WAND/SET THEORIES: A REALIZATION OF CONWAY'S MATHEMATICIANS' LIBERATION MOVEMENT, WITH AN APPLICATION TO CHURCH'S SET THEORY WITH A UNIVERSAL SET

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Abstract. Consider a variant of the usual story about the iterative conception of sets. As usual, at every stage, you find all the (bland) sets of objects which you found earlier. But you also find the result of tapping any earlier-found object with any magic wand (from a given stock of magic wands).

By varying the number and behaviour of the wands, we can flesh out this idea in many different ways. This paper's main Theorem is that any loosely constructive way of fleshing out this idea is synonymous with a ZF-like theory.

This Theorem has rich applications; it realizes John Conway's (1976) Mathematicians' Liberation Movement; and it connects with a lovely idea due to Alonzo Church (1974).

Here is a template for introducing mathematical objects.

> *The Wand/Set Template*. Objects are found in stages. For every stage S:
>
> - for any things found before S, you find at S the bland set whose members are exactly those things;
> - for anything, a, which was found before S, and for any magic wand, w, you find at S the result of tapping a with w (if tapping a with w yields something other than a bland set).
>
> You find nothing else at S.

As I will explain in Section 1: this Template has rich applications; it realizes John Conway's [7] Mathematicians' Liberation Movement; and it generalizes a lovely idea due to Alonzo Church [6].

Some parts of this Template are familiar: the bit about bland sets is just the ordinary story we tell about the cumulative iterative conception of set. But this talk of "magic wands" is new and different. Moreover, parts of the story are left unspecified. We know very little about: the sequence of stages; what the wands are; the circumstances under which tapping something with a wand will yield anything; nor what will thereby be revealed.

This under-specification is deliberate: the Template is left *as* a template precisely so that we can flesh it out in umpteen different ways. Nevertheless, the Main Theorem

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of the paper is that *any loosely constructive way of fleshing out the Template is synonymous with a* ZF*-like theory*. (I define "loosely constructive" in Section 4, and "ZF-like" in Section 6.)

§1. Motivations. I will start by explaining why the Wand/Set Template is worthy of attention. Briefly, this is because it provides us with an extremely rich framework for introducing mathematical objects (see Section 1.1). I hope this is enough to make it interesting. But I am not above appealing to authorities; in this case, Conway (Section 1.2) and Church (Section 1.3).

1.1. Simple examples. The Wand/Set Template is very abstract. So it will help to start with some concrete instances of the Template. This will also illustrate the scope of my Main Theorem.

Example 1.1 (Conway-games). Conway [7] famously provided a general theory of (hereditary) two-player games. A Conway-game has *left-options* and *right-options*, and Conway-games are identical iff they have the same left- and right-options.

Here is a sketch of how to provide a wand/set theory of Conway-games. We have a single wand, game. Tapping an ordered-pair, $\langle a, b\rangle$, of bland sets with game (typically) yields a new object; we deem this to be a Conway-game whose left-options are exactly a's members and whose right-options are exactly b's members. By courtesy, a bland set, a, is the Conway-game with no right-options and whose left-options are exactly a's members.

Here is the idea in a bit more detail. Reserve curly-brackets for bland sets, and $\in$ for membership of the same. With $\langle a, b\rangle := \{\{a\}, \{a, b\}\}$, stipulate that game only acts on objects of the form $\langle a, b\rangle$, where both a and b are bland sets. Then $\mathsf{game}(\langle a, \emptyset\rangle) := a$, and if $b \neq \emptyset$ then $\mathsf{game}(\langle a, b\rangle)$ is an object which is not a bland set; and games are governed by this rule:

$$\mathsf{game}(\langle a, b\rangle) = \mathsf{game}(\langle a', b'\rangle) \rightarrow (a = a' \wedge b = b').$$

Reading "$x \circ_{\mathrm{L}} y$" as "x is a left-option of y", and similarly for "$x \circ_{\mathrm{R}} y$", explicitly define:

$$x \circ_{\mathrm{L}} y :\equiv \exists a \exists b(y = \mathsf{game}(\langle a, b\rangle) \wedge x \in a).$$
$$x \circ_{\mathrm{R}} y :\equiv \exists a \exists b(y = \mathsf{game}(\langle a, b\rangle) \wedge x \in b).$$

Properly implemented, we have a theory whose objects are exactly the Conway-games. By my Main Theorem, this is synonymous with a ZF-like theory.[1]

Example 1.2 (Partial functions). Consider a cumulative iterative hierarchy of one-place partial functions: we start with the function which is undefined for every

[1]This essentially subsumes Cox and Kaye's [8] result concerning Amphi-ZF. Note a difference: Cox and Kaye take $\circ_{\mathrm{L}}$ and $\circ_{\mathrm{R}}$ as primitives; I take *Bland* as a primitive and (arbitrarily) treat bland sets as Conway-games with no right-options. Cox and Kaye's approach is more natural; mine lets me press the theory of Conway-games into the mould of a wand/set theory (see Definition 5.1) without positing bland sets as objects which are *not* themselves Conway-games (see the end of Section 1.2).

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input; at every layer,[2] we find all possible one-place partial functions whose domain-and-range are exhausted by objects we found earlier.[3]

Here is a sketch of how to put this into the form of a wand/set theory. We have a single wand, fun. We use bland sets to encode functions in the usual way; tapping such a bland set with fun (typically) yields a new object; we deem this to be a function of the appropriate sort. By courtesy, a bland set, a, will be the identity function defined only over a's members. Of course, more detail can be offered (as in Example 1.1); but this ultimately yields a theory whose objects are exactly the (hereditary) partial one-place functions. By the Main Theorem, this is synonymous with a ZF-like theory.[4]

We can easily overcome the restriction to *one*-place partial functions by positing a countable infinity of wands. For each n, tapping appropriate bland sets with fun_n yields the n-place partial functions whose domain-and-range are exhausted by what we have already found. Again: the result is synonymous with a ZF-like theory.

Example 1.3 (Multisets). Consider a cumulative iterative hierarchy of multisets: we start with the empty multiset; at every layer, we find all possible multisets whose members and associated multiplicities (when the multiplicity is > 1) were found earlier. So for any x: we first find multisets with just one copy of x immediately after we have found x; and for each cardinal $\mathfrak{a} > 1$, we first find multisets with $\mathfrak{a}$ copies of x immediately after we have found both x and $\mathfrak{a}$.

To render this as a wand/set theory, just treat a multiset, s, as a one-place partial function, where $s(x) = \mathfrak{a}$ indicates that s has $\mathfrak{a}$ copies of x. Then the theory of multisets is just a slight variant of Example 1.2. The only differences are: (i) our functions' values must always be cardinals (in the bland sense), and (ii) each bland set, a, is treated as the multiset which contains exactly 1 copy of each of a's members. Again: the result is synonymous with a ZF-like theory.[5]

Example 1.4 (Accessible pointed graphs). Aczel [1] thinks of "sets" in terms of accessible pointed graphs (APGs), whose "members" are equivalents of those objects which the APG's point can "see". Skimming over details, this is easy to render as a wand/set theory. We can encode the definition of an APG in terms of bland sets. Then we have a single wand, set, which behaves thus: when a is a bland set which encodes an APG, $\mathsf{set}(a)$ is an object whose "APG-members" are those things, x, such that a encodes an edge from a's point to an entity equivalent to x.

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[2]Such *layers* are not precisely the *stages* mentioned in the Wand/Set Template. As we will see: $\emptyset$ is (by courtesy) the function which is undefined for any input; $\{\emptyset\}$ is (by courtesy) the identity function which just maps $\emptyset \mapsto \emptyset$; now let f be the function which maps $\{\emptyset\} \mapsto \emptyset$ and is undefined for all other inputs. So f is found at the *third layer* in the explanation at the start of this example; but $f = \mathsf{fun}(\{\langle\{\emptyset\}, \emptyset\rangle\})$ is found at the *sixth stage* of the Wand/Set implementation. This mismatch between layers and stages is ultimately unimportant. Provided that there is no last stage or layer (see WeakHeight in Section 5.2), all the same objects will be found, whether we tell the story in terms of layers or stages; moreover, we can easily define the layer-ordering by a recursion on the stage-ordering, and vice versa (and any wand/set theory, in the sense of Definition 5.1, has a suitable recursion theorem).

[3]This idea was outlined by Robinson [18].

[4]This essentially subsumes Meadows' [16] and my [5] independently obtained synonymy results about iterative theories of functions.

[5]So this goes well beyond, e.g., Blizard's [2] embedding result.

This example is somewhat different from Examples 1.1–1.3. Before, I explained how to treat any bland set as an object of the relevant sort (a Conway-game; a partial function; a multiset). But I have given no uniform way to treat any bland set *as* an APG, and indeed this is problematic. To see why, consider this APG:  . It depicts a (unique) ill-founded set, Ω, whose sole "member" is Ω itself.[6] But the bland singleton of Ω—which we should find at the next stage after we have found Ω—has exactly one member, Ω; so, by extensionality, that bland set should *be* Ω itself; and so some bland set would be self-membered, contrary to what the Template tells us.[7]

Consequently, *this* approach does not give us a theory whose objects are exactly the APGs. Instead, it gives us a theory whose objects are exactly the APGs and bland sets thereof. Nonetheless, the result is synonymous with a ZF-like theory.[8]

1.2. Conway's liberationism. Further instances of the Wand/Set Template could be offered, but I hope that Examples 1.1–1.4 already illustrate the Template's power. I now want to move beyond piecemeal examples and explore a more principled reason for investigating the Template.

Immediately after outlining his theory of games (see Example 1.1), Conway advocated for a Mathematicians' Liberation Movement. His liberationism held that:

(i) "Objects may be created from earlier objects in any reasonably constructive fashion.
(ii) Equality among the created objects can be any desired equivalence relation."[9]

He also expected that there would be a meta-theorem, which would show that any theory whose objects are created in such a "reasonably constructive" fashion can be embedded within (some extension of) ZF.

As the examples of Section 1.1 suggest, my Wand/Set Template realizes Conway's proposed liberationism. Regarding (i): to implement the idea of creating some object "from earlier objects", all you need to do is form their bland set, and then tap that set with an appropriate wand (or perhaps with the right wands in the right order). Regarding (ii): when we specify the behaviour of our wands, we can allow "any desired equivalence relation" to dictate when the result of tapping a with wand w is equal to the result of tapping b with wand u. And my Main Theorem exceeds Conway's expectations: all loosely constructive implementations of the Wand/Set Template are not merely *embeddable* in (some extension of) ZF, but *synonymous* with a ZF-like theory.

Admittedly, my Template may not fully exhaust the scope of Conway's liberationism. This is because Conway's (i) and (ii) give us *permissions*, not *obligations*; they say what we *may* create, without dictating what we *must* create. My Template, by contrast, *insists* on creating[10] bland sets at each stage. However, my insistence is unlikely to impinge seriously on Conway's proposed liberationism. For one thing: whenever Conway creates entities of some kind, K, he is confident that they will

[6] See Aczel [1, pp. 6–7].
[7] Specifically, it contradicts Lemma 7.4(7).
[8] Thanks to Luca Incurvati for getting me thinking about treating Aczel's APGs in terms of a wand/set theory (see also his [15, Chapter 7]).
[9] Conway [7, p. 66].
[10] I prefer to speak of "finding" (than "creating") objects; it seems marginally preferable to tell a story which is compatible with platonism.

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be embeddable into a ZF-style hierarchy; so it is no great burden to request that, when he creates the Ks, he also create the ZF-like bland-sets into which his Ks will be embedded. For another: if we *really* want our theory to be a theory of Ks only, we can often do this by creatively (and courteously) regarding bland sets as Ks of a particular sort: we saw this in Examples 1.1–1.3, though not in Example 1.4.

1.3. Church's set theory with a universal set. I have presented some examples of the Wand/Set Template, and explained how the Template relates to Conway's Liberationism. I now want to consider a more surprising connection: we can regard Church's [6] *set theory with a universal set* as both an instance of the Wand/Set Template and as an exercise of Conway's liberationism.[11]

Church's [6] idea is roughly this. We have a ZF-like hierarchy of bland sets. We also have some wands: a complement wand and, for each positive natural n, a $\mathsf{cardinal}_n$ wand. These wands should behave as follows:

(0) Tapping any object, a, with complement yields a's absolute complement; that is, the set whose members are exactly the things not in a.

(n) Tapping some bland set, a, with $\mathsf{cardinal}_n$, yields the set of all bland sets *n-equivalent* with a (if there are any; if there are none, this yields nothing; and tapping anything other than a bland set with $\mathsf{cardinal}_n$ yields nothing).

This explanation mentions *n-equivalence*. This is an explicitly defined equivalence relation (defined recursively on n; see Definition 11.1). Its details are largely irrelevant for now; what matters is just that Church's theory can be thought of as invoking a countable infinity of wands with explicitly defined actions.

What is particularly interesting about *Church's* theory, however, is that—unlike Examples 1.1–1.4—it does not obviously seem like an instance of *Conway's* liberationism. Conway's (ii) says that "Equality among the created objects can be any desired equivalence relation"; this is fine, since Church's wands are based around explicitly defined equivalence relations. But Conway's (i) says that "Objects may be created from earlier objects in any reasonably constructive fashion"; Church's wands seem to break this condition. After all: following Church, tapping $\emptyset$ with complement must reveal the universal set, V; but things we have not *yet* discovered must be members of V; so one might well doubt whether this is "reasonably constructive"!

Crucially, this doubt can be addressed. The complement of a bland set is never itself bland. So, where $\in$ is the primitive membership relation we use for bland sets, we can explicitly define an extended sense of "membership" which is suitable for both bland sets and their complements, along these lines:

$$x \mathrel{\varepsilon} y :\equiv (\mathit{Bland}(y) \land x \in y) \lor (\exists c : \mathit{Bland})(y = \mathsf{complement}(c) \land x \notin c).$$

Now, $\emptyset$ is bland, and *nothing* is in$_\varepsilon$ it; and $\emptyset$'s complement, V, is not bland, and *everything* is in$_\varepsilon$ it. By definition, though, the latter claim is exactly as constructive as the former. Doubt resolved!

[11] I have Forster [10] to thank for approaching Church in essentially this way, and for making the link to Conway. I should note that, whilst I think it is *cleanest* to understand Church in this way, it is not exactly his own presentation; for details, and some important caveats, see Section 11.2. This section builds on my [4], which essentially covers the case without any $\mathsf{cardinal}_n$ wands, so that my Main Theorem subsumes the synonymy result in my [4].

Of course, this is only a quick sketch of how to make Church's [6] theory look "reasonably constructive": it says nothing about the $\mathsf{cardinal}_n$ wands, and omits a lot of other details. Still, it gives us all the right ideas, and the rest really is *just* details (for which, see Section 11). The surprising upshot is that Church's theory can be regarded as a loosely constructive implementation of the Wand/Set Template. By my Main Theorem, it is therefore synonymous with a ZF-like theory.[12]

Having tackled Church's theory, it is natural to ask whether we can grapple with other theories which are—at least at first blush—deeply non-constructive. Specifically, we might observe that Quine's NF can be given a finite axiomatization which is based on the idea that there are certain "starter" sets, and that the other sets are formed from these by simple operations, such as taking complements.[13] Regarding the operations as wand-actions, we might then hope to be able to regard NF *itself* as an implementation of the Wand/Set Template. This might even lead to a consistency proof for NF, or—as Church suggested—a "synthesis or partial synthesis of" ZF with NF.[14]

Clearly, the Wand/Set Template opens up some giddying possibilities. Inspired by them, my aim is to set the Template on a formal footing, and then establish my Main Theorem: *any loosely constructive way of fleshing out the Wand/Set Template is synonymous with a ZF-like theory.* (For better or worse, this Theorem ultimately restores some sobriety to our giddy state, since it seems to undermine the possibility of saying much about NF using the Wand/Set Template.[15])

I will explain this synonymy result more precisely as I go. After some simple preliminaries (Sections 2 and 3), I will explicate what it takes to flesh out the Wand/Set Template in a *loosely constructive* way (Sections 4 and 5). This will allow me to present a general definition of an arbitrary wand/set theory (Definition 5.1). I will then explain how any wand/set theory mechanically generates a theory which is "ZF-like" in a precise sense (Section 6). Having done this, I will prove the Main Theorem (Sections 7–10). I conclude by applying my wand/set apparatus to Church's [6] theory (Section 11).

§2. Notation. In what follows, I restrict my attention to first-order theories.[16] For readability, I concatenate infix conjunctions, writing e.g., $a \subseteq r \in s \in t$ for $a \subseteq r \land r \in s \land s \in t$. I also use some simple abbreviations (where Ψ can be any

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[12] Unlike Examples 1.1–1.4, this synonymy result is genuinely novel to this paper.

[13] See, e.g., Holmes's [14] axiomatization. The "starter sets" are given by the axioms of: Empty Set, Diagonal, Projections, Inclusion. The operations are given by: Complements, (Boolean) Unions, Set Union, Singletons, Ordered Pairs, Cartesian Products, Converses, Relative Products, Domains, Singleton Images. Admittedly, some of the operations require two inputs (e.g., ordered pairs), whereas the wands in my Template act on just one input. However, the Template could easily be tweaked to allow for this. Moreover, the adjustment is scarcely likely to be necessary: if we wanted to input a and b into some "binary" wand, we could instead input the bland set $\{\{a\}, \{a, b\}\}$ into some "unary" wand (cf. Example 1.1).

[14] Church [6, p. 308]; Church mentions Hailperin's [13] axioms rather than Holmes's [14].

[15] See Forster and Holmes's [11] argument for Sadie Kaye's Conjecture, that "No extension of NF is synonymous with any theory of wellfounded sets". Their argument uses my Main Theorem; indeed, I wrote this paper in part to provide fuel for their argument.

[16] If desired, both the idea of wand/set theories and the Main Theorem easily carry over to the second-order case; we simply replace schemes with axioms in the most obvious way.

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predicate with x free, and $\lhd$ can be any infix predicate):

$$(\forall x : \Psi)\phi := \forall x(\Psi(x) \to \phi), \qquad (\forall x \lhd y)\phi := \forall x(x \lhd y \to \phi),$$
$$(\exists x : \Psi)\phi := \exists x(\Psi(x) \land \phi), \qquad (\exists x \lhd y)\phi := \exists x(x \lhd y \land \phi).$$

When I announce a result or a definition, I will list in brackets the axioms which I am assuming.

Over the next few sections, my aim is to define the general idea of a wand/set theory. All such theories will have the following signature:

Bland: a one-place predicate; intuitively, this picks out the bland sets.
Wand: a one-place predicate; intuitively, this picks out the wands.
$\in$: a two-place (infix) predicate; intuitively, $x \in a$ says that a is a bland set with x as a member.
Tap: a three-place relation; intuitively, $Tap(w/a, c)$ says that c is found by tapping a with w. (Nothing special is indicated by separating arguments with a slash rather than a comma; it simply improves readability.)

When working in this signature, I use set-builder notation exclusively for *bland sets*; so $\{x : \phi\}$ is to be the bland set (if there is one) whose members are exactly those x such that ϕ. Similarly, $\emptyset$ will be the empty bland set; $\{\emptyset\}$ will be the bland set whose unique member is $\emptyset$, etc. For brevity, I will often describe something as simply "bland", to mean that it is a bland set.

I now introduce three quasi-notational axioms:[17]

InNB $x \in a \to Bland(a)$.
TapNB $Tap(w/a, c) \to \big(Wand(w) \land \neg Bland(c)\big)$.
TapFun $\big(Tap(w/a, c) \land Tap(w/a, d)\big) \to c = d$.

Axiom InNB ensures that we reserve $\in$ for the notion of membership according to which bland sets have members. It does not follow that non-bland things have *no* members; it just means that, if we want to allow non-bland things to have members, then we will have to define some richer notion of membership. (I foreshadowed this point in Section 1.3, when I outlined "ε", and I develop this in detail in Sections 11.2 and 11.3.)

Axiom TapNB is similar. It accords with the Template's prescriptions that (1) we only ever tap things with *wands*, and that (2) bland things are never *found* by tapping something with a wand but are always (and familiarly) found just by waiting until the stage immediately after you have found all their members. Again: this does not force us to deny that tapping something with a wand might ever "yield" something bland; we can allow this by defining a richer notion of wand-tapping. (Indeed, Church's [6] theory allows that tapping something with complement can *yield* a bland entity, but it will always be a bland entity which was *found* at an earlier stage; I discuss this in detail in Section 11.2.)

Finally, TapFun makes explicit the implicit point that *Tap* is functional (where defined). This licenses another bit of helpful notation: in what follows, I write $\boxed{w}a$ for the (necessarily unique) object, c, if there is one, such that $Tap(w/a, c)$.

[17] For readability, I often omit outermost universal quantifiers on axioms; these should be read as "the universal closure of...".

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§3. Core axioms governing wevels. My first serious task is to formalize the key idea of the Wand/Set Template: that, at each stage, we find (1) all bland sets of earlier-found objects, and (2) the result of tapping any earlier-found object with a wand (if the result is not bland).

This key idea mentions stages. So I could start by positing *stages* as special objects of a primitive sort, and laying down axioms about them. However, the tools and techniques developed in my [3] presentation of LT, Level Theory, allow me to skip this step. Instead, I can just define some special bland sets which will go proxy for the Template's stages. These stage-proxies will be called *wand-levels*, or *wevels* for short.

To build up to the definition of a wevel, I begin with some basic axioms, which are the obvious versions of Extensionality and Separation for bland sets:

Ext $(\forall a : \mathit{Bland})(\forall b : \mathit{Bland})\big(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b\big)$.
Sep $(\forall a : \mathit{Bland})(\exists b : \mathit{Bland})\forall x\big(x \in b \leftrightarrow (x \in a \wedge \phi)\big)$;
schema for any ϕ not containing 'b'.

The next idea requires more explanation. Recall that wevels will be bland sets which go proxy for stages. Suppose we can define a formal predicate "$\mathit{Wev}(x)$", for *x is a wevel*, and a formal predicate "$x \trianglelefteq r$", for *x is found at wevel r*. Then we will want a wevel to be precisely the bland set of all earlier-found things, i.e.,

$$s = \{x : (\exists r : \mathit{Wev})(x \trianglelefteq r \in s)\}.$$

That is our target; it just remains to give the explicit definition. Here it is:[18]

DEFINITION 3.1 (InNB, TapNB). Define:

$$x \trianglelefteq r :\equiv \big(\mathit{Bland}(x) \wedge x \subseteq r\big) \vee \exists w(\exists b \in r)\mathit{Tap}(w/b, x),$$
$$x \triangleleft a :\equiv \exists r(x \trianglelefteq r \in a).$$

Let $\mathbb{P}a$ be $\{x : x \triangleleft a\}$.[19] Say that h is a wistory, $\mathit{Wis}(h)$, iff $\mathit{Bland}(h) \wedge (\forall a \in h)a = \mathbb{P}(a \cap h)$.[20] Say that s is a wevel, $\mathit{Wev}(s)$, iff $(\exists h : \mathit{Wis})s = \mathbb{P}h$.

To see why this is the correct definition of $\trianglelefteq$, recall that, at any wevel, we should find both all bland subsets of earlier wevels, and all wand-taps of members of earlier wevels (unless this yields a bland set), and nothing else. The definition of *Wev* is less immediately intuitive,[21] but it is vindicated by these results (see Section 7 for more).

Lemma 7.2. s is a wevel iff $s = \mathbb{P}\{r \in s : \mathit{Wev}(r)\}$.
Theorem 7.3. The wevels are well-ordered by $\in$.

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[18] Here, I highlight the use of two quasi-notational axioms. These ensure that, in the second disjunct, x is non-bland and r is bland and w is a wand.

[19] We do not assume at the outset that $\mathbb{P}a$ exists; but Lemma 7.4 proves it does. Note that $\mathbb{P}a$ is bland by definition.

[20] As usual, we define $a \cap h = \{x \in a : x \in h\}$, i.e., it is bland by definition; by Sep, it will always exist.

[21] For those familiar with LT (see [3]), here is a quick explanation of why Definition 3.1 works. In ordinary LT, where we have no wands, we replace $\trianglelefteq$ with $\subseteq$, since a level should just comprise all (bland) subsets of earlier levels. The remaining definitions are the same, using $\subseteq$ in place of $\trianglelefteq$.

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Unpacking the definition of $\mathbb{P}$, Lemma 7.2 says that a wevel is *exactly* the bland set of all earlier-found things, which was precisely our target a couple of paragraphs above. Then Theorem 7.3 tells us that wevels are very nicely behaved.

The two results just mentioned can be proved using a very weak theory, whose axioms are just InNB, TapNB, Ext, and all instances of Sep. I will call this theory CoreWev, since it proves the core results about wevels.

The Wand/Set Template itself, though, tells us to go beyond CoreWev. It says that every object is found at some stage. Since wevels are proxies for stages, this becomes the following axiom:

Strat $\forall a(\exists s : Wev)a \trianglelefteq s$.

Since wevels are well-ordered, we can define the wevel "at which a is first found":

Definition 3.2 (CoreWev, Strat). For each a, let $\mathbb{L}a$ be the $\in$-least wevel s such that $a \trianglelefteq s$. So: $a \trianglelefteq \mathbb{L}a$ and $\neg(\exists r : Wev)a \trianglelefteq r \in \mathbb{L}a$. Let $\mathrm{R}a$ be the ordinal β such that $\mathbb{L}a = \mathbb{L}\beta$.[22]

Using just these axioms, we can prove several unsurprising but reassuring results about wevels; for example, that the αth wevel is always $\mathbb{L}\alpha$.

§4. Loose constructivism. We have seen that CoreWev + Strat ensures that our objects are arranged in a well-ordered hierarchy. However, this says nothing, yet, about the hierarchy's height, nor about the effects of tapping various objects with our various wands.

This is deliberate. Given my aims (see Section 1), I want to consider many different instances of the Wand/Set Template. Since different instances may describe hierarchies of various heights, or employ different wands, I cannot afford to be *too* specific.

That said, I do want to be a *bit* more specific. Following Conway's lead (see (i) of Section 1.2), I will restrict my attention to *loosely constructive* ways of fleshing out the Wand/Set Template. The idea of loose constructivism is both conceptually subtle and formally fiddly, so the point of this section is to motivate and explicate the idea.

4.1. The Picture. I will start with some very soft imagery. Imagine we are about to tap some object, a, with a wand, w, for the first time. We are trying to calculate what, if anything, we will find as a result of this wand-tap, i.e., whether $\boxed{w}a$ will exist and what it will be like. Now: *what resources should we be allowed to invoke in our calculations?*

Evidently, we should be allowed to refer to any objects which we have *already* found. They are, after all, lying around for us to inspect. But if we attempt to consider objects which have not *yet* been found, we start to run the risk of vicious circularity. After all, suppose we think that whether $\boxed{w}a$ will exist depends upon which objects will exist *later*. Well, what exists *later* depends upon whether $\boxed{w}a$ will exist after the wand-tap we are about to perform, and we have run into circularity.

Such concerns are completely familiar from discussions of (im)predicativity in set theory. But they need not force us to adopt a fully predicative standpoint. After all,

[22] Here and throughout I define ordinals in the usual, von Neumann, fashion, i.e., as transitive (bland) sets well-ordered by $\in$; and I reserve the first few lower-case Greek letters to refer to ordinals.

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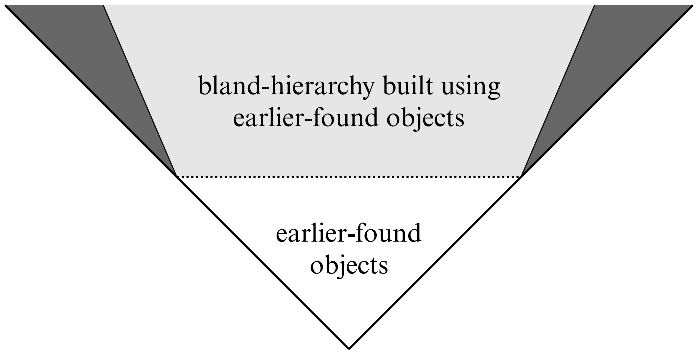
most of us are perfectly relaxed about the amount of impredicativity licensed by ZF. So I propose that we be allowed *exactly* that amount of impredicativity.

Let me spell out what this proposal amounts to. Our Template tells us that we find new bland sets at every stage. Moreover, these bland sets behave just like ZF-style sets. So, whatever objects we have already found (via whatever esoteric use of wands), we know that there *will be* a "bland-hierarchy" which is built from them, intuitively treating the already-found objects as urelements and collecting them together into (bland) sets without any *further* deployment of our wands. This "bland-hierarchy" is exactly as impredicative as ordinary ZF; so we should be allowed to make reference to it, but nothing outside it.

Since things are getting complicated, a picture may help. (Compare this with the big "V" we draw to describe the "ordinary" set-theoretic hierarchy.)



The unshaded area represents what we have already found; we are allowed to refer to this part of the hierarchy in our calculations, since it is perfectly transparent to us. The lightly shaded area represents the bland-hierarchy constructed from the earlier-found objects. Although this bland-hierarchy is undiscovered as yet, its impredicativity is exactly of a piece with that of ordinary ZF; we are allowed to refer to this part of the hierarchy in our calculations, since its behaviour has already been tamed. But the darkly shaded area represents objects we will find (only) by future uses of wand-taps; we are barred from referring to this part of the hierarchy in our calculations since, as legend warns, here be viciously circular dragons.

In what follows, I refer to this image as the *Picture of Loose Constructivism*. This Picture guides me through the next couple of sections. But my immediate task is to make this (informal) Picture more precise, so that we can embed it within our formal theorizing.

4.2. Loosely constructive formulas. I start by explicitly defining the "bland-hierarchy built from u", where u is any set of objects which we will treat as urelements. Intuitively, we treat u as given; we then resist the temptation to deploy any wands, but keep repeatedly forming all possible bland sets.

Evidently, this "bland-hierarchy" should be arranged into well-ordered layers, which we might call *levels*$_{\mathrm{u}}$. Indeed, writing V_{u}^{α} for the αth level$_{\mathrm{u}}$, a little reflection

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on the "intuitive" idea just sketched indicates that the levels$_{\mathrm{u}}$ should behave thus:[23]

$$V_{\mathrm{u}}^{\alpha} = \{x : \mathit{Bland}(x) \wedge (\exists\beta < \alpha)x \subseteq V_{\mathrm{u}}^{\beta}\} \cup \mathrm{u}.$$

This identity would work perfectly well as a recursive *definition* of the levels$_{\mathrm{u}}$.[24] However, it will be slightly easier for us if we instead, officially, define the levels$_{\mathrm{u}}$ non-recursively (noting that this is equivalent to the recursive approach, by Lemma 7.12):[25]

DEFINITION 4.1. For any u, say that:[26]

$$x \lhd_{\mathrm{u}} a :\equiv \big(\mathit{Bland}(x) \wedge \exists c(x \subseteq c \in a)\big) \vee x \in \mathrm{u}.$$

Let $¶_{\mathrm{u}}(a) := \{x : x \lhd_{\mathrm{u}} a\}$. Say that $\mathit{Hist}_{\mathrm{u}}(h)$ iff $\mathit{Bland}(h) \wedge (\forall x \in h)x = ¶_{\mathrm{u}}(x \cap h)$. Say that t is a level$_{\mathrm{u}}$, written $\mathit{Lev}_{\mathrm{u}}(t)$, iff $(\exists h : \mathit{Hist}_{\mathrm{u}})t = ¶_{\mathrm{u}}(h)$. Say that $\mathbf{V}_{\mathrm{u}}(a)$ iff $(\exists t : \mathit{Lev}_{\mathrm{u}})a \in t$.

Now the formal claim that x satisfies $\mathbf{V}_{\mathrm{u}}$ precisely explicates the intuitive idea that x falls somewhere in the "bland-hierarchy built from u".

I can now explain how to embed the Picture of Loose Constructivism within our formal theorizing. Suppose we are at stage s, and some formula, ϕ, is supposed to tell us something about what will happen later. So ϕ must deploy (only) resources which count as loosely constructive at s. In this case, I will say that ϕ is *loosely bound* by s; and I will now define this notion precisely.

Given the Picture, ϕ will be loosely bound by s iff ϕ considers (only) the bland-hierarchy built from the objects found at s, i.e., the bland-hierarchy built from the objects *in* s^+, where s^+ is the level immediately after s.[27] So ϕ should range (only) over $\mathbf{V}_{s^+}$. Moreover, if ϕ mentions any wand-taps, i.e., if ϕ contains expressions of the form "$\boxed{w}x$", then, since $\boxed{w}x$ is never bland (by InNB), ϕ must contain a guarantee that $\boxed{w}x$ is found no later than s; this is equivalent to insisting that x itself is found *strictly before* s,[28] i.e., to insisting that $x \in s$.

These ideas are rolled together in the following definition.

DEFINITION 4.2. Let s be any wevel and ϕ be any formula.

An instance of a quantifier in ϕ is *loosely bound* by s iff it is restricted to $\mathbf{V}_{s^+}$. So such instances are of the form $(\forall x : \mathbf{V}_{s^+})\psi$ or $(\exists x : \mathbf{V}_{s^+})\psi$.

An instance of a free variable b is *loosely bound* by s in ϕ iff ϕ is a conjunction and $\mathbf{V}_{s^+}(b)$ is one of its conjuncts.

An instance of $\mathit{Tap}(w/a, x)$ is *loosely bound* by s in ϕ iff it occurs in a binary conjunction whose other conjunct is $a \in s$, i.e., $(\mathit{Tap}(w/a, x) \wedge a \in s)$.

A formula is *loosely bound* by s iff all its quantifier-instances, free variables, and *Tap*-instances are loosely bound by s.

---

[23] Equivalently we could say: $V_{\mathrm{u}}^{0} = \mathrm{u}$; and $V_{\mathrm{u}}^{\alpha+1} = \wp(V_{\mathrm{u}}^{\alpha}) \cup \mathrm{u}$; and $V_{\mathrm{u}}^{\alpha} = \bigcup_{\beta<\alpha} V_{\mathrm{u}}^{\beta}$ for limit α.

[24] See Section 9.2 for a little more about the use of recursion definitions in the level-theoretic context.

[25] Compare this with Definition 3.1 and my [3, appendix A] Level Theory with Urelements.

[26] Equivalently: $x \lhd_{\mathrm{u}} a$ iff $x \lhd \mathrm{u} \vee \big(\mathit{Bland}(x) \wedge x \lhd a\big)$.

[27] Recall: Theorem 7.3 tells us that the wevels are well-ordered. Lemma 7.2 tells us that $x \in s^+$ iff $x \trianglelefteq s$, i.e., iff x is found at s. (And by the same Lemma, if x is found strictly before s, i.e., if there is a wevel r such that $x \trianglelefteq r \in s$, then x is found at s, i.e., $x \trianglelefteq s$.)

[28] See Lemma 7.7.

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To round off this discussion, it is worth giving particular consideration to the bland-hierarchy which is built by treating *nothing* as an urelement, i.e., the denizens of $\mathbf{V}_\emptyset$. These are bland sets, whose members are bland, whose members of members are bland ... and so on, all the way down. In a word, they are *hereditarily bland*. These are important because, given the Picture, hereditarily bland sets are precisely the objects which are available to consider "at the outset". It turns out (Lemma 7.15) that we can equivalently define hereditarily bland sets without mentioning $\mathbf{V}_\emptyset$, and this will be my official definition:

DEFINITION 4.3. Say that a is *hereditarily bland*, written $Heb(a)$, iff both $Bland(a)$ and $(\exists c \supseteq a)(\forall x \in c)(x \subseteq c \wedge Bland(x))$.

§5. General definition of a wand/set theory. In the last section, I formalized the notion of loose constructivism (Definition 4.2). Using this, I can at last define what it takes for a theory to provide us with a loosely constructive implementation of the Wand/Set Template.

5.1. The wands are hereditarily bland. I have said that any wand/set theory will need to have the three quasi-notational axioms (InNB, TapNB, and TapFun from Section 2) and the three axioms which arrange everything into well-ordered wevels (Ext, Sep, and Strat from Section 3). But I have said nothing, yet, about the wands. This obviously needs to addressed.

Different instances of the Template will have different wands. Still, we can still make a general comment. The Template tells us to tap everything we can, at every stage, with every wand. This means that every wand must be "available" to us at the outset. But only hereditarily bland sets are "available" to us at the outset (see the end of Section 4.2). So every wand must be hereditarily bland. We therefore adopt the axiom:

HebWands $(\forall w : Wand)Heb(w)$.

5.2. Weak claims about the hierarchy's height. Next, I lay down an axiom which makes a weak claim about the hierarchy's height:

WeakHeight $\{w : Wand(w)\}$ exists; and for every ordinal α both $\alpha \cup \{\alpha\}$ and $\mathrm{R}(\{w : Wand(w)\}) + \alpha$ exist.[29]

Given the ambient axioms, to say that $\{w : Wand(w)\}$ exists is to say that the hierarchy is sufficiently tall that all the wands are (eventually) found together at some wevel. The statement about an ordinal-sum existing just makes a further claim about the hierarchy's height. After all, each ordinal β is first found at the βth wevel (see Section 3). So we can control the hierarchy's height just by insisting that certain ordinals exist; and that is what WeakHeight does.

Having understood roughly *what* WeakHeight says, we need to ask *why* we should adopt it. Candidly, WeakHeight is *proof-generated*: it is the weakest height-principle which licenses my proof of synonymy![30] I concede that it goes beyond anything

[29]More fully spelled out, this would say: there is an ordinal which is isomorphic to the well-order obtained by taking a 0-indexed copy of $\mathrm{R}(\{w : Wand(w)\})$ and appending a 1-indexed copy of α.

[30]Specifically, it is exactly what Definition 9.4 requires; see Lemmas 9.6 and 9.7.

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mentioned in the Wand/Set Template—the Template itself allows that we might have precisely *forty-two* stages, for example—but I note that it really is *extraordinarily* weak for most mathematical purposes.[31] I adopt it going forward.

5.3. Variable axioms for hereditarily bland sets. So far, I have specified some *generic* axioms which will hold for any wand/set theory. I now need to start considering axioms which vary between theories.

To illustrate: Example 1.1 only required a single wand (see Section 1.1), but Church's [6] theory treated ω as the set of wands (see Section 1.3). Saying that ω exists makes a specific demand on the height of the hierarchy, which is independent of the axioms we have laid down so far. But it is a demand which, with Church, we may well want to make.

We can make demands on the height of the hierarchy in other ways too. For example: suppose we add ZF's Replacement scheme, restricting all the quantifiers in each scheme-instance to hereditarily bland sets; via perfectly familiar reasoning, this will ensure that our hierarchy is strongly inaccessible.

We might want to say this; we might not; it all depends on exactly how we want to implement the Wand/Set Template. In order to be maximally accommodating, I allow the following. A wand/set theory can take *any* further sentences as axioms, provided that they are the result of taking a sentence, ϕ, in the signature $\{Wand, \in\}$, and then restricting all of ϕ's quantifiers to hereditarily bland sets. (In what follows, I use the notation ϕ^{Heb} for this.)

Such axioms allow us to characterize the height of the hierarchy, precisely by characterizing the height of its hereditarily bland inner model. And any axiom which has been laid down in this way automatically conforms with loose constructivism. After all, the hereditarily bland sets are all available at the outset (see the end of Section 4.2).

Such axioms will also allow us to say that the set of wands is ω, or the least strongly inaccessible ordinal, or what-have-you. Of course, such axioms might lead us to inconsistency; there is no foolproof method for guarding against that. But nor should we expect such a method, any more than we expect one when we consider extensions of ZF.

5.4. Axioms for wand-taps. Finally, we need some axioms which tell us how wand-taps *behave*. Different wand/set theories will, of course, say different things at this point; but we can still offer some useful general principles.

[31] WeakHeight is immediately acceptable to anyone tempted by the vague but intuitive idea that the hierarchy "shouldn't stop too early". A good way to cash out this intuitive idea is via the precise claim: *there is no unbounded map from any bland set into the hierarchy's levels*. This entails a version of Replacement (see Shoenfield [20, pp. 323–326]; Incurvati [15, pp. 93–95]; Button [3, Section 7]), and Replacement immediately entails that the sum of any two ordinals exists. It also yields a compelling argument for the existence of $\{w : Wand(w)\}$. Suppose, for reductio, that $\{w : Wand(w)\}$ does not exist; so, via our newly minted Replacement-principle, there are proper-class many wands. Now consider an instance of the Template according to which tapping any object with any wand always yields a new object. So: at the outset, we find $\emptyset$. At the next stage, we find $\boxed{w}\emptyset$ for every wand w; by supposition, then, we find proper-class many entities. At the next stage we find all possible bland sets comprising only $\emptyset$ and the various $\boxed{w}\emptyset$'s; hence we find two-to-the-proper-class many bland sets; and this is a contradiction.

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Nothing I have said yet tells us whether (and when) we should find some object by tapping a with w, i.e., whether $\boxed{\scriptstyle w}a$ exists. We will address this by asking whether a is in w's "domain of action", written $Dom(w/a)$. Of course, different theories will specify different "domains" for different wands. So, to specify a wand/set theory, we must explicitly define some formula, $Dom(w/a)$, with all free variables displayed, and we then stipulate that Dom is a necessary and sufficient condition for finding something by wand-tapping:

Making $Dom(w/a) \leftrightarrow \exists c\ Tap(w/a, c)$.

We also need to know whether (and when) entities found by wand-tapping are identical, i.e., whether $\boxed{\scriptstyle w}a = \boxed{\scriptstyle u}b$. We will address this by asking whether w and a (in that order) are "equivalent" to u and b (in that order); we write this $Eq(w/a, u/b)$. Of course, different theories will use different notions of equivalence. So, to specify a wand/set theory, we must explicitly define some formula, $Eq(w/a, u/b)$, with all free variables displayed, and we then lay down a kind of quotienting principle:

Equating $Tap(w/a, c) \rightarrow \big(Tap(u/b, c) \leftrightarrow Eq(w/a, u/b)\big)$.

Otherwise put: if $\boxed{\scriptstyle w}a$ exists, then $\boxed{\scriptstyle w}a = \boxed{\scriptstyle u}b$ iff w, a is equivalent to u, b.

As remarked: the axioms Making and Equating invoke make use of formulas, Dom and Eq, which a wand/set theory must explicitly define. So the last question is: *What sorts of definitions are allowed?* Unsurprisingly, I want to maximize flexibility. So I want to say that any definitions are allowed, provided they meet these six constraints:

(dom1) $Dom(w/a)$ entails $Wand(w)$;
(dom2) $Dom(w/a)$ is loosely bound by $\mathbb{L}a$;
(dom3) Dom is preserved under equivalence, in that we always have:

$$(Eq(w/a, u/b) \wedge Dom(w/a)) \rightarrow Dom(u/b);$$

(eq1) $Eq(w/a, u/b)$ entails $Wand(w)$ and $Wand(u)$;
(eq2) $Eq(w/a, u/b)$ is loosely bound by $\mathbb{L}a \cup \mathbb{L}b$;
(eq3) Eq describes an equivalence relation, in that we always have:

$$Wand(w) \rightarrow Eq(w/a, w/a),$$
$$(Eq(w/a, u/b) \wedge Eq(w/a, v/c)) \rightarrow Eq(u/b, v/c).$$

These six constraints really just make explicit some of our implicit commitments. We insist on (dom1) and (eq1) because we are considering tapping things with wands, and only with wands. The Picture of Loose Constructivism forces (dom2) and (eq2) upon us: these constraints ensure that, when we (first) ask whether we should find anything by tapping a with w, or whether w, a is equivalent to u, b, we offer answers using only resources which are "available" to us there and then. The need for (eq3) is wholly obvious: we want Eq to define a kind of *equivalence*, which governs *identity* between wand-taps via Equating. The most interesting constraint is (dom3). To see why we need it: suppose that $Dom(w/a)$, so that tapping $\boxed{\scriptstyle w}a$ should indeed exist, by Making. Suppose also that w, a is equivalent to u, b; then $\boxed{\scriptstyle w}a = \boxed{\scriptstyle u}b$ by Equating. But then, by Making, we must also have that $Dom(u/b)$.

This explains *why* I have imposed conditions (dom1)–(dom3) and (eq1)–(eq3); it remains to explain *how* I will impose them.

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Intuitively, the idea is as follows. We can lay down any "attempted" definitions of *Dom* and *Eq* that we like; but if our "attempted" definitions ever stop behaving well, then we (thereafter) "default" to something which *is* guaranteed to behave well. So, this does not limit our ability to define wand/set theories; it simply ensures that *Dom* and *Eq* will satisfy (dom1)–(dom3) and (eq1)–(eq3), even if our "attempt" misses the mark.

Here are the details. Let $D(w/a)$ be any formula which is loosely bound by $\mathbb{L}a$, and let $E(w/a, u/b)$ be any formula which is loosely bound by $\mathbb{L}a \cup \mathbb{L}b$. (Think of these as our "attempted" definitions of *Dom* and *Eq*.) Now we define[32]

$$Dom(w/a) :\equiv Wand(w) \wedge D(w/a)$$

and

$$\begin{aligned} Eq(w/a, u/b) :\equiv & Wand(w) \text{ and } Wand(u) \text{ and either} \\ & w = u \wedge a = b; \text{ or} \qquad (1) \\ & E(w/a, u/b) \wedge \qquad (2) \\ & \quad D \text{ restricted to } \mathbb{L}a \cup \mathbb{L}b \text{ is preserved under } E \wedge \\ & \quad E \text{ restricted to } \mathbb{L}a \cup \mathbb{L}b \text{ is an equivalence relation.} \end{aligned}$$

By (1) and (2): if D or E stops behaving well at some level, then *Eq* collapses to strict identity at all higher levels. This all works as required to ensure that (dom1)–(eq3) hold (see Lemma 7.5). So our only constraint is this: we insist that our theory's predicates *Dom* and *Eq* are defined in this way, from suitable predicates D and E.

5.5. Definition of a wand/set theory. I am now in a position to define the general notion of a *wand/set theory*. This makes precise the idea of a loosely constructive instance of the Wand/Set Template.

DEFINITION 5.1. A wand/set theory, **S**, is any theory in the signature $\{Bland, Wand, \in, Tap\}$ which is axiomatized with all these axioms:

(1) InNB, TapNB, and TapFun (see Section 2);
(2) Ext, Sep, and Strat (see Section 3);
(3) HebWands and WeakHeight (see Sections 5.1 and 5.2);
(4) Equating and Making (see Section 5.4), which use two explicitly defined formulas, *Dom* and *Eq*, that meet (dom1)–(dom3) and (eq1)–(eq3).

Moreover, if **S** is axiomatized with any *further* axioms, then these are of the form ϕ^{Heb}, where ϕ is some sentence in the signature $\{Wand, \in\}$, and ϕ^{Heb} simply involves restricting all quantifiers in ϕ to *Heb* (see Section 5.3).

To recap: I have made some notational choices which impose a certain shape of formalism upon us; but these impose no substantial restrictions on our ability to flesh out the Template however we like (see, e.g., Section 2). My method for arranging entities into a hierarchy of wevels does nothing more than eliminate the need to talk about stages in our formalism (see Section 3). The Picture of Loose

[32] This is what it means, to say that E restricted to s is an equivalence relation: for any wands x, y, z, and for any $c, d, e \trianglelefteq s$, both $E(x/c, x/c)$ and also $(E(x/c, y/d) \wedge E(x/c, z/e)) \rightarrow E(y/d, z/e)$. This is what it means, to say that D restricted to s is preserved under E: for any wands x, y, and for any $c, d \trianglelefteq s$, we have $(D(x/c) \wedge E(x/c, y/d)) \rightarrow D(y/d)$.

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Constructivism yields a well-motivated explication of what it is to be "reasonably constructive" (see Section 4). The constraints on the notion of a wand's "domain of action", and of "equivalence" between wand-taps, are as permissive as they can be (see Section 5.4). I admit that I have gone *slightly* beyond the Template in making certain claims about the height of the hierarchy via WeakHeight, but these are *extremely* weak claims (see Section 5.2). Summing all this up: when we study all wand/set theories, in the sense of Definition 5.1, we study exactly the "reasonably constructive" ways to flesh out the Wand/Set Template (which accept a very weak principle about the hierarchy's height).

At this point, readers may wish to check that all of the examples discussed in Section 1 can be rendered *as* wand/set theories. I leave Examples 1.1–1.4 to the reader, but exhaustively deal with Church's [6] theory in Section 11.

§6. A ZF-like theory, LT_{S}. The Main Theorem of this paper is that any wand/set theory is synonymous with a ZF-like theory. I have explained what a wand/set theory is; I now need to articulate the "ZF-like theory" in question.

Of course, no single ZF-like theory is synonymous with *all* wand/set theories. Recall that wand/set theories can have any (arbitrary) axioms whose quantifiers are all restricted to hereditarily bland sets (see Section 5.3). What we need, then, is a canonical way to associate each wand/set theory, **S**, with a particular ZF-like theory, which I will call LT_{S}.

The choice of name references LT, or Level Theory. The crucial point is that LT is a pure theory of sets which says *exactly* enough to ensure that the sets are arranged into well-ordered levels. This is achieved with a rather terse definition:[33]

DEFINITION 6.1. Define $x \blacktriangleleft a$ iff $\exists r(x \subseteq r \in a)$. Say that a is potent iff $(\forall x \blacktriangleleft a)x \in a$. Let $¶a$ be $\{x : x \blacktriangleleft a\}$. Say that h is a history, $Hist(h)$, iff $(\forall a \in h)a = ¶(a \cap h)$. Say that s is a level, $Lev(s)$, iff $(\exists h : Hist)s = ¶h$. The axioms of (first-order) LT are then:

Extensionality $\forall a \forall b\big(\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b\big)$.

Separation $\forall a \exists b \forall x\big(x \in b \leftrightarrow (\phi(x) \wedge x \in a)\big)$, for any ϕ not containing 'b'.

Stratification $\forall a(\exists s : Lev)a \subseteq s$.

Now, recall that I want to associate each wand/set theory, **S**, with some theory, LT_{S}. This will be an extension of LT, which makes the same claims about the height of the hierarchy as **S**, whilst using a new predicate to single out certain special objects as proxies for **S**'s wands; we might as well just re-use the predicate "*Wand*" for this purpose. Here is the formal definition:

DEFINITION 6.2. Where **S** is any wand/set theory, the theory LT_{S} is defined as the theory with signature $\{\in, Wand\}$ and exactly these axioms:

(1) Each of the axioms of LT (allowing Separation-instances to include "*Wand*").
(2) An axiom, Pseudo-WeakHeight, which states: $\{w : Wand(w)\}$ exists; and for any ordinal α, both $\alpha \cup \{\alpha\}$ and $\mathrm{R}(\{w : Wand(w)\}) + \alpha$ exist.[34]
(3) An axiom, ϕ, for every **S**-axiom of the form ϕ^{Heb} (see Definition 5.1).

[33] See my [3]; and cf. Definition 3.1 and **S**'s axioms Ext, Sep, and Strat.

[34] This is not *verbatim* **S**'s axiom WeakHeight, since (for example) **S**'s axiom uses the predicate "*Bland*".

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My target result is then that every wand/set theory is synonymous with its associated ZF-like theory. More precisely:

MAIN THEOREM. LT_{S} *and* S *are synonymous, for any wand/set theory,* S.

What follows in the next four sections is nothing but a proof of the Main Theorem, without any further pauses for reflection. Throughout, I take S to be some fixed, arbitrary, wand/set theory. Here is my proof strategy.

In Section 7, I will provide some characteristic results concerning S.

In Section 8, I will define a translation $\flat : \mathsf{LT}_{\mathsf{S}} \longrightarrow \mathsf{S}$ and prove that it is an interpretation.

In Section 9, I will define a translation $\sharp : \mathsf{S} \longrightarrow \mathsf{LT}_{\mathsf{S}}$ and prove that it is an interpretation.

In Section 10, I will show that these two interpretations constitute a bi-interpretation. By the Friedman–Visser Theorem, it follows that LT_{S} and S are synonymous.

§7. Elementary results within S.

The aim of this section is to prove generic results about S. I start with basic results about wevels (Section 7.1), then consider some similar results about levels$_{\mathrm{u}}$ (Section 7.2).

7.1. Results about wevels.

The following definition is helpful for reasoning about wevels:

DEFINITION 7.1. Say that a is *wand-potent* iff $(\forall x \lhd a)x \in a$. Say that a is *wand-transitive* iff $(\forall x \in a)x \trianglelefteq a$.

Using this, we can replicate in S a sequence of elementary results which hold for LT (with tiny adjustments).[35] I leave the details to the reader, but they culminate in two central results:

LEMMA 7.2 (CoreWev). *s is a wevel iff $s = \mathbb{P}\{r \in s : Wev(r)\}$.*

THEOREM 7.3 (CoreWev). *The wevels are well-ordered by $\in$.*

We can also show that the wevels behave nicely:

LEMMA 7.4 (CoreWev, Strat). *For all a, b, and all wevels r, s:*

(1) *$\mathbb{L}a$ and $\mathbb{P}a$ exist, with $\mathbb{P}a \subseteq \mathbb{L}a$,*
(2) *$a \notin \mathbb{L}a$,*
(3) *$r \subseteq s$ iff $s \notin r$,*
(4) *$s = \mathbb{L}s$,*
(5) *if $b \subseteq a$ and both are bland, then $\mathbb{L}b \subseteq \mathbb{L}a$,*
(6) *if $b \in a$, then $\mathbb{L}b \in \mathbb{L}a$,*
(7) *$a \notin a$.*

Arranging everything into well-ordered wevels has many nice consequences. For example, recall that wand-tapping is controlled by the defined predicates *Dom* and *Eq* (see Section 5.4); now simple reasoning about wevels shows that my method for

[35] See Button [3, Results 3.2–3.10].

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ensuring the "good behaviour" of *Dom* and *Eq* works just as intended (I leave the proof to the reader):

LEMMA 7.5 (CoreWev, Strat). *Since Dom and Eq are defined as in Section 5.4, all of* (dom1)–(dom3) *and* (eq1)–(eq3) *hold.*

Here is another important but simple consequence. If c is not bland, there is some rank-minimal a such that c is obtained by tapping a with some wand:

LEMMA 7.6 (CoreWev, Strat). *If c is not bland, there are w and a such $Tap(w/a, c)$ and $\forall u \forall b(Tap(u/b, c) \rightarrow \mathrm{R}a \leq \mathrm{R}b)$. In that case, $\mathrm{R}a + 1 = \mathrm{R}c$.*

It will later be helpful to restate Lemma 7.6 in terms of *Eq* rather than *Tap*:

LEMMA 7.7 (CoreWev, Strat, Equating). *Say $Mini(w/a) :\equiv \forall u \forall b\big(Eq(w/a, u/b) \rightarrow \mathrm{R}a \leq \mathrm{R}b\big)$. If c is not bland, there are w and a such $\boxed{w}a = c$ and $Mini(w/a)$. In such case, $\mathrm{R}a + 1 = \mathrm{R}c$.*

This also yields an expected result: anything non-bland can be obtained from something bland using only finitely many wand taps:

LEMMA 7.8 (CoreWev, Strat, TapFun). *For any a, there is some n such that $(\exists b : Bland)a = \boxed{w_n} \dots \boxed{w_1}b$.*

PROOF. Here is the key idea. If a is bland, we are done. Otherwise, $a = \boxed{w}b$ for some b with $\mathrm{R}b + 1 = \mathrm{R}a$, using Lemma 7.6. If some such b is bland, we are done; otherwise, keep going...this involves a strictly descending chain of ordinals, and hence we will find some bland set after finitely many steps.

The only complication, in fact, comes from the need to define "$\boxed{w_n} \dots \boxed{w_1}b$".[36] As a first pass: where e is a function whose domain is a positive natural, define $BigTap(e, 0) := e(0)$ and $BigTap(e, n + 1) := \boxed{e(n+1)}(BigTap(e, n))$. Writing $e(i)$ as e_i, we can then think of $BigTap(e, n)$ as $\boxed{e_n} \dots \boxed{e_1}e_0$. ⊣

Note that there is no *uniqueness* condition associated with Lemma 7.8; there may be many (equally short) paths to the same object.

7.2. Results about levels$_\mathrm{u}$ and hereditary-blandness. I now move from considering wevels to considering levels$_\mathrm{u}$. Unsurprisingly, given their similar definitions, wevels and levels$_\mathrm{u}$ behave very similarly, provided we assume that u has no self-membered elements.[37] In detail, let CoreLev$_\mathrm{u}$ be the theory whose axioms are Ext, all instances of Sep, and $(\forall x \in \mathrm{u})\mathrm{u} \notin \mathrm{u}$. Next, we use a new definition (cf. Definition 7.1):

DEFINITION 7.9. Say that a is *potent*$_\mathrm{u}$ iff $(\forall x : Bland)((\exists c \notin \mathrm{u})(x \subseteq c \in a) \rightarrow x \in a)$. Say that a is *transitive*$_\mathrm{u}$ iff $(\forall x \notin \mathrm{u})(x \in a \rightarrow x \subseteq a)$.

---

[36]This needs an explicit definition, since we are reasoning about "finitely many steps" in the *object* language. See Section 9.2 for comments on recursive definitions in this sort of setting. What follows is a "first pass" definition, because its success in general requires that there be no last wevel. This can be secured using WeakHeight; but we can avoid the need for this by coding a few iterated wand-taps "manually" (cf. my [4, footnote 37]). For example, we can ask directly whether there are $v_0, \dots, v_4$ such that $\boxed{v_4}\boxed{v_3}\boxed{v_2}\boxed{v_1}v_0 = a$ and $\mathrm{R}v_0 + 4 = \mathrm{R}a$; and then search for a function e and some n with $BigTap(e, n) = v_0$. (Note also that the proof requires no version of Choice.)

[37]I could ensure this via Lemma 7.4(7). However, that would require the axiom Strat, and it is worth noting that we do not need the full strength of that axiom.

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The enthusiastic reader will now find it easy to prove these key facts:

LEMMA 7.10 (CoreLev$_{\text{u}}$). *s is a level$_{\text{u}}$ iff $s = \P_{\text{u}}\{r \in s : Lev_{\text{u}}(r)\}$.*

LEMMA 7.11 (CoreLev$_{\text{u}}$). *The levels$_u$ are well-ordered by $\in$.*

These results allow us to talk about *the* αth level$_{\text{u}}$, or V_{u}^{α},[38] which immediately vindicates the recursive characterization of the levels$_{\text{u}}$ offered in Section 4.2:

LEMMA 7.12 (CoreLev$_{\text{u}}$). $V_{\text{u}}^{\alpha} = \{x : Bland(x) \wedge (\exists \beta < \alpha) x \subseteq V_{\text{u}}^{\beta}\} \cup \text{u}$, *for any α.*

In Section 4.2, I also "unofficially" defined the hereditarily bland sets as the denizens of $\mathbf{V}_{\emptyset}$, i.e., as members of any $V_{\emptyset}^{\alpha}$, whilst "officially" offering Definition 4.3. To see that these definitions are equivalent, we just need to introduce the idea of the hereditarily bland part of a set; the rest is simple:

DEFINITION 7.13. Let $\breve{a} := \{x \in a : Heb(x)\}$.

LEMMA 7.14 (CoreWev, Strat). *a is hereditarily bland iff a is bland and every member of a is hereditarily bland.*

PROOF. *Left to right.* When c witnesses that a is hereditarily bland, also c witnesses that $x \in a$ is hereditarily bland. *Right to left.* In this case, $\breve{s}$ witnesses that a is hereditarily bland, for any wevel $s \supseteq a$, using Lemma 7.2. ⊣

LEMMA 7.15 (CoreWev, Strat). *a is hereditarily bland iff $\mathbf{V}_{\emptyset}(a)$.*

PROOF. By induction, using Lemmas 7.12 and 7.14. ⊣

§8. Interpreting LT$_{\mathsf{S}}$ in S. With some understanding of how **S** behaves, I will now show how to interpret LT$_{\mathsf{S}}$ in **S**. We defined LT$_{\mathsf{S}}$ precisely so that **S**'s hereditarily bland sets will interpret LT$_{\mathsf{S}}$, but here are the details. I start by defining the translation:

DEFINITION 8.1. Let $\flat$ be an identity-preserving translation, whose domain formula is *Heb*, and with atomic clauses:[39]

$$x \in_{\flat} y :\equiv x \in y \wedge Heb(y),$$
$$Wand_{\flat}(x) :\equiv Wand(x).$$

By HebWands, WeakHeight, and Lemma 7.14, **S** delivers Pseudo-WeakHeight$^{\flat}$ (see Definition 6.2(2)). Moreover, where ϕ^{Heb} is any **S**-axiom (see Definition 6.2(3)), ϕ is an LT$_{\mathsf{S}}$-axiom, and $\mathsf{S} \vdash \phi^{\flat} \leftrightarrow \phi^{Heb}$ by definition and Lemma 7.14. So it just remains to show that **S** proves Extensionality$^{\flat}$, Separation$^{\flat}$, and Stratification$^{\flat}$ (see Definition 6.2(1)). The first step is easy: by repeatedly applying Lemma 7.14 to **S**'s axioms Ext and Sep, we find:

LEMMA 8.2 (**S**). *Both Extensionality$^{\flat}$ and Separation$^{\flat}$ hold.*

The key step to establishing Stratification$^{\flat}$ is to show that the levels$^{\flat}$ are the hereditarily bland parts of wevels (recall Definition 7.13); I leave this to the reader:[40]

---

[38] We do not assume this exists for every α and u; the point is that V_{u}^{α} is unique *if* it exists.

[39] Note that **S** proves $Wand_{\flat}(x) \rightarrow Heb(x)$ by HebWands, and $x \in_{\flat} y \rightarrow Heb(x)$ by Lemma 7.14.

[40] The proof is almost exactly as in my [4, Lemmas C.4 and C.5].

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LEMMA 8.3 (S). *a is a level$^\flat$ iff* $(\exists s : Wev)a = \breve{s}$.

Now Stratification$^\flat$ follows immediately via Strat, so that:

PROPOSITION 8.4 (S). $\flat : \mathsf{LT}_{\mathsf{S}} \longrightarrow \mathsf{S}$ *is an interpretation.*

§9. Interpreting S in LT_{S}. I now show how to interpret S in LT_{S}. This is much more challenging. The hardest part is to define the right translation (see Section 9.1); then it is just a matter of laboriously checking that the definition works as intended (see Sections 9.2–9.4).

9.1. Defining the translation. It will help to start with an informal overview of how I will define the translation. I start by defining the things which will serve as the domain of interpretation; I call these things *conches*, as a rough portmanteau of "*Con*way" and "*Ch*urch". Here are basic ideas about conches.

– We need a way record the result of "tapping" some earlier-found object, a, with some "wand", w. (Scare-quotes are needed, since this is all under interpretation.) We do this by considering the pair $\langle w, a\rangle$.
– We need to quotient such pairs under an equivalence relation (the interpretation of Eq); so a conch will typically be a set of such equivalent pairs. To ensure that we have a *set* of such pairs, we use a version of Scott's trick, taking only the equivalent pairs of minimal rank (for some suitable notion of rank).
– Finally, we need a way to consider "bland" entities; we take them to have the form $\{\langle\emptyset, a\rangle\}$, which I write $\uparrow a$. (So I will insist that $\emptyset$ is not a "wand", but a marker of "blandness".)

Where $\sharp$ is the translation that I will define, I will set things up so that $x \in_\sharp \uparrow a$ iff $x \in a$. This allows me to compress most of the above into the following statement of intent (which I ultimately prove in Section 9.4):

Corollary 9.27. c is a conch iff either:

(1) $c = \uparrow s$, for some s whose members are all conches; or
(2) $c = \{\langle w, a\rangle : Tap_\sharp(w/a, c) \wedge Mini^\sharp(w/a)\} \neq \emptyset$.

Condition (1) will hold iff c is bland$_\sharp$. Condition (2) will hold iff c is non-bland$_\sharp$, whereupon $\langle w, a\rangle \in c$ iff c is found by "tapping" a with "wand" w (and not by "tapping" anything of lower rank than a).

It should be clear that, if we are careful, we will indeed have interpreted S in LT_{S}. It just remains to thrash through the details.

In S, the hereditarily bland sets form a sort of "spine" around which the rest of the hierarchy is constructed. So I will start by explaining how to simulate S's hereditarily bland sets within LT_{S}. In fact, this simulation is dictated by the choices I made above:

DEFINITION 9.1 (LT_{S}). Let $\uparrow a := \{\langle\emptyset, a\rangle\}$. When $a = \{\langle\emptyset, c\rangle\} = \uparrow c$, let $\downarrow a := c$, so that $\downarrow\uparrow c = c$. Recursively let $\Theta a := \uparrow\{\Theta x : x \in a\}$.

Note that LT_{S} proves both that Θ is injective and that $\emptyset \neq \Theta x$ for any x (see Lemma 9.6). So, having used $\emptyset$ to code "bland" conches, I can say that a "wand" is any Θ-image of a *Wand*. This motivates the following:

Definition 9.2. For any w, x, y:

$$Bland_*(y) :\equiv \exists z\ y = \uparrow z.$$
$$x \in_* y :\equiv Bland_*(y) \wedge x \in \downarrow y.$$
$$Wand_*(w) :\equiv (\exists v : Wand) w = \Theta v.$$

This definition forms the heart of my translation $\sharp : \mathsf{LT_S} \longrightarrow \mathbf{S}$. However, I cannot complete that translation until I have defined a suitable domain formula, i.e., until I have said what the conches will be. Unfortunately, this will takes some time.

In Section 4, I explicated the Picture of Loose Constructivism. The general idea was to treat the objects found at some stage of the hierarchy as urelements, and then to build a "bland-hierarchy" from those urelements. My first job is to translate this idea into $\mathsf{LT_S}$; I do this by recursively defining $\mathbf{U}_\mathrm{u}$ (cf. Definition 4.1 and Lemma 7.12):

Definition 9.3 ($\mathsf{LT_S}$). For any set u, recursively define (where this exists):

$$U^\alpha_\mathrm{u} := \{\uparrow x : (\exists \beta < \alpha) x \subseteq U^\beta_\mathrm{u}\} \cup \mathrm{u}.$$

Say that $\mathbf{U}_\mathrm{u}(x)$ iff $\exists \alpha\ x \in U^\alpha_\mathrm{u}$.

The next task is to specify what the stages of our "**S**-hierarchy" will look like. Specifically, I must define the conches and associate them with some notion of *rank* according, intuitively, to the stage at which they are first found.

To effect this, I define, for each ordinal σ, the sets C_σ, D_σ, and E_σ. Here are the guiding ideas:

- C_σ is a set whose members are the conches of rank $\leq \sigma$ (mnemonically: Conch).
- B_σ is a whose members are the conches of rank $< \sigma$ (mnemonically: Below).
- I write $|a| = \sigma$ to indicate that a is a conch of rank σ.
- D_σ and E_σ respectively simulate *Dom* and *Eq* for conches of rank $\leq \sigma$. Indeed, we will ultimately set things up so that:

$$\langle w, a\rangle \in \mathrm{D}_\sigma \text{ iff } |a| \leq \sigma \text{ and } Dom^\sharp(w/a),$$
$$\langle w, a, u, b\rangle \in \mathrm{E}_\sigma \text{ iff } |a|, |b| \leq \sigma \text{ and } Eq^\sharp(w/a, u/b).$$

The guiding principles are quite clear; alas, the formal definition itself is quite long. Indeed, it is long enough for me to need to interrupt it with commentary to make it intelligible. Here is how it starts.

Definition 9.4 ($\mathsf{LT_S}$). For each ordinal σ, let

$$\mathrm{C}_\sigma := \{\uparrow s : s \subseteq \mathrm{B}_\sigma\} \cup \{c : (\exists \tau < \sigma)(c \text{ is a } \tau\text{- tap})\},$$

where I write $\mathrm{B}_\sigma := \bigcup_{\tau<\sigma} \mathrm{C}_\tau$. Say that x is a conch, written $Conch(x)$, iff $\exists \sigma\ x \in \mathrm{C}_\sigma$. For each conch x, let $|x|$ be the least σ such that $x \in \mathrm{C}_\sigma$. TBC

This formal definition is incomplete (hence the "To be continued"). In particular, I have used the phrase "a τ-tap", which I have not yet defined. However, I should pause to explain the definition for C_σ. The component $\{\uparrow s : s \subseteq \mathrm{B}_\sigma\}$ collects "bland" conches: when s is a set of conches, $\uparrow s$ will act as a "bland" conch with exactly the members of s as "members", and $\uparrow s$'s rank will be the supremum of its "members". Then $\{c : (\exists \tau < \sigma)(c \text{ is a } \tau\text{-tap})\}$ will collect the "non-bland" conches.

To complete Definition 9.4, then, I must define what I mean by a τ-tap; or, changing index for convenience, a σ-tap. Roughly speaking, a σ-tap should be a "non-bland" conch of rank $\sigma + 1$ which was obtained by tapping some conch of rank σ with some "wand". More precisely:

DEFINITION (9.4, CONT.): For each ordinal σ, say that c is a σ-tap iff c is a non-empty set of ordered pairs such that, for all $\langle w, a\rangle \in c$, these three conditions hold:

$$|a| = \sigma \text{ and if } \langle w, a, u, b\rangle \in \mathrm{E}_\sigma, \text{ then } |b| = \sigma. \tag{tapr}$$

$$\langle w, a\rangle \in \mathrm{D}_\sigma. \tag{tapd}$$

$$\langle w, a, u, b\rangle \in \mathrm{E}_\sigma \text{ iff } \langle u, b\rangle \in c. \tag{tape}$$

TBC

Again, I will pause to explain this. Roughly: (tapr) says that a has rank σ and that this rank is as small as possible; (tapd) says that a is within w's "domain of action"; and (tape) says that c is an equivalence class of minimally ranked entities. But of course these clauses can only have this effect if we have appropriately defined D_σ and E_σ. So we offer:

DEFINITION (9.4, CONT.): For each ordinal σ:

$$\begin{aligned}
\mathrm{D}_\sigma &:= \{\langle w, a\rangle : |a| \leq \sigma \text{ and } \mathit{Wand}_{\tilde{\sigma}}(w) \text{ and } \mathit{Dom}^{\tilde{\sigma}}(w/a)\}.\\
\mathrm{E}_\sigma &:= \{\langle w, a, u, b\rangle : |a|, |b| \leq \sigma \text{ and } \mathit{Wand}_{\tilde{\sigma}}(w) \text{ and } \mathit{Wand}_{\tilde{\sigma}}(u) \text{ and}\\
&\qquad \mathit{Eq}^{\tilde{\sigma}}(w/a, u/b) \vee (w = u \wedge a = b)\}.
\end{aligned}$$

TBC

Interrupting again: whatever the exact meaning of the $\tilde{\sigma}$-translation, it should be clear how the definitions of D_σ and E_σ are supposed to fit with our intentions (see the comments just before the start of Definition 9.4). So it just remains to define each $\tilde{\sigma}$:

DEFINITION (9.4, FINISHED.): For each ordinal, σ, define an identity-preserving translation $\tilde{\sigma}$ as follows (using Δ_I for I's domain formula):

$$\begin{aligned}
\Delta_{\tilde{\sigma}}(x) &:\equiv \mathbf{U}_{\mathrm{C}_\sigma}(x),\\
\mathit{Bland}_{\tilde{\sigma}}(y) &:\equiv \mathit{Bland}_*(y) \wedge \Delta_{\tilde{\sigma}}(y),\\
x \in_{\tilde{\sigma}} y &:\equiv x \in_* y \wedge \Delta_{\tilde{\sigma}}(y),\\
\mathit{Wand}_{\tilde{\sigma}}(w) &:\equiv \mathit{Wand}_*(w) \wedge \mathbf{U}_\emptyset(w),\\
\mathit{Tap}_{\tilde{\sigma}}(w/a, c) &:\equiv \langle w, a\rangle \in \mathrm{D}_{|a|} \wedge |a| < \sigma \wedge |c| \leq \sigma \wedge\\
&\qquad \exists u \exists b \big(\langle w, a, u, b\rangle \in \mathrm{E}_{|a|} \wedge \langle u, b\rangle \in c\big).
\end{aligned}$$

The domain formula, $\mathbf{U}_{\mathrm{C}_\sigma}$, ensures that $\tilde{\sigma}$ quantifies only over entities which are "available" at the σth "wevel". We then translate *Bland* and $\in$ as suggested by Definition 9.2, whilst ensuring that we are dealing with entities in the intended "domain". The clause for $\mathit{Wand}_{\tilde{\sigma}}$ is similar but more restrictive, and reflects the intention that "wands are hereditarily bland" (see Proposition 9.13 for more). Finally, the idea for $\mathit{Tap}_{\tilde{\sigma}}$ is as follows: if c is found by "tapping" a with w, then c is a τ-tap, for some $\tau < \sigma$.

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Definition 9.4 is finally complete. And I can use it to define the $\sharp$-translation as the "limit" of the $\tilde{\sigma}$-translations.

DEFINITION 9.5 ($\mathsf{LT_S}$). Define $\sharp$ as an identity-preserving translation as follows:

$$\begin{aligned}
\Delta_\sharp(x) &:\equiv \mathit{Conch}(x),\\
\mathit{Bland}_\sharp(y) &:\equiv \mathit{Bland}_*(y) \wedge \Delta_\sharp(y),\\
x \in_\sharp y &:\equiv x \in_* y \wedge \Delta_\sharp(y),\\
\mathit{Wand}_\sharp(w) &:\equiv \mathit{Wand}_*(w) \wedge \mathbf{U}_\emptyset(w),\\
\mathit{Tap}_\sharp(w/a, c) &:\equiv \langle w, a\rangle \in \mathrm{D}_{|a|} \wedge \Delta_\sharp(c) \wedge\\
&\qquad \exists u \exists b\big(\langle w, a, u, b\rangle \in \mathrm{E}_{|a|} \wedge \langle u, b\rangle \in c\big).
\end{aligned}$$

I hope that my informal explanations make it plausible that $\sharp$ is, indeed, an interpretation. The remainder of this section simply verifies this fact.

9.2. Confirming the recursions. I must start by commenting on my use of recursive definitions. In brief: recursive definitions make good sense in $\mathsf{LT_S}$, but the defined term-function may be *partial*. Here is a slightly fuller explanation of this point.

Given a term, $\mathbf{t}$, we can stipulate (as usual) that f is an α-approximation to $\mathbf{t}$ iff $\mathrm{dom}(f) = \alpha$ and $(\forall \beta < \alpha) f(\beta) = \mathbf{t}(f{\restriction_\beta})$. Ordinal induction will establish that α-approximations agree for any shared arguments, so we can write $f^{\mathbf{t}}_\alpha$ for the unique α-approximation, if any exists. Where we then explicitly define

$$\mathbf{r}(\alpha) \coloneqq f^{\mathbf{t}}_{\alpha+1}(\alpha)$$

we can prove a version of the general recursion theorem:

$$\begin{aligned}
&\mathbf{r}(\alpha) = \mathbf{t}(\{\langle \beta, \mathbf{r}(\beta)\rangle : \beta < \alpha\}), && \text{if } f^{\mathbf{t}}_{\alpha+1} \text{exists},\\
&\mathbf{r}(\alpha) \text{ is undefined}, && \text{otherwise.}
\end{aligned}$$

All of this is as in ZF. The sole difference is that ZF proves that $f^{\mathbf{t}}_{\alpha+1}$ must exist for any α, and $\mathsf{LT_S}$ might not prove this. So $\mathbf{r}$ may be *partial*.

It is, then, important to confirm that many of the notions defined in Section 9.1 are *total*. To do this, we can take a slight shortcut. Pseudo-WeakHeight ensures that there is no last level. So, where $\mathbf{r}$ is recursively defined from $\mathbf{t}$ as above, to show that every $\mathbf{r}(\alpha)$ exists, we need only show that there must always be an ordinal corresponding to "what $\mathbf{r}(\alpha)$'s rank would have to be".

We will first use this strategy to show that $U^\alpha_\emptyset$ and Θ are total.

LEMMA 9.6 ($\mathsf{LT_S}$). *For any α, $U^\alpha_\emptyset$ exists. For any a, both $\Theta a \in U^{\mathrm{R}a+1}_\emptyset$ and $\mathrm{R}a = |\Theta a|$. Indeed, the map $\Theta : V \longrightarrow \mathbf{U}_\emptyset$ is a total bijection.*

PROOF. Where α is any ordinal, write it as $\alpha = \lambda_\alpha + n_\alpha$, with λ_α either a limit ordinal or 0, and n_α a natural number. By induction, $\mathrm{R}(U^\alpha_\emptyset) = \lambda_\alpha + 4n_\alpha$, so each $U^\alpha_\emptyset$ exists.

A simple $\in$-induction shows that Θ is injective. Suppose for induction that $b \in U^{\mathrm{R}b+1}_\emptyset$ whenever $b \in a$. As $\mathrm{R}a = \mathrm{lsub}_{b \in a} \mathrm{R}b$, we have ${\downarrow}\Theta a \subseteq U^{\mathrm{R}a}_\emptyset$, so $\Theta a \in U^{\mathrm{R}a+1}_\emptyset$. Suppose also that $\mathrm{R}b = |\Theta b|$ whenever $b \in a$; then

$$\mathrm{R}a = \operatorname*{lsub}_{b \in a} \mathrm{R}b = \operatorname*{lsub}_{b \in a} |\Theta b| = \operatorname*{lsub}_{x \in {\downarrow}\Theta a} |x| = |\Theta a|.$$

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It follows that $\Theta : V \longrightarrow \mathbf{U}_\emptyset$ is total. For surjectivity, suppose for induction that if $x \in U^\gamma_\emptyset$ then $\Theta^{-1}x$ exists, whenever $\gamma < \beta$. Let $b \in U^\beta_\emptyset$; then $a = \{\Theta^{-1}x : x \in \downarrow b\}$ exists by Separation, and and $\Theta a = \uparrow\downarrow b = b$. ⊣

Note that $\mathsf{LT}_\mathbf{S}$ might *not* prove that U^α_u exists for every α and every u. This is not a problem; we do not in general require that it does. (Similarly: $\mathbf{S}$ might not prove that V^α_u exists for every u and α.) Unsurprisingly, though, we *do* require that C_α exists for every α. And it provably does. Indeed (Pseudo-)WeakHeight was chosen precisely to secure this fact.

Lemma 9.7 ($\mathsf{LT}_\mathbf{S}$). *For any α, each of $\mathrm{B}_\alpha, \mathrm{C}_\alpha, \mathrm{D}_\alpha$, and E_α exists.*

Proof. It suffices to establish the result for C_α. Let $\Omega = \mathrm{lub}_{w \text{ a wand}} \mathrm{R}(\Theta w)$. Writing each α as $\lambda_\alpha + n_\alpha$, I claim that

$$\mathrm{RC}_\alpha \leq \Omega + \lambda_\alpha + 4n_\alpha + 4.$$

By Lemma 9.6, there is m such that $\Omega = \mathrm{R}(\{w : Wand(w)\}) + m$; so C_α's required rank exists by my claim and Pseudo-WeakHeight.

It just remains to establish my claim. This is by induction. Trivially, $\mathrm{RC}_0 = 4$.

Successor case. Suppose $\mathrm{RC}_\alpha \leq \Omega + \lambda_\alpha + 4n_\alpha + 4$. If $s \subseteq \mathrm{B}_{\alpha+1} = \mathrm{C}_\alpha$, then

$$\mathrm{R}(\uparrow s) \leq \Omega + \lambda_\alpha + 4n_\alpha + 7.$$

If c is a β-tap with $\beta \leq \alpha$, then c's members are of the form $\langle \Theta w, d\rangle$, with w a wand and $\mathrm{R}d \leq \Omega + \lambda_\alpha + 4n_\alpha + 3$, and since $\mathrm{R}(\langle \Theta w, d\rangle) = \max(\mathrm{R}\Theta w, \mathrm{R}d) + 2$, we obtain:

$$\mathrm{R}c \leq \Omega + \lambda_\alpha + 4n_\alpha + 6.$$

So $\mathrm{RC}_\alpha \leq \Omega + \lambda_\alpha + 4n_\alpha + 8$, as required.

Limit case. Using the induction hypothesis, if $s \subseteq \mathrm{B}_\alpha$ then $\mathrm{R}(\uparrow s) \leq \mathrm{lub}_{\beta<\alpha}(\Omega + \lambda_\beta + 4n_\beta + 4) + 3 = \Omega + \alpha + 3$. So $\mathrm{R}(\mathrm{C}_\alpha) \leq \Omega + \alpha + 4$, as required. ⊣

9.3. $\star$-translations and blandness$_\star$. I will now look at the $\sharp$ and $\tilde{\sigma}$ translations. For readability, I use $\star$ to stand ambiguously for either $\sharp$ or any $\tilde{\sigma}$. So if I assert that $\phi^\star$, I mean that $\phi^\sharp$ and $\phi^{\tilde{\sigma}}$ for any σ. Recall that $\Delta_\star$ is always $\star$'s domain-formula.

I start with some elementary results about bland$_\star$ conches. These results are pretty immediate consequences of the definitions (and previous results) and I leave their proofs to the reader:

Lemma 9.8 ($\mathsf{LT}_\mathbf{S}$). *Both InNB* and TapNB* hold.*

Lemma 9.9 ($\mathsf{LT}_\mathbf{S}$). *If $\Delta_\star(c)$, then: c is not bland$_\star$ iff c is some σ-tap.*

Lemma 9.10 ($\mathsf{LT}_\mathbf{S}$). *If Bland$_\star(a)$ then $\Delta_\star(a)$. If $b \in_\star a$, then $\Delta_\star(b)$ and $\Delta_\star(a)$. If Tap$_\star(w/a, c)$, then $\Delta_\star(w)$, $\Delta_\star(a)$ and $\Delta_\star(c)$. So $\star$ is a translation.*

Lemma 9.11 ($\mathsf{LT}_\mathbf{S}$). *If $\Delta_\star(\uparrow a)$, then:*

(1) $|\uparrow a| = \mathrm{lsub}_{x \in a} |x|$.
(2) *$b \in a$ iff $b \in_\star \uparrow a$.*
(3) *$c \subseteq a$ iff $\uparrow c \subseteq^\star \uparrow a \wedge \Delta_\star(\uparrow c)$.*

Lemma 9.12 ($\mathsf{LT}_\mathbf{S}$). *Ext* and Sep* (scheme) hold.*

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I now want to consider hereditarily bland⋆ conches. In particular, I want to show that Θ is an "isomorphism" to the hereditarily bland⋆ conches:

PROPOSITION 9.13 ($\mathsf{LT_S}$). *For any a, b, w:*

(1) *$Heb^\star(b)$ iff $\mathbf{U}_\emptyset(b)$, so that $\Theta : V \longrightarrow Heb^\star$ is a total bijection.*
(2) *$a \in b$ iff $\Theta a \in^{\flat\star} \Theta b$ iff $\Theta a \in_\star \Theta b$.*
(3) *$Wand(w)$ iff $Wand^{\flat\star}(\Theta w)$ iff $Wand_\star(\Theta w)$.*
(4) *b is an ordinal⋆ iff $\exists \alpha\, b = \Theta\alpha$, whereupon $\alpha = |b|$.*

PROOF. *(1)* If $Heb^\star(b)$ then $b \in U_\emptyset^{|b|+1}$ by induction on rank using Lemma 9.11. For the converse, suppose $\mathbf{U}_\emptyset(b)$. By construction, $Bland_\star(b)$ and $\downarrow b \subseteq U_\emptyset^\beta$ for some β. Now $\uparrow U_\emptyset^\beta$ witnesses that $Heb^\star(b)$. After all, $b \subseteq^\star \uparrow U_\emptyset^\beta$ by Lemma 9.11. And if $x \in_\star \uparrow U_\emptyset^\beta$, i.e., $x \in U_\emptyset^\beta$, then $Bland_\star(x)$ and $\downarrow x \subseteq U_\emptyset^\beta$ so that $x \subseteq^\star \uparrow U_\emptyset^\beta$ by Lemma 9.11.

This establishes the biconditional; "so that" holds via Lemma 9.6.

(2)–(3) Immediate from (1).

(4) By induction, using Θ's injectivity and (2), if α is any ordinal, then $\Theta\alpha$ is an ordinal⋆. Furthermore, $\alpha = \mathrm{R}\alpha = |\Theta\alpha|$ by Lemma 9.6. Now let c be an ordinal⋆ and let $d = \Theta(|c|)$. So $|c| = |d|$ as above, so that $c \notin_\star d$ and $d \notin_\star c$ by Lemma 9.11; so $c = d = \Theta(|c|)$. ⊣

It follows that hereditary blandness⋆ does not depend on the choice of $\star$. We also obtain the $\star$-translations of any S-axioms which only concern hereditarily bland sets:

LEMMA 9.14 ($\mathsf{LT_S}$). *HebWands⋆ holds, as does $(\phi^{Heb})^\star$, for any S-axiom ϕ^{Heb}.*

PROOF. By Proposition 9.13(1), every wand$_\star$ is hereditarily bland⋆. Next, by Definition 6.2: whenever ϕ^{Heb} is an S-axiom, ϕ is an $\mathsf{LT_S}$-axiom. Now note that, for any sentence ϕ in $\mathsf{LT_S}$'s signature, $\phi \leftrightarrow (\phi^{Heb})^\star$; this holds by an induction on complexity using Proposition 9.13. ⊣

9.4. Wevels⋆ and levels$^\star_\mathbf{u}$. I will now move on to considering "wand taps", starting with some easy facts:

LEMMA 9.15 ($\mathsf{LT_S}$). *If $\Delta_\star(c)$ and c is an α-tap, then:*

(1) *$|c| = \alpha + 1$, and $|a| = \alpha$ for every $\langle w, a\rangle \in c$; if also $\star = \tilde{\sigma}$ then $\alpha + 1 \leq \sigma$.*
(2) *If $Tap_\star(w/a, c)$, then $|c| \leq |a| + 1$.*
(3) *If $\langle w, a\rangle \in c$, then $Tap_\star(w/a, c)$.*

This allows us to describe the moment when a conch is "first found":

LEMMA 9.16 ($\mathsf{LT_S}$). *If $\Delta_\star(c)$ and $\Delta_\star(\uparrow \mathrm{B}_\alpha)$, then $|c| \leq \alpha$ iff $c \trianglelefteq^\star \uparrow \mathrm{B}_\alpha$.*

PROOF. If c is bland$_\star$, use Lemma 9.11. Otherwise, c is some σ-tap by Lemma 9.9; now use Lemmas 9.11 and 9.15. ⊣

The next challenge is characterize the wevels⋆. This is easier than we might fear. The basic facts about wevels follow from CoreWev (see Sections 3 and 7.1); so the $\star$-translations of those basic facts hold by Lemmas 9.8 and 9.12. Specifically: $\in_\star$ well-orders the wevels⋆ (by Theorem 7.3⋆), and a wevel⋆ is exactly the bland$_\star$ set

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of everything found-earlier* (by Lemma 7.2*). Once we combine this with Lemma 9.16, we can characterize the wevels* in terms of the B_α's:

LEMMA 9.17 (LT_S). *If $\Delta_\star(\mathord{\uparrow}\mathrm{B}_\alpha)$, then $\mathord{\uparrow}\mathrm{B}_\alpha$ is the αth wevel*.*

PROOF. Whenever $\Delta_\star(c)$, Lemma 9.16 tells us that $|c| < \alpha$ iff $(\exists\beta < \alpha)c \trianglelefteq^\star \mathord{\uparrow}\mathrm{B}_\beta$. So suppose for induction that $\mathord{\uparrow}\mathrm{B}_\beta$ is the βth wevel*, whenever $\beta < \alpha$. Then by Lemma 9.11, the earlier biconditional becomes

$$c \in_\star \mathord{\uparrow}\mathrm{B}_\alpha \text{ iff } (\exists r : \mathit{Wev}^\star)c \trianglelefteq^\star r \in_\star \mathord{\uparrow}\mathrm{B}_\alpha.$$

So $\mathord{\uparrow}\mathrm{B}_\alpha$ is the αth wevel*, by Lemma 7.2*. Now use induction, Theorem 7.3*. ⊣

Combining Lemmas 9.16 and 9.17 allows us to characterize $\mathbb{L}^\star$:

LEMMA 9.18 (LT_S). *If $\Delta_\star(c)$ and $\Delta_\star(\mathord{\uparrow}\mathrm{B}_{|c|})$, then $\mathbb{L}^\star(c) = \mathord{\uparrow}\mathrm{B}_{|c|}$ and $\mathrm{R}^\star(c) = \Theta(|c|)$.*

PROOF. In this case, $c \trianglelefteq^\star \mathord{\uparrow}\mathrm{B}_{|c|}$, and $|c|$ is minimal here by Lemma 9.16. By Lemma 9.17, $\mathord{\uparrow}\mathrm{B}_{|c|}$ is the $|c|$th wevel*. By Proposition 9.13(4), the $|c|$th ordinal* is $\Theta(|c|)$, i.e., $\mathrm{R}^\star(c) = \Theta(|c|)$. ⊣

This delivers the $\sharp$-translation of two of **S**'s axioms:

LEMMA 9.19 (LT_S). *Strat$^\sharp$ and WeakHeight$^\sharp$ hold.*

PROOF. *Strat*$^\sharp$. By Lemma 9.18, as $\Delta_\sharp(\mathord{\uparrow}\mathrm{B}_{|c|})$ for any conch c.

WeakHeight$^\sharp$. Using Pseudo-WeakHeight and Lemma 9.6, let $a = \{w : \mathit{Wand}(w)\}$, and let $b = \Theta a = (\{w : \mathit{Wand}(w)\})^\sharp$. Now $\Theta(\mathrm{R}a) = \Theta(|b|) = \mathrm{R}^\sharp b$ by Lemmas 9.6 and 9.18. Also, $\mathrm{R}a + \alpha$ exists for any ordinal α, by Pseudo-WeakHeight. So $\Theta(\mathrm{R}a + \alpha) = \Theta(\mathrm{R}a) +^\sharp \Theta\alpha$ is an ordinal$^\sharp$ by Proposition 9.13; so $\mathrm{R}^\sharp b +^\sharp x$ exists for any ordinal$^\sharp$ x by Proposition 9.13(4). ⊣

Note that this only gives us the $\sharp$-translations of Strat and WeakHeight. Indeed, we should *not* expect their $\tilde{\sigma}$-translations to hold: denizens of $\Delta_{\tilde{\sigma}}$ must sit at some level$^{\tilde{\sigma}}_{\uparrow\mathrm{C}_\sigma}$ but may come after any wevel$^{\tilde{\sigma}}$. Our next job, then, must be to characterize the levels$^\star_{\uparrow\mathrm{u}}$. This can be done along the same line as the wevels*, mimicking the proof of Lemma 9.17:

LEMMA 9.20 (LT_S). *If $\Delta_\star(\mathord{\uparrow}\mathrm{u})$, then $\mathord{\uparrow}U^\alpha_\mathrm{u}$ is the αth level$^\star_{\uparrow\mathrm{u}}$.*

PROOF. By Lemmas 9.11, 9.16, 7.10*, and 7.11*. ⊣

Now that we understand wevels* and levels$^\star_{\uparrow\mathrm{u}}$, we can $\star$-translate the central notion used in the Picture of Loose Constructivism, i.e., $\mathbf{V}_\mathrm{u}$:

LEMMA 9.21 (LT_S). *If $\Delta_\star(\mathord{\uparrow}\mathrm{B}_\alpha)$, these are equivalent predicates:*

$$\mathbf{U}_{\mathrm{B}_\alpha}(x) \qquad \exists w\big(w = \mathord{\uparrow}\mathrm{B}_\alpha \wedge (\mathbf{V}_w(x))^\star\big) \qquad \exists w\big(|w| = \alpha \wedge (\mathbf{V}_{\mathbb{L}w}(x))^\star\big).$$

PROOF. Use Lemmas 9.18 and 9.20 respectively for the two equivalences. ⊣

This delivers a key fact: satisfaction of loosely bound formulas is preserved as we expand our interpretations, moving (with $\sigma < \tau$) from $\tilde{\sigma}$ to $\tilde{\tau}$ and ultimately to $\sharp$:

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Proposition 9.22 ($\mathsf{LT_S}$, scheme). *Let* $\phi(v_1, \dots, v_n)$ *be loosely bound by* $\max(\mathbb{L}v_1, \dots, \mathbb{L}v_n)$. *When* $\max(|c_1|, \dots, |c_n|) \leq \sigma < \tau$, *these are equivalent*:

$$\phi^{\tilde{\sigma}}(c_1, \dots, c_n) \qquad \phi^{\tilde{\tau}}(c_1, \dots, c_n) \qquad \phi^{\sharp}(c_1, \dots, c_n).$$

Proof. Let $n = 1$, and consider $v = v_1$ and $c = c_1$ with $|c| = \gamma \leq \sigma$. (This is for readability; the same proof works for $n > 1$.) Since $\phi(v)$ is loosely bound by $\mathbb{L}v$:

(1) $\phi(v)$'s free and bound variables are restricted to $\mathbf{V}_{(\mathbb{L}v)^+}$.
(2) Any instance of $Tap(w/a, x)$ in $\phi(v)$ is conjoined to $a \in \mathbb{L}v$.

By (1), $\phi^{\tilde{\sigma}}(v)$'s free and bound variables are restricted to $(\mathbf{V}_{(\mathbb{L}v)^+})^{\tilde{\sigma}}$. Since $\gamma \leq \sigma$, we have $\Delta_{\tilde{\sigma}}(\uparrow\mathrm{C}_\gamma)$; so, substituting c for v, by Lemma 9.21, $\phi^{\tilde{\sigma}}(c)$'s free and bound variables are, equivalently, restricted to $\mathbf{U}_{\mathrm{C}_\gamma}$. Now, $\tilde{\sigma}$-translation also restricts all quantifiers to $\mathbf{U}_{\mathrm{C}_\sigma}$; but since $\gamma \leq \sigma$, this imposes no *additional* restriction. So we have (up to equivalence):

(1c) $\phi^{\tilde{\sigma}}(c)$'s free and bound variables are restricted to $\mathbf{U}_{\mathrm{C}_\gamma}$.

Similar reasoning using Lemma 9.18 gives us (up to equivalence):

(2c) any instance of $Tap_{\tilde{\sigma}}(w/a, x)$ in $\phi^{\tilde{\sigma}}(c)$ is conjoined to $|a| < \gamma$.

Finally, recall that $\tilde{\sigma}$, $\tilde{\tau}$ and $\sharp$ explicitly agree on everything, except on any mentions of $\Delta_\star$ and on their rendering of wand-taps. But the differences concerning $\Delta_\star$ are irrelevant here, since $\gamma \leq \sigma < \tau$, so that $\tilde{\tau}$ and $\sharp$ yield the same restrictions (1c)–(2c). Then Lemma 9.15 and (2c) ensure they agree about wand-taps. So they fully agree. ⊣

Corollary 9.23 ($\mathsf{LT_S}$). *Lemma* 7.5$^\sharp$ *holds, so that, e.g.,* $Eq^\sharp$ *is an equivalence relation. Furthermore, these formulas are equivalent* (*row-wise*), *when* $|a|, |b| \leq \sigma$:

$$\begin{array}{ccc} Dom^\sharp(w/a) & Dom^{\tilde{\sigma}}(w/a) & \langle w, a\rangle \in \mathrm{D}_{|a|} \\ Eq^\sharp(w/a, u/b) & Eq^{\tilde{\sigma}}(w/a, u/b) & \langle w, a, u, b\rangle \in \mathrm{E}_{\max(|a|,|b|)}. \end{array}$$

Proof. Lemma 7.5$^\sharp$ holds by Lemmas 9.12 and 9.18. This gives us $(\mathsf{dom1})^\sharp$–$(\mathsf{dom3})^\sharp$ and $(\mathsf{eq1})^\sharp$–$(\mathsf{eq3})^\sharp$. Now Proposition 9.22 yields the equivalences. ⊣

Using Corollary 9.23 freely and frequently from this point onwards, we are at last in a position to show that $\sharp$ is an interpretation:

Lemma 9.24 ($\mathsf{LT_S}$). *TapFun*$^\sharp$, *Making*$^\sharp$, *and Equating*$^\sharp$ *hold.*

Proof. *TapFun*$^\sharp$. Suppose $Tap_\sharp(w/a, c)$ and $Tap_\sharp(w/a, d)$. So there is $\langle w_1, a_1\rangle \in c$ with $Eq^\sharp(w/a, w_1/a_1)$ and $\langle w_2, a_2\rangle \in d$ with $Eq^\sharp(w/a, w_2/a_2)$. Now $Eq^\sharp(w_1/a_1, w_2/a_2)$, so $|a_2| = |a_1|$ by (tapr). Hence $\langle w_1, a_1\rangle \in d$ and $\langle w_2, a_2\rangle \in c$ by (tape). Since c and d are overlapping equivalence classes of minimal rank, it is easy to show that $c = d$.

Making$^\sharp$. Similar but easier.

Equating$^\sharp$. Similar, using Making$^\sharp$. ⊣

Assembling all the $\sharp$-translations of **S**'s axioms, we have what we required:

Proposition 9.25. $\sharp : \mathbf{S} \longrightarrow \mathsf{LT_S}$ *is an interpretation.*

We can also obtain the Corollary which I stated as a target in Section 9.1, via an easy lemma:

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Lemma 9.26 ($\mathsf{LT_S}$). *If* $Tap_\sharp(w/a, c)$, *then* $c = \{\langle u, b\rangle : Eq^\sharp(w/a, u/b) \wedge Mini^\sharp(u/b)\}$.

Proof. Suppose $Tap_\sharp(w/a, c)$ and let

$$e = \{\langle u, b\rangle : Eq^\sharp(w/a, u/b) \wedge Mini^\sharp(u/b)\}.$$

To see $c \subseteq e$, fix $\langle u, b\rangle \in c$. Now $Tap_\sharp(u/b, c)$ by Lemma 9.15(3), and so $Eq^\sharp(w/a, u/b)$ by Equating$^\sharp$. Also, if $Eq^\sharp(u/b, v/c)$, then $|b| \leq |c|$ by (tapr), i.e., $\mathrm{R}^\sharp b \leq^\sharp \mathrm{R}^\sharp c$ by Lemma 9.18, so $Mini^\sharp(u/b)$. To see $e \subseteq c$, keep $\langle u, b\rangle \in c \neq \emptyset$ as before, and suppose $\langle u', b'\rangle \in e$. Now $Eq^\sharp(u/b, u'/b')$, and since $Mini^\sharp(u'/b')$ we have $\langle u', b'\rangle \in c$ by (tape). ⊣

Corollary 9.27 ($\mathsf{LT_S}$). *c is a conch iff either*:

(1) $c = {\uparrow}s$, *for some s whose members are all conches*; *or*
(2) $c = \{\langle w, a\rangle : Tap_\sharp(w/a, c) \wedge Mini^\sharp(w/a)\} \neq \emptyset$.

§10. Bi-interpretation and synonymy. In the past two sections, I have laid down two interpretations

$$\flat : \mathsf{LT_S} \rightleftarrows \mathsf{S} : \sharp.$$

Furthermore, Proposition 9.13 tells us that $\flat\sharp$ is a self-embedding. I now want to show that $\sharp\flat$ is too, so that $\flat$ and $\sharp$ form a bi-interpretation.

Inspired by Corollary 9.27, I will recursively define the map, Ξ, which will witness that $\sharp\flat$ is a self-embedding:

Definition 10.1 (S). For each c, define Ξc as a bland set of ordered pairs thus:

$$\begin{aligned} \Xi c &:= {\uparrow}\{\Xi x : x \in c\}, && \text{if } Bland(c),\\ \Xi c &:= \{\langle \Xi w, \Xi a\rangle : Tap(w/a, c) \wedge Mini(w/a)\}, && \text{if } \neg Bland(c), \end{aligned}$$

where ${\uparrow}s = \{\langle \emptyset, s\rangle\}$, as before.

This recursive definition has a twist. We *first* define Ξc, just for hereditarily bland c; this ensures that Ξw is well-defined for each wand w. *Then* we augment this to a definition which covers everything else. This second step does not disrupt the definition for hereditarily bland entities, so it is well-defined. We can immediately establish some useful properties (using induction and Lemma 7.14):

Lemma 10.2 (S). *Ξ is an injective map $V \longrightarrow Heb$.*

My goal, though, is to show that Ξ is an isomorphism $V \longrightarrow Conch^\flat$. My strategy is to use ordinal induction within S to show that, for each ordinal α, Ξ's restriction is a structure-preserving bijection from $\mathbf{V}_{\mathbb{L}\alpha}$ to $(\mathbf{V}_{\mathbb{L}\alpha})^{\sharp\flat}$.

Let me start by unpacking the last expression. By Proposition 9.13(4)$^\flat$ and Lemma 9.21$^\flat$, we know that $(\mathbf{V}_{\mathbb{L}\alpha}(x))^{\sharp\flat}$ is equivalent to $(\mathbf{U}_{\mathrm{B}_\alpha}(x))^\flat$. Next, note that $\flat$ translates all predicates verbatim, but restricts our purview to Heb; but $\mathbf{U}_{\mathrm{B}_\alpha}(x)$, as defined in S, is clearly solely concerned with hereditarily bland entities in any case; so we can

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just *ignore* $\flat$'s action.[41] My aim, then, is to show that Ξ's restriction is a structure-preserving bijection $\mathbf{V}_{\mathbb{L}\alpha} \longrightarrow \mathbf{U}_{\mathrm{B}_\alpha}$.

In fact, it is easy to show (without using induction) that Ξ preserves *Wand*:

LEMMA 10.3 (**S**). *$Wand^{\sharp\flat}(x) \rightarrow (\exists w : Wand)x = \Xi w$, and $Wand(w) \leftrightarrow Wand^{\sharp\flat}(\Xi w)$*

PROOF. Using HebWands, note that $Wand^{\sharp\flat}(x)$ iff $(\exists v : Wand)x = \Theta^\flat v \wedge \mathbf{U}_\emptyset(x) \wedge Conch^\flat(x)$. Now Ξ, restricted to hereditarily bland things, is $\Theta^\flat$. This gives the first claim. For the second: $\mathbf{U}_\emptyset(x)$ entails $Conch^\flat(x)$; and Ξ is injective; so $Wand^{\sharp\flat}(\Xi w)$ iff $Wand(w) \wedge \mathbf{U}_\emptyset(\Xi w)$. To complete the proof, just note that when $Wand(w)$, so that $Heb(w)$, we have $\mathbf{U}_\emptyset(\Xi w)$. $\dashv$

However, it will take more effort to secure the preservation of LT_S's other primitives. Indeed, it requires some thought about what we *want* to preserve. I will set up my induction hypothesis to (try to) preserve *Bland* and $\in$ in the most obvious way. However, recalling the Picture of Loose Constructivism from Section 4, when considering entities of rank $< \alpha$, we must remember that wand-tapping an entity whose rank is *immediately* below rank α will take us *outside* of $\mathbf{V}_{\mathbb{L}\alpha}$. For this reason, my induction hypothesis considers preservation of *Tap* for entities found at least one clear rank before α.

With all that in mind, here is my big induction hypothesis, BIH, for all $\beta < \alpha$:

$$\begin{array}{rl}
\text{If } \mathbf{V}_{\mathbb{L}\beta}(a) \text{ then} & \mathbf{U}_{\mathrm{B}_\beta}(\Xi a) \text{ and } \mathrm{R}a = |\Xi a|.\\
\text{If } \mathbf{U}_{\mathrm{B}_\beta}(b) \text{ then} & b = \Xi a \text{ for some } a \text{ such that } \mathbf{V}_{\mathbb{L}\beta}(a).\\
\text{If } \mathbf{V}_{\mathbb{L}\beta}(a) \text{ then} & Bland(a) \leftrightarrow Bland^{\sharp\flat}(\Xi a).\\
\text{If } \mathbf{V}_{\mathbb{L}\beta}(a) \text{ and } \mathbf{V}_{\mathbb{L}\beta}(b) \text{ then} & a \in b \leftrightarrow \Xi a \in^{\sharp\flat} \Xi b.\\
\text{If } \mathrm{R}a + 1 < \beta \text{ then} & Tap(w/a, c) \leftrightarrow Tap^{\sharp\flat}(\Xi w/\Xi a, \Xi c).
\end{array}$$

Note: *From here until the end of the section, I reserve α and β for use in this induction hypothesis, BIH.* Assuming BIH, I will show that each of the claims holds with α in place of β. By ordinal induction, the claims will hold for every ordinal, so that $\Xi : V \longrightarrow Conch^\flat$ is isomorphism, as required.

My first result ensures preservation of loosely bound formulas:

LEMMA 10.4 (**S**, BIH). *Let $\phi(v_1, \dots, v_n)$ be loosely bound by* $\max(\mathbb{L}v_1, \dots, \mathbb{L}v_n)$. *Let* $\max(\mathrm{R}a_1, \dots, \mathrm{R}a_n) < \beta$. *Then $\phi(a_1, \dots, a_n) \leftrightarrow \phi^{\sharp\flat}(\Xi a_1, \dots, \Xi a_n)$. So in particular, this holds for Dom, Eq, and Mini.*

PROOF. Where $\gamma = \max(\mathrm{R}a_1, \dots, \mathrm{R}a_n) < \beta$, bound and free variables in $\phi(a_1, \dots, a_n)$ are restricted to $\mathbf{V}_{(\mathbb{L}\gamma)^+}$, and any instance of "$Tap(x/y, z)$" is conjoined with "$x \in \mathbb{L}\gamma$", i.e., in effect the claim that $\mathrm{R}x < \gamma$. By BIH, Ξ's restriction is a structure-preserving bijection $\mathbf{V}_{\mathbb{L}\beta} \longrightarrow \mathbf{U}_{\mathrm{B}_\beta}$, with the caveat that *Tap* is preserved

[41] Strictly speaking, there is a *tiny* abuse of notation here: to characterize $\emptyset$ in **S**, we say "the *bland* thing with no members", rather than "the thing with no members" (as in LT_S). So, strictly speaking, a slightly different definition of "$\mathbf{U}_{\mathrm{B}_\alpha}$" is required. But the difference would evidently make no real difference. I repeatedly rely on this harmless abuse of notation throughout this section.

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when tapping entities of rank $< \gamma < \beta$, which aligns perfectly with the conjunctive-restriction. So the biconditional holds by induction on complexity. $\dashv$

Using this, I can show that *Tap* is preserved in the required way:

LEMMA 10.5 (**S**, BIH).

(1) *If* $\mathrm{R}a + 1 < \alpha$ *and* $Tap(w/a, c)$, *then* $Tap^{\sharp\flat}(\Xi w/\Xi a, \Xi c)$ *and* $\mathrm{R}c = |\Xi c|$.
(2) *If* $|e| < \alpha$ *and* $\neg Bland^{\sharp\flat}(e)$, *then* $e = \Xi c$ *for some non-bland* c, *and* $\mathrm{R}c = |\Xi c|$.
(3) *If* $\mathrm{R}a + 1 < \alpha$ *and* $Tap^{\sharp\flat}(\Xi w/\Xi a, \Xi c)$, *then* $Tap(w/a, c)$.

PROOF. *(1)* Let $\mathrm{R}a + 1 < \alpha$ and $Tap(w/a, c)$. So $Dom(w/a)$ by Making, and $Dom^{\sharp\flat}(\Xi w/\Xi a)$ by Lemma 10.4. By Making$^{\sharp\flat}$, there is e such that $Tap^{\sharp\flat}(\Xi w/\Xi a, e)$. By Lemma 9.26$^\flat$:

$$e = \{\langle x, d\rangle : Eq^{\sharp\flat}(\Xi w/\Xi a, x/d) \wedge Mini^{\sharp\flat}(x/d)\}.$$

Fix $\langle x, d\rangle \in e$; so $|d| \leq |\Xi a|$; but $|\Xi a| = \mathrm{R}a$ by BIH; so $|d| + 1 < \alpha$. By BIH again, $d = \Xi b$ for some b with $\mathrm{R}b = |d|$. Also, $x = \Xi u$ for some u by Lemma 10.3. So

$$e = \{\langle \Xi u, \Xi b\rangle : Eq^{\sharp\flat}(\Xi w/\Xi a, \Xi u/\Xi b) \wedge Mini^{\sharp\flat}(\Xi u/\Xi b)\}$$

and therefore, by Lemma 10.4 and Equating,

$$e = \{\langle \Xi u, \Xi b\rangle : Tap(u/b, c) \wedge Mini(u/b)\} = \Xi c.$$

To compute rank, fix $\langle \Xi u, \Xi b\rangle \in e$. Since $Mini(u/b)$, both $\mathrm{R}b + 1 = \mathrm{R}c$ by Lemma 7.7, and also $\mathrm{R}b \leq \mathrm{R}a$ and hence $\mathrm{R}b = |\Xi b|$ by BIH. Since $|\Xi b| + 1 = |e|$, we get $\mathrm{R}c = |e|$.

(2) With $|e| < \alpha$ and e non-bland$^{\sharp\flat}$, Corollary 9.27$^\flat$ tells us

$$e = \{\langle x, d\rangle : Tap^{\flat\sharp}(x/d, e) \wedge Mini^{\sharp\flat}(x/d)\} \neq \emptyset.$$

Fix $\langle x, d\rangle \in e$. So $Tap^{\sharp\flat}(x/d, e)$ and $|d| < |e| < \alpha$ so $Dom^{\sharp\flat}(x/d)$ by Making$^{\sharp\flat}$. By BIH, $d = \Xi a$ for some a with $\mathrm{R}a = |d|$. Using Lemma 10.3, let $x = \Xi w$; now $Dom(w/a)$ and $Mini(w/a)$ by Lemma 10.4; so by Making there is c such that $Tap(w/a, c)$. Since $\mathrm{R}a + 1 < \alpha$, now $\mathrm{R}c = |\Xi c|$ and $Tap^{\sharp\flat}(\Xi w/\Xi a, \Xi c)$ by (1), i.e., $Tap^{\sharp\flat}(x/d, \Xi c)$; so $e = \Xi c$ by TapFun$^{\sharp\flat}$.

(3) Let $\mathrm{R}a + 1 < \alpha$ and $Tap^{\sharp\flat}(\Xi w/\Xi a, \Xi c)$. Now $|\Xi c| \leq |\Xi a| + 1$ by Lemma 9.15(2)$^\flat$. Also $\mathrm{R}a = |\Xi a|$ by BIH; so $|\Xi c| < \alpha$ and hence c is not bland and $\mathrm{R}c = |\Xi c| < \alpha$ by (2). Using Lemma 7.6, fix u, b with $Tap(u/b, c)$ and $\mathrm{R}b + 1 = \mathrm{R}c < \alpha$; now $Tap^{\sharp\flat}(\Xi u/\Xi b, \Xi c)$ by (1), so $Eq^{\sharp\flat}(\Xi w/\Xi a, \Xi u/\Xi b)$ by Equating$^{\sharp\flat}$, and $Tap(w/a, c)$ by Lemma 10.4 and Equating. $\dashv$

With wand-taps taken care of, I can now show that Ξ's restriction is the bijection we wanted, and that it preserves blandness as hoped:

LEMMA 10.6 (**S**, BIH). *For any* γ:

(1) *If* $a \in V^\gamma_{\mathbb{L}\alpha}$, *then* $\mathrm{R}a = |\Xi a|$ *and* $\Xi a \in U^\gamma_{B_\alpha}$ *and* $Bland(a) \leftrightarrow Bland^{\sharp\flat}(\Xi a)$.
(2) *If* $b \in U^\gamma_{B_\alpha}$, *then* $b = \Xi a$ *for some* $a \in V^\gamma_{\mathbb{L}\alpha}$.

Hence Ξ*'s restriction is a bijection* $\mathbf{V}_{\mathbb{L}\alpha} \longrightarrow \mathbf{U}_{B_\alpha}$, *and if* $\mathbf{V}_{\mathbb{L}\alpha}(a)$ *then* $Bland(a) \leftrightarrow Bland^{\sharp\flat}(\Xi a)$.

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PROOF. Both claims hold by induction on γ; "hence" then follows from Ξ's injectivity and the definitions of $\mathbf{V}_{\mathbb{L}\alpha}$ and $\mathbf{U}_{\mathrm{B}_\alpha}$. In what follows, I use the fact that $\mathit{Bland}^{\sharp\flat}(v)$ iff $(\exists e : \mathit{Heb})v = \uparrow e \wedge \mathit{Conch}^{\flat}(v)$.

Induction case when $\gamma = 0$. For (1), suppose $a \in V^0_{\mathbb{L}\alpha} = \mathbb{L}\alpha$, i.e., $\mathrm{R}a < \alpha$. First, suppose a is not bland. Using Lemma 7.6, fix w and b such that $\mathit{Tap}(w/b, a)$ and $\mathrm{R}b < \mathrm{R}a$; now $|\Xi a| = \mathrm{R}a$ and $\neg \mathit{Bland}^{\sharp\flat}(\Xi a)$ by Lemma 10.5(1).

Next, suppose a is bland. If $x \in a$, then $\mathrm{R}x < \mathrm{R}a < \alpha$, so $\mathit{Conch}^{\flat}(\Xi x)$ by BIH. Let $e = \{\Xi x : x \in a\}$; this witnesses that $\mathit{Bland}^{\sharp\flat}(\Xi a)$, because $\mathit{Heb}(e)$ by Lemma 10.2, and $\mathit{Conch}^{\flat}(\uparrow e)$, and $\uparrow e = \Xi a$. To compute $|\Xi a|$, using BIH and Lemma 9.11(1)$^{\flat}$,

$$\mathrm{R}a = \operatorname*{lsub}_{x \in a} \mathrm{R}x = \operatorname*{lsub}_{x \in a} |\Xi x| = |\Xi a|.$$

For (2), suppose $b \in U^0_{\mathrm{B}_\alpha} = \mathrm{B}_\alpha$, i.e., $|b| < \alpha$. If $\neg \mathit{Bland}^{\sharp\flat}(b)$, use Lemma 10.5(2); so suppose instead that $\mathit{Bland}^{\sharp\flat}(b)$; let e witness this, so that $b = \uparrow e$. If $z \in e$ then $|z| < |b|$ by Lemma 9.11(1)$^{\flat}$, so that $\mathrm{R}(\Xi^{-1}z) = |z|$ by BIH. Let $a = \{\Xi^{-1}z : z \in e\}$; now $b = \uparrow e = \Xi a$, and indeed $\mathrm{R}a = |\Xi a|$ by the same computation as before.

Induction case when $\gamma > 0$. Similar, using the induction hypothesis for γ rather than BIH. ⊣

It follows straightforwardly that membership is preserved:

LEMMA 10.7 (**S**, BIH). *If $\mathbf{V}_{\mathbb{L}\alpha}(a)$ and $\mathbf{V}_{\mathbb{L}\alpha}(b)$, then $a \in b \leftrightarrow \Xi a \in^{\sharp\flat} \Xi b$.*

PROOF. Note that $\Xi a \in^{\sharp\flat} \Xi b$ iff $\mathit{Bland}^{\sharp\flat}(\Xi b) \wedge \Xi a \in \downarrow\Xi b$. Now use Ξ's injectivity and Lemma 10.6. ⊣

And this was the last piece of the puzzle. We have proved the induction step; now by ordinal induction and Lemma 10.2, we immediately obtain what we wanted:

PROPOSITION 10.8 (**S**). *$\Xi : V \longrightarrow \mathit{Conch}^{\flat}$ witnesses that the map $\sharp\flat$ is a self-embedding.*

Assembling all of this, we obtain the Main Theorem, that $\mathsf{LT}_{\mathbf{S}}$ and **S** are synonymous.

PROOF OF MAIN THEOREM. Propositions 8.4 and 9.25 tell us that we have interpretations $\flat : \mathsf{LT}_{\mathbf{S}} \rightleftarrows \mathbf{S} : \sharp$. By Propositions 9.13 and 10.8, these witness an identity-preserving bi-interpretation. Since $\mathsf{LT}_{\mathbf{S}}$ is sequential, $\mathsf{LT}_{\mathbf{S}}$ and **S** are synonymous by the Friedman–Visser Theorem.[42] ⊣

§11. Church's [6] theory as a wand/set theory. My Main Theorem deals with wand/set theories in full generality. I will close with something much more specific. In Section 1.3 and Section 5.5, I claimed that Church's [6] set theory with a universal set can be presented as a wand/set theory (so that it is synonymous with a ZF-like theory). The point of this section is to make good on this claim.

Once you have defined the notion of a wevel, Church's [6] theory can be given a perspicuous axiomatization; I cover this in Sections 11.1–11.3. However, it is not

[42]That is, Friedman and Visser [12, Corollary 5.5].

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immediately obvious that the perspicuous axiomatization is a wand/set theory, in the sense of Definition 5.1; I show that it is in Section 11.4.

11.1. Axiomatizing Church's [6] theory as CUS. In this sub-section, I define CUS, my version of Church's Universal Set theory. The basic idea is to help myself to the notion of a wevel, and then write down the axioms that would be obviously suggested by the discussion in Section 1.3.

CUS's signature is the same as any wand/set theory. It has the quasi-notational axioms InNB, TapNB, and TapFun. It also has the axioms which arrange everything into well-ordered wevels—Ext, Sep, and Strat—and the axiom WeakHeight.

As indicated in Section 1.3, I want to treat 0 as the complement wand, and each $n > 0$ as the cardinal$_n$ wand. So Church's wands will be the natural numbers:

Wands$_c$ $\{w : Wand(w)\} = \omega$.

Note that this entails HebWands, and it is formulated solely in terms of hereditarily bland sets and *Wand* (see Section 5.3). So far, then, CUS is on track to be a wand/set theory.

I now need to provide axioms which specify how the wands behave. To explain the cardinal$_n$ wands, I need to introduce Church's notion of n-equivalence (though see *Caveat 5* of Section 11.2):[43]

Definition 11.1 (CoreWev, Strat, WeakHeight). Recursively define $\bigcup^0 a = a$ and $\bigcup^{n+1} a = \bigcup \bigcup^n a$.[44] For any n, say that a and b are n-equivalent, written $a \approx_n b$, iff:

(1) both $\bigcup^i a$ and $\bigcup^i b$ are non-empty, and all their members are bland, for each $0 \leq i < n$;
(2) there is a bijection $f_{n-1} : \bigcup^{n-1} a \longrightarrow \bigcup^{n-1} b$; and
(3) setting $f_i(x) := \{f_{i+1}(y) : y \in x\}$ defines a bijection $f_i : \bigcup^i a \longrightarrow \bigcup^i b$, for each $0 \leq i < n - 1$.[45]

Note that $a \approx_1 b$ iff a and b are (non-empty, bland) sets which are equinumerous in the usual sense.

I now need axioms to say exactly when $\boxed{n}a$ should exist, for any wand n and any object a. We want to say: tapping anything with complement yields an object; tapping anything bland with any cardinal$_n$ yields an object (if the n-equivalence class would be non-empty); and tapping anything non-bland with any cardinal$_n$ yields nothing. But there is a wrinkle. Suppose b is bland; since we are treating 0 as complement, we expect that $\boxed{0}\boxed{0}b = b$;[46] but this would violate TapNB.

To retain TapNB, we will say that we *find* an object by tapping a with complement iff a is not itself the complement of anything bland. (As foreshadowed in Section 2, I will eventually define a more expansive notion, according to which tapping $\boxed{0}b$ with 0 will *yield* b itself; see Definition 11.2.) We effect this by stipulating:

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[43] Church [6, p. 302]. Sheridan provides a version of Church's set theory which invokes a slight variant of $\approx_n$ (see [19, pp. 109–111]).

[44] This is well-defined, for the reasons sketched in Section 9.2; and it is total since the axioms mentioned prove that $\bigcup x$ exists for any x.

[45] Talk of bijections here is understood using the usual Kuratowski implementation for (bland) sets.

[46] Note: $\boxed{w}\boxed{u}a$ is always to be read as $\boxed{w}(\boxed{u}a)$.

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Making$_{\mathbf{C}}$ $\exists c\ Tap(n/a, c) \leftrightarrow \big((n = 0 \wedge (\forall b : Bland)a \neq \boxed{0}b) \vee (0 < n < \omega \wedge a \approx_n a)\big)$.

Note that this does not have the same shape as Making, so we have departed from the shape of axioms stated in Definition 5.1. (I return to this in Section 11.4.)

Our next axioms govern the complement wand:

Comp1 if $\boxed{0}a = \boxed{0}b$, then $a = b$.
Comp2 if $\neg Bland(a) \wedge (\forall b : Bland)a \neq \boxed{0}b$, then $a = \boxed{0}\boxed{0}a$.

These say that $\boxed{0}$ is injective, and that double-$\boxed{0}\boxed{0}$ is identity (provided this does not require taking $\boxed{0}\boxed{0}b$ for any bland b). Last, we have axioms governing the cardinal$_n$ wands:

if $0 < m < \omega$ and $0 < n < \omega$, then:
Card1 $\boxed{m}a = \boxed{n}b \leftrightarrow (m = n \wedge a \approx_m b)$,
Card2 $(\forall b : Bland)\boxed{m}a \neq \boxed{0}b$, and
Card3 $\boxed{m}a \neq \boxed{0}\boxed{n}b$.

So a and b have the same cardinality$_n$ iff a and b are n-equivalent; if $n \neq m$ then no cardinal$_n$ is a cardinal$_m$; and no cardinal$_n$ is the complement of anything bland or of any cardinal$_m$.

We now define CUS as the theory whose axioms are: InNB, TapNB, TapFun, Ext, Sep, Strat, WeakHeight, Wands$_{\mathrm{C}}$, Making$_{\mathrm{C}}$, Comp1–Comp2, and Card1–Card3.

11.2. Caveats and more expansive notions. With CUS, we have a cleanly presented version of Church's [6] theory. Or so I claim; I should now offer some small caveats.

Caveat 1: Church had ZF in mind. More specifically, Church wanted (what I call) the hereditarily bland sets to obey ZF. This causes no concern: if we want to follow Church here, we can augment CUS with the axiom scheme of Replacement, restricted to hereditarily bland sets. This will not affect our synonymy result; rather, the resulting theory will be synonymous with ZF itself.

Caveat 2: Church did not speak about wands. In a sense this is immaterial; talking about wands is only a helpful heuristic. But there is an important formal point: CUS's signature has primitives *Bland*, $\in$, *Wand* and *Tap*; Church's own theory has just one primitive, $\in$. Fortunately, this is incidental: in Section 11.3, I show that we can reformulate CUS using just $\in$.

Caveat 3: I have a restrictive notion of tapping. As explained: if b is bland, then CUS proves that $\boxed{0}\boxed{0}b$ does not exist, but we intuitively want tapping b twice with 0 to yield b. To address this, I simply introduce a more expansive notion of tapping (whose good behaviour is confirmed in Lemma 11.7):

DEFINITION 11.2 (CUS). Let $\blacksquare_n a := \boxed{n}a$ if $\boxed{n}a$ exists. Let $\blacksquare_0\boxed{0}b := b$ if $\boxed{0}b$ exists but $\boxed{0}\boxed{0}b$ does not. In all other cases, let $\blacksquare_n a$ be undefined.

Caveat 4: I have a restrictive notion of membership. I have been glossing $\boxed{0}a$ as "a's complement". However, I have said nothing, yet, which indicates that $\boxed{0}a$'s members should be exactly those x such that $x \notin a$. Worse: whilst $\boxed{0}\emptyset$ should be "the universal set", CUS's axioms TapNB and InNB entail that $\forall x\ x \notin \boxed{0}\emptyset$. Evidently, I need to define a more expansive notion of membership; here it is:

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DEFINITION 11.3 (CUS). When c is bland and $0 < n < \omega$, define:

$$x \mathrel{\varepsilon} c :\equiv x \in c, \qquad x \mathrel{\varepsilon} \boxed{0}c :\equiv x \notin c,$$
$$x \mathrel{\varepsilon} \boxed{n}c :\equiv x \approx_n c, \qquad x \mathrel{\varepsilon} \boxed{0}\boxed{n}c :\equiv x \not\approx_n c.$$

We confirm this covers all cases in Lemma 11.6.

Caveat 5: Church's definition of $\approx_n$ is not quite my own. Relatedly: Church defined $\approx_n$ using an "expansive" notion of membership, according to which everything is a member of the (non-bland) universal set. I defined $\approx_n$ using the "restrictive" notion of membership, $\in$, according to which only bland sets have members. (I am forced to do things this way, because I invoke $\approx_n$ when I define the "expansive" notion of membership, ε, in Definition 11.3.)

Again, the difference is unimportant. Church's definition of $\approx_j$ only shows up in his axiomatic theory in the following axiom:[47]

L_j: if a is well founded, then $\{x : a \approx_j x\}$ is a set.

Now suppose that a is well-founded and that $a \approx_j x$. Then x's members are bland,[48] and so are the members of members, and so on, digging j-levels deep through membership chains. So, as it occurs in L_j, either definition would have the same effect.

However, there is a more subtle difference. Church's principle L_j is not a single axiom; instead, he has one axiom for each natural number $j > 0$, with numbers supplied in the metalanguage. This suggests that he has defined j-equivalence by *meta*linguistic recursion rather than *object*-linguistic recursion. By contrast, my Definition 11.1 is fully object-linguistic, and the success of Definition 11.3 requires this fact.

I conjecture that Church relied on metalinguistic recursion because he had not proved a suitable object-linguistic recursion theorem.[49] Fortunately, I have such a theorem (cf. Section 9.2), and I am not afraid to use it.

Caveat 6: Church had no Beschränktheitsaxiom.[50] The axiom Strat acts as an axiom of restriction, arranging everything into a well-founded hierarchy. Indeed, CUS proves Lemma 7.8, which states that anything non-bland is obtained by starting with something bland and tapping it finitely many times with some wands (i.e., some object-linguistic natural numbers). By contrast, Church's theory is open-ended: for all we know, its universe contains objects obtained by using hitherto-unmentioned wands, or beasts which float around outside of any hierarchy.

Deleting Strat from CUS would certainly destroy my proof-strategy for obtaining synonymy. (Indeed, I suspect it would actually surrender synonymy.) So: *including Strat genuinely changes Church's theory*.

That said: this change to the *theory* does not not flag a departure from Church's *approach*. Church established the consistency of his [6] theory by outlining what

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47 Church [6, p. 305].

48 Church [6, p. 298] would have said "low" rather than "bland", but the point remains.

49 Church [6, p. 305] expresses the desire for a "possibly more general axiom or axiom schema" than L_j.

50 Thanks to Thomas Forster for emphasizing the importance of *Beschränktheitsaxiome*.

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is now called a "Church–Oswald model",[51] and the denizens of Church–Oswald models are always arranged into a nicely well-founded hierarchy. So the model theory of Church-style approaches has always implicitly invoked a *Beschränktheitsprinzip*, even if his formal theory lacked an explicit *Beschränktheitsaxiom*. Indeed: my framework of wand/set theories really just provides a framework for turning a description of a Church–Oswald model into a formal theory with a *Beschränktheitsaxiom*, which is then (by my Main Theorem) synonymous with a ZF-like theory.

11.3. CUS as a theory of nothing but sets. I now want to show that CUS does what we want it to do. I begin with two trivial results concerning $\boxed{0}$, whose proof I leave to the reader:

Lemma 11.4 (Making$_{\mathrm{C}}$, Comp2). *If* $\boxed{0}\boxed{0}a$ *exists, then both* $a = \boxed{0}\boxed{0}a$.

Lemma 11.5 (Making$_{\mathrm{C}}$, Card2, Card3). *If* $m > 0$ *and* $\boxed{m}a$ *exists, then* $\boxed{0}\boxed{m}a$ *and* $\boxed{0}\boxed{0}\boxed{m}a$ *exist.*

The next result says that, in CUS, everything is (exclusively) either: bland, the complement of something bland, some cardinality$_n$ of something bland, or the complement of some such cardinality$_n$.

Lemma 11.6 (CUS). *Any a belongs to exactly one of these three kinds*:

(1) *a is bland*;
(2) $a = \boxed{n}c$ *for some bland c and* $n < \omega$;
(3) $a = \boxed{0}\boxed{n}c$ *for some bland c and* $0 < n < \omega$.

Moreover, if (2) *or* (3) *obtains, then n is unique.*

Proof. Using Theorem 7.8, let $a = \boxed{n_i}\dots\boxed{n_1}c$, with c bland and $i < \omega$ as small as possible. I now note three facts.

First: there are no adjacent zeros; i.e., there is no j with $n_{j+1} = n_j = 0$. For if $\boxed{0}\boxed{0}b$ exists then $b = \boxed{0}\boxed{0}b$ by Lemma 11.4, but i is minimal.

Second: if $n_1 > 0$, *then either* $i = 1$, or $i = 2$ *and* $n_2 = 0$. Suppose n_2 exists; since $\boxed{n_1}c$ is not bland, $\boxed{n_1}c \not\approx_m \boxed{n_1}c$ for any wand m; so $n_2 = 0$ by Making$_{\mathrm{C}}$. Repeating this reasoning, the only wand m such that $\boxed{m}\boxed{0}\boxed{n_1}c$ exists is $m = 0$; now recall the First fact.

Third: if $n_{j+1} > 0$, *then* $j = 0$. Since if $m > 0$ and $\boxed{m}x$ exists, x is bland by Making$_{\mathrm{C}}$.

These facts entail that kinds (1)–(3) are exhaustive. It is now easy to show that kinds (1)–(3) are exclusive, and to establish the "Moreover" remark. ⊣

This result vindicates my expansive notion of "membership". Roughly, we want to say that x is in a's cardinal$_n$ iff x and a are n-equivalent; and that x is in a's complement iff x is not in a; we stipulate this with Definition 11.3, and can see by Lemma 11.6 that this exhausts all cases.

[51] See Church [6, pp. 305–307], Oswald [17], Forster [9, Section 2], and Sheridan [19].

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It is obvious from Definition 11.3 that each $n > 0$ behaves *as* the $\mathsf{cardinal}_n$ wand. It is only slightly less obvious that 0 behaves *as* the $\mathsf{complement}$ wand. Specifically, using Definition 11.2 and writing $x \not\varepsilon\ y$ for $\neg(x\ \varepsilon\ y)$:

LEMMA 11.7 (CUS). *$x\ \varepsilon\ a$ iff $x \not\varepsilon\ \boxed{0}a$.*

PROOF. By Lemma 11.6, we only need to check a few cases. This is easy, using Making_{C}, TapNB, Comp1, and Card2. ⊣

This yields a generalized version of extensionality:

LEMMA 11.8 (CUS). *If $\forall x(x\ \varepsilon\ a \leftrightarrow x\ \varepsilon\ b)$, then $a = b$.*

PROOF. By Lemma 11.6, we need only check a few cases. This is easy, using Lemma 11.7 and elementary (CUS-provable) facts about $\approx_n$. ⊣

Since the objects of CUS obey this generalized extensionality principle, we can treat CUS as a theory of *nothing but sets*. Indeed, as claimed in *Caveat 2* of Section 11.2, we can use Lemmas 11.6–11.8 to reformulate CUS using *only* a membership-predicate.

Here is one way to do this. Start by defining an identity-preserving translation with these atomic clauses (and an unrestricted domain-formula):

$$\begin{aligned}
\mathit{Bland}_\bullet(a) &:\equiv a \notin a \wedge \exists b \forall x(x \in b \leftrightarrow (x \in a \vee \forall z\ z \notin x)).\\
\mathit{Wand}_\bullet(n) &:\equiv n \in \omega.\\
b \in_\bullet a &:\equiv b \in a \wedge \mathit{Bland}_\bullet(a).\\
\mathit{Tap}_\bullet(n/a, c) &:\equiv \big(n = 0 \wedge (\forall d : \mathit{Bland}_\bullet)\exists x(x \in d \leftrightarrow x \in a) \wedge \forall x(x \in c \leftrightarrow x \notin a)\big) \vee\\
&\qquad \big(0 < n < \omega \wedge a \approx_n^\bullet a \wedge \forall x(x \in c \leftrightarrow x \approx_n^\bullet a)\big).
\end{aligned}$$

The translation is well-defined, since $\approx_n$ is defined without using the primitive *Tap*. Now $\in$ is the only primitive of $\text{CUS}^\bullet$, i.e., the $\bullet$-translation of CUS.

Trivially, $\bullet$ is an interpretation CUS $\longrightarrow$ $\text{CUS}^\bullet$. We can also define a translation, $\circ$, in the opposite direction, via $x \in^\circ a :\equiv x\ \varepsilon\ a$. By running these translations back-to-back, it is easy to show that CUS and $\text{CUS}^\bullet$ are synonymous. Hence CUS can be reformulated, without gain or loss, as a theory whose only primitive is $\in$, and which obeys Extensionality.

I leave the proof of this claim to keen readers, but here are the main steps. Using Lemmas 11.6 and 11.8, show that CUS proves: $\mathit{Bland}(a) \leftrightarrow \mathit{Bland}^{\bullet\circ}(a)$; and $x \in a \leftrightarrow x \in^{\bullet\circ} a$; and $\mathit{Wand}(w) = \mathit{Wand}^{\bullet\circ}(w)$; and $\mathit{Tap}(w/a, c) \leftrightarrow \mathit{Tap}^{\bullet\circ}(w/a, c)$. Then use Lemmas $11.6^\bullet$ and $11.8^\bullet$ to show that $\text{CUS}^\bullet$ proves $x \in a \leftrightarrow x \in^{\circ\bullet} a$. This establishes synonymy, and $\text{CUS}^\bullet$ proves Extensionality by Lemma $11.8^\bullet$.

11.4. CUS is a wand/set theory. We have seen that CUS does everything we wanted it to. But my last task is to show that CUS is a wand/set theory, in the sense of Definition 5.1. To this end, I will close by presenting a wand/set theory, WSC, for Wand/Set Church theory, and showing that it is just an alternative axiomatization of CUS.

WSC's first few axioms are: InNB, TapNB, TapFun, Ext, Sep, Strat, WeakHeight, and Wands_{C}. These are shared with CUS. WSC's remaining axioms, Making and Equating, arise by explicitly defining two formulas, D and E (see Section 5.4). To

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define D, I simply insert a loose bound on the formula in the right-hand-side of Making$_C$:[52]

$$D(n/a) :\equiv (n = 0 \wedge (\forall x \in \mathbb{L}a)(\mathit{Bland}(x) \rightarrow a \neq \boxed{0}x)) \text{ or } (0 < n < \omega \wedge a \approx_n a).$$

To define E, I simply list the few cases under which we would want to identify wand-taps, whilst inserting some loose bounds when required:

$$E(m/a, n/b) :\equiv m, n \in \omega \text{ and:}$$

$$(m = n \wedge a = b) \text{ or} \tag{e1}$$

$$\big(0 < m = n \wedge a \approx_m b\big) \text{ or} \tag{e2}$$

$$\big(0 = m < n \wedge (\exists d \approx_n b)(a = \boxed{0}\boxed{n}d \wedge d, \boxed{n}d \in \mathbb{L}a)\big) \text{ or} \tag{e3}$$

$$\big(0 = n < m \wedge (\exists d \approx_m a)(b = \boxed{0}\boxed{m}d \wedge d, \boxed{m}d \in \mathbb{L}b)\big). \tag{e4}$$

Note: *From here onwards, I reserve D and E for these defined predicates, and I reserve Dom and Eq for the predicates we obtain from D and E as in Section 5.4.*

It just remains to show that WSC = CUS. The crucial insight is that, because everything is arranged neatly into wevels, the explicitly stipulated loose bounds have no real effect. Concerning D: if x is bland and $a = \boxed{0}x$, then we "know" that x is found before a. Concerning E: if there is some $d \approx_n b$ with $a = \boxed{0}\boxed{n}d$, then there is some $e \approx_n d$ of minimal rank; so $a = \boxed{0}\boxed{n}e$ and we "know" that both e and $\boxed{n}e$ are found before a.

It just remains to prove what I have claimed to "know". My approach is to build up a reservoir of principles shared by WSC and CUS, until the theories come to coincide.

LEMMA 11.9 (WSC/CUS). *Comp1 holds.*

PROOF. This is an axiom of CUS. For WSC: if $\boxed{0}a = \boxed{0}b$, then $Eq(0/a, 0/b)$ by Equating; so $a = b$, either directly via Eq's definition or via (e1) if $E(0/a, 0/b)$. ⊣

LEMMA 11.10 (WSC/CUS). *For any wevel s:*

(1) *E restricted to s is an equivalence relation.*
(2) *D restricted to s is preserved under E.*

Hence: $Dom(n/a) \leftrightarrow D(n/a)$ *and* $Eq(m/a, n/b) \leftrightarrow E(m/a, n/b)$.

PROOF. *(1)* Fix $m, n, i \in \omega$, and $a, b, c \trianglelefteq s$. Evidently $E(m/a, m/a)$ by (e1). Now suppose $E(m/a, n/b)$ and $E(m/a, i/c)$; we show that $E(n/b, i/c)$. Most of this holds by elementary reasoning concerning n-equivalence; only two cases merit comment.

Case when $E(m/a, n/b)$ and $E(m/a, i/c)$, via disjunct (e3) both times. So $0 = m < n, i$, there is $d \approx_n b$ such that $a = \boxed{0}\boxed{n}d$, and there is $e \approx_i c$ such that $a = \boxed{0}\boxed{i}e$. Now $\boxed{n}d = \boxed{i}e$ by Comp1, so that $n = i$ and $d \approx_n e$. (In CUS, this holds by Card1; in WSC, note that $Eq(n/d, i/e)$ by Equating, so $n = i$ and either $d = e$ or $d \approx_n e$, either directly via Eq's definition, or via (e1)–(e2).) Hence $b \approx_n c$, so that $E(n/b, i/c)$ via (e2).

[52] Note: whilst the definition of $x \approx_n y$ conceals some quantifiers, the axioms used in its definition (see Definition 11.1) entail that the quantifiers can be restricted to $\mathbf{V}_{(\mathbb{L}x \cup \mathbb{L}y)^+}$ without loss. So both D and E are provably equivalent (using those axioms) to suitably loosely bounded formulas.

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Case when $E(m/a, n/b)$ and $E(m/a, i/c)$, via disjunct (e4) *both times.* Similar.
(2) Reason through the cases (e1)–(e4), using Comp1 in the last.
Hence: the claim for *Dom* is trivial; the claim for *Eq* holds by (1)–(2). ⊣

From here on, I will invoke Lemma 11.10 without further comment.

Lemma 11.11 (WSC/CUS). *Card1–Card3 hold.*

Proof. These are axioms of CUS; so work in WSC, and fix positive m and n.
For Card1. Use Making, Equating, and conditions (e1)–(e2).
For Card2. Use Equating and (e4).
For Card3. For reductio, suppose $\boxed{m}a = \boxed{0}\boxed{n}b$. Now $Eq(m/a, 0/\boxed{n}b)$ by Equating, so by (e4) there is d with $\boxed{n}b = \boxed{0}\boxed{m}d$ and $\mathrm{R}\boxed{m}d < \mathrm{R}\boxed{n}b$. Now $Eq(n/b, 0/\boxed{m}d)$, so by (e4) there is $e \approx_n b$ with $\mathrm{R}\boxed{n}e < \mathrm{R}\boxed{m}d$. By Card1, $\boxed{n}b = \boxed{n}e$. Now $\mathrm{R}\boxed{n}b < \mathrm{R}\boxed{n}b$, a contradiction. ⊣

Lemma 11.12 (WSC/CUS). *If b is bland, then* $\mathrm{R}b < \mathrm{R}\boxed{0}b$.

Proof. In this case, if $\boxed{0}b = \boxed{n}c$, then $n = 0$ by Card2, so $b = c$ by Comp1. Hence $\mathrm{R}b + 1 = \mathrm{R}\boxed{0}b$ by Lemma 7.6. ⊣

Lemma 11.13 (WSC/CUS). *$\mathit{Making}_{\mathrm{C}}$ and Making hold.*

Proof. Use Making/$\mathrm{Making}_{\mathrm{C}}$ and Lemmas 11.10 and 11.12. ⊣

Lemma 11.14 (WSC/CUS). *If $m > 0$ and $a \approx_m a$, then:*

(1) $\mathrm{R}\boxed{m}a < \mathrm{R}\boxed{0}\boxed{m}a$.
(2) *If* $\boxed{m}a = \boxed{0}\boxed{0}c$, *then* $\boxed{m}a = c$.
(3) *If* $(\forall c \approx_m a)\mathrm{R}a \leq \mathrm{R}c$, *then* $\mathrm{R}a < \mathrm{R}\boxed{m}a$.
(4) $\boxed{m}a = \boxed{0}\boxed{0}\boxed{m}a$.

Proof. *(1)* Now $\boxed{0}\boxed{m}a$ exists by Lemma 11.5. Using Lemma 7.6, fix n and b such that $\boxed{n}b = \boxed{0}\boxed{m}a$ and $\mathrm{R}b < \mathrm{R}\boxed{0}\boxed{m}a$. Since a is bland, $n = 0$ by Card3, so $b = \boxed{m}a$ by Comp1.

(2) In CUS, use Lemma 11.4. In WSC: by Equating, $Eq(m/a, 0/\boxed{0}c)$, so via (e4) there is $d \approx_m a$ with $\boxed{0}c = \boxed{0}\boxed{m}d$, hence $c = \boxed{m}d = \boxed{m}a$ by Card1 and Comp1.

(3) Using Lemma 7.6, fix b and n such that $\boxed{n}b = \boxed{m}a$ and $\mathrm{R}b < \mathrm{R}\boxed{m}a$. Suppose for reductio that b is not bland. Now $n = 0$ by $\mathrm{Making}_{\mathrm{C}}$. Since b is not bland, use Lemma 7.6 again to fix c and i such that $\boxed{i}c = b$ and $\mathrm{R}c < \mathrm{R}b$. Now $\boxed{m}a = \boxed{0}b = \boxed{0}\boxed{i}c$, so $i = 0$ by Card3, and therefore $\boxed{m}a = c$ by (2). So $\mathrm{R}c < \mathrm{R}b < \mathrm{R}c$, a contradiction. So b must be bland after all. Now $n \neq 0$ by Card2, so that $m = n$ and $a \approx_m b$ by Card1; hence $\mathrm{R}a \leq \mathrm{R}b$ given our assumptions about a.

(4) In CUS, use Lemmas 11.4 and 11.5. In WSC: fix $d \approx_m a$ of minimal rank, so that $\boxed{m}a = \boxed{m}d$ by Card1 and $\mathrm{R}d < \mathrm{R}\boxed{m}d < \mathrm{R}\boxed{0}\boxed{m}d$ by (3) and (1). Now d witnesses that $Eq(m/a, 0/\boxed{0}\boxed{m}d)$ via (e4). So $\boxed{m}a = \boxed{0}\boxed{0}\boxed{m}d = \boxed{0}\boxed{0}\boxed{m}a$ by Equating. ⊣

Lemma 11.15 (WSC/CUS). *Comp2 holds.*

Proof. This is an axiom of CUS. For WSC, we argue by induction. Suppose Comp2 holds whenever $\mathrm{R}a < \alpha$; now let $\mathrm{R}a = \alpha$, and suppose $\neg \mathit{Bland}(a)$ and $(\forall x : \mathit{Bland})a \neq \boxed{0}x$. Using Lemma 7.6, fix n and b with $\boxed{n}b = a$ and $\mathrm{R}b < \mathrm{R}a$. If $n > 0$, then $a = \boxed{0}\boxed{0}a$ by Lemma 11.14(4). So suppose instead that $n = 0$,

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i.e., $a = \boxed{0}b$. Now $\neg \mathit{Bland}(b)$, by Comp1 as $(\forall x : \mathit{Bland})\boxed{0}b \neq \boxed{0}x$. Also $(\forall x : \mathit{Bland})b \neq \boxed{0}x$, by Making$_{\mathrm{C}}$ since $\boxed{0}b$ exists. So $b = \boxed{0}\boxed{0}b$ by the induction hypothesis, and $a = \boxed{0}b = \boxed{0}\boxed{0}\boxed{0}b = \boxed{0}\boxed{0}a$. ⊣

LEMMA 11.16 (WSC/CUS). *Equating holds.*

PROOF. This is an axiom of WSC. For CUS, fix $\boxed{m}a$ and $\boxed{n}b$.

Right to left. Suppose $Eq(m/a, n/b)$. I claim that $\boxed{m}a = \boxed{n}b$. If (e1), this is immediate. If (e2), use Card1. If (e3) then $0 = m < n$ and $a = \boxed{0}\boxed{n}d$ for some $d \approx_n b$; now $\boxed{m}a = \boxed{0}\boxed{0}\boxed{n}d = \boxed{n}d = \boxed{n}b$ by Lemma 11.4 and Card1. If (e4), reason similarly.

Left to right. Suppose $\boxed{m}a = \boxed{n}b$. I claim that $Eq(m/a, n/b)$. If either $m = n = 0$, or both $m, n > 0$, this is easy; so suppose $m = 0 < n$ (the case when $0 = m < n$ is similar). Now $b \approx_n b$ by Making$_{\mathrm{C}}$. Fix $d \approx_n b$ of minimal rank; now $\boxed{n}b = \boxed{n}d$ by Card1, and $\mathrm{R}d < \mathrm{R}\boxed{n}d < \mathrm{R}\boxed{0}\boxed{n}d$ by Lemma 11.14, and $\boxed{0}\boxed{n}d = \boxed{0}\boxed{0}a = a$ by Lemma 11.4. So d witnesses that $Eq(m/a, n/b)$ via (e3). ⊣

Assembling everything, WSC = CUS. So CUS is a wand/set theory.

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REFERENCES

[1] P. ACZEL, ***Non-Well-Founded Sets***, Center for the Study of Language and Information, Stanford, 1988.

[2] W. BLIZARD, *Multiset theory*. ***Notre Dame Journal of Formal Logic***, vol. 30 (1989), no. 1, pp. 36–66.

[3] T. BUTTON, *Level theory, part 1: Axiomatizing the bare idea of a cumulative hierarchy of sets*. ***Bulletin of Symbolic Logic***, vol. 27 (2021), no. 4, pp. 436–460.

[4] ———, *Level theory, part 3: A Boolean algebra of sets arranged in well-ordered levels*. ***Bulletin of Symbolic Logic***, vol. 28 (2022), no. 1, pp. 1–26.

[5] ———, *The iterative conception of function and the iterative conception of set*. In: C. Antos, N. Barton, and G. Venturi (eds), *Palgrave Companion to the Philosophy of Set Theory*. Palgrave.

[6] A. CHURCH, *Set theory with a universal set*, ***Proceedings of the Tarski Symposium*** (L. Henkin, editor), Providence, RI, 1974, pp. 297–308.

[7] J. H. CONWAY, ***On Numbers and Games***, Academic Press, London, 1976.

[8] M. COX and R. KAYE, *Amphi-ZF: Axioms for Conway games*. ***Archive for Mathematical Logic***, vol. 51 (2012), pp. 353–371.

[9] T. FORSTER, *Church's set theory with a universal set*, ***Logic, Meaning and Computation*** (C. A. Anderson and M. Zelëny, editors), Springer, Dordrecht, 2001, pp. 109–138.

[10] ———, *The iterative conception of set*. ***Review of Symbolic Logic***, vol. 1 (2008), no. 1, pp. 97–110.

[11] T. FORSTER and R. HOLMES, *Synonymy questions concerning the Quine systems*. (MS).

[12] H. M. FRIEDMAN and A. VISSER, *When bi-interpretability implies synonymy*, Logic Group Preprint Series, 320, 2014, pp. 1–19.

[13] T. HAILPERIN, *A set of axioms for logic*, this JOURNAL, vol. 9 (1944), no. 1, pp. 1–19.

[14] R. HOLMES, ***Elementary Set Theory with a Universal Set***, ***vol. 10,*** Cahiers du Centre de logique, Université catholique de Louvain, Ottignies-Louvain-la-Neuve, 1998.

[15] L. INCURVATI, ***Conceptions of Set and the Foundations of Mathematics***, Cambridge University Press, Cambridge, 2020.

https://doi.org/10.1017/jsl.2024.21 Published online by Cambridge University Press

[16] T. Meadows, *What set theory could not be about*. In: C. Antos, N. Barton, and G. Venturi (eds), *Palgrave Companion to the Philosophy of Set Theory*. Palgrave.

[17] U. Oswald, ***Fragmente von "New Foundations" Und Typentheorie***, Ph.D. thesis, ETH Zürich, 1976.

[18] R. M. Robinson, *The theory of classes: A modification of von Neumann's system*, this Journal, vol. 2 (1937), no. 1, pp. 29–36.

[19] F. Sheridan, *A variant of Church's set theory with a universal set in which the singleton function is a set*. ***Logique et Analyse***, vol. 59 (2016), no. 233, pp. 81–131.

[20] J. R. Shoenfield, *Axioms of set theory*, ***Handbook of Mathematical Logic*** (J. Barwise, editor), North-Holland, London, 1977, pp. 321–344.

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