{1,1}$	$(0, \lambda + 1, 1)$
$A_{2,2}$	$(\lambda + 1, 0, 1)$
$A_{3,3}$	$(\lambda + 1, 1, 0)$.

We proceed by computing the equations of the cevians whose points of intersection are M and N. The cevian $A_2A_{2,2}$ has the equation

$$\begin{vmatrix} 0 & 1 & 0 \\ \lambda + 1 & 0 & 1 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0$$

which is equivalent to $t_1 - (\lambda + 1)t_3 = 0$. The cevian $A_1A_{1,1}$ has the equation

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \lambda + 1 & 1 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0$$

which is equivalent to $-t_2 + (\lambda + 1)t_3 = 0$, and the cevian $A_3A_{3,1}$ has the equation

$$\begin{vmatrix} 0 & 0 & 1 \\ \lambda + 1 & 1 & 0 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0$$

which is equivalent to $-t_1 + (\lambda + 1)t_2 = 0$. Hence, the point M can be computed by the following system of equations;

$$\begin{cases} t_1 & & -(\lambda + 1)t_3 = 0 \\ & -t_1 + (\lambda + 1)t_2 & = 0 \end{cases}.$$

Substituting $t_2 = 1$ we get

$$M = (\lambda + 1, 1, 1).$$

Similarly, the point N is obtained from the following equations:

$$\begin{cases} t_1 & -(\lambda + 1)t_3 = 0 \\ & -t_1 + (\lambda + 1)t_3 = 0 \end{cases}.$$

Substituting $t_3 = 1$ we get

$$N = (\lambda + 1, \lambda + 1, 1).$$

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Consequently, the ratio of the area of the triangle *GMN* to the area of the triangle $A_2A_3A_1$ is equal to the determinant

$$\begin{vmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \lambda + 1 & 1 & 1 \\ \lambda + 1 & \lambda + 1 & 1 \end{vmatrix} = \frac{\lambda^2}{3}$$

divided by the product of the sums of the rows:

$$(\lambda + 3)(2\lambda + 3).$$

Therefore, the ratio of the area of the hexagon *IJKLMN* to the area of the triangle $A_2A_3A_1$ is

$$\frac{\frac{6\lambda^2}{3}}{(\lambda + 3)(2\lambda + 3)} = \frac{2\lambda^2}{(\lambda + 3)(2\lambda + 3)},$$

in agreement with the statement of the theorem.

2.1 *Special cases*

1. If $\lambda = 1$ then the ratio of the area of the hexagon to the area of the triangle is $\frac{1}{10}$. This special case is referred to as Marion Walter's theorem [5] which states the following: *If the trisection points of the sides of any triangle are connected to the opposite vertices, the resulting hexagon has one-tenth the area of the original triangle.*
2. If n is odd, $n = 2k + 1$, then taking $\lambda = \frac{1}{k}$ implies that the ratio of the area of the hexagon to the area of the triangle is

$$\frac{2}{(3k + 1)(3k + 2)} = \frac{8}{9n^2 - 1}.$$

This special case is referred to as Morgan's theorem, which was proved by T. Watanabe, R. Hanson and F. D. Nowosielski [6] using the Routh-Steiner theorem several times.

3. If n is even, then taking $\lambda = n - 2$ implies a new result which was not mentioned in the previous discussion. In this case, we have the following:

The ratio of the area of the hexagon to the area of the triangle is

$$\frac{2(n - 2)^2}{(n + 1)(2n - 1)}.$$

Note that, if each side of the triangle is divided into n equal parts, then for odd n, $n = 2k + 1$, the hexagon is the sub-polygon in Theorem 2, formed by the cevians $\{A_iA_{k,i}, A_iA_{k+1,i}\}$, $1 \leqslant i \leqslant 3$. While for even n, the hexagon is formed by the cevians $\{A_iA_{1,i}, A_iA_{n-1,i}\}$, $1 \leqslant i \leqslant 3$.

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3 *Parallelograms*

We can generalise the Routh-Steiner theorem to parallelograms in the following manner:

Theorem 4: If the sides BC, CD, DA, AB of a parallelogram $ABCD$ are divided at K, L, M, N in the respective ratios $\kappa : 1$, $\lambda : 1$, $\mu : 1$, $\nu : 1$, the cevians AK, BL, CM, DN form a quadrilateral whose area is

$$\frac{1}{2}\frac{\left(\frac{\mu}{1+\nu} + \frac{1+\mu}{\lambda}\right)\left(1 + \frac{\kappa}{1+\mu} + \frac{\kappa+1}{\nu(1+\mu)}\right)}{\left(2 + \frac{1}{\lambda} - \frac{1}{1+\mu}\right)\left(1 + \mu + \frac{\nu\mu}{1+\nu}\right)\left(1 + 2k + \frac{\kappa+1}{\nu}\right)} + \frac{1}{2}\frac{\left(\frac{\kappa+1}{\nu} + \frac{\kappa}{1+\lambda}\right)\left(1 + \frac{1}{\lambda} + \frac{\kappa}{1+\mu}\right)}{\left(2 + \frac{1}{\lambda} - \frac{1}{1+\mu}\right)\left(1 + \kappa + \frac{\lambda\kappa}{1+\lambda}\right)\left(1 + 2k + \frac{\kappa+1}{\nu}\right)} \qquad (1)$$

times that of $ABCD$.

Proof: If the barycentric coordinates of A, B, C are (0,1,0), (0,0,1), (1,0,0) respectively, then the barycentric coordinates of D are (1,1,−1). This is true since the diagonals in the parallelogram bisect each other and the barycentric coordinates of the midpoint of AC are $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ (see Figure 4).

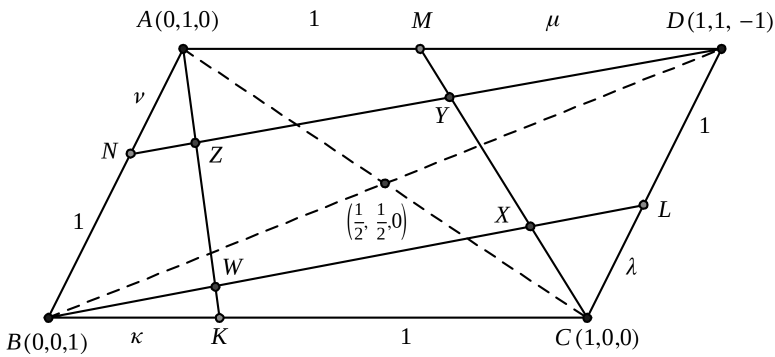

FIGURE 4: The sides of a parallelogram are divided in the ratios $\kappa : 1$, $\lambda : 1$, $\mu : 1$, $\nu : 1$

The barycentric coordinates of K, L, M, N are shown in the following table:

point	*barycentric coordinates*
K	$(\kappa, 0, 1)$
L	$(\lambda + 1, \lambda, -\lambda)$
M	$(1, \mu + 1, -1)$
N	$(0, 1, \nu)$.

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Proceeding as in the proof of Theorem 3, the cevians *AK*, *BL*, *CM*, *DN* have the following equations:

cevian	*barycentric equation*
AK	$-t_1 + \kappa t_3 = 0$
BL	$-\lambda t_1 + (\lambda + 1) t_2 = 0$
CM	$t_2 + (\mu + 1) t_3 = 0$
DN	$(1 + \nu) t_1 - \nu t_2 + t_3 = 0$

Now, let X, Y, Z, W be the points of intersections of pairs of cevians *BL* and *CM*, *CM* and *DN*, *DN* and *AK*, *AK* and *BL*, respectively (see Figure 4). Then the barycentric coordinates of these points are given in the following table:

point	*barycentric coordinates*
X	$\left(\frac{\lambda + 1}{\lambda},\ 1,\ \frac{-1}{\mu + 1}\right)$
Y	$\left(-\frac{1 + \nu + \nu\mu}{1 + \nu},\ -\mu - 1,\ 1\right)$
Z	$\left(\kappa,\ \frac{1 + \kappa + \kappa\nu}{\nu},\ 1\right)$
W	$\left(\kappa,\ \frac{\lambda\kappa}{\lambda + 1},\ 1\right)$.

The area of the quadrilateral $XYZW$ equals the sum of the areas of the triangles XYZ and ZWX. Normalising the barycentric coordinates of X, Y, Z, W and dividing by 2 which is the area of the parallelogram *ABCD*, we find that the ratio of the area of the quadrilateral $XYZW$ to the area of the parallelogram *ABCD* equals $\frac{1}{2}(r_1 + r_2)$ where

$$r_1 = -\frac{\begin{vmatrix} \frac{\lambda+1}{\lambda} & 1 & \frac{-1}{\mu+1} \\ -\frac{1+\nu+\nu\mu}{1+\nu} & -\mu - 1 & 1 \\ \kappa & \frac{1+\kappa+\kappa\nu}{\nu} & 1 \end{vmatrix}}{\left(2 + \frac{1}{\lambda} - \frac{1}{\mu+1}\right)\left(1 + \mu + \frac{\nu\mu}{1+\nu}\right)\left(1 + 2\kappa + \frac{\kappa+1}{\nu}\right)} \tag{2}$$

and

$$r_2 = -\frac{\begin{vmatrix} \kappa & \frac{1+\kappa+\kappa\nu}{\nu} & 1 \\ \kappa & \frac{\lambda\kappa}{\lambda+1} & 1 \\ \frac{\lambda+1}{\lambda} & 1 & \frac{-1}{\mu+1} \end{vmatrix}}{\left(2 + \frac{1}{\lambda} - \frac{1}{1+\mu}\right)\left(1 + \kappa + \frac{\lambda\kappa}{1+\lambda}\right)\left(1 + 2\kappa + \frac{\kappa+1}{\nu}\right)}. \tag{3}$$

Note that

$$\begin{vmatrix} \frac{\lambda+1}{\lambda} & 1 & \frac{-1}{\mu+1} \\ -\frac{1+\nu+\nu\mu}{1+\nu} & -\mu - 1 & 1 \\ \kappa & \frac{1+\kappa+\kappa\nu}{\nu} & 1 \end{vmatrix} = \begin{vmatrix} \frac{\lambda+1}{\lambda} & 1 & \frac{-1}{\mu+1} \\ -\frac{1+\nu+\nu\mu}{1+\nu} + (\mu + 1)\frac{\lambda+1}{\lambda} & 0 & 0 \\ \kappa & \frac{1+\kappa+\kappa\nu}{\nu} & 1 \end{vmatrix}$$

and

$$\begin{vmatrix} \kappa & \frac{1+\kappa+\kappa\nu}{\nu} & 1 \\ \kappa & \frac{\lambda\kappa}{\lambda+1} & 1 \\ \frac{\lambda+1}{\lambda} & 1 & \frac{-1}{\mu+1} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1+\kappa+\kappa\nu}{\nu} & 1 \\ 0 & \frac{\lambda\kappa}{\lambda+1} & 1 \\ \frac{\lambda+1}{\lambda} + \frac{\kappa}{\mu+1} & 1 & \frac{-1}{\mu+1} \end{vmatrix}.$$

After evaluating the determinants and substituting in (2) and (3) the result follows.

In particular, if $\kappa = \lambda = \mu = \nu$ then the expressions in the denominators of (1) are equal to

$$\frac{\left(2\lambda^2 + 2\lambda + 1\right)^3}{\lambda^2(\lambda + 1)^2},$$

while the expressions in the numerators of (1) are equal to

$$\frac{\left(2\lambda^2 + 2\lambda + 1\right)^2}{\lambda^2(\lambda + 1)^2}.$$

Hence we have the following:

Corollary 5: If the sides *BC*, *CD*, *DA*, *AB* of a parallelogram *ABCD* are divided at *K*, *L*, *M*, *N* in the ratio $\lambda : 1$, then the cevians *AK*, *BL*, *CM*, *DN* form a quadrilateral whose area is

$$\frac{1}{2\lambda^2 + 2\lambda + 1}$$

times that of *ABCD*.

Substituting $\lambda = \frac{1}{p-1}$, $p \geqslant 2$, we get the equivalent expression $\frac{p^2 - 2p + 1}{p^2 + 1}$, which was discovered by M. De Villiers [7]. He did not provide any proof but pointed out the following: 'Since a square is affinely equivalent to a parallelogram, the easiest way to derive and prove this formula is to consider the special case of a square.'

We leave to the reader to see what happens in another particular case of the theorem: $\kappa = \mu$ and $\lambda = \nu$.

Finally, the fascinating theorem of Routh-Steiner attracted the interest of mathematics education researchers, probably because of the use of dynamic geometry software to rediscover the theorem, but it still attracts the interest of mathematics researchers as well.

Acknowledgement: The author is indebted to the anonymous referee for his valuable comments. This research was supported by Beit Berl College.

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doi: 10.1017/mag.2014.6

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Nemo continued from Page 10

3. "What age were you when you went to L------ ?"

   "About ten."

   "And you stayed there eight years: you are now, then, eighteen?"

   I assented.

   "Arithmetic, you see, is useful; without its aid, I should hardly have been able to guess your age.“

4. Zounds, a dog, a rat, a mouse, a cat to scratch a man to death! A braggart, a rogue, a villain that fights by the book of arithmetic!

5. Our Estimates a Scheme —
   Our Ultimates a Sham —
   We let go all of Time without
   Arithmetic of him —

6. The Memory, that of any former thing
   Could character the poise, the form, the size,
   The impress of its shape upon the air,
   And now, forgetting its blithe energies,
   Lies drowsing in the sun, or, as it lies,
   Repeats a fond arithmetic of sighs