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AN ACTUARIAL THEORY OF OPTION PRICING

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ABSTRACT

Using an empirical approach to capital market returns analogous to that used for mortality rates by Halley more than three centuries ago to establish life assurance on a sound and scientific footing, a theory of option pricing is built up in terms of the same three key components as for life assurance premiums, namely the expected cost of claims, an allowance for expenses and a contingency margin as a reserve against the risk of insolvency. The dimensionality of the process describing security returns to any future point in time is increased from two to three by the addition of systematic variability around 'central values' to the standard descriptors of expected return and variance of return. It is shown that this approach, which involves only common sense principles and elementary mathematics, has important theoretical, practical and regulatory advantages over the Black-Scholes and related methodologies of modern finance theory.

KEYWORDS

Option Pricing; Black-Scholes Formula; Cox, Ross & Rubinstein; Diffusion Processes; Dynamic Equilibrium Model; Stochastic Calculus; Equilibrium; Efficient Market Hypothesis; Risk; Uncertainty; Central Values

The use of even the most sophisticated forms of mathematics can never be considered as a guarantee of quality. Mathematics is, and can only be, a means of expressing and reasoning. The real substance on which the economist works remains economic and social. Indeed, one must avoid the development of a complex mathematical apparatus whenever it is not strictly indispensible. Genuine progress never consists in a purely formal exposition, but always in the discovery of the guiding ideas which underlie any proof. It is these basic ideas which must be explicitly stated and discussed.

Maurice Allais, Nobel Lecture, 9 December 1988

1. INTRODUCTION AND OVERVIEW

1.1 *The New Approach*

1.1.1 This paper is very much a personal viewpoint, based on around a quarter of a century's practical experience of institutional investment. To a mathematician who has not been closely involved in the day-to-day vicissitudes of our complex and often perplexing financial world, it might appear that the many simplifying assumptions inherent in currently accepted theories of option pricing are relatively innocuous. As explained below, particularly in Section 5, my personal experience has led me to a quite different perspective.

1.1.2 There are very strong parallels between the transaction of life assurance business and the writing of options on securities; both involve a contractual liability to make future payments determined by outcomes that cannot be predicted with certainty. In the former case, where average rates of mortality for homogeneous subgroups have been found to be remarkably stable over time, the appropriate premium comprises three quite distinct components — the expected cost of claims paid out to the policyholder, an allowance for expenses of management, and a contingency margin as a reserve against the risk of insolvency. Provided that a sufficiently robust probability model for the amount to be paid out to the option holder is available, the premium for a call or put option on a security can similarly be regarded as the sum of the same three basic components, namely the expected cost of payments to the option holder, an allowance for expenses of management and a contingency margin as a reserve against the risk of insolvency.

1.1.3 Consider a company which writes a block of N identical European call (or put) options of life t years. Let P be the option premium per contract, let q be the expected amount per contract to be paid out on the expiry date, let Q be the higher of the amount per contract to be held at the end of the period on prudential grounds (i.e. on the initiative of the company) or required to be held for regulatory purposes, let E be the expenses of management per contract (assumed to be paid out half way through the life of the contracts), let R be the target rate of return on capital employed, and let r be the return on admissible assets (i.e. the lower of that in line with the internal guidelines of the company or as prescribed by regulation). Then, if C is the additional amount of capital employed per contract required at the beginning of the period, NC when added to NP and accumulated at rate r must be able, first of all, to pay the aggregate expenses NE and then to meet the 'worst case' total NQ of payments to option holders. This accumulation must also be sufficient, after paying the aggregate expenses NE and the expected cost of payments Nq, to give a return of R on the capital employed of NC. The two resulting equations of value are:

$$(NC+NP)(1+r)^t = NQ + NE(1+r)^{\frac{t}{2}}$$

$$= Nq + NE(1+r)^{\frac{t}{2}} + NC(1+R)^t.$$

Eliminating P gives the capital employed per contract C as:

$$C = \frac{(Q-q)}{(1+R)^t}$$

while eliminating C gives the option premium P as:

$$P = \frac{q}{(1+r)^t} + \frac{E}{(1+r)^{\frac{t}{2}}} + (Q-q)\left(\frac{1}{(1+r)^t} - \frac{1}{(1+R)^t}\right)$$

$$= \frac{q}{(1+r)^t} + \frac{E}{(1+r)^{\frac{t}{2}}} + C\left(\frac{(1+R)^t - (1+r)^t}{(1+r)^t}\right).$$

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which comprises three quite distinct components, namely an expected cost of payments component, an expenses component and a contingency margin against the risk of further capital having to be injected to avoid insolvency. The expression for C and the latter expression for P can be justified very easily by the familiar actuarial verification technique of 'general reasoning':

> "The reserve to be set up at the end of the contract is $Q-q$, and accordingly the amount of capital to be set aside initially is this reserve discounted at the target rate R. The 'net premium' corresponding to the amount per contract to be set aside initially to meet the expected payment is q discounted at the rate r earned on the invested assets, and this has to be supplemented by two further amounts, namely the expenses E discounted for half the period at the rate r earned on the invested assets and such an amount as is sufficient, when accumulated at the rate r earned on the invested assets, to make up the difference between the accumulation of C at the target rate R and the accumulation of C at the rate r actually earned on invested assets."

1.1.4 The only difficult step in this new approach is to find a tractable mathematical description of future share prices which not only encapsulates the general characteristics of real world behaviour, but can also accommodate relevant judgements as to future financial trends that can be made by experienced practitioners. I believe that there are two quite distinct causal mechanisms behind capital market prices. First, there is a very strong 'central value' component, the present day equivalent of Adam Smith's 'invisible hand', which tends to drive prices towards what are perceived to be 'sensible' values. Two obvious examples are that one unit of current earnings per share will be valued more highly the higher a company's perceived future rate of profits growth, and that the 'yield ratio' of the long-dated gilts yield to the dividend yield on equities will increase with the perceived growth prospects for corporate earnings in the aggregate. Second, there is a significant 'systematic over-reaction' component which causes prices to swing back and forward through their central values as the result of inherent instabilities in consensus investor sentiment. A very clear description of this latter effect and the need to take explicit account of it in prudent financial management is given by Keynes (1936):

> "Expert professionals are concerned, not with what an investment is really worth to a man who buys it 'for keeps', but with what the market will value it at, under the influence of mass psychology, three months or a year hence. For it is not sensible to pay 25 for an investment of which you believe the prospective yield to justify a value of 30, if you also believe that the market will value it at 20 three months hence."

1.1.5 Both of these components are comprehensively documented in the United Kingdom actuarial literature. Jamieson (1959), Weaver & Hall (1967) and Clarkson (1981) describe practical approaches for the estimation of central values, while Plymen & Prevett (1972) and Clarkson (1978, 1981) describe the Mean Absolute Deviation approach, which can be used to great practical effect in the analysis of security prices relative to their central values. The new approach incorporates compound share price distributions which are the product of two simpler probability distributions, one relating to the 'central value' component and the other to the 'systematic over-reaction' component.

1.1.6 Consider a European call option of life one year on an equity share that is expected to outperform the index by 5% over the year, with the index expected to rise by 3% over the year. The exercise price of the option is 100p, and the present share price is 95p. Then, from Table 2, as described in more detail in Section 6, the amount q expected to be paid out to the option holder on expiry is 8.42p. Taking E and Q as $0.05q$ and $2q$ respectively, and using values of 8% p.a. and 15% p.a. respectively for r and R, gives the call option premium as:

$$P = 7.796 + 0.405 + 0.475 \quad \text{i.e. } 8.68\text{p.}$$

For the corresponding European put option with identical assumptions, the amount q expected to be paid out to the option holder is, from Table 4, 4.49p, giving the put option premium as:

$$P = 4.157 + 0.216 + 0.253 \quad \text{i.e. } 4.63\text{p.}$$

1.2 *The Inertia of Old Ideas*

1.2.1 Adam Smith in his *History of Astronomy*, published posthumously in 1795, was the first philosopher to direct attention to a very serious limitation of the human intellect, namely the inability to adapt readily to new and better ideas for guiding practical action once familiar old ideas have become deeply ingrained through many years of teaching. Keynes (1936) describes the seriousness of this problem in the final two sentences of the Preface to his *General Theory*:

> "The ideas which are here expressed so laboriously are extremely simple and should be obvious. The difficulty lies, not in the new ideas, but in escaping from the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds."

Allais (1989) is no less forthright in describing the consequential risk to society as a whole, namely that 'better', and often simpler, ideas will, in general, be vehemently resisted by the establishment.

1.2.2 In the context of the present paper, the 'old ideas' are three of the cornerstones of so-called 'modern finance theory', namely the Efficient Market Hypothesis, the presumption of equilibrium, and the variance of return paradigm of risk. The 'new ideas' are by no means new in the chronological sense, but relate to insights drawn from the successful practice of life assurance and pensions business by actuaries over the past three centuries.

1.3 *The Black-Scholes Formula and Recent Criticisms*

1.3.1 The pricing formula for European options, derived in Black & Scholes (1973), was, from the very beginning, regarded by experienced investment professionals as being based on simplifying assumptions that bore no resemblance to the real financial world. In particular, in the Black-Scholes world it is assumed

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that capital markets are frictionless, that security prices are always in equilibrium, and that risk can, in some miraculous manner, be 'diversified away'. In recent years, this instinctive distrust of the Black-Scholes methodology has been justified by an increasing number of highly critical assessments, six of which are summarised below.

1.3.2 After a detailed and highly perceptive analysis of option prices in the U.K., Nisbet (1990) concludes that "either the efficiency of the London Traded Options Market or the equilibrium between the option and share markets must be questioned."

1.3.3 In evidence to the United States House Banking Committee, Soros (1994) describes how 'dynamic hedging' strategies derived from the mathematics of the Black-Scholes methodology can, in certain circumstances, destabilise market levels, and thereby greatly increase the inherent level of financial risk:

> "If there is an overwhelming amount of dynamic hedging to be done in the same direction, price movements may become discontinuous. This raises the specter of financial dislocation. Those who need to engage in dynamic hedging, but cannot execute their orders, may suffer catastrophic losses .... That is what happened in the stock market crash of 1987. In short, attempts to 'rebalance' portfolios on either a sharp fall or a sharp rise in the market could shatter the theoretical 'equilibrium' on which the rebalancing strategy was based."

1.3.4 Bouchaud & Sornette (1994) and Bouchaud, Iori & Sornette (1996) criticise the price behaviour assumptions of the Black-Scholes formula and show that, in both the French and the U.K. traded options markets, variance does not increase linearly with time, as assumed in Black & Scholes (1973) and in other currently accepted methodologies, such as Cox, Ross & Rubinstein (1979). They suggest that a radically different type of price distribution is required, and also replace the equilibrium-based 'risk-free' approach with a risk framework broadly consistent with the familiar mean-variance methodology of Markowitz (1952, 1959). A crucial practical consequence of their proposed alternative framework is that higher solvency reserves than those based on the Black-Scholes formula are required. In a recent letter in a U.S. physics periodical, Bouchaud & Sornette (1996) claim that their approach is preferable to currently accepted methodologies. In particular, they suggest that the Black-Scholes approach involves mathematics well beyond the grasp of most business school graduates, with the result that many users of the Black-Scholes approach are relying on a 'black box' system which they do not understand.

1.3.5 Bartels (1995) examines the hypotheses underlying option pricing theories, and, in particular, discusses the criticisms of the Black-Scholes methodology made by Bergman (1982). In one of the very few direct attacks on the mathematics used, Bergman suggests, not only that the application of stochastic calculus is erroneous, but also that the hedge portfolio is not, in fact, self-financing, and concludes that 'the Black-Scholes derivation is an example of two wrongs which make a (most important) right'. Bartels also cites empirical evidence for the rejection of the Black-Scholes model, and agrees with the

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conclusions set out in Föllmer (1991) regarding the unsatisfactory nature of the underlying diffusion process used for price fluctuations:

> "In any case, there seems to be a need for a thorough look at the probabilistic structure of basic price fluctuations. This is a challenging program in itself. But it would also lead to a reconsideration of hedging strategies. Furthermore, it would be a crucial step towards a more rigorous analysis of the impact of such strategies on the underlying price process."

1.3.6 In the third of a series of increasingly comprehensive investigations of the foundations of currently accepted option pricing approaches, Jousseaume (1996) shows that the Black-Scholes methodology leads to numerous paradoxical results which suggest that there are serious inconsistencies in the underlying assumptions. Jousseaume rejects any approach based on a continuously rebalanced 'hedge portfolio', and puts forward as a possible alternative an 'actuarial' framework based on expected values.

1.3.7 Geman & Ané (1996) highlight two well known problems that raise serious doubts as to the validity of the assumption that geometric Brownian motion over time is the stochastic process which drives asset prices, namely that distributions of asset returns are typically far more peaked than the corresponding fitted Gaussian distribution, and the 'volatility smile' effect of the prices of deep in-the-money and deep out-of-the money options, implying higher volatilities than those which are at-the-money. They explain how a 'stochastic clock' may cast light on the origins of both non-normal asset returns and stochastic volatility, and use this approach to estimate a probability density function for Standard & Poors (S&P) 500 returns at timescales of one minute, fifteen minutes, thirty minutes, one hour and one day. For one-minute returns, the density function obtained exhibits two very pronounced features, namely a much higher peak than the fitted Gaussian distribution, and 'shoulders' at around one standard deviation above and below the mean where the density function is roughly constant at about one third of the value at the peak.

1.3.8 A less specific, but nevertheless important, criticism of the Black-Scholes methodology relates to the general perception of many investment professionals and regulators that, while it may, in most circumstances, give reasonable results in the context of the short-term financial risks typically traded by banks, it is seen as far less satisfactory as a framework for assessing longer-term financial risks. In particular, the costs of longer-term options often appear excessive to those who wish to use them for prudential purposes in a life assurance or pension fund context, and amongst regulators there is an instinctive distrust of its robustness in the face of extreme financial circumstances of the type exemplified by the 'Crash of 1987'.

1.4 *The Dangers of Mathematics in Economics*

Allais (1954) draws a very clear distinction between the selection of realistic simplifying assumptions that are required before a formal approach can be developed and the use of mathematics as a rigorous logical framework to develop these simplifying assumptions into hypotheses or theories about real world behaviour. He also suggests that only those who have extensive practical experience gained over a period of many years should attempt to formulate economic models. Towards the end of his Nobel Lecture given on 9 December 1988 (from which the introductory quotation of this paper is taken) Allais reiterates his serious concerns about the misguided use of mathematics, and stresses that the validity of even highly fashionable methodologies must, on occasion, be questioned, regardless of the consequences:

> "I have never hesitated to question commonly accepted theories when they appeared to me to be founded on hypotheses which implied consequences incompatible with observed data. Dominant ideas, however erroneous they may be, end up, simply through continual repetition, by acquiring the quality of established truths which cannot be questioned without confronting the active ostracism of the establishment."

1.5 *Equilibrium, Efficiency and Risk*

1.5.1 The Black-Scholes assumptions regarding equilibrium, efficiency and risk, while perhaps in line with the embryonic state that modern finance theory had reached at the time Black and Scholes were finalising their seminal 1973 paper, are seriously inconsistent with the experience of most professional investors. Equilibrium implies that returns on all securities, perhaps adjusted in some manner for risk, are equal. The gilts model described in Clarkson (1978) improved the return on a portfolio of long-dated stocks by around 0.4% p.a. through virtually risk-free anomaly switching, while the equity model described in Clarkson (1981) could achieve, after one year, consistent differentials between the top 20% and bottom 20% of shares as ranked by apparent attractiveness. The results of the latter model also destroy the credibility of any notion of stockmarket efficiency. In the Black-Scholes world it is assumed, not only that risk is equivalent to the variance of return (which is furthermore assumed to be constant over time), but also that, in some miraculous way, it can be 'diversified away' to create a 'risk-free' asset. The alternative downside framework for risk, first outlined in Clarkson & Plymen (1988) and subsequently developed in Clarkson (1989, 1990, 1996a), is, I believe, much closer to reality, and at the 6th AFIR International Colloquium, held in 1996 in Nuremberg, it was obvious, both from the presented papers and the discussions thereon, that the downside or shortfall approach to risk is gaining in acceptance over the variance of return paradigm, first propounded in Markowitz (1952).

1.5.2 An excellent starting point as regards the concept of equilibrium is the analysis by Adam Smith at the beginning of his investigation into rates of wages and profit in his *Wealth of Nations*:

> "The whole of the advantages and disadvantages of the different employments of labour and stock must, in the same neighbourhood, be either perfectly equal or continually tending to equality. If in the same neighbourhood, there was any employment evidently either more or less advantageous than the rest, so many people would crowd into it in the one case, and so many would desert it in the other, that its advantages would soon return to the level of other

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> employments. This at least would be the case in a society where things were left to follow their natural course, where there was perfect liberty, and where every man was perfectly free both to choose what occupation he thought proper, and to change it as often as he thought proper. Every man's interest would prompt him to seek the advantageous, and to shun the disadvantageous employment."

He then observes, on the basis of very extensive empirical evidence, that "pecuniary wages and profit, indeed, are everywhere in Europe extremely different according to the different employments of labour and stock", and explains these differences in terms of three quite distinct factors. First, there are genuine differences in relevant attributes which lead to eminently sensible variations in wage and profit rates. Second, there are perceived or psychological differences in attributes which exist only 'in the imaginations of men'. These include, in particular, what I describe above as 'systematic over-reaction' behaviour. Third, there is government intervention (such as restrictions on entering certain trades) "which nowhere leaves things at perfect liberty".

1.5.3 Consider now the Clarkson (1981) equity model, where the guiding principle is that, after adjustment for other important attributes such as the dividend payout ratio, the price that should be paid for unit amount of current earnings per share increases as the perceived future growth rate of earnings per share increases. This growth rate and other important attributes, such as the dividend payout ratio, correspond to Smith's 'genuine differences'. Misconceptions as to a realistic future growth rate, as vividly exemplified by 'speculative bubbles' such as the boom and subsequent bust in Poseidon and other Australian nickel shares around a quarter of a century ago, correspond to Smith's 'perceived differences'. Legislative and internal restrictions on the admissibility of certain investments (e.g. U.K. smaller companies unit trusts will not invest in FT-SE 100 constituents other than to a very limited degree) are very strong frictional forces which correspond to the government intervention described by Smith. The empirical results, which show a steadily increasing differential in performance for periods of up to a year as between 'buys' and 'sells' selected by the model, are consistent with equity share prices 'continually tending towards equality' rather than being 'perfectly equal' in terms of expected future returns. Accordingly, equity share prices are not in equilibrium.

1.5.4 Financial economists, on the other hand, tend to take equilibrium for granted. A classic example is Jensen (1968), one of the cornerstones of the so-called scientific evidence of stockmarket efficiency. The final crucial assumption required to set up the Capital Asset Pricing Model as the measuring rod for detecting any apparent inefficiencies is introduced in a somewhat casual manner, and with no further justification or discussion whatsoever, as:

> "Given the additional assumption that the capital market is in equilibrium, ..."

1.5.5 The formalisation of the concept of stockmarket efficiency in terms of the Efficient Market Hypothesis, propounded by Fama (1970), can be regarded as being equivalent to the statement that security prices fully and instantaneously reflect all available information, with efficiency being classified as 'weak', 'semi-strong', or 'strong', depending on whether the available information is respectively purely historical, publicly available, or all information whether publicly available or not. There are, however, very fundamental conceptual difficulties involved in devising tests of efficiency that are compatible with mainstream modern finance theory. In particular, in ¶1.5.3, I explain why the highly profitable differentials over time, in the aggregate performance of different sub-groups of shares identified by my equity model, show that the equity market is not in equilibrium in the generally accepted sense, whereas the Capital Asset Pricing Model measurement tool set up by Jensen to test for 'strong level' efficiency assumes that the equity market is, in fact, in equilibrium. The Jensen (1968) methodology of testing for 'strong level' efficiency is, accordingly, as nonsensical as looking through a red filter to test whether visible light towards the blue/indigo/violet end of the spectrum is present.

1.5.6 To circumvent these conceptual difficulties, I suggest, in Clarkson (1996a), a 'Turing test' for equity selection models. For some suitably large universe of shares, such as the FT-SE 100 or FT-SE Mid-Cap 250 constituents (to ensure that the results are not unduly distorted by mere random chance), rank the shares in order of attractiveness to identify the 20% of shares which are most attractive (the 'buys') and the 20% of shares which are least attractive (the 'sells'). At weekly intervals from the selection date, calculate the average performance relative to the universe of shares for:

- $M(B)$ — the 'buys' selected by the model;
- $M(S)$ — the 'sells' selected by the model;
- $U(B)$ — the best performing 20% of shares in the universe from the selection date; and
- $U(S)$ — the worst performing 20% of shares in the universe from the selection date.

Then calculate:

(1) the performance index I as:

$$I = \frac{M(B) - M(S)}{2}$$

(2) the '100% hindsight' performance P as:

$$P = \frac{U(B) - U(S)}{2}$$

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(3) the added value ratio V as:

$$V = \frac{I}{P}.$$

The Efficient Market Hypothesis will be contradicted if the performance index I is consistently in excess of that required to cover the level of switching costs appropriate to an institutional investor. The results of my equity model around 20 years ago give typical values for I of around 0.03 after three months and around 0.045 after six months, which translates, after 2% 'round trip' expenses for an institutional investor, into a highly satisfactory outperformance of 6% p.a. on the optimal switching period of around four months.

1.5.7 These results from my equity model some twenty years ago are unlikely to be regarded as a serious breach of the Efficient Market Hypothesis, since the situation relates to 'strong level' efficiency results that were not verified or subjected to statistical tests by any external party. However, the publicly available U.K. equity share selection technique, based primarily on the ratio of the prospective price-earnings ratio to the prospective growth in earnings per share over the next twelve months, as described in Slater (1996), not only appears to demonstrate an impressive performance record on back-testing, but can also be regarded as a simplified special case of my equity model. If, as seems highly likely, my 'Turing test' can show that significant and steady differentials in performance exist for all selection dates tested, the case, not only for rejecting the Efficient Market Hypothesis at both the 'semi-strong' and 'strong' levels, but also for abandoning the concept of capital market equilibrium, would appear to be overwhelming.

1.5.8 Even if it is accepted in principle that a downside approach to risk is required, there remains the very difficult problem of identifying realistic probability distributions for future security prices. Despite the pioneering work of Mandelbrot (1963), more than thirty years ago, normal (i.e. Gaussian) or lognormal (as in the Black-Scholes formula) distributions still dominate modern finance theory. This pursuit of mathematical and statistical tractability, at the possible expense of suppressing reality, is reminiscent of remarks by Keynes (1921):

> "The statistical result is so attractive in its definiteness that it leads us to forget the more vague though more important considerations which may be, in a particular case, within our knowledge. To a stranger the probability that I shall send a letter to the post unstamped may be derived from the statistics of the Post Office; for me those figures would have but the slightest bearing upon the question."

Similarly, I regard the empirical results of Mandelbrot (1963), Peters (1991), Bouchaud & Sornette (1994), Walter (1995) and Geman & Ané (1996), all of which refute the existence of underlying normal or lognormal distributions, as being "important considerations .... within our knowledge". The obvious way

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forward is to endeavour to find some probabilistic model of future security prices that is not only consistent with these empirical studies, but is also amenable to practical implementation.

1.6 *Four Levels of Behaviour*

1.6.1 The observation by Keynes, that his behaviour will not follow that of some simplistic stereotype which might be convenient to model in statistical terms, corresponds to my belief, as set out in Clarkson (1996a, 1996b), that it is essential to abandon the assumption that all investors behave 'rationally' (in the omniscient and faultless manner as formulated by economists), and to introduce, instead, four quite distinct stereotypes of behaviour.

1.6.2 'Intelligent' investors are those who assess the future prospects of a security against its current relative rating, while 'unintelligent' investors are those who buy or sell on 'good' or 'bad' news respectively, irrespective of the current relative rating. Then, in line with the observation by Keynes (1936) that 'it is not sensible to pay 25 for an investment of which you believe the prospective yield to justify a value of 30, if you also believe that the market will value it at 20 three months hence', we introduce a third kind of investor, namely 'optimal' investors, who not only understand the actions of both 'intelligent' and 'unintelligent' investors, but also endeavour, by studying historic cyclical patterns as well as current sentiment and fundamentals, to exploit the deviations from 'fair value' resulting from the actions of 'unintelligent' investors. Finally, interpreting 'optimal' behaviour in a static sense, as relating to current levels of skills and achievement, we introduce 'rational' investors as those who not only act 'optimally', but also endeavour, through conscious choice, to improve their level of achievement through appropriate training and practical experience.

1.7 *Four Kinds of Uncertainty*

1.7.1 Guided both by the postulated existence of these four levels of investor behaviour and by parallels with life assurance, which for several centuries has represented by far the most successful financial application of probability theory, I believe that it is very useful to regard the probability distributions underlying any assessment of risk as comprising four quite different kinds of uncertainty. 'Uncertainty of the first kind' relates to the existence of differing behaviour as between different sub-groups of individuals. In the case of life assurance, rates of mortality in general vary markedly in three quite distinct ways — as between assured lives and annuitants, as between males and females, and as a function of the elapsed time since 'selection' in that 'select' mortality rates are lower than, but converge fairly rapidly towards, 'ultimate' rates. 'Uncertainty of the second kind' relates to the expected future values of primary fundamental variables. In the case of life assurance, this relates to the force of mortality as a function of attained age on the basis of past experience extrapolated to take account of any systematic trends over time. 'Uncertainty of the third kind' is the aleatory (i.e. pure random chance) variability of actual deaths for a given underlying force of

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mortality, on the assumption that for different lives the probabilities of death in a given period are independent. 'Uncertainty of the fourth kind' relates to the existence of any causal factors which could lead to a model based solely on the previous three kinds of uncertainty breaking down as a framework for portraying behaviour in the real world. In life assurance, for instance, the 'third kind' independence assumption would not apply in the case of a group life scheme covering employees in a large chemicals factory where there was a significant risk of explosion or fire. Also, essentially unpredictable future events, such as an AIDS epidemic or a cure for cancer, could lead to significant losses on assured life and annuitant business respectively, where the premium rates were based essentially on past experience.

1.7.2 In the case of capital market behaviour, some parallels are reasonably obvious. 'Uncertainty of the second kind' relates to fundamental attributes, such as earnings per share and future growth rates in the case of individual companies, and to interest rates, dividend yields, and aggregate rates of dividend growth in the case of equity market levels, while 'uncertainty of the fourth kind' relates to extreme and apparently unpredictable price fluctuations such as the 'Crash of 1987'. As regards 'uncertainty of the third kind', I suggest, in Clarkson (1996a, 1996b), a Dynamic Equilibrium Model, the resultant of the actions of the different levels of behaviour described in Section 1.6, which is characterised by simple harmonic motion of prices (with a random term superimposed) around 'central' values. In mainstream modern finance theory, however, it is assumed that all investors act 'rationally' in terms of faultless and omniscient behaviour, and accordingly that no such systematic deviations from 'fair value' can occur.

1.8 *Structure of the Paper*

1.8.1 The 'rise and fall' of the Black-Scholes methodology from around a quarter of a century ago is strongly reminiscent of the situation as regards 'laws of mortality' (see Neill, 1977) in the development of life assurance practice from around a century and a half earlier, with the diffusion of security returns to some future point in time corresponding to a simple mathematical expression for the force of mortality as a function of age. Gompertz Law, propounded in 1825, expresses the force of mortality μ_x at age x as:

$$\mu_x = Bc^x$$

where c is the crucial fitted parameter and B is a relatively unimportant scaling constant. This formulation bears an uncanny functional resemblance to the Black-Scholes formula, where the crucial fitted parameter is volatility (as encapsulated by the variance of return paradigm of risk) and the relatively unimportant subsidiary variable is the so-called 'risk-free' rate of return.

1.8.2 Although Gompertz Law was a highly significant breakthrough at that time, in terms of the sound and scientific practice of life assurance, various painstaking empirical studies showed that systematic deviations from predicted

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patterns of mortality could be detected. These studies led to the formulation of more realistic laws of mortality, such as, in 1860, Makeham's Law, which generalises Gompertz' Law by the addition of a constant, and, in 1867, the Double Geometric Law, which permits much more complex patterns of variation by age.

1.8.3 The discussions in Sections 1.3, 1.5, 1.6 and 1.7 suggest that either some fairly radical modification to the Black-Scholes methodology or a completely new approach is necessary if we are to find a satisfactory formulation of the probability distributions of future security returns. In Section 2, after examining the Black & Scholes (1973) methodology from three different perspectives, namely commonsense principles, empirical evidence and mathematical principles, I conclude that we have to abandon this methodology completely, and return to first principles to build, from the ground upwards, a new approach incorporating a much more realistic formulation of capital market behaviour. Section 3, which develops the crucial insight in Bernoulli (1738) that led to what we now know as utility theory, sets out some very general principles of human behaviour in the highly uncertain financial world, and then Section 4 discusses further parallels with life assurance and, in particular, develops a compound distribution approach which can address what I call 'uncertainty of the first, second and third kinds'. After this preparatory groundwork, Section 5 discusses the dynamics of capital market prices, and suggests how risk, particularly that arising from 'uncertainty of the fourth kind', can be taken into account.

1.8.4 If this new theory of option pricing is to have any likelihood of being regarded as a serious rival to current methodologies, it is essential that a practical method of computing the compound distributions can be demonstrated. This is addressed in Section 6 using a 'commutation function' approach, similar to that used in life contingencies. The crucial computations are summarised in terms of a series of tables for both European call options and European put options, with, as very special cases, one table of each series giving the values as calculated in terms of the familiar Black-Scholes formula. Thereafter, Section 7 draws brief comparisons with current methodologies and the suggested approaches of Bouchaud & Sornette (1994, 1996) and Geman & Ané (1996), and Section 8 sets out general conclusions at to the merits of the new approach.

2. Guiding Principles

2.1 *Parallels with Gilts*

In the highly successful non-linear gilts model described in Clarkson (1978), the crucial fundamental relationship, which was deduced from a 'common sense' examination of the opportunities available to investors, is that a certain second order partial derivative is negative. In the old paradigm, characterised by redemption yield models of the type described in Pepper (1964), which had been

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successful until gilt yields rose markedly in the late 1960s and early 1970s, price was a linear function of coupon, and accordingly this partial derivative was identically equal to zero.

2.2 *A Fundamental Relationship*

Suppose now that we have a class of functions for share returns to some future point in time that involves, possibly amongst other independent variables, the expected return (or 'drift') r and the standard deviation s. Consider, from the point of view of the buyer of a European call option, the amount q expected to be received from the option writer at the end of the contract. Elementary considerations show that, for a given value of s, q will increase with r. On the important proviso that 'all other things are equal', the price that the option buyer will be prepared to pay will also increase with r. Irrespective of the manner in which the option writer may choose to 'reinsure' or 'hedge' the uncertain future liability to make a payment to the option holder, it seems highly likely that the general pattern of option prices P, as described in ¶1.1.3, will also increase with r, giving the crucial functional relationship:

$$\frac{\partial P}{\partial r} > 0.$$

In the Black-Scholes paradigm, however, this partial derivative is 'proved' to be identically equal to zero. Parallels with the paradigm shift in gilt-edged mathematics around twenty years ago suggest that any new and better paradigm of option pricing will embrace the 'commonsense' approach, in which this partial derivative is strictly positive in the case of call options and strictly negative in the case of put options.

2.3 *Empirical Evidence*

2.3.1 Many of the empirical studies referred to earlier, particularly Mandelbrot (1963), Bouchaud & Sornette (1994), Walter (1995) and Geman & Ané (1996), are very strong evidence that the class of function required to describe share returns will have to include further independent variables in addition to the 'drift' and 'variance' factors used both in the continuous time and (mathematically equivalent) discrete time processes underlying current methodologies. In recent years, more general investigations by Peters and others have used the diagnostic tools of chaos theory to arrive at the same fundamental conclusion, namely that diffusion processes of the type first suggested by Bachelier (1900) cannot come anywhere near a realistic portrayal of observed capital market behaviour.

2.3.2 Peters (1991) uses the methodology of chaos theory, and in particular the Hurst exponent (which is the reciprocal of Mandelbrot's fractal dimension) to show that, not only individual share prices, but also market indices, have Hurst exponents well in excess of the value of 0.5 implied by a random diffusion

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process. This proves the existence of a significant 'long-term memory' effect which is totally inconsistent, not only with the teachings of the Efficient Market Hypothesis, but also with the presumption of equilibrium.

2.4 *The Need for a New Paradigm*

2.4.1 Kuhn (1970) describes how the general pattern of scientific progress is for one temporarily successful paradigm (i.e. conceptual approach) to be superseded, usually after acrimonious debate within the relevant scientific community, by a new and better paradigm, which is not only inconsistent with the old paradigm, but can also explain empirical evidence that stubbornly refuses to conform with the teachings of the old paradigm.

2.4.2 In the light of:

(1) all the criticisms (many by exceptionally able mathematicians) of the Black-Scholes methodology as summarised in Section 1.3;
(2) the analogy, in Section 1.8, of Gompertz' Law as a temporarily successful law of mortality falling out of favour once it was shown that it could not portray real world experience with sufficient accuracy;
(3) the analogy, in ¶2.1, of a new non-linear paradigm of gilt-edged mathematics being successful in replacing the old paradigm based on linear methodologies; and
(4) the empirical evidence, as summarised in Section 2.3, to the effect that geometric diffusion processes, whether in continuous time or discrete time, bear little resemblance to observed behaviour in the real financial world;

it would seem unscientific in the extreme not to conclude that a completely new paradigm of option pricing is urgently required.

3. BERNOULLI — THE POOR MAN AND THE RICH MAN

3.1 *The Principle of Differing Values*

3.1.1 Bernoulli (1738), after describing the then current paradigm of risk in the following terms:

> "Since there is no reason to assume that of two persons encountering identical risk, either should expect to have his desires more closely fulfilled, the risks anticipated by each must be deemed equal in value,"

observes that the logical deduction that no characteristic of the persons themselves ought to be taken into consideration is inconsistent with the following example:

> "Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats?"

Bernoulli suggests that the poor man would be ill-advised not to sell it for nine thousand ducats, while a rich man would be ill-advised to refuse to buy it for nine thousand ducats, and concludes that the previous paradigm, in which all men use the same rule to evaluate the risky situation, must be discarded.

3.1.2 Bernoulli then generalises the argument to say that the value of an item must not be based on its price, but rather on the 'utility' it yields. The price of an item is dependent only on the thing itself, and is equal for everyone, whereas the 'utility' is dependent on the particular circumstances of the individual making the assessment. This entirely new hypothesis, namely that 'no valid measurement of a risk can be obtained without consideration being given to ... the utility ... to the individual', remained essentially dormant until it was formalised in precise mathematical terms by von Neumann & Morgenstern (1944). Its subsequent axiomatisation by U.S. mathematicians such as Savage (1954) resulted in utility theory becoming one of the central pillars of modern finance theory.

3.2 *Lack of Uniqueness*

3.2.1 The 'willing buyer, willing seller' price of nine thousand ducats is, in general, only one possible price. If, for example, the poor man is prepared to accept any price not less than eight thousand ducats, while the rich man is prepared to accept any price not more than nine and a half thousand ducats, the price at which the bargain is struck could be anywhere within this range, depending on the interaction of a multitude of essentially extraneous circumstances, such as the negotiating skills of both participants and the perceived existence and preferences of alternative buyers and sellers of similar risky propositions.

3.2.2 An immediate corollary is that the actual price cannot, in general, be expressed uniquely as a single-valued function of the characteristics and preferences of the buyer and the seller. This lack of uniqueness destroys all possibility of the price being (as is generally assumed by financial economists) a continuous or differentiable function of variables relating to the circumstances and preferences of the buyer and seller and to general external financial conditions. Also, this fundamental principle that 'price' and 'value' are normally different is consistent with comments by Smith (1776):

> "The natural price, therefore, is, as it were, the central price, to which the prices of all commodities are continually gravitating. Different accidents may sometimes keep them suspended a good deal above it, and sometimes force them down even somewhat below it. But whatever may be the obstacles which hinder them from settling in this centre of repose and continuance, they are constantly tending towards it."

3.3 *Risk, Risk Aversion and Divisibility*

3.3.1 It is instructive to discuss, in general terms, the underlying reasons as to why both the poor man and the rich man perceive that their situations can be improved by exchanging the lottery ticket at a price of around 9,000 ducats.

3.3.2 The poor man, on the implicit assumption that any similar opportunity is unlikely to occur, knows that there is no 'middle ground' as regards the

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outcome of the lottery, and he also knows that, if his ticket loses, it will cause him considerable mental anguish, possibly for many years, to realise that he has thrown away the opportunity of achieving, with certainty, a wealth level vastly in excess of what previously seemed possible. He can visualise very clearly the much enhanced quality of life that 9,000 ducats would bring, and he regards as unacceptable the great risk of seeing these aspirations shattered.

3.3.3 As regards the rich man, on the other hand, the implicit assumptions are that the unfavourable outcome of receiving nothing causes him no meaningful mental anguish, and that there are available to him a large number of similar, but probabilistically independent, risky ventures. Accordingly, he can confidently visualise the likelihood of his average out-turn over this large number of such risky ventures leading to a satisfactory return on the capital employed. This divisibility of part of his wealth ensures that he would be highly unlucky not to achieve an overall out-turn close to the 'middle ground' expectation, the chance of a significant proportionate shortfall from his expectation is accordingly very low, and hence he perceives a very low level of risk.

3.3.4 Three generalisations seem appropriate. First, risk relates both to the severity of the consequences of an outcome significantly less advantageous than some 'comfort level' benchmark and to the perceived probability of occurrence of this adverse outcome. Second, a prudent individual will be risk averse in the sense that, provided the likely cost is not unreasonable, a course of action will be pursued, if available, which eliminates the possibility of a psychologically distressing shortfall below some achievable 'comfort level'. Third, when a collective risky event is divisible into a number of subsidiary risky events with known and independent probability distributions, the risk decreases towards zero as the amount committed to each subsidiary risky event decreases to zero, provided that the expected value is at least equal to the 'comfort level' benchmark.

3.4 *Order and Disorder in the Financial Markets*

3.4.1 Bernoulli's 'poor man, rich man' example is static as regards the dimension of time, in that the price of, say, 9,000 ducats at which the two participants are prepared to deal relates to a specific instant in time. While the mathematical formulation is simplified immeasurably if it is assumed that the basis on which the price of the bargain is struck does not vary over time, a little thought shows that this is a totally unrealistic assumption.

3.4.2 Consider the present day 'poor man, rich man' situation of an individual investor of modest means and a unit trust management company. Very often the purchase or sale of units is done on a historic price basis using the previous daily valuation of the fund, which could be up to 24 hours out of date. Provided that market prices have not moved significantly since the previous valuation point, which is usually the case, the convenience and psychological comfort to the investor from knowing in advance the price at which units are bought or sold becomes, in the aggregate, a perceived marketing advantage to the company that usually outweighs the risk of loss through unit-holders trying to profit from known market movements since the previous valuation point. In periods of disorderly markets, on the other hand, when wild swings in prices occur, the company will move to a forward pricing basis to ensure equity as between those investors who wish to deal and continuing unitholders, and thereby avoid a possible loss through selection on the part of some investors. This 'commonsense' discontinuity in behaviour on the part of the unit trust company in the face of disorderly markets has, for a number of years, been formalised in terms of U.K. financial regulation, in that unit trust companies must move to a forward pricing basis whenever it is believed possible that a new valuation could result in a price more than 2% different from that at the previous valuation point. Similarly, while most unit trust and unit-linked life and pensions business is transacted on an 'offer price' basis, companies will move to a 'bid price' basis whenever a significant outflow of funds is anticipated.

3.4.3 In the context of a general theory of option pricing, the assumption that a 'costless risk-free hedge' can be set up at any instant in time is, accordingly, inappropriate in the extreme. In periods of disorderly market behaviour, not only will 'normal' patterns of behaviour tend to break down, but dealing in some investment instruments may be suspended by the market authorities. In the 'Crash of 1987', for instance, the New York Stock Exchange did not allow trading in S & P futures at any time on 'Black Monday'.

3.4.4 It is instructive to note that Bernoulli discussed, but chose to ignore in his subsequent mathematical development, the possible existence of occasions when 'normal' trading patterns between his poor man and his rich man would break down.

4. ADAM SMITH AND THE OPTION WRITER

4.1 *Self-Interest as the Driving Force*

4.1.1 Smith (1776) describes very vividly how providers of everyday services to the general public are motivated, not by altruistic or philanthropic instincts, but by the expectation of profit:

> "It is not from the benevolence of the butcher, the baker, or the brewer that we expect our dinner, but from their regard to their own interest."

4.1.2 Similarly, present day providers of financial services, such as the writing of options on securities, will, in general, aim to earn a return that is satisfactory in comparison to the returns available from possible alternative uses of the capital employed, the skills and experience of the workforce, and the infrastructure and systems currently in place. The minimum level of option prices which results from such considerations, under the influence of free competition, provides a convenient starting point for the construction of a theoretical framework for option prices.

4.2 *Parallels with Life Assurance*

4.2.1 Smith observes that the minimum price an individual can expect to pay to have a particular risk insured is made up of three components, namely the expected cost of claims, the expenses of management, and an appropriate profit margin to give a return on capital equal to that available from other trades. Smith also observes that, in general, the average rate of return should increase with the inherent level of risk. The following simplified example shows how these elementary principles can be translated into the theoretical framework that led, by direct analogy, to the corresponding framework for option pricing as set out in ¶1.1.3.

4.2.2 Consider a life assurance company which writes a block of N identical term assurance contracts, of term t years, for unit sum assured (with any claim being paid out at the end of the period) on lives of the same age whose mortality is assumed to be independent. Then, as in ¶1.1.3, let P be the single premium per contract, let q be the probability of a claim arising, let Q be the higher of the amount per contract to be held at the end of the period on prudential grounds (i.e. on the initiative of the company) or required to be held for regulatory purposes, let E be the expenses of management per contract (assumed to be paid out half way through the life of the contracts), let R be the target rate of return on capital employed, and let r be the return on admissible assets (i.e. the lower of that in line with the internal guidelines of the company or as prescribed by regulation). Then, if C+P is the reserve per contract to be set up initially, precisely the same logic as in ¶1.1.3 gives:

$$C = \frac{(Q-q)}{(1+R)^t}$$

and

$$\begin{aligned} P &= \frac{q}{(1+r)^t} + \frac{E}{(1+r)^{\frac{t}{2}}} + (Q-q)\left(\frac{1}{(1+r)^t} - \frac{1}{(1+R)^t}\right) \\ &= \frac{q}{(1+r)^t} + \frac{E}{(1+r)^{\frac{t}{2}}} + C\left(\frac{(1+R)^t - (1+r)^t}{(1+r)^t}\right). \end{aligned}$$

4.2.3 Taking N = 100, we now discuss the assessment of q and Q using the framework of the four kinds of uncertainty referred to in ¶1.7.1.

4.2.4 'Uncertainty of the first kind' is addressed by using mortality rates appropriate to the experience of assured lives rather than, for instance, annuitant or 'population mortality' rates.

4.2.5 'Uncertainty of the second kind' requires the identification of a mortality basis which describes, not only the expected value, but also the

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underlying variability of the 'true' rate of mortality. Suppose, for illustration, that the value of q is taken as 0.1, with an estimation error equivalent to a standard deviation of 0.01. Assuming a normal distribution for the 'true' value of q, it can be represented sufficiently accurately by the following discrete probability distribution:

q	Probability
0.08	0.0625
0.09	0.25
0.1	0.375
0.11	0.25
0.12	0.0625

4.2.6 'Uncertainty of the third kind' relates to the aleatory (i.e. pure random chance) variability of the actual number of claims for any given 'true' underlying value of q and, given the independence assumption, involves only elementary probability theory.

4.2.7 Suppose that we wish to define Q by specifying that the probability of the number of claims exceeding $100Q$ is not to exceed 1%. If q was constant at 0.1, the claims distribution would have a variance of 100 x 0.9 x 0.1, i.e. 9, and a standard deviation of 3. The relevant '1% confidence level', based on a normal distribution would be:

$$10 + 100 \times 0.01 \times 3 \times 2.326, \quad \text{i.e. about } 17.0.$$

Since the distribution is skewed slightly to the upside, and a further upward bias results from the variability of the 'true' value of q, we can take 0.18 as a reasonable 'actuarial' estimate of Q before allowing, if appropriate, for 'uncertainty of the fourth kind'.

4.2.8 In this simplified example it is, of course, possible to obtain a more accurate value of Q by calculating, for each possible number of claims, the associated probability as the weighted average over the different values of q for that number of claims. For example, for 15 claims, the respective probabilities for the five different values of q are 0.00745, 0.01721, 0.03268, 0.05281 and 0.07455, and weighting these by the appropriate probabilities gives the overall probability of 0.03488. These probabilities for different numbers of claims give the following cumulative probabilities:

n	Probability of n or more claims
17	0.02669
18	0.01395
19	0.00692

Interpreting Q as essentially continuous rather than discrete and taking the probability of '18 or more claims' on this continuous basis as half the probability of 18 claims plus the probability of 19 or more claims on the discrete basis gives:

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$$0.5\,(0.01395 - 0.000692) + 0.00692, \quad \text{i.e. } 0.01044$$

from which we conclude that, to three significant figures, a more accurate value of Q is 0.180.

4.2.9 It is often said that the traditional actuarial approach of being 'conservative', by incorporating margins based on practical experience, is unscientific as compared to a formalised risk approach using, for example, mean-variance analysis. The exact converse is, I believe, the case. In the above example, the actuary identifies two quite distinct mechanisms that could lead to the central expectation being wrong — an estimation error regarding the true underlying probability of a claim, and the aleatory (i.e. pure random chance) mechanism relating to the number of claims. The combination of appropriate margins in each separately is equivalent to specifying an acceptably small probability of 1% of the 'worst case' scenario of 18 rather than 10 claims arising, in which case additional capital would be required to avoid insolvency.

4.2.10 More generally, the probability of any outcome is described by an easily understood compound distribution which is, in effect, the product of two simpler distributions, a normal distribution for 'uncertainty of the second kind' and a binomial distribution for 'uncertainty of the third kind'.

4.2.11 Let us suppose that, after considering possible scenarios that could lead to future mortality experience being significantly higher than that calculated on the basis of historic experience, it is concluded that no further special reserve is required. Then 'uncertainty of the fourth kind', although having been taken into account, involves no adjustment in this case.

4.2.12 Further insights into how actuaries have learned to manage uncertainty in the area of mortality can be obtained from a study of the following values of q_x extracted from three different life tables contained in the 1980 edition of *Formulae and Tables for Actuarial Examinations*:

	a(55) males		A 1967-70		E.L.T. No.12 males
x	select	ultimate	select	ultimate	
60	0.0084	0.0140	0.0067	0.0144	0.0229
70	0.0227	0.0378	0.0136	0.0391	0.0557
80	0.0621	0.0986	0.0253	0.0970	0.1275

The ultimate annuitant and assured lives rates (i.e. a(55) and A 1967-70 respectively) are very similar and significantly lower than the population (E.L.T. No. 12) rates, because of the selection processes (by the purchaser in the case of annuitant mortality and by the life office in the case of assured life mortality) that exclude most of the seriously impaired lives within a random sample from the population at large. The select rates are, furthermore, significantly lower than the ultimate rates. Suppose now that, for some curious reason, we have estimates of the variance of the actual mortality experience, but no direct knowledge of the average rate of mortality for a given age. For a large number N of lives of a

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given age, who might be either annuitants or assured lives, and either 'select' or 'ultimate', and an underlying probability of death over one year of q, the standard deviation V (taking 'alive' at the end of the year as 0 and 'dead' as 1) is:

$$V = \sqrt{Nq(1-q)}$$

from which we can (discarding the quadratic root close to 1) calculate an estimate of q. If we have no reason to believe that there is any material difference between annuitant and assured life mortality or between select and ultimate rates, it would be possible to transact life and annuity business on reasonably sound lines using this universal value of q appropriate to each age, provided that the poorest quality lives proposing for life assurance had been identified in the underwriting process and had either been declined or accepted at suitably higher than normal rates.

4.2.13 A life assurance industry run on these lines, while perhaps less risky (as regards the likelihood of insolvency) than, say, general insurance, would involve numerous elements of inequity. In particular, for short-term contracts the universal value of q, being largely 'ultimate' experience, would be too generous to annuitants and unfairly expensive to assured lives. Also, if, for example, annuitant mortality was systematically lighter than assured life mortality, companies would incur mortality losses on both annuity and assured life business, leading to a significant risk of insolvency.

4.2.14 From our present day actuarial perspective, we would regard such a rudimentary approach to mortality as unscientific in the extreme. However, a little thought shows that the Black-Scholes formula, which encapsulates nothing more than a 'universal' 'risk-free' rate of return and a volatility parameter estimated from historic experience, represents the option pricing equivalent of this rudimentary approach to mortality.

4.3 *Differing Risk Profiles*

4.3.1 We now use the 'four kinds of uncertainty' framework to examine the quite different risk profiles inherent in option pricing. The most obvious difference is that, unlike the life assurance case, diversification over a large number of similar contracts does not effectively eliminate the uncertainty of the aggregate out-turn. Consider a company which writes a block of 100 one-year call options, one on each of the FT-SE 100 constituents. The key determinant of the amount to be paid out on expiry is clearly the level of the FT-SE 100 Index at that time. The uncertainty of the outcome cannot be 'diversified away' within the one-year term of the options. Over a series of five one-year periods, on the highly implausible assumption that similar blocks of options were written each year, a reasonable degree of stability may be observed in the five-year average. However, human nature is such that undue attention is paid to the financial results in the latest reporting period; the long-term average is rarely even mentioned.

4.3.2 'Uncertainty of the first kind' relates to systematic differences in experience as between different homogeneous subgroups. Consider the amounts paid out on expiry to holders of call options on FT-SE 100 constituents. These amounts can be regarded as being determined by two components, the level of the FT-SE 100 Index and the relative performance against the FT-SE 100 Index of shares on which investors have chosen to buy call options. The same applies for put options. There is clearly selection against the option writer, in that buyers of call options will tend to select underlying shares that they expect to show a strong relative performance, whereas buyers of put options will tend to select shares that are expected to underperform in relative terms. This suggests that, for shares of the same historic volatility, the call option prices that the writer charges should be based on higher expected returns than for put option prices. It is interesting to note that Black & Scholes (1972), as cited in the concluding paragraphs of Black & Scholes (1973), contains empirical results consistent with these differential returns, namely that buyers of call options consistently pay prices higher than those based on the Black-Scholes methodology.

4.3.3 By analogy with the life assurance case, 'uncertainty of the second kind' relates to estimates of central values of fundamental variables relating to future share prices. Both 'best estimates' and assessments of dispersion are required in two different areas, the future level of a relevant market index and the relative performance against this index. Since we are dealing with skilled judgements rather than all possible future scenarios, normal distributions are likely to be reasonably satisfactory in both cases, and — at least as a working approximation — we can assume that these two normal distributions are independent.

4.3.4 The Dynamic Equilibrium Model, which, in effect, states that share prices tend to move in simple harmonic motion around central values with random noise superimposed, will be used to represent 'uncertainty of the third kind'. Again it is not unreasonable to assume, in the first instance, that this random noise follows a normal distribution which is independent of the other two normal distributions.

4.3.5 The resulting share price model which combines 'uncertainty of the first, second and third kinds' is, accordingly, a compound distribution which is the product of a normal distribution (with variance equal to the sum of the variances of the three subsidiary normal distributions) and simple harmonic motion. The conceptual parallels with the life assurance case are obvious. However, in the case of share prices the aleatory 'third kind' variability results from the statistical description (over time) of deviations from central values as the result of aggregate investor behaviour, whereas, in the case of life assurance, this aleatory variability represents 'pure random chance' in terms of elementary probability theory as applied to a repeated binary process. Various commonsense arguments suggest that financial market returns are multiplicative rather than additive over time, corresponding, in actuarial science, to the force of mortality and the force of interest being the primary underlying forces. Also, in empirical terms it has been found that lognormal distributions are closer to (if not fully in accord with) reality than are normal distributions. Accordingly, for the new compound distribution I use lognormal distributions and the corresponding logarithmic form of simple harmonic motion, which I shall henceforth describe as 'logharmonic'.

4.3.6 The following worked example may be more helpful than a formal mathematical description in showing how these concepts can be translated very easily into a practical numerical framework. A share which is a FT-SE 100 constituent, and whose upper and lower control limits in a mean absolute deviation chart are typically 15% apart, is expected to outperform the index by 5% over the next year, while the index itself is expected to rise by 3%. The standard deviation of the aggregate lognormal process, representing 'uncertainty of the second kind', is of the order of 15% of the expected end period value. For a current share price of 95p and an option striking price of 100p, calculate the values q expected to be paid to the buyer on expiry for a one-year European call option and a one-year European put option.

4.3.7 The 9-point discrete distribution where the probabilities are the binomial coefficients 1, 8, 28, 56, 70, 56, 28, 8 and 1, divided by 256, provides a very accurate approximation to the (continuous) normal distribution with the same mean and standard deviation. For unit spacing between the points, this discrete distribution has a standard deviation of $\sqrt{2}$. Accordingly, the spacing to give a standard deviation of 0.15 is $0.15 \div \sqrt{2}$, i.e. 0.10607. To convert to a lognormal distribution, we use powers of 1.10607 rather than a linear spacing of points 0.10607 apart. For unit 'central value', this gives the 9-point lognormal approximation:

x	$(1.10607)^x$	Probability %
−4	0.6681	0.39
−3	0.7390	3.12
−2	0.8124	10.94
−1	0.9041	21.88
0	1.0000	27.34
1	1.1061	21.88
2	1.2234	10.94
3	1.3532	3.12
4	1.4967	0.39

4.3.8 It is a straightforward matter to obtain a 9-point logharmonic approximation using similar principles. Since the sine function of simple harmonic motion varies from −1 to + 1 the required 9 points are −0.8889, −0.6667, −0.4444, −0.2222, 0, 0.2222, 0.4444, 0.6667 and 0.8889. The respective probabilities are obtained by considering the inverse sine function. For example, the inverse sine functions of 0.1111 and 0.3333 are 6.38 degrees and 19.47 degrees respectively, giving a range of 13.09 degrees. Dividing by 180 (since the sine function varies from −1 to +1 over 180 degrees) gives a probability of

0.0727 that a value selected at random falls within the range 0.1111 to 0.3333. Since the deviation from the central value to each control limit in the mean absolute deviation chart is 7.5% of the central value, we replace 0.2222 by 0.2222 x 0.075, i.e. 0.01667. Ignoring the very slight convexity, we then attach a probability of 0.0727 to the midpoint value of 0.01667. Proceeding similarly for the other values, and replacing, for instance, 1.0333 by $(1.01667)^2$, we obtain the 9-point logharmonic approximation:

y	$(1.01667)^y$	Probability %
−4	0.9360	21.63
−3	0.9516	9.62
−2	0.9675	7.93
−1	0.9836	7.27
0	1.0000	7.10
1	1.0167	7.27
2	1.0336	7.93
3	1.0508	9.62
4	1.0684	21.63

4.3.9 After a year, it can be assumed that the position in the mean absolute deviation chart is independent of the initial value. Accordingly, the lognormal and logharmonic distributions are independent, and the compound probabilities are the products of the individual probabilities. Thus, for example, the probability that $x = 1$ and $y = 3$ is 0.2188 x 0.0962, i.e. 0.0210.

4.3.10 The above values relate to unit 'central value', whereas the expected share price on expiry is 95p x 1.03 x 1.05, i.e. 102.74p. The element of the compound distribution described by $x = 1$ and $y = 3$ corresponds to a value on expiry of 102.74p x 1.10607 x 1.0508, i.e. 119.41p. This is 19.41p above the strike price of 100p, with an associated probability of 0.0210, giving a contribution of 0.408p to the value of q for the call option and a nil contribution to the value of q for the put option. Repeating this for all 81 combinations of x and y gives values for q of 8.36p in the case of the call option and 4.44p in the case of the put option.

4.3.11 Since the U-shaped distribution, over time, of the logharmonic component bears no resemblance to the types of distribution employed by financial economists, it is appropriate to examine some of the implications.

4.3.12 It is interesting to note that, early this century, there seemed to be a growing awareness amongst statisticians that, despite their mathematical elegance, the normal distribution and related mean-variance estimation techniques were often of little practical relevance. Keynes (1921) summarises the situation as follows:

> "Apart, however, from theoretical refutations, statisticians now recognise that the arithmetic mean and the normal law of error can only be applied to certain special classes of phenomena. Quetelet was, I think, the first to point this out. In England, Galton drew attention to the fact

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many years ago, and Professor Pearson has shown 'that the Gaussian - Laplace normal distribution is very far from being a general law of frequency distribution either for errors of observation or for the distribution of deviations from type such as occur in organic populations .... It is not even approximately correct, for example, in the distribution of barometric variations, or grades of fertility and incidence of disease.'"

4.3.13 The logharmonic deviations of price from a central value are particular examples of 'deviations from type' and represent the observed real world consequences of my 'Systematic Over-reaction Hypothesis'. Such behaviour, however, is inconsistent with the standard assumptions of modern finance theory.

4.3.14 The compound distribution of the lognormal and logharmonic distributions combined has a far more complex shape than the lognormal diffusion process of known and constant variance assumed in the Black-Scholes formula. Accordingly, for differing strike prices there will be systematic divergences between 'new framework' prices and Black-Scholes prices.

4.3.15 To determine the general nature of these divergences, consider the exaggerated example of a 5-point binomial approximation to a normal distribution and a binary distribution, with the mean and standard deviation being the same in both cases. For a current share price of 100p, take the expected price on expiry as 110p, to reflect the effect of either the 'risk-free' rate or of a more realistic expected rate of return. Then a standard deviation of 10p gives, for the 5-point distribution, values of 90p, 100p, 110p, 120p and 130p with probabilities of 0.0625, 0.25, 0.375, 0.25 and 0.0625 respectively, and, for the binary distribution, values of 100p and 120p each with probability 0.5. For different strike prices, the expected values of the proceeds on expiry for the binomial and binary processes, together with their ratios and the inverses of these ratios, are shown below:

Strike price	Binomial value	Binary value	Ratio	Inverse
85p	25.00p	25.00p	1.000	1.000
90p	20.00p	20.00p	1.000	1.000
95p	15.31p	15.00p	1.021	0.980
100p	10.62p	10.00p	1.062	0.941
105p	7.19p	7.50p	0.958	1.043
110p	3.75p	5.00p	0.750	1.333
115p	2.19p	2.50p	0.875	1.143

Taking the binomial and binary values as crude proxies for Black-Scholes and 'new framework' prices respectively, we conclude that for 'at-the-money' call options the Black-Scholes price is too high, and that to replicate the 'new framework' prices a lower Black-Scholes volatility has to be used for 'at-the-money' options than for options which are either 'in-the-money' or 'out-of-the-money'.

4.3.16 This systematic pattern is, of course, precisely the 'smile effect' that is a well known feature of real world option prices. The obvious inference is that option pricing practice has already moved ahead of current option pricing theory.

4.4 *Conflicts of Self-Interest*

4.4.1 The self-interest of profit-oriented financial organisations is, at least in the short term, to maximise the volume of business that can be transacted for a given level of capital, thereby exerting a strong downward force on the value of Q. However, the serious shortcomings of the variance of return paradigm of risk in terms of internal prudential controls are vividly described by G.T. Pepper in the discussion at the Faculty of Actuaries on Clarkson & Plymen (1988) and Clarkson (1989):

> "When a large Securities House is dealing in many products, ranging right across the foreign exchange markets and debt markets, from overnight money to long-term bonds, from sterling bonds to dollar bonds and deutchmark bonds, and including equities, there is a very strong case for having a consistent system of risk control and for setting limits. The initial approach in each of the markets was to examine the percentage daily changes in price and from these figures to calculate the standard deviations. An extension of that was to do so on a rolling basis. I understand in the middle of 1987 it was common practice in the United States primary government bond market to measure the standard deviation of bonds over the last 30 business days. In the middle of 1987 some US bond houses were arguing that the bond market had become a lot less volatile in the last year and, therefore, they could run much larger positions on the same amount of capital than the year previously without incurring any greater risk. The extraordinary rise in the bond market when the equity market crashed in October 1987 then occurred."

Houses that were short of bonds (which are low risk, if not 'risk-free' investments, according to the teachings of modern finance theory) suffered massive losses of the same order of magnitude as the losses suffered by some houses who were long of equities. Pepper then observes that, in terms of 'genuine risk', the least important period is the last 30 days, and concludes with a plea for more attention on examining the characteristics of the tails of the distribution and not so much on the centre of the distribution.

4.4.2 The self-interest of regulators, on the other hand, is to err on the side of caution, by setting stringent reserving requirements in an attempt to minimise the likelihood of severe adverse consequences, such as individuals or corporate identities suffering extremely serious losses, or the complete collapse of part or all of the financial system. This exerts a strong upwards force on the value of Q. In the options field, the Bank of International Settlements has decreed that banks can either follow a specified formula for market risk or use an in-house model approved by the authorities. The Basle Committee has built in a very large additional margin by stipulating that the amount calculated (usually along essentially variance of return lines) by an in-house model must be multiplied by three.

4.4.3 If the regulators, distrustful of currently accepted methodologies for assessing risk, set the capital adequacy requirements at too high a level, banks will find it unprofitable to transact options business, and, accordingly, will withdraw from the provision of certain financial products which, if used properly, can reduce the overall level of financial risk to society as a whole. If, on the other hand, the regulators set too low a standard, not only are many financial companies likely to suffer serious losses, but there is also the possibility that a 'domino effect' could lead to the collapse of the entire financial system.

4.4.4 The current teachings of modern finance theory offer no guidance on these crucial matters. Accordingly, in the next section I extend the applicability of the Dynamic Equilibrium Model in an attempt to obtain a robust theoretical framework within which it might be possible to calculate appropriate values of Q in terms of the new framework.

5. THE DYNAMICS OF CAPITAL MARKET PRICES

5.1 *Extending the Dynamic Equilibrium Model*

5.1.1 The Dynamic Equilibrium Model, derived in Clarkson (1996a, 1996b), is clearly only a first step, although possibly an important one, towards finding a more realistic model for share returns than the Capital Asset Pricing Model or geometric diffusion processes. The general principles underlying its derivation are extended below, using four aspects of capital market behaviour in the U.K. over the past few decades.

5.1.2 During the second half of 1975, when the U.K. equity selection model, described in Clarkson (1981), was being implemented by a life office for the first time, it appeared that the very high inflation rates at that time would have an exceptionally adverse effect on the long-term growth prospects of capital-intensive companies such as general engineering companies. Since this assessment was confirmed by the fact that most major engineering companies had recently required fairly heavy rights issues to replenish their working capital, nearly all of the life office's engineering portfolio was sold and reinvested in companies expected to fare far better in an environment of high inflation. However, the dominant market sentiment as regards engineering shares was 'buy for recovery', with the result that over the next six months the shares sold outperformed the shares bought (which themselves outperformed the market by around 5%) by no less than 35%. Only thereafter did the underlying fundamentals (which had been assessed correctly) become dominant; breakeven in performance terms was achieved by the middle of 1977, and by the autumn of 1980, some five years after the switch was carried out, the purchases were around three and a half times the value of the sales.

5.1.3 The strong outperformance of U.K. smaller companies for many years in the 1980s, followed by equally dramatic underperformance from 1989 onwards, became widely known as the 'small company effect', but analyses thereof have tended to be qualitative in nature and have thrown little light on the underlying causal forces. However, Bowie & Clarkson (1996) provides a suitable numerical framework by fitting a smooth curve to the weightings by market capitalisation rank of constituents of the FT-SE 100 Index in precisely the same manner that Halley (1693) first smoothed observed mortality rates at different ages. The relative market capitalisation of the constituent ranked 75th in terms of size varied as shown in the following table from 1984 to 1994:

Year	% of FT-SE 100 capitalisation	Year	% of FT-SE 100 capitalisation
1984	0.40	1990	0.42
1985	0.42	1991	0.35
1986	0.44	1992	0.35
1987	0.48	1993	0.40
1988	0.47	1994	0.40
1989	0.44		

This general pattern can be seen as a ten-year cycle around a central value of 0.40, with causal mechanisms that will be familiar to professional investors. Despite considerable scepticism in the early 1980s, strong U.K. economic growth led to very strong volume growth, and hence profits growth, for medium-sized and smaller companies that had much more operational flexibility than very large companies. The superior relative performance of smaller companies led to much more interest on the part of institutional investors, more stockbrokers' research, and the setting up of many smaller companies unit trusts. All these factors contributed to a feedback process which reinforced the outperformance of smaller companies, leading to very demanding ratings. When a much harsher economic environment unfolded towards the end of the 1980s, the combination of lower profits, poorer management controls and demanding ratings led to a reversal of the process; strategic policy decisions on the part of many institutional investors to reduce the proportion invested in smaller companies led to an acceleration of the downtrend, taking the relative ratings of smaller companies to a nadir in 1991, before recovering by 1993 to more normal levels.

5.1.4 To investment professionals it is intuitively obvious that the two key determinants of equity market levels are the yield on long-term government bonds and the expected rate of dividend growth. This perception is strongly supported by the results of the co-integration vector analysis set out in Mills (1991). Using what used to be known as the 'confidence indicator', namely the ratio of the long-term gilts yield to the equity market dividend yield, Mills constructs standard error bounds that act as 'resistance lines' and notes that "these bounds are strikingly reminiscent of the control limits obtained by Clarkson (1978, 1981) from models analysing individual gilt-edged stocks and equity prices respectively". Further confirmation that capital market levels do, indeed, follow non-random behavioural patterns is given by Peters (1991), who shows that market indices as well as individual securities have Hurst exponents significantly in excess of 0.5, indicating a 'long-term memory' effect. Practical techniques for estimating the central values of equity and fixed-interest indices are outlined in Clarkson (1995b).

5.1.5 The 'Crash of 1987', in which the U.K. equity market soared far above the upper 'resistance line', as shown in Mills (1991), before its dramatic collapse to more normal ratings on 19 and 20 October 1987, is the most obvious example in recent times of 'disorderly' capital market behaviour. In many respects this episode is the broad equity market equivalent of the more localised 'small company effect' described in ¶5.1.3. The combination of strong economic growth and levels of inflation that were very low in comparison to the previous decade led to several years of strong outperformance of equities over gilts. Institutional investors who, traditionally, had a significant proportion in fixed-interest securities, were, year after year, turning in poorer performances than so-called 'balanced managers', whose strategy was to be essentially equity based. Although a few managers held a significant proportion in cash as equity ratings became more and more demanding, houses with very high equity exposures translated their superior relative performance into the winning of new management contracts, with a consequent switch into equities from some or all of the fixed-interest content of the portfolios transferred from the previous managers. Some other managers, on 'commercial matching' grounds, tended to move to higher benchmark proportions for equities, despite their very high ratings. In short, the normal stabilising mechanism of a reasonable balance between buyers and sellers of equities ceased to operate, leading to market ratings which, with hindsight, were dangerously vulnerable to the slightest change of sentiment.

5.1.6 The above examples suggest that segments of equity markets and equity markets themselves exhibit cyclical behaviour over time, but generally with a much longer (but still variable) cycle period than that of individual equities about their central values, where the cycle period (see, for instance, the mean absolute deviation chart for Whitbread shares in Clarkson, 1981) is typically of the order of six months. This similarity of pattern, with an essentially continuous spectrum of cycle periods, suggests that the fractal nature of capital market behaviour, as first comprehensively documented by Mandelbrot (1963), and recently confirmed by even more extensive empirical testing by Walter (1995), is caused by the interaction over continuously varying timescales of two quite different general forces, a fairly strong centralising force which tends to take prices towards central values that are 'sensible' on a very long-term perspective, and a very strong 'momentum' or 'inertia' force which represents a feedback effect of recent price movements. As Keynes (1936) describes so vividly in Chapter 12 ('The state of long-term expectation') of his *General Theory*, this very strong feedback effect, combined with the 'short-termism' introduced whenever investment performance is reviewed by 'committees or boards or banks', tends to force professional investors to protect, not only their immediate self-interest in terms of management fees, but also their very survival, by pursuing strategies which seem likely to put them 'ahead of the crowd' rather than by pursuing strategies which reflect the 'true' long-term value of the investment vehicles they have been contracted to manage on behalf of their clients. An immediate, but very distressing, corollary is that, when equity markets are becoming seriously over-extended, the safest investment from the perspective of the investment clients, namely cash, is the riskiest investment from the perspective of the investment managers.

5.1.7 The 'financial avalanches' that resulted from the 'Crash of 1987' and from the aftermath of the 'small company effect' of the 1980s point to a common causal mechanism, namely a run of 'successful' years generating, through a false perception of security, a sufficiently strong feedback effect to eliminate the usual stabilising mechanism of investment strategies based on 'true' long-term value. This avalanche metaphor appears, indeed, to be highly appropriate. In the physical world, a series of small or moderate snowfalls spread out over time leads to a consolidated and reasonably stable snow structure. An unusually high snowfall over a very short period of time, however, results in a large and dangerously unstable slab of unconsolidated new snow that can avalanche at the slightest physical stimulus, such as a sudden rise in temperature.

5.1.8 By presuming equilibrium and (at least as a good first approximation) efficiency, current finance theory states that 'price' and 'value' are, for all practical purposes, always identical, whereas the above discussion suggests that strong dynamic patterns can be identified and exploited by skilled investment professionals. The results of the empirical studies set out in Shiller (1989) are also consistent with a dynamic rather than a static framework, in that observed levels of volatility are very much higher than what would be expected if price changes resulted only from 'new information'. The equilibrium or 'no arbitrage' assumption is crucial to stochastic calculus, where (see, for instance Lamberton & Lapeyre, 1996) it is assumed that markets are 'viable', in the sense that discounted values of entitlements from securities are always exactly equal to their prices. Since the standard assumption of equilibrium is supposedly justified by the Efficient Market Hypothesis not having been convincingly refuted since its formulation more than a quarter of a century ago, I suggest, in the next section, two new hypotheses which address the general types of 'inefficiency' that investment professionals strive to understand and exploit.

5.2 *Replacing the Efficient Market Hypothesis*

5.2.1 Consider the interaction of the behavioural patterns implied by the following two observations by Smith (1776), the first in the context of theoretical equilibrium and the second in the context of his analysis of actual human behaviour:

> "If in the same neighbourhood, there was any employment evidently either more or less advantageous than the rest, so many people would crowd into it in the one case, and so many would desert it in the other, that its advantages would soon return to the level of other employments",

and

> "The natural effort of every individual to better his own condition, when suffered to exert itself with freedom and security, is so powerful, that it is alone, and without any assistance, not only capable of carrying on the society to wealth and prosperity, but of surmounting a hundred impertinent obstructions with which the folly of human laws too often encumbers its operations."

5.2.2 The Dynamic Equilibrium Model recognises, in the context of security prices relative to a market index, that 'the natural effort of every individual to better his own condition' is not invariant in nature (as in the laws of physics), but is a reflection of his training, skill and experience. In particular, some individuals will see 'trend-chasing' behaviour as being in their best interests, whereas others who believe that they can identify 'fundamental value' will see 'centralising' behaviour as being in their best interests. As discussed in ¶5.1.6, similar principles appear to apply in the case of general market levels.

5.2.3 Suppose now that we introduce a third observation by Smith, namely that human behaviour often falls short of the 'omniscient' and 'faultless' manner generally assumed by present day economists, but is, in part, guided instead by factors which exist only 'in the imaginations of men'. The empirical evidence discussed in Section 5.1 suggests that 'short-termism', in the specific sense that observed behaviour in the recent past is taken as a proxy for likely future behaviour, is the most significant cause of 'unintelligent' trend-chasing behaviour. This conjecture is consistent with comments by Keynes (1936) as to how we should assess the 'long-term expectation':

> "It would be foolish, in forming our expectations, to attach great weight to matters which are very uncertain. It is reasonable, therefore, to be guided to a considerable degree by the facts about which we feel somewhat confident, even though they may be less decisively relevant to the issue than other facts about which our knowledge is vague and scanty. For this reason the facts of the existing situation enter, in a sense disproportionately, into the formation of our long-term expectations; our usual practice being to take the existing situation and to project it into the future, modified only to the extent that we have more or less definite reasons for expecting a change."

5.2.4 In the light of the above discussion, the obvious way forward is to abandon the Efficient Market Hypothesis and to replace it by two new hypotheses, the 'Central Value Hypothesis' and the 'Systematic Over-Reaction Hypothesis'. The Central Value Hypothesis states that there exist central values, based on perceived underlying fundamental variables, towards which individual security prices and capital market levels tend to gravitate. The Systematic Over-Reaction Hypothesis states that, as a result of differing levels of human behaviour attributable to differing levels of training, skill, and practical experience, security prices relative to their central values, and both sets of central values, all exhibit cyclical variation over time which, as a first approximation, can be represented by simple harmonic motion of variable periodicity with a superimposed random error term. Further elaboration on these two new hypotheses is clearly outwith the scope of the present paper.

5.3 *An 'Ultimate' Compound Process*

5.3.1 It may be suggested by some proponents of modern finance theory that there is no conclusive evidence that capital market behaviour is inconsistent with the standard assumptions in the areas of equilibrium, efficiency and risk. However, if a new paradigm can immediately explain anomalous behaviour that stubbornly defies inclusion in even ingenious modifications of the old paradigm, then this is powerful evidence in favour of the new paradigm. A classic such example in the physical sciences (see Weinberg, 1993) was Einstein's demonstration, in 1915, that his new theory of General Relativity immediately explained an anomaly relating to the perihelion of Mercury's orbit that had troubled astronomers for more than half a century. One of the effects that contributes to this reconciliation is that in Einstein's theory there is the additional gravitation field produced by the energy in the gravitation field itself. In the classical theory formulated by Newton, gravitation is produced by mass alone, not energy.

5.3.2 There are obvious parallels with my formulation of the Dynamic Equilibrium Model, which adds the inertia ('energy') of price movements to the equilibrium-producing arbitrage ('gravitation') of classical economic theory. In addition, I recognise the existence of occasional periods of 'disorderly' behaviour caused by the breakdown of the usual dynamic equilibrium resulting from the interaction of these 'trend-chasing' and 'centralising' forces of investor behaviour. Since the shorter the observation period the less the empirical results will be distorted by 'disorderly' behaviour, the obvious place to look for the capital market equivalent of the Mercury anomaly is in very-short-term security returns.

5.3.3 The very recent work by Geman & Ané (1996), referred to in ¶1.3.7, uses a 'stochastic clock' approach to graduate high-frequency data for S&P 500 returns over periods of one minute, fifteen minutes, thirty minutes, one hour, and one day. For all five timescales, the frequency distributions differ significantly from those for Gaussian or lognormal processes, and, in particular, the graph of the one-minute distributions shows a far higher central peak and 'shoulders' around one standard deviation above and below the mean, where the distribution is roughly constant at about one third of the value at the central peak. These features correspond to very serious anomalies in the context of the lognormal processes underlying modern finance theory in general, and option pricing theory in particular.

5.3.4 Whereas the new approach is 'select', in the sense that the initial state of the share price is known in terms of its position in a mean absolute deviation chart, the old paradigm, by not recognising the possibility that 'price' and 'value' may differ, is 'ultimate', in the sense that the position in the mean absolute deviation chart is unknown. Ignoring the lognormal component and any 'drift', the initial and final share prices are $(1 + b)^x$ and $(1 + b)^y$ respectively, and the ratio of the final price to the initial price is $(1 + b)^{y-x}$, giving a return of $(1 + b)^{y-x} - 1$ over the period. When the initial mean absolute deviation position is unknown, x and y are independent, and each can take any integral value from -4 to 4 inclusive, with the U-shaped probability distribution derived in ¶4.3.8. A value of 6 for $y - x$ can arise in three ways: $x = -2$ and $y = 4$; $x = -3$ and $y = 3$; and $x = -4$ and $y = 2$. Since x and y are independent, the probability that $y-x$ is equal to 6 is:

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$$0.079 \times 0.216 + 0.096 \times 0.096 + 0.216 \times 0.079, \quad \text{i.e. } 0.0436.$$

Proceeding similarly for other possible values of x and y gives a probability distribution for the return $(1 + b)^{y-x} - 1$ that is strikingly similar to the results for one-minute S&P 500 returns obtained by Geman & Ané (1996):

$y-x$	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8
Probability x 100	4.7	4.2	4.4	4.6	5.1	5.6	6.5	7.9	14.0	7.9	6.5	5.6	5.1	4.6	4.4	4.2	4.7

In short, the new approach immediately explains the two best known departures from classical Gaussian or lognormal diffusion processes, namely the very high central peak and the 'fatter tails'. It is interesting to note that anomalous behaviour of this general type was first reported more than eighty years ago by Mitchell (1915), in precisely the same year that Einstein realised his new physical theory could explain the apparent anomaly in the perihelion of Mercury's orbit.

5.4 *Risk Assessments and Hedging*

5.4.1 This paper is the options equivalent of life contingencies rather than of life office practice, in that it addresses, in the main, only the broad concepts and related numerical techniques of the new framework. However, in the areas of 'uncertainty of the first, second, third and fourth kinds', some very general comments as to the implications of the new approach are appropriate.

5.4.2 Unlike the Black-Scholes methodology, the new framework forces option writers to make judgements (under 'uncertainty of the first, second and third kinds') about future levels of stockmarket indices and about the performance of individual securities relative to these indices. Ignoring expenses and the investment returns on the premiums received and capital reserves set up, the profitability of writing options will be determined by the accuracy or otherwise of the underlying investment assumptions in the light of subsequent experience. As a consequence of option prices being based on 'best estimates' of future experience rather than on 'all possible outcomes', as based primarily on historic volatility of returns, it is obvious that, unless the 'worst case' outcome Q is very significantly higher than the 'best estimate' q, option prices under the new framework will be consistently lower, particularly over other than very-short-term periods, than those calculated under current methodologies. There is also likely to be a much greater variation in the prices quoted by different option writers. An obvious life assurance parallel is where a particular tax situation makes it possible to write a block of business on more attractive terms than those available in the market generally.

5.4.3 It might be argued by some that the new framework is unsound, in that the only 'safe' approach to writing options is to use 'no loss' arbitrage principles in the same way that a bookmaker will generally set his odds for different horses in a race on the basis of amounts of bets received rather than on his judgements as to which horses are likely to win. This simplistic argument ignores the inevitability of adverse selection under 'uncertainty of the first kind', when a lower cost producer of part of a product range emerges to challenge the previous apparently impenetrable monopoly. Suppose that call option prices, as quoted by a 'new framework' writer for options on three shares which are correctly assessed as likely to outperform by 5%, perform in line, and underperform by 5%, are 125, 100 and 75 respectively. Suppose also that the 'historic volatility' Black-Scholes price is 100 in all three cases. Then the buyer and the 'new framework' writer are engaging in a 'fair game' not dissimilar to that played between portfolio managers and market-makers in the management of institutional portfolios, where the actual gains or losses to each party are the resultant of their share selection (or arbitraging) skills and the expenses incurred. Similarly, if the buyer transacts equal volumes of business on all three shares with a Black-Scholes writer, a form of 'fair game' still results, even although some options are consistently overpriced and others are consistently underpriced. But when the option buyer can go to the 'new framework' writer for the option valued and priced at 75, and still go to the Black-Scholes writer for the option priced at 100, but with a value of 125, the 'new framework' writer can still achieve his target rate of return on capital employed, whereas the Black-Scholes writer will be unable to 'hedge away' the shortfall of 25 and accordingly will suffer unsustainable losses. The precedents of 'banc-assurance' and of previously non-financial U.K. companies setting up as lower cost producers of unit trusts, personal equity plans and pensions products suggest that similar radical restructurings will occur in a 'new framework' options world.

5.4.4 Under the new framework, risk to the option writer in relation to 'uncertainty of the fourth kind' is not static in terms of some time-invariant measure (such as a 'jump process') of either historic volatility or even of the relative frequencies of 'extreme' movements in the past, but is, instead, dynamic in that both the likelihood and severity of a 'financial avalanche' increase in line with deviations of 'confidence ratios' from their observed historic central values. A plausible 'new framework' approach for prudential or regulatory control of the writing of call options on equities might be to specify that the 'worst case' value Q has to be a certain multiple, such as twice, the 'best estimate' value of q for 'normal' values of the yield ratio of the long-term conventional gilts yield to the equity market dividend yield, but with appropriately higher multiples whenever this ratio was significantly higher. Thus, a typical yield ratio of around 2 would correspond to a green light ('proceed with caution'), a yield ratio of 2.5 would be an amber light warning of significantly increased risk, while a yield ratio of 3 would be an unmistakable red light signifying acute financial risk. Again the avalanche metaphor is appropriate, in that at both the bottom and top of major access lifts at continental ski resorts, where off-piste skiing is practised, the general level of avalanche risk is displayed on a numerical scale similar to the Beaufort scale for wind speed.

5.4.5 Modern theories of finance and economics do not recognise that risk to an individual tends to decrease as his levels of skill and experience improve. This can be viewed as an important consequence of the powerful 'division of labour' mechanism described by Adam Smith. In the same way as a professional musician will develop a sharper ear than most others, a professional investor should be able to make far more detailed and successful judgements about likely future returns than those with no practical experience of finance and investment. This is a financial world equivalent of the example used by Einstein (1920) to illustrate his theory of Special Relativity. If he drops a stone from a moving train, then, as seen by an observer on the embankment, it traces out (ignoring air resistance) a parabola, whereas to Einstein himself it appears to fall in a vertical straight line. Bernoulli (1738) observes that risk is a function of the circumstances of the individual, but the same is also true of expected returns.

5.4.6 The extreme complexity (or, some would say, chaotic nature) of capital market behaviour can be explained very easily in conceptual terms as the interaction of adaptive strategies based on differing perceptions of expected return, and hence of risk. As discussed in Section 5.2, the introduction of a dynamic equilibrium between the quite distinct 'gravitation' and 'inertia' forces provides a mathematical framework which can be used to measure and control financial risk. Current theories of finance, however, cannot accommodate the observed complexity of real world financial behaviour, and hence are seriously incomplete as regards a scientific framework for measuring or controlling financial risk. In particular, the Black-Scholes differential equation has no term corresponding to the 'inertia' force and accordingly turns the scientific calendar back several centuries to the social sciences equivalent of Newtonian mechanics.

5.4.7 The new framework does not depend on the setting up and maintenance of a 'hedge' portfolio on arbitrage trading principles, and accordingly can be applied immediately to options on highly illiquid asset classes such as property (i.e. real estate). This apparent disregard for the arbitrage pricing approach that has dominated both the theory and the practice of option pricing since the Black-Scholes revolution in 1973 may, at first sight, be perceived by proponents of the Black-Scholes methodology as a fatal flaw of the new approach. The reality, I suggest, is the exact opposite.

5.4.8 Consider the purely hypothetical example of a very large proprietary life assurance company with ample free reserves, which, in addition to transacting long-term life and pensions business, has diversified into the provision of a range of purely investment products such as unit trusts and personal equity plans for individuals, and discretionary portfolio management arrangements for pension funds and charities. Its shareholders' funds are managed in such a way as to maximise the long-term return to shareholders, since there are more than adequate reserves to absorb any short-term stockmarket fluctuations. The directors of this hypothetical company decide that it would be highly beneficial, not only to their shareholders, but also to society at large, if they provided, on a 'risk premium' basis similar to that used to reinsure the 'death strain at risk' for life assurance companies unwilling to assume certain mortality risks, a large (but nevertheless limited) volume of investment guarantees that banks and securities houses could use as the essential building blocks for options contracts. In particular, rather than selling apparently overpriced securities for the shareholders' or other funds, they could finesse their investment strategy by writing call options on these securities, and thereby effectively sell them at well above the ruling market prices.

5.4.9 The banks and securities houses who retail the options do not take on any investment risk, but are, instead, only assemblers of financial packages where the essential ingredients are bought in from the wholesaler, namely the large insurance company. Provided that no other financial institution is able to underwrite the investment risks, the price an option buyer pays could be far in excess of the underlying 'value'. If, for example, the insurance company estimates the 'true' investment cost as 100, it could increase this, for expense, profit and solvency margins, to 125 in terms of the price charged to the retailer, with the retailer, in turn, adding on further expense and profit margins, and charging the final buyer 150. The present arbitrage-dominated options industry is not unlike the above situation, but with investment markets as a whole playing the role of the large insurance company, and the costs of setting up and maintaining the 'risk-free' hedge corresponding to the 125 charged to the retailer. However, this financial world equivalent of 'pass the parcel' has two fundamental disadvantages. First, the prices of longer-term options tend to be higher than most prospective buyers are prepared to pay. Second, and more serious, the mechanics of the packaging process break down when investment markets are in a state of acute disorder, thereby increasing the general level of systemic risk within the banking industry worldwide.

5.4.10 In the new framework, since option prices (other than in 'over-extended' market conditions) can be regarded as smooth functions of the underlying share price and of time to expiry, precisely the same arbitrage approach as in a Black-Scholes world could still be used. However, even allowing for healthy profit and solvency margins, the new approach is likely, for the reasons set out in ¶5.4.2, to lead to consistently lower option prices except when market levels deviate significantly from their central value parities. A Black-Scholes writer who did not adapt to the new and generally lower price framework would soon be forced out of business.

5.4.11 This general discussion on hedging also suggests that regulatory controls of the options industry should be along the lines of those in place in the U.K. for long-term insurance business, with particular emphasis on 'resilience testing' of possible combinations of extreme circumstances, rather than along the essentially short-term matching controls of banking supervision, where the consequences of highly disorderly financial conditions cannot be assessed in any scientific manner.

5.4.12 The suggested application of the arbitrage approach for the construction of a supposedly 'risk-free' hedge in the case of long-term portfolios has led to 'delta hedging' strategies generally known as 'portfolio insurance', whereby, on a fall in equities, sales of equities have to be carried out to rebalance the portfolio. Again the avalanche metaphor comes immediately to mind. When ski mountaineers have to cross a slope where there is a significant risk of avalanche, the party spreads out to around fifty to a hundred yards apart for three commonsense reasons. The additional stress exerted on the potentially unstable snow is reduced to the minimum, only one of the party is likely to be buried should an avalanche occur, and the others would be sufficiently close at hand to carry out an immediate and probably successful rescue operation. The financial parallels need no elaboration, and, accordingly, it is difficult to avoid the conclusion that current methodologies in the area of financial risk are not only seriously incomplete in theoretical terms, but are also dangerously unsound in some suggested practical applications.

6. Computational Aspects

6.1 *Parallels with Commutation Functions*

6.1.1 With a European call option with a strike price of k, the payout for a given value x of the underlying security on expiry is $x - k$, which increases linearly, and with unit gradient, from a value of nil when x equals k. This is similar to the 'functional shape' of a deferred increasing assurance where the sum assured is 1 in year $n + 1$, 2 in year $n + 2$, 3 in year $n + 3$, etc., and which can be evaluated very easily in terms of standard commutation functions as:

$$_{n|}(IA)_x = \frac{R_{x+n}}{D_x}.$$

R_{x+n} and D_x are functions of the underlying mortality rates and of the interest rate assumed. Also, values of R_{x+n} can be calculated from the recurrence relationship:

$$\begin{aligned} R_{x+n} &= R_{x+n+1} + M_{x+n} \\ &= R_{x+n+1} + M_{x+n+1} + C_{x+n}. \end{aligned}$$

In the options case, we can regard the probability distribution of the price of the underlying security on expiry as the equivalent of the probabilities of death in different years, and we can regard the logharmonic parameter b as the equivalent of the interest rate. Accordingly, it is intuitively obvious that a similar recurrence relationship will provide a very convenient means of computing option prices for different values of the strike price.

6.1.2 For the expected amount q paid out on expiry under a European call option, there are, in the first instance, four variables:

(1) the lognormal parameter a;
(2) the logharmonic parameter b;
(3) the 'expected' share price y on expiry; and
(4) the strike price k.

Clearly we can calculate q on the basis that y is equal to 1, use a strike price of

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$k \div y$, and then multiply at the end by a scaling factor of y. Analogies with tables of commutation functions suggest the construction of a series of tables, one for each value of the logharmonic parameter b, where the columns correspond to different values of the lognormal parameter a and the rows correspond to different values of the strike price k. The table for $b = 0$ corresponds to the Black-Scholes formula.

6.1.3 The appropriate recurrence relationship can be found very easily by considering the case for the basis used in ¶4.3.6, namely $a = 0.10607$ and $b = 0.01667$. The expiry values of the underlying security in excess of 1.4, together with the corresponding probabilities, are as below:

x	y	Expiry value	Probability %
4	4	1.599	0.08
4	3	1.573	0.04
4	2	1.547	0.03
4	1	1.522	0.03
4	0	1.497	0.03
4	−1	1.472	0.03
4	−2	1.448	0.03
4	−3	1.424	0.04
4	−4	1.401	0.08
3	4	1.446	0.67
3	3	1.422	0.30

For a strike price of 1.5, the value of $100q$ is:

$$0.099 \times 0.08 + 0.073 \times 0.04 + 0.047 \times 0.03 + 0.022 \times 0.03, \quad \text{i.e. } 0.0129.$$

For a strike price of 1.4, the contribution to $100q$ from the last seven expiry values is similarly 0.0450. The contribution to $100q$ from the first four expiry values is the previous value of 0.0129, increased by 0.1, multiplied by the cumulative (percentage) probability of 0.18 over these four expiry values. Accordingly, for a strike price of 1.4, the value of $100q$ is:

New value	= old value	+ 0.1 x cumulative probability	+ new product
i.e.	0.0129	+ 0.1 x 0.18	+ 0.0450
i.e.	0.0759.		

6.1.4 The schedule of computations (using more significant figures) is thus:

Strike price	Probability %	Cumulative probability %	New product	$100q$
1.5	0.181	0.181	0.0132	0.0132
1.4	1.184	1.365	0.0446	0.0759
1.3	3.536	4.901	0.0792	0.2916
1.2	5.262	10.163	0.2730	1.0547
1.1	16.003	26.166	0.8882	2.9592

Each successive value of q is the previous value, plus the new product, plus one tenth of the previous cumulative probability. By symmetry, a virtually identical computational scheme can be used for put options. Although, in practice, it may be desirable to use a personal computer to evaluate q, these computational schedules serve two useful purposes. First, they demonstrate the transparency of the new framework; once an appropriate basis has been chosen, evaluation (as in life contingencies) is a matter of simple arithmetic. Second, the logic underlying the schedules allows the necessary computer programs to be structured in a highly efficient manner as regards both running time and cost. Computer-based calculations would clearly use a smaller 'step-length' for both the lognormal and logharmonic distributions.

6.2 *Illustrative Tables*

6.2.1 For day-to-day practical use, a set of tables for the values of q for European call and put options for a 'central value' on expiry of 100p might cover all values of the logharmonic deviation from nil to 10% at intervals of 0.5% and all values of the strike price from 50p to 200p at 1p intervals. The Appendix contains, for purely illustrative purposes, four tables — Tables 1 and 2 for call options with logharmonic deviations of nil and 7.5% respectively, and the corresponding Tables 3 and 4 for put options. Values of 50p to 150p, at 10p intervals, are used for the strike price, and values of nil, 5%, 10%, 15%, 20% and 25% are used for the lognormal standard deviation.

6.2.2 For the examples referred to in ¶4.3.6, the strike price of 100p and the 'central value' of 95p x 1.03 x 1.05, i.e. 102.74p, on expiry, translate into a scaling factor of 1.0274 and a normalised strike price of 100p ÷ 1.0274, i.e. 97.33p. Using second difference interpolation in Table 2, the value of q for the call option is 8.1924p x 1.0274, i.e. 8.42p. Similarly, from Table 4, the corresponding value of q for the put option is 4.3724p x 1.0274, i.e. 4.49p. The slight differences from the values in ¶4.3.10 are due to rounding and interpolation errors.

6.2.3 The option writer will then, using appropriate values for the other variables E, r, R and Q, translate these values of q into quoted market prices using the formula in ¶1.1.3. The option buyer can calculate the implied return over the term of the option by comparing the expected proceeds q on expiry with the best quote available in the market.

6.2.4 The 'net' call and put option prices before adding margins for expenses (i.e. the values of q divided by $(1 + r)$) are 7.80p and 4.16p respectively. It is instructive to compare these with the Black-Scholes prices relating to, say, a risk-free rate of return of 6% p.a. By the same logic as above, the implied central price on expiry of 95p x 1.06, i.e. 100.7p, and the strike price of 100p translate into a scaling factor of 1.007 and a normalised strike price of 99.3p. For lognormal standard deviations of 15% and 20%, the call option values of q are, from Table 1, 6.47p and 8.62p respectively, and dividing by 1.06 gives the Black-Scholes formula values of 6.10p and 8.13p respectively. First difference

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interpolation gives an 'implied volatility' of 19.2% as that corresponding to the 'new framework' net price of 7.80p. Similarly, from Table 3 we can calculate the 'implied volatility' of the put option as 14.2%. This significant divergence is largely due to the 'new framework' expected rate of return, at 8% p.a., being greater than the Black-Scholes rate of 6%.

6.2.5 Tables 1, 2, 3 and 4 can also be used to highlight the well-known 'smile' and 'skew' effects. Using only the values of q, the Black-Scholes 'implied volatilities' for call options for a logharmonic deviation of 7.5% vary with the strike price as shown below:

	Implied volatility %	
Strike price	10% lognormal standard deviation	20% lognormal standard deviation
0.5	11.18	20.71
0.6	11.18	20.69
0.7	11.16	20.60
0.8	11.09	20.52
0.9	11.26	20.70
1.0	11.68	21.59
1.1	11.28	20.52
1.2	10.83	20.42

6.2.6 The ability to evaluate Black-Scholes prices using only one simple table for call options and a similar simple table for put options appears to have been overlooked in the voluminous, and often highly mathematical, literature that has arisen on option theory. For example, Bernstein (1992) refers to the inherent complexity of the Black-Scholes formula in the following terms:

> "Soon people were going about with little hand-held calculators that had been programmed to perform the necessary calculations once the inputs had been punched in. Many options traders operate with powerful computers at their beck and call. Here, too, they have no choice. The formula is not exactly designed for quick calculations on the back of an envelope. By the time anyone figures out the answer by hand .... the world will have moved on so far that the whole result will be obsolete even before the calculations are done."

It is difficult to avoid the conclusion that the use of Itô's lemma and the other 'results' of continuous time finance to 'prove' what is, in effect, a one-parameter graduation formula, has misled many people, particularly mathematicians with little previous practical experience of finance, as to the true nature of capital market behaviour. Those who are distrustful of, or fearful for practical or professional reasons of their unfamiliarity with, the continuous time approach of Itô's lemma, in particular, or stochastic calculus, in general, are in exceptionally distinguished company; Markowitz (see Bernstein, 1992) wrote, in 1985, that:

> "Itô's lemma turned out to be a cornucopia of interesting results, and .... has become central to much of the modern theory of finance. The one thing that bothers me about continuous portfolio selection is that I don't really understand it."

6.3 *Possible Extensions*

6.3.1 The new framework for option pricing set out in this paper is consistent with all four of the general principles envisaged in Section 11 of Clarkson (1995a). However, further work of a technical nature will be required in numerous areas before the full potential of the new approach can be realised, and three such areas are described below.

6.3.2 The logharmonic distribution assumes that the final position in the mean absolute deviation chart is random, and hence uncorrelated with the initial position. For options on equity shares with terms of, say, more than six months, this assumption of randomness is not unreasonable, but for shorter terms, particularly of three months or less, these initial and final positions will be correlated. This can be dealt with by a diffusion process under which the mean absolute deviation position fans out over, say, six months from the known initial value to the aleatory logharmonic distribution.

6.3.3 For American options, where exercise can occur at any time, and for more complex options, appropriate evaluation procedures will have to be developed. However, given the transparency of the new framework, no insuperable problems are likely to be encountered. Also, a specific adjustment for dividends on a 'best estimate' basis can be incorporated very easily.

6.3.4 For other than very short terms, estimates of future returns on an equity index can be made using the same principles as for estimating the performance of equity shares relative to a market index. Estimates of fixed-interest yields and aggregate dividend growth are required to address 'uncertainty of the second kind', while both Clarkson (1995b) and Mills (1991) provide frameworks for describing the aleatory nature of equity market levels relative to particular values of these primary attributes, thereby addressing 'uncertainty of the third kind'. This general approach of studying stochastic (or pure random chance) variability around 'best estimate' descriptions of future financial conditions is, of course, already an established and very powerful actuarial technique.

6.4 *Parallels with Halley (1693)*

6.4.1 Earlier attempts to produce mortality rates for London and Dublin from the recorded numbers of deaths had been affected by three deficiencies: the sizes of the populations were unknown; the ages at death were unknown; and there were so many 'incomers' that the numbers of deaths greatly exceeded the numbers of births, making any 'stable population' approach totally inappropriate. The monthly data for the German city of Breslau for the years 1687 to 1691, on the other hand, related to an essentially self-contained population and included details of the ages at death. The numbers of deaths at different ages varied in a somewhat erratic manner, but Halley's crucial insight that led to the construction of his pioneering life table was that there was an underlying smooth progression of mortality rates as a function of age, so that the observed irregularities could be 'attributed to chance' (what I describe as 'uncertainty of the third kind'), and "would rectify themselves were the number of years much more considerable, as 20 instead of 5."

6.4.2 The first use that Halley describes for his mortality table is to show the proportion in any population of men able to bear arms ("fencible men, as the Scotch call them"), which he identifies as those between the ages of 18 and 56. Halley suggests that his estimate of $\frac{9}{34}$, or slightly more than a quarter, "may perhaps pass as a rule for all other places". This aggregate statistic bears functional similarities to the graduated statistics for high-frequency investment returns produced by Geman & Ané (1996).

6.4.3 The second, and most fundamental, use that Halley describes is to obtain the differing probabilities of death or survivorship over a particular period as a function of initial age. For example, the probability that a man aged 40 lives for 7 years is the ratio of those alive at 47 to those alive at 40, namely 377 divided by 445, or 0.85. Similarly, a central feature of the new framework for option pricing is the introduction of as accurate as possible an estimate of the likely return on the underlying security.

6.4.4 The third use described by Halley is to calculate the median expectation of life by finding the future age to which it is an 'even lay' (i.e. a 50:50 chance) that a life of a given age survives. This future age is found as the age in the life table at which the number alive is half that at the initial age. This approach, which regards the median as a more fundamental central measure than the mean (i.e. expected value) is mirrored, not only in my evaluation of option prices, but also, much more generally, in the use of median, quartile and percentile rankings as the most meaningful descriptors of relative investment performance.

6.4.5 Halley's fourth use is determining the appropriate price for term assurance in line with the amount expected to be paid out, which varies markedly with age, "it being 100 to 1 that a man of 20 dies not in a year, and but 38 to 1 for a man of 50 years of age". The analogy, in the new framework, is that the premium for a put option varies with the expected return on the underlying security.

6.4.6 The fifth and most important application envisaged by Halley is the valuation of annuities, where "it is plain that the purchaser ought to pay for only such a part of the value of the annuity as he has chances that he is living; and this ought to be computed yearly, and the sum of all those yearly values being added together will amount to the value of the annuity for the life of the person proposed." It is illuminating to consider Halley's comments about this crucial application as showing "how to make a certain estimate of the value of annuities for lives, which hitherto has been only done by an imaginary valuation". While, in modern usage, the phrase 'certain estimate' represents an obvious contradiction in terms, Halley's use of the epithet 'certain' is clearly a much abbreviated version of the adjectival phrase "ascertained as a sound framework for practical implementation by the scientific analysis of highly erratic empirical data", and I interpret the *Certum ex Incertis* motto of the Institute of Actuaries accordingly. Halley's crucial insight was to realise that a graduation process that eliminated 'uncertainty of the third kind' led to mortality estimates in the area of 'uncertainty of the second kind' which resulted in a sound and scientific basis for the transaction of life assurance and annuity business. Halley's reference to the valuation of annuities having hitherto "been only done by an imaginary valuation" is essentially a reference to what is described in Dunlop (1992) as the 'arbitrary and unsound prices' at which the English government sold annuities, even after the publication of Halley's pioneering paper. The unmistakable parallel in option pricing is that the new framework, unlike the Black-Scholes formula, takes explicit account of the best available estimates of future returns on the underlying securities.

6.4.7 Halley calculates illustrative annuity values using a compound interest rate of 6% p.a., and he includes a compound interest table of present values on this basis. However, he stresses that particular regard has to be paid to the actual return achieved, since the values of annuities decrease markedly as the assumed rate of interest increases. The parallels in the new framework are the specific recognition of the expected rate of return on admissible assets and of the target rate of return on capital employed. The Black-Scholes formula, on the other hand, incorporates only the so-called 'risk-free' rate of return.

6.4.8 Halley observes that his method of obtaining the 'true value' of an annuity "will without doubt appear to be a most laborious calculation". However, given the importance of this application, and having found some 'compendia' (literally approaches whereby successive computations would 'hang together') he 'took the pains' to compute a table showing annuity values at quinquennial ages up to 70. Similarly, the calculations set out in Sections 6.1 and 6.2 are included mainly to show that computations within the new framework involve only simple arithmetic and can be structured very efficiently.

6.4.9 The sixth use described by Halley relates to functions involving two lives, where, on the assumption (which is not stated explicitly) that the lives are independent, a probability involving the combined status of the two lives is the product of the relevant single life probabilities. This gives a means of calculating, in particular, "what value ought to be paid for the reversion of one life after another, as in the case of providing for clergy-men's widows and others by such reversions". The parallel here is that the value of q for complex options, such as 'barrier' options, can be evaluated using a straightforward extension of the principles used for European options.

6.4.10 An outstandingly successful application of Halley's new scientific approach was the setting up, in 1743, of the Scottish Ministers' Widows' Fund. Halley's Breslau life table was used both to determine contribution levels and for periodic valuations of the fund. It is widely believed (see Dunlop, 1992) that this was the earliest actuarially-based fund in the world, extending Halley's static 'true value' approach as regards the correct price of life assurance and annuities to 'fairness' to all parties in the long-term management of funds, as epitomised by the *Ad Finem Fidelis* motto of the Faculty of Actuaries. The obvious parallels in the transaction of options business are that a similar forward-looking valuation

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process to that inherent in the calculation of the premium charged should be used to assess 'surplus' or 'profit' and to test for solvency.

6.4.11 Halley's seventh use, the extension of his principles to three or more lives, includes an observation that is of crucial practical significance. He notes that it will not be necessary to compute the values at yearly intervals, since "in most cases every 4th or 5th year may suffice", with interpolation then being used for the intermediate years. Hardy's '39a' and similar approximate integration (i.e. numerical quadrature) formulae follow precisely this principle, and allowed actuaries of the pre-computer era to translate their conceptual approach as regards fundamental principles into practical answers to real world financial problems. More generally, no unrealistic assumptions have to be introduced to ensure mathematical tractability, and accordingly mathematics is the servant, not the master, of actuarial science. Common sense suggests that the same should be true of any 'scientific' approach to option pricing.

6.4.12 Contrast this with the mean-variance optimisation approach of modern finance theory, as first propounded in Markowitz (1952), then expounded in much more detail in Markowitz (1959), and thereafter translated by financial economists into the simplistic 'risk equals variance of return' paradigm from which the Black-Scholes formula was first derived, using the mathematics of the heat diffusion process in physics. Suppose that we equate a portfolio of 10 securities to the single life case, a portfolio of around 25 securities to the joint life case, and (on an equal geometric increase) a portfolio of 60 or more shares to the case of three or more lives. While Markowitz (1959) includes an evaluation of the 10 securities case, it is only in the 'personal notes' appended to the 1991 second edition that Markowitz elaborates on the mathematical intractability of his theoretical approach:

> "I had planned a 25-security analysis for the book, using analysts' beliefs as inputs. H. Isleib, Investment Officer of Yale University, and R. Halsey, Jr., Assistant Investment Office, filled out the forms in figures 6 and 7 of chapter II for the 25 securities. Jim Tobin, Professor A. J. J. Van Woerkom of the Astronomy Department and I drove one day to IBM's T. J. Watson Research Laboratory where Van Woerkom attempted to program the whole analysis for the 650. Jim Tobin tells me that Professor Van Woerkom is a computer genius who wired two IBM 602As together to facilitate the computation of precise planetary orbits. He was the only one of us who knew how to program the 650. It is clear to me now, and eventually became clear to me then, that the task was beyond the programming time I could ask Professor Van Woerkom to volunteer, as well as the time available to perform the analysis on the 604A. I decided that the book would have to do without the 25-security analysis."

In short, more than a quarter of a millenium after Halley (1693), some seven years after his pioneering 1952 paper, and some two years after the birth of the space age (as marked by the launch of the first Sputnik satellite in 1957), Markowitz was unable to provide even a computer-based computational implementation for a typical institutional investment portfolio. Both within the general mean-variance framework of modern finance theory and within suggested derived applications such as the Black-Scholes option pricing formula, mathematics is unquestionably the master, not the servant, of the financial community, and hence of society as a whole.

6.4.13 While Halley readily admits that his Breslau life table cannot be used unthinkingly as a universal standard, he suggests that it is, perhaps, as good a standard as could be found at that time, in that London mortality was very similar in terms of both infant rates and the proportion (around one thirtieth) of the population who died in any year. His final sentence is an exhortation for similar investigations of mortality to be carried out in other cities:

> "At least 'tis desired that in imitation hereof the curious in other cities would attempt something of the same nature, than which nothing perhaps can be more useful."

The Continuous Mortality Investigation Bureau of the Institute and Faculty of Actuaries has performed precisely this function on a formalised basis for more than seventy years. Previously, various *ad hoc* actuarial committees studied specific aspects of mortality experience and then reported their findings to the profession.

6.4.14 In this case the parallels with the new options framework are less obvious, since expected returns on securities are not absolute, but are a function of the skill and experience of those assessing the expected returns. For U.K. equities, the equivalent of Halley's Breslau life table can be regarded as the FTSE Actuaries All-Share Index; the principles underlying this index and the technical details of its design and construction are set out in Haycocks & Plymen (1956, 1964). Day, Green & Plymen (1994) describe assessment techniques using this or a similar market index, whereby more meaningful judgements of manager skill can be made than with traditional investment performance techniques. Largely as a result of the pervasive influence of the Efficient Market Hypothesis, it is often believed that such assessments of manager skill will not reveal any significant consistent departures from average performance in line with a relevant market index. However, the comprehensive study of potentially profitable investment strategies, set out in O'Shaughnessy (1996), shows that significant and reasonably consistent outperformance can be achieved without increasing the general level of investment risk.

6.4.15 Clearly the profitability of either writing or buying options on individual equity shares will increase with the accuracy of the estimates of relative performance, and practical experience of the share selection models, described in Weaver & Hall (1967) and Clarkson (1981), shows that the average outperformance or underperformance of a group of shares, identified as very cheap or very dear, respectively, can be of the order of 5% over a year. This suggests that a new type of investment service is likely to develop and flourish, namely proprietary equity selection models, which rank shares in order of relative attractiveness. At present, largely because the basic Black-Scholes formula does not involve the expected return on the underlying security, all the emphasis is on estimating volatility.

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6.4.16 It has been suggested, even by some actuaries, that life assurance is nothing more than a specialised application of options theory, and, accordingly, that the currently accepted methodologies of option pricing lead to a 'more scientific' framework than that provided by the traditional actuarial approach. The above commentary supports a diametrically opposite viewpoint, namely that, in terms of understanding and managing uncertainty in the financial world, the crucial scientific breakthrough came in 1693, not 1973.

7. COMPARISONS WITH OTHER APPROACHES

7.1 *Black & Scholes*

7.1.1 Although there are numerous isolated references to the Black-Scholes methodology throughout the paper, it may be helpful to draw together all these and other strands of argument by considering the perspectives and self-interests of: first of all the mathematician; then the trader; then the risk manager or regulator; and finally society as a whole.

7.1.2 A mathematician with no practical experience of investment will be impressed by the manner in which an elegant option pricing formula, which has been found to be immensely useful in practical applications, can be derived from a second order differential equation obtained by considering a particular 'risk-free' hedge portfolio, and will take it for granted that the underlying assumptions, if not completely realistic in all respects, are unlikely to lead to theoretical models which are potentially unsound as a guide to practical action by traders, risk managers, or regulators.

7.1.3 A few highly perceptive mathematicians have concluded that, despite some apparent internal inconsistencies, the Black-Scholes methodology, nevertheless, provides a sound numerical framework. Bergman (1982), for instance, describes it as "an example of two wrongs which make a (most important) right".

7.1.4 My viewpoint, as both a mathematician and also an investment actuary who has endeavoured to build a more realistic framework for the assessment and management of risk, is that, while the Black-Scholes formula represents, on certain very restrictive and often unrealistic assumptions, a remarkably accurate one-parameter graduation formula akin to Gompertz' Law of mortality, its formal mathematical derivation is completely detached from reality. In Clarkson (1990) I calculate, from the perspective of an option buyer who adopts my suggested approach to risk, the following prices of a one-year European call option for different values of the strike price k as a proportion of the initial share price.

k	Clarkson (1990)	Black-Scholes
0.8	0.281	0.281
0.9	0.212	0.212
1.0	0.157	0.156
1.1	0.113	0.111
1.2	0.080	0.078

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With a little mathematical ingenuity, it can be shown that my expression for the option premium P can be written as:

$$P = A + B - C$$

where B is the Black-Scholes value, A and C are an order of magnitude smaller than B, and — on the very restrictive assumptions I use in this example — A is approximately equal to C. The fundamental problem is that a mathematician who has chosen to live and work in the Black-Scholes world will have no reason to suspect that there might be real world scenarios in which A and C do not conveniently cancel out.

7.1.5 From the perspective of a trader or market-maker in options, the soundness, or otherwise, of the Black-Scholes methodology from a mathematical point of view is of no consequence whatsoever. What matters is its usefulness in guiding day-to-day trading decisions in the pursuit of a superior profits performance that will be rewarded very directly through higher basic salary, higher performance bonuses, and enhanced status within the organisation. While practical experience on the part of successful traders has led to various essentially trial-and-error modifications to the Black-Scholes formula (including, in particular, an awareness of the 'volatility smile' effect), its deceptive simplicity can also act as a straitjacket by effectively ruling out potentially profitable strategies that traders could exploit within the framework of the new approach.

7.1.6 The first limitation relates to the symmetric nature of volatility of return as the descriptor of risk within the Black-Scholes formula. Suppose that a trader correctly anticipates that the equity market will break out very strongly upwards from its recent trading range, and wishes to know the consequences for traded call and put options as compared to their present quoted prices. In a Black-Scholes world, the expectation of a significant upwards move would translate, first of all, into higher volatility and then into symmetric increases in the underlying values of both call and put options. In the new approach, this expectation of a significant upwards move would translate directly into an upwards revision of the value of a call option and a downwards revision of the value of a put option. The Black-Scholes approach is equivalent to arguing, in a life assurance context, that a strongly improving trend of mortality has the same impact on the profitability of both assured life and annuitant business.

7.1.7 The second limitation relates to potentially profitable trading strategies on the part of an institutional investor. The new approach allows a buyer of options to calculate the value q of the expected proceeds from an option using realistic estimates of expected returns. The Black-Scholes formula, on the other hand, incorporates only the 'risk-free' return, which is generally equated to a short-term interest rate.

7.1.8 The third, and possibly most costly, limitation relates to the presumption, on the part of the writer of an option, that the 'failsafe position' from a risk management point of view is to create a 'costless' and 'risk-free' hedge portfolio which is then 'rebalanced continuously' on the assumption that capital market behaviour is continuous in nature. The 'frictionless and costless' and 'continuous' assumptions, while indispensable features of the Black-Scholes world, play no central part in the new framework. Furthermore, there are (at least to actuaries) very obvious, but apparently hitherto unnoticed, parallels with life assurance. A life office may choose to reinsure, through the 'risk premium' mechanism, the 'death strain at risk' for certain policies such as those involving impaired lives, where it is judged that the aggregate costs involved (including both underlying mortality rates and expense loadings) are acceptable in terms of reducing the volatility of mortality profits. However, there are significant costs involved, and a life office, which has sufficient reserves to withstand the variability of mortality profits without having to resort to reinsurance, will experience a higher long-term rate of return on capital employed. Similarly, the not inconsiderable costs involved in pursuing a 'continuously rebalanced' hedge portfolio, in the context of options trading, could result in a quite unnecessary reduction in profitability as compared to a strategy of absorbing short-term fluctuations in profits within a sufficiently strong balance sheet.

7.1.9 From the perspective of a risk manager within a financial organisation, the over-riding imperative is to reduce to an acceptably low level the likelihood of occurrence of some concatenation of circumstances which threatens either the financial viability of the organisation or the credibility of its senior management. Similarly, a regulator will strive to minimise the likelihood of the occurrence of a whole spectrum of 'financial accidents', from unsophisticated members of the public suffering politically embarrassing financial losses to the entire collapse of the financial system in some 'melt-down' scenario. Several considerations suggest that the new approach is far superior to the Black-Scholes methodology as regards these prudential aspects.

7.1.10 The new approach is transparent to risk managers and regulators, whereas the Black-Scholes methodology, which involves the relatively unfamiliar mathematical tool of stochastic calculus (see, for instance, Lamberton & Lapeyre, 1996), is complex in the extreme.

7.1.11 The new approach expresses the premium that an option writer should receive in terms of three basic components. Given the power and cost-effectiveness of modern computers, it is, in principle, straightforward to carry out an analysis of surplus on the expiry of each contract to investigate the difference between the actual and target profit. The aggregate results would provide valuable information to the risk manager and senior management about the underlying profitability of the business and the accuracy of the assumptions used to calculate option premiums. Furthermore, regulators would be able to judge, from these management systems, whether or not the senior management of the organisation had a sufficiently firm grasp of the nature of the business they were transacting.

7.1.12 Perhaps the most important advantage of the new approach is that it recognises that the 'safety margin' represented by Q should vary over time with the general likelihood of an extreme market movement. In 'normal' circumstances, for instance, a regulator could require Q to be a minimum of $2q$, i.e. a 100% margin through earmarked capital. When the gilt/equity yield ratio was, say, 2.5 or higher, a much larger minimum margin would be prescribed for writing put options on equities or call options on gilts. It is well known that the Black-Scholes formula cannot take explicit account of extreme market movements.

7.1.13 It is generally believed that options and other derivatives on securities can play a valuable role in reducing financial risk, and thereby be of considerable benefit to society as a whole. There are three very general areas where the new approach appears to offer advantages over the Black-Scholes methodology in terms of these wider benefits.

7.1.14 First, the impenetrable nature of the advanced mathematics of current methodologies and the difficulty of setting up appropriate control systems are strong deterrents to the more widespread use of derivatives. The transparency of the new approach and the many parallels with life assurance should help to overcome these problems. In particular, the new approach draws specific attention to the importance of investigating what might happen in extreme circumstances rather than following a 'penny wise, pound foolish' approach of simply estimating volatility from recent experience which will often not include extreme events.

7.1.15 Second, the discussion, in Section 5, of the dynamics of capital markets is very strong evidence to the effect that, precisely as described by Soros (1994), unthinking rebalancing strategies of the 'delta hedging' variety will destabilise markets at critical times, thereby increasing very significantly the likelihood of an extreme event and increasing the level of systemic financial risk to possibly catastrophic levels. Modern theories of finance, however, have completely failed to recognise that trading strategies devised to reduce risk could have precisely the opposite effect. The new approach also has the powerful inbuilt stabilising mechanism of increasing, at critical times, the general level of solvency reserves that a writer of options has to hold, thereby reducing the volume of potentially destabilising speculative activity.

7.1.16 The third, and perhaps most important, feature of the new theory, as regards the benefits to society as a whole, is that it abandons completely the pseudo-scientific approach which assumes that mathematical models (such as Brownian motion) that are appropriate in the physical sciences can be unthinkingly applied to the interactive behaviour of millions of fallible human beings. The serious consequences of putting theory before reality in this manner are described by Hayek (1975), in his Nobel Memorial Lecture entitled 'The Pretence of Knowledge'. After describing the then very serious threat of accelerating inflation as having been "brought about by policies which the majority of economists recommended and even urged governments to pursue", he comments as follows:

> "It seems to me that this failure of the economists to guide policy more successfully is closely connected with their propensity to imitate as closely as possible the procedures of the

brilliantly successful physical sciences - an attempt which in our field may lead to outright error. It is an approach which has come to be described as the 'scientistic' attitude - an attitude which, as I defined it some thirty years ago, 'is decidedly unscientific in the true sense of the word, since it involves a mechanical and uncritical application of habits of thought to fields different from those in which they have been formed'. I want today to begin by explaining how some of the gravest errors of recent economic policy are a direct consequence of this scientific error."

In the context of option pricing theory, it seems to me that the gravest error is the adoption of an abstract mathematical approach which leads to the dangerously false belief that, in some miraculous manner, risk can be 'diversified away' and reduced to zero. The many financial disasters that have arisen from derivatives trading over the past few years tell a different story. There are no magic wands that we can wave to eliminate financial risk.

7.2 *Cox, Ross & Rubinstein*

7.2.1 The multiplicative binomial process, described in Cox, Ross & Rubinstein (1979), is a discrete time diffusion process which, in the limiting case where the time interval tends to zero, gives the Black-Scholes formula for European options as first derived through the application of stochastic calculus. Precisely the same economic mechanisms, namely the construction and maintenance of a 'risk-free' hedge portfolio, are used as in the Black-Scholes methodology. The discrete time approach is, in many ways, easier to understand than the continuous time approach, and it also has the advantage of being able to cope with American and other options, where exercise before the end of the contract may be optimal.

7.2.2 Although the formal mathematical development in Cox, Ross & Rubinstein (1979) proceeds along quite different 'first principles' lines, an application of Taylor's Theorem, very similar to that used in Clarkson (1978) to obtain the crucial second order partial differential inequality for gilts, leads to precisely the same second order partial differential equation that Black & Scholes (1973) derive using stochastic calculus. It is, therefore, not surprising that the Cox, Ross & Rubinstein methodology can be regarded, for all practical purposes, as being mathematically equivalent to the Black-Scholes methodology. Given this mathematical equivalence, all the comparisons with the Black-Scholes methodology, described in Section 7.1, apply also in the Cox, Ross & Rubinstein case, except that the latter approach is far more transparent and hence far easier for traders, risk managers and regulators to understand and to apply in practice.

7.3 *Bouchaud & Sornette*

7.3.1 The Bouchaud & Sornette approach has two important features in common with the new approach. No 'risk-free' hedge portfolio needs to be set up, and, in the process describing security returns to any future point in time, it is recognised that something quite different from a normal or lognormal diffusion process is required.

7.3.2 The 'truncated Lévy' process for security returns is remarkably similar, in terms of general shape, to the 'lognormal plus logharmonic' compound process of the new approach. However, the crucial truncation element is justified only on empirical grounds rather than, as in the new approach, in economic terms as the resultant of two simpler processes, one relating to the 'Central Value Hypothesis' and the other to the 'Systematic Over-Reaction Hypothesis'.

7.3.3 Unlike the new theory, the Bouchaud & Sornette methodology is essentially static, in that it assumes that the likelihood of an extreme movement is constant over time. Accordingly, while it leads to option prices that are higher than Black-Scholes prices through the incorporation of a solvency margin, this margin may be too high most of the time, but not nearly high enough when market levels have diverged very significantly from their central parities.

7.3.4 By following the spirit of the Markowitz mean-variance approach, the risk framework that has been built into the Bouchaud & Sornette methodology is that of utility theory. As explained in Clarkson (1990, 1996a), utility theory is inconsistent with the downside approach to risk that has been used to build the new theory of option pricing. A crucial point is that, as mentioned in ¶3.4.4, utility theory, as first developed in Bernoulli (1738), has no relevance in other than 'normal circumstances', and accordingly cannot be regarded as a credible mathematical framework for financial risk. Even in 'normal circumstances' the relevance of utility theory has been questioned by some eminent economists. In particular, Allais (1953) states, quite explicitly, that:

> "Whatever their attraction might be, none of the fundamental postulates leading to the Bernoulli principle as formulated by the American school can withstand analysis. All are based on false evidence."

7.4 *Geman & Ané*

7.4.1 The Geman & Ané approach, which is essentially a refinement of the Black-Scholes methodology rather than a completely new theory, recognises that the variance of return of a security does not increase linearly over time (as is the standard assumption), but can be shown to be closely correlated to the number of trades. However, the stochastic volatility, identified by this 'stochastic clock' of the number of transactions made in the market, is essentially an observed 'all-worlds' measure rather than, as in the case of the new framework, the resultant of two processes which can be studied separately.

7.4.2 Perhaps the most important feature of the Geman & Ané approach is that, as shown in Section 5.3, the empirical results are consistent with the logharmonic process of the new framework.

8. Conclusions

8.1 *Standard Criteria for Evaluating New Theories*

8.1.1 Kuhn (1977) sets out the five criteria, namely accuracy, consistency,

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scope, simplicity and fruitfulness, that he believes should be used when a choice has to be made between an established theory and an 'upstart competitor':

> "First, a theory should be accurate: within its domain, that is, consequences deducible from a theory should be in demonstrated agreement with the results of existing experiments and observations. Second, a theory should be consistent, not only internally or with itself, but also with other currently accepted theories applicable to related aspects of nature. Third, it should have broad scope: in particular, a theory's consequences should extend far beyond the particular observations, laws, or subtheories it was initially designed to explain. Fourth, and closely related, it should be simple, bringing order to phenomena that in its absence would be individually isolated and, as a set, confused. Fifth - a somewhat less standard item, but one of special importance to actual scientific decisions - a theory should be fruitful of new research findings: it should, that is, disclose new phenomena or previously unnoted relationships among those already known."

8.1.2 Kuhn also draws attention to two difficulties often encountered by those trying to apply these criteria. First, as a result of the imprecision of the criteria, individuals may legitimately differ in their assessments in specific cases. Second, conflicting assessments often arise from two or more criteria; accuracy may, for instance, point to the adoption of one theory while simplicity points to the adoption of its competitor.

8.2 *Assessing the New Theory*

8.2.1 As regards accuracy, the new theory not only allows judgements about expected returns to be incorporated, but also recognises that the general level of financial risk, and hence the magnitude of an appropriate 'safety margin' for both prudential and regulatory purposes, varies markedly over time. Furthermore, it is consistent with numerous empirical studies (such as Peters, 1991; Bouchaud & Sornette, 1994; Walter, 1995; and Geman & Ané, 1996) that cannot be explained by current methodologies. In particular, the 'volatility smile' effect is a prediction of the new theory.

8.2.2 Interpreting life assurance as by far the most successful example of 'other currently accepted theories applicable to related aspects of nature', the new theory clearly achieves a high rating on general considerations of consistency. It also scores highly on the consistency criterion within the domain of option pricing, in that the Black-Scholes formula is a very much simplified special case in which important features of the new approach are suppressed. However, individuals whose background is essentially mathematical rather than practical will see the new approach as markedly inconsistent with 'currently accepted theories', in that it abandons three of the cornerstones of modern finance theory, namely the presumption of equilibrium, the applicability (at least as a good first approximation) of the Efficient Market Hypothesis, and the variance of return paradigm of risk.

8.2.3 In terms of scope, and extending far beyond the initial remit of option pricing theory, two positive features emerge. First, the underlying compound process for security returns to any future point in time can be translated very easily into other important application areas such as asset/liability models for pension funds, life assurance and general insurance. Second, a better understanding of the causal mechanisms that drive capital market behaviour would lead to a marked reduction in the level of systemic financial risk, and thereby reduce very significantly the likelihood of some 'nightmare scenario', as feared by some very senior regulators such as Mr Eddie George, Governor of the Bank of England.

8.2.4 In terms of being simple and bringing order to a series of apparently disorderly facets of behaviour, several points can be made in favour of the new approach. The crucial, but nevertheless simple, new idea is that the dimensionality of the process generating future security returns must be increased from two to three by adding systematic variability around central values to the 'drift' and 'volatility' descriptors of current methodologies. Also, the resulting compound process (essentially the addition of simple harmonic motion to a Gaussian distribution), which has very general applicability, is shown to have a simple causal mechanism (the interaction of differing levels of investor behaviour) that can be regarded as the first plausible economic explanation of the fractal nature of capital markets, first comprehensively described more than thirty years ago by Mandelbrot (1963), and convincingly confirmed by Walter (1995).

8.2.5 On the negative side, as regards the criterion of simplicity, the new approach requires additional inputs, in particular the expected return on the underlying security, the 'worst case' outcome for the purposes of risk management, and the target rate of return on capital employed. Common sense, however, suggests that any approach which fails to take explicit account of all these factors is not only theoretically incomplete, but is also potentially unsound as a guide to practical action.

8.2.6 Clearly, much more work remains to be done to fine tune the new approach in the same way that many practitioners have adapted the Black-Scholes formula to meet their own specific requirements. In particular, different compound distributions might be used for option terms of, say, less than three months, and modifications to deal with American and other more complex options would be required. Accordingly, while the Black-Scholes methodology has been analysed and refined for more than twenty years, the new approach is still in its infancy, and accordingly should be 'fruitful of new research findings'.

8.3 *The Assessment of Risk and Mathematical Tractability*

8.3.1 This application of Kuhn's criteria suggests that the most serious perceived failing of the new approach might be that it is inconsistent with the variance of return paradigm of risk, in which it is often assumed (as in the derivation of the Black-Scholes formula) that risk can somehow be 'diversified away' to create a 'risk-free' asset. However, it is instructive to trace the 'rise and fall' of variance of return as a measure of risk from its first appearance in Markowitz (1952) to the present day.

8.3.2 Markowitz (1959) contains numerous warnings about the unrealistic nature of some of the crucial simplifying assumptions regarding rational behaviour, risk and utility theory. Also, the respective merits of variance and semi-variance are debated at great length, and the choice in favour of variance is made on the grounds that, when expected utility is calculated, it generally gives similar results as for semi-variance (which is more plausible on commonsense grounds), but is easier to understand and faster and more cost-effective to run on a computer. These balanced arguments tended to be forgotten by the disciples of Markowitz, who seized on the mathematical tractability of mean-variance analysis and enthusiastically promoted it as the only 'scientific' approach to investment under the name of 'Modern Portfolio Theory'. Even in influential textbooks, such as Rudd & Clasing (1982), it is stated, without any discussion whatsoever, that "risk equals variance (or standard deviation) of return".

8.3.3 In a new note on semi-variance in the second edition of Markowitz (1959) (the 'advocate of semi-variance' referred to is the present author), Markowitz suggests that, with the greatly increased speeds and greatly reduced costs of modern computers, it might perhaps be appropriate to look again at semi-variance. Within a few years Markowitz was using semi-variance in proprietary packages for portfolio optimisation, as described in Markowitz, Todd, Xu & Yamane (1993).

8.3.4 I refer, in ¶1.5.1, to the increasing number of papers being written on downside or shortfall approaches to risk. Also, the Value At Risk (VAR) approach, which is increasing rapidly in popularity amongst traders and risk managers, involves the systematic study of possible shortfall scenarios, and hence is identical in principle to my 'four kinds of uncertainty' approach. In short, a downside approach to risk, incorporating a structured analysis of all factors capable of leading to unsatisfactory outcomes, is rapidly becoming the new paradigm; the old variance of return paradigm has failed to give sufficiently convincing answers in many important applications.

8.4 *Concluding Remarks*

8.4.1 In the concluding paragraphs of Clarkson (1996a) I suggest that many current theories of financial economics are incomplete in one or more of three crucial areas: no general recognition that financial behaviour in the real world is far more complex than that implied by the standard simplifying assumptions; no recognition of differing levels of behaviour as a result of differing levels of skill and experience; and the absence of a credible theory of risk. I also express my belief that a much better theory of finance, and, moreover, one that is essentially actuarial in nature, is well within our grasp.

8.4.2 The present paper addresses, in the very important field of option pricing, each of these three areas of incompleteness, and then constructs, using actuarial principles and only elementary mathematics, a new theory of option pricing which appears to have important theoretical, practical and regulatory advantages over the Black-Scholes and related methodologies of modern finance theory.

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8.4.3 The author apologises for the inordinate length of the paper as compared to others in the financial literature that set out 'new ideas'. However, because of the widespread acceptance of the 'old ideas' represented by Modern Portfolio Theory, the Efficient Market Hypothesis and the Black-Scholes methodology, it has been necessary to devote a large proportion of the paper to repeating the numerous reasons why many investment practitioners have never accepted these theories and summarising some of the very many papers by leading actuaries, economists and physicists that draw attention to the numerous faults and failures of these theories. For those who are prepared to clear their minds of the pseudo-scientific distractions of Modern Portfolio Theory and its related methodologies, the new paradigm offers a scientific, robust and practical framework for valuing options.

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APPENDIX

Table 1. Value of q for European call options; logharmonic deviation nil

	Lognormal standard deviation					
Strike price	0%	5%	10%	15%	20%	25%
50	50.00	50.12	50.47	51.02	51.76	52.68
60	40.00	40.12	40.47	41.02	41.77	42.71
70	30.00	30.12	30.47	31.03	31.89	33.02
80	20.00	20.12	20.48	21.26	22.60	24.22
90	10.00	10.13	11.09	12.52	14.57	16.77
100	0.00	1.96	3.98	6.06	8.20	10.40
110	0.00	0.05	0.99	2.43	4.57	6.77
120	0.00	0.00	0.13	0.85	2.21	3.64
130	0.00	0.00	0.01	0.24	0.77	2.20
140	0.00	0.00	0.00	0.04	0.39	0.92
150	0.00	0.00	0.00	0.00	0.08	0.57

Table 2. Value of q for European call options; logharmonic deviation 7.5%

	Lognormal standard deviation					
Strike price	0%	5%	10%	15%	20%	25%
50	50.13	50.25	50.60	51.15	51.89	52.82
60	40.13	40.25	40.60	41.15	41.90	42.87
70	30.13	30.25	30.60	31.17	32.03	33.23
80	20.13	20.25	20.65	21.50	22.77	24.37
90	10.13	10.40	11.45	13.03	14.88	16.90
100	2.36	3.02	4.68	6.76	8.90	11.11
110	0.00	0.29	1.36	2.96	4.80	6.83
120	0.00	0.00	0.25	1.05	2.33	4.05
130	0.00	0.00	0.02	0.29	1.09	2.24
140	0.00	0.00	0.00	0.08	0.40	1.21
150	0.00	0.00	0.00	0.01	0.16	0.58

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Table 3. Value of q for European put options; logharmonic deviation nil

	Lognormal standard deviation					
Strike price	0%	5%	10%	15%	20%	25%
50	0.00	0.00	0.00	0.00	0.00	0.00
60	0.00	0.00	0.00	0.00	0.00	0.03
70	0.00	0.00	0.00	0.01	0.13	0.34
80	0.00	0.00	0.02	0.24	0.83	1.54
90	0.00	0.01	0.62	1.50	2.80	4.08
100	0.00	1.84	3.51	5.04	6.44	7.72
110	10.00	9.93	10.52	11.41	12.80	14.09
120	20.00	19.88	19.66	19.83	20.45	20.96
130	30.00	29.88	29.54	29.22	29.01	29.51
140	40.00	39.88	39.53	39.02	38.63	38.23
150	50.00	49.88	49.53	48.98	48.31	47.89

Table 4. Value of q for European put options; logharmonic deviation 7.5%

	Lognormal standard deviation					
Strike price	0%	5%	10%	15%	20%	25%
50	0.00	0.00	0.00	0.00	0.00	0.00
60	0.00	0.00	0.00	0.00	0.01	0.05
70	0.00	0.00	0.00	0.02	0.14	0.40
80	0.00	0.00	0.05	0.35	0.88	1.55
90	0.00	0.15	0.85	1.88	2.99	4.08
100	2.23	2.77	4.08	5.61	7.01	8.29
110	9.87	10.04	10.76	11.81	12.91	14.01
120	19.87	19.75	19.65	19.90	20.44	21.23
130	29.87	29.75	29.43	29.14	29.20	29.42
140	39.87	39.75	39.40	38.93	38.51	38.39
150	49.87	49.75	49.40	48.86	48.28	47.76

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ABSTRACT OF THE DISCUSSION

HELD BY THE FACULTY OF ACTUARIES

The text of the paper 'Actuaries and Derivatives', by M. H. D. Kemp, M.A., F.I.A., together with the abstract of the discussion on it held by the Institute of Actuaries on 28 October 1996, are printed in *British Actuarial Journal*, **3**, I, pages 51–180.

At the meeting of the Faculty of Actuaries on 20 January 1997, both papers, 'Actuaries and Derivatives' and 'An Actuarial Theory of Option Pricing', were discussed.

The President (Mr P. H. Grace, F.F.A.): It is my pleasure to extend a warm welcome to Professor Hans Bühlmann of the Swiss Actuarial Association, a guest of the Faculty this evening.

Mr M. H. D. Kemp, F.I.A. (introducing his paper): We have two papers to discuss, which take somewhat differing perspectives. In my paper I attempt to argue that there is much that the actuarial profession can learn from derivatives practitioners who are not actuaries, and vice-versa. Professor Clarkson's paper argues that there is a much wider difference between these two camps.

Rather than formally introducing my paper, I would like to discuss whether we are actually talking about a narrow or a wide gap. I have therefore entitled this presentation, 'The Financial Economics Approach Versus The Actuarial Approach — Is this the Right Debate?'

Let us forget about options to start with; instead, let us first look at how, as actuaries, we might value a fairly simple sort of investment, e.g. a series of cash flows, such as a bond. Suppose that the cash flows involve us receiving three payments of 10 in years 1, 2 and 3, and a payment of 20 in year 4.

Those who remember compound interest examinations will remember that the way actuaries work out the value of such an instrument is simply to add together the present values of the four components. Fundamental to all actuarial thought is this ability to add numbers together. I do not, of course, mean the ability to do it on a calculator. Instead I mean that the values satisfy some sort of 'additivity' rule, so that the process of addition gives the correct answer. This 'additivity' seems rather obvious, indeed one might even say axiomatic. However, if we ask why should we be able to produce the correct value by adding the four components together, we realise that this question is rather deeper than it looks at first sight.

From a financial economist's perspective, 'additivity' of values requires what is called 'the principle of no arbitrage'. An arbitrage is said to exist if we can find an investment of zero value which, somehow, we can guarantee that, at some future date, will have always grown to at least zero value, and, in some circumstances, will be strictly greater than zero in value. Thus an arbitrage will generate a strictly positive value, even though it costs nothing to invest in now.

Returning to the example I talked about; if arbitrage exists, then the value of the four cash flows plus the arbitrage ought to be greater than the value of the four cash flows without the arbitrage, even though, at outset, we can rearrange the two portfolios to be identical. Hence also 'value' becomes indeterminate, depending on how much of the arbitrage we invest in. Adding values together becomes meaningless. Thus, no arbitrage is fundamental to the concept that we can add values together.

What has this to do with valuing an option? With the previous example I sliced up the cash flows into time buckets. With an option I can, instead, slice the cash flow into pay-off buckets. Take, for example, a call option. Let us assume that it has a maturity of one year. We can slice the option into many different components, each one of which pays out, depending on whether or not the share price at maturity is within a certain bracket. It seems obvious to me that we can, in principle, decompose an option in this fashion. It is, therefore, reasonable to suggest that the present value of the total of all these components (i.e. the value of the call option as a whole) ought to be equal to the sum of the parts. Again, we require no arbitrage, but this does not seem unreasonable.

We now have to revert to some more complicated mathematics, but I shall try to explain all the steps involved. Suppose I want to value any pay-off $Z(S)$, which is a function of the share price at maturity. In the example of a call option with strike price 10, the pay-off Z is zero if S is below 10, and then increases uniformly as S increases above 10.

As actuaries, you will remember that the way to value such a compound set of events is to work out the probability that each will occur, and to multiply this probability by the amount you will actually receive if the event occurs. You also need to discount at some suitable discount rate. Thus we can determine the present value of the pay-off as:

$$\text{PV}(Z) = \frac{1}{(1+j_Z)}\sum p(E)Z(E)$$

where $p(E)$ is the probability of S being between E and $E+dE$ at maturity, $Z(E)$ is the pay-off of the option if S is between E and $E+dE$ at maturity and j_Z is some suitable discount rate; I will leave the question of what j_Z is for a few seconds.

Using the no arbitrage principle, we can also value the pay-off in a different fashion as follows:

$$\text{PV}(Z) = \sum V(E)Z(E)$$

where $V(E)$ is the value of a derivative paying 1 if S is between E and $E+dE$ at maturity, or 0 otherwise.

Applying the principle of no arbitrage again, we can also conclude that these two expressions give identical answers.

What does this mean in practice? Let us consider an unusual derivative which provides a pay-off of one, wherever the share price ends up. It does not require a lot of effort to see that this is the same as a zero coupon bond that will pay one (in a year's time). However, we know its present value; it is that of a zero coupon bond. If we do some re-organisation for that special case, we find that:

$$\sum(1 + j_{\text{zero coupon}})V(E) = \sum p(E) = 1.$$

Let us define a new quantity called p^*, defined as follows:

$$p^*(E) = (1 + j_{\text{zero coupon}})\ V(E).$$

It follows immediately from what I have just said that the sum of all $p^*(E)$ is equal to one. We can also note that all the $p^*(E)$ must be at least zero, so that we can, therefore, treat $p^*(E)$ just like a probability distribution.

Thus the present value of Z, whatever Z is, is given by the formula:

$$\text{PV}(Z) = \frac{\sum p^*(E)Z(E)}{(1 + j_{\text{zero coupon}})}.$$

This means that, as long as no arbitrage applies, we can always value such an option by calculating its expected value, discounting at $(1 + j_{\text{zero coupon}})$, using what is called the 'risk neutral probability distribution'.

What does this mean in practice? Well, the standard financial economist's approach is just as I have outlined. It involves valuing a derivative by taking expectations using a risk neutral distribution and discounting at the 'risk free' rate.

This explanation helps to clear up several common misconceptions relating to the Black-Scholes formula and with option pricing in general. Do all investors need to agree what risk free means? No. If we go back to my formula, we note that the correct rate is, in fact, the zero coupon rate. As long as there is a unique value for the zero coupon bond, then the discount rate implied by it is the correct rate to use in the above calculation, whatever anybody thinks is 'risk free', or even how they define 'risk'.

Do markets need to be efficient? Again the answer is no; the above analysis contains no mention whatsoever of market efficiency. All I required was the ability to add values together. No arbitrage is a very much stronger constraint than whether or not we believe that markets are fairly or unfairly priced.

Do we need to measure risk using standard deviations? Again, the answer is no. Standard deviations and variances are not mentioned above. If people disagree with Black-Scholes or financial economists' approaches to option pricing in general because they believe that risk is not measured by variance or standard deviations, they are missing the key essence of financial economics.

Indeed, a point which is, perhaps, not so obvious in the above analysis is that there is no dependence on whether markets show mean reversion. Markets can revert as much as anyone likes, but the ability to use a risk-neutral distribution and a risk-neutral rate to value options still applies. Again, the idea that markets cannot show mean reversion if they are to be consistent with Black-Scholes and financial economics, more generally, is erroneous.

Does that mean that you should take Black-Scholes as being true as long as no arbitrage applies? No. If you look at what Black-Scholes actually depends on, there are certain other assumptions on which it relies. Within the paper I have tried to develop what these are. The three areas where Black-Scholes (or a suitable generalisation) can break down, other than in the presence of arbitrage, are:

(a) if markets jump;
(b) if volatility of markets is uncertain; and
(c) if there are transaction costs.

In summary, it is very important to focus on the principle of no arbitrage when discussing whether financial economists' and actuaries' approaches to option pricing are different. Believing that there can be arbitrage, it seems to me, is effectively saying that there is a block of investors who guarantee to write a cheque to another block of investors without any recompense. That is an inherently implausible happening in financial markets to any great extent, and would invalidate the traditional actuarial approach to valuing an investment by adding up the values of the individual cash flows.

I was trying to think of a suitable analogy for belief in arbitrage, and considered that of money growing on trees. However, as those of you who remember *The Hitch Hiker's Guide to the Galaxy* will know that at the end of that story money does, indeed, grow on trees, because the leaf had been declared legal tender. So I suggest another analogy — belief in Father Christmas. We all know that, although Father Christmas may not exist, there are many people who make money out of him, and not just those who wander around in grottoes wearing white beards. I would not want you to read into my analogy that arbitrages never exist, but I do not believe that they are easy to come by or that it is wise to assume that they will always be available to bail you out of problems.

Thus the debate should not be between financial economists and actuaries, as both are, I believe, on the same side, believing in the importance of arbitrage-free models.

Professor R. S. Clarkson, F.F.A. (introducing his paper): In this paper, I try to translate the combination of my mathematical and actuarial training and my practical investment experience into a new and better theoretical framework for option pricing.

Shortly after finishing my paper, there were three quite different developments which gave me considerable reassurance as to the practical relevance of my new approach. On what has been called 'Frantic Friday', a few seemingly innocuous words from Mr Alan Greenspan, Chairman of the United

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States Federal Reserve, as to potentially irrational ratings, sent equity markets world-wide into a tail-spin. This behaviour, while fully consistent with my observations, in Section 5.1, that, when equity valuations are stretched, the slightest stimulus can result in acute instability, bears no resemblance whatsoever to the equilibrium and rational behaviour paradigms of modern finance theory.

The second development came a week later, when the Basle Committee dismissed as self-interested commercial rhetoric the bleatings of some of the London banking community as to the high reserves that had been prescribed as solvency margins in the writing of options business. This is precisely the predictable conflict between financial regulators and self-interested profit maximisers that I describe in Section 4.4.

Third, and possibly most significant, I learned that Mr John Pemberton, having, like myself, researched the philosophical and mathematical background in some detail, had written a paper on option pricing for presentation to the Staple Inn Actuarial Society, which, in conceptual terms, is virtually indistinguishable from mine. In particular, he regards the use of elaborate mathematics as unhelpful, and recommends the use of step functions, which are equivalent to my discrete distributions. He stresses, as I do, the parallels with life assurance; and he stresses, also, that the next step might be to produce standard tables for option pricing similar to those used in actuarial life and pensions work. These are precisely the type of tables that I show examples of in the Appendix to my paper.

Albert Einstein once said that a new physical theory should be so simple that even a child can understand it. I have done my best to follow this excellent advice by developing my new formula for option pricing in ¶1.1.3, and giving the time-honoured actuarial 'general reasoning' verification in ¶1.1.4.

To add to Mr Kemp's comments, as well as assuming no arbitrage, you also need to assume that markets are complete in a very strict sense. The Black-Scholes continuous time approach may work very well in orderly markets. However, what happens when we have disorderly markets? On occasions you can have the market computer systems running behind executed trades. That, to me, is an example where markets are not complete and where risks can be very high. Even more serious, as I mention in the paper, the New York Stock Exchange authorities did not allow trading in the S&P futures at any time on Black Monday.

Dr D. C. Bowie, F.F.A. (opening the discussion): The authors have provided us with a large amount of material on a topic which has become increasingly important in the actuarial context. They agree on one thing at least, that actuaries and derivatives are a natural fit. However, they do have quite different opinions on how the profession is to take up its role. For whatever reasons, the actuarial profession is now a long way behind some of the other financial professions in terms of contributing to the derivatives field. The profession urgently needs to make up that ground, and these two papers are an ideal catalyst for focusing the debate as to how that should be achieved.

The sheer length of the papers will mean that any discussion is likely to be far ranging, with discussants bringing with them their own particular viewpoints to the debate. My own background leads me to identify two issues which impinge on the way actuaries can, and should, proceed in this area. The first is whether or not the basic ideas encompassed in financial economics (such as hedging and risk-neutral pricing) are useful additions to the actuarial methods of financial modelling. The second is whether or not the mathematical framework, which is so closely allied with financial economics, is a useful addition to the actuarial technical toolkit.

Mr Kemp's paper contends that the actuarial profession already accepts, albeit implicitly or under different guises, many of the concepts and ideas inherent in the financial economic approach to derivative pricing. His thesis is that actuaries do have the appropriate skills and methodologies to extend the financial economic models, provided that they are willing to put in the effort of catching up on the mathematics. Professor Clarkson's views are that the profession should ignore the fuss associated with both the mathematics and the ideas. He asserts that only a return to time-honoured actuarial ideas and mathematics will put the analysis of derivatives onto a sound footing worthy of actuarial involvement.

On balance, I prefer the Kemp approach (exemplified in Sections 9 and 10 of his paper) of beginning the model building with financial economic ideas and then interleaving actuarial ideas, to the Clarkson approach (explained in Section 2 of his paper) of explicitly rejecting everything financial and economical and starting from scratch. I prefer the Kemp approach, because I am convinced that financial economic ideas are good ones which add tremendous insight into how investments are valued relative to each other. I am also convinced by Kemp's and others' arguments that there is room enough in the financial economic models for actuaries to add substantially, so that important actuarial concepts, such as risk reserving, can be included.

The apparent naivety of financial economics in general and the derivative pricing models in particular, is more an artefact of poorly informed critics rather than a weakness in the science itself. Financial economics comes very richly endowed with empirical tests and assessments of its models. Indeed, the very active interchange between the theory and practice of financial economics has spawned a truly formidable literature. Financial economists were not ignorant of taxation and did not just forget about transaction costs when they developed their models. The key thing about building models is that, if you are wanting to understand a process (such as pricing derivatives), it is rather pointless to start off with a model so complicated that you lose sight of the big picture. If simplicity in the first stage of the modelling process is a criticism, then actuarial science is also at fault. Actuaries have built up simple models of complicated businesses for centuries, and then have added to the simple models so that they become more realistic. When extending and modifying the models mathematically became impracticable or undesirable, judgement and experience were interwoven into the decision-making process. It is the same with financial economics. Simple models have been built. Modifications and extensions to these models have been developed, and nothing stops experienced financial practitioners from adding their expertise and experience to these models. The historical development of actuarial science can be used to accelerate the learning process in a relatively young science such as financial economics, but it is nothing short of misplaced arrogance to repudiate a perfectly respectable and successful science on the basis of a criticism to which our own profession is also subject.

Both authors identify elements from the literature which point out empirical problems with the theory of Black & Scholes. The non-normality of stockmarket returns is a particular example. The existence of jumps in the price process, rather than only diffusions, implies that the dynamic hedging required in Black–Scholes will not be possible, even theoretically. Professor Clarkson sees this as an insurmountable problem with the Black-Scholes framework, and uses it as a prompt for developing his alternative discounted expected cash flow method. The Kemp approach is to try to allow for the jumps by costing for them separately. Moreover, Mr Kemp argues, in ¶7.3.9, that estimating the appropriate risk discount rates for an approach such as Professor Clarkson's is very much a non-trivial problem, which makes the hedging approach more desirable.

The Clarkson model may have an apparent structural and mathematical simplicity, but it obscures much highly subjective work in estimating the model parameters. He argues that estimation of the parameters is where actuaries get to demonstrate their experience and judgement in valuing the instrument, but I think that that leaves the model rather far short of being useful for accounting purposes or producing trading strategies. It may be useful for managing long-term funds to distinguish between actuarial value and market price, but the distinction is less useful when it comes to investments which need to be marked to market.

The mathematics of derivative pricing, which is, essentially, just a small subset of stochastic calculus, is alien to many actuaries because it is very different from the training that we receive in the current and previous actuarial syllabuses. I would argue that it is unwise for the profession to eschew the mathematics, either on the rather weak grounds that it is unfamiliar, or because it happens to be associated with financial economics (i.e. not 'actuarial'). The mathematics is not even the messenger of financial economics, it is merely the language. It should be noted that, despite the apparent implications of ¶1.3.5 in Clarkson's paper there is absolutely nothing incorrect or sinister in the mathematics of financial economics. Indeed, the language of stochastic calculus has already been adopted in other aspects of actuarial work: mortality models and, more recently, the modelling of life contingencies, for example. The use of stochastic calculus does not remove any actuarial expertise or denigrate the mathematical work of the past at all. It is a language which is explicitly designed to

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describe how random events vary with time, which seems to make it ideal as the language for the development of actuarial concepts. Given the mathematical heritage of actuarial science, it would be a great sadness if the profession were to deny future generations of actuaries the ability to read and write this language. All our good ideas need to be expressed unambiguously and succinctly, using a suitable language. In extending his theory of special relativity to the more general case, Albert Einstein noted this need for understanding difficult mathematics. I quote:

> "But one thing is certain, in all my life I have laboured not nearly as hard, and I have become imbued with great respect for mathematics, the subtler part of which I had in my simple mindedness regarded as pure luxury until now."

Tackling the mathematics is hard work, but it brings rewards with it.

Financial economics has adopted the language of mathematics wholeheartedly. Professor Clarkson, in ¶6.2.6, warns that the use of proofs involving advanced mathematics will mislead all but the most experienced investors into believing that financial economic models are fundamental truths. I would hope that a numerate profession such as ours would be unlikely to be duped into muddling models and reality, and would see all mathematics for what it is, namely a language for saying mathematical things unambiguously. The proofs in financial economics, as with all scientific pursuits, are fundamental to the understanding and sensible application of the computational tools which emerge from the models.

The benefit of the mathematical layout of financial economic theories has been that the assumptions are made crystal clear. They are clearly stated in the development of the model for all to see. Thereafter, mathematical logic takes the reader inexorably from these model assumptions through various key results to the final computational tool. If the computational tool, for example a pricing formula, does not produce realistic results, it is, conceptually at least, a simple matter to alter the assumptions to make the model useful. The transparent nature of the process positively encourages the modification and extension of the models. However, I think that the very structured presentation of financial economic models is also one of the reasons for the intolerance which many actuaries display towards them. In particular, the assumptions tend to be the one part of any financial economic paper which is written in non-technical language, and, unless the reader is willing to work through all the mathematics, it is very difficult to spot how important each of the assumptions is. Some of the assumptions may be there merely to ensure mathematical elegance of the development, or may be overkill, in the sense that they are not necessary in their current strong form. Other assumptions will be absolutely fundamental to the model. The fact that some assumptions can be relaxed or modified to be more realistic will escape the notice of those who criticise models purely on their assumptions. The discussion in Section 8 of Mr Kemp's paper is particularly pertinent in this regard, and Section 8.3 is a useful checklist for when Black-Scholes will or will not work.

Mr M. W. Jones, F.F.A.: Mr Kemp's paper is an excellent overview of the topic of derivatives. It introduces us to the main types of derivatives and their diverse applications in the financial world. It leads us through pricing methodologies and their underlying assumptions, this including the presumed focus of debate, the pricing of options. Finally, we are introduced to the regulatory requirements for such instruments and the importance of control procedures. Surely this is the paper to read for those actuaries wishing to venture into this specialised financial world. There are many points that I found worthy of highlight.

Section 11 contains many sensible guidelines, including Section 11.5, that gives five commonsense values that should be at the heart of any investment risk-management operation. Hindsight has shown that the Barings fiasco could have been avoided had proper control procedures been in place and adhered to. Unfortunately, failures such as Barings will contribute to increases in regulation which will result in the use of derivatives being considerably more cumbersome in the future.

Asset allocation is an important use of derivatives. However, in the real world, the use of futures for instantaneous switching of assets from one region to another is more grey than the black and white that the author presents. There are risks involved in switching when one market is live and the other

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is closed. Consider, for example, selling U.K. exposure and buying the equivalent Japanese exposure. One can buy Japanese exposure in U.K. time, but, in essence, you have bought time risk. The Japanese market will not open for a few hours, and perhaps something will happen in the intervening time that will result in the Nikkei opening at a substantially different level than that at which one has bought. Also, how would one buy Far East exposure? On top of time risk, there is the problem that not all of the underlying stock markets of that region have liquid futures markets.

I must admit to a wry smile when reading Section 4.5, where the author refers to: "actuaries focus on value, whereas financial economists focus on price". It was one of Oscar Wilde's famous lines that cynics know the price of everything and the value of nothing. I am not quite sure whether too many financial economists would like to be thought of in this light. The difference between price and value is a valid one, and certainly this is an area where the actuary can make a contribution. However, I would caution against any broad adoption of utility theory. While this theory does look great on paper, I am sceptical whether many people know what their investment utility function is: differential; non-linear; or otherwise. Behavioural considerations are a key input into investment decisions; I am just not sure that utility theory is the best way to describe them.

Thus, the paper introduces actuaries to a new field of expertise. While there is some overlap with skills that the actuary has built up to now, for many of us there will be a step change required to achieve the higher levels of skills demanded by derivatives. The role of the actuary in this field is clear. It is one job generating numbers; it is wholly another to interpret them and to communicate the information in a clear, concise manner to clients. There is a saying: "quantitative work does not make things more right, just more convincing". We must be able to understand the complexity of derivatives.

Professor Clarkson's paper introduces us to a new methodology in the field of option pricing. The approach is more pragmatic than academic, and the author uses history to display weakness in current theory. The author recommends a model which is stated to be commonsense and free of the shackles of financial economics, with the Black-Scholes methodology coming in for some particularly damning treatment. As an investment practitioner, I admit to having some sympathy with the author's views on the weaknesses underlying Black-Scholes and related methodologies. However, I believe that he is incorrect to reject such approaches because of these drawbacks. Let me put things into context. When the Black-Scholes methodology was first introduced in the early 1970s, it introduced a pricing discipline that hitherto had not existed in the world of options. That it is still used nearly 25 years after its introduction is a major reason to believe that the approach is a robust one. The financial world does not accept every theory that the academics present. Certainly Black-Scholes has its holes, but it has provided a numerical framework within which the financial world has been content to operate. I think that most financial economists would agree that there are weaknesses inherent in the initial construction of the model, but countless improvements have been made to option pricing techniques since.

The author presents a new methodology, and as such it merits attention. My view as a practitioner is simple: if it helps make money, I will use it; but it should not be seen as replacing the robust theories that have developed over the years. My wish would be for the debate to discuss the value of more pragmatic-based approaches as compared to those of academia. My fear is that the language of Clarkson's paper will invite only tit-for-tat exchanges between two entrenched schools of thought, academia versus non-academia.

We should look to embrace a multitude of broad ideas as we endeavour to develop a marketable skill in the field of derivatives. At the moment we are standing still. We seem more intent in squabbling over who is right and who is wrong. This is a war from which there is only one clear loser — the actuarial profession.

Mr A. D. Smith: The Black-Scholes model, despite its known shortcomings, is still by far the most widely used option pricing model in the financial markets. We have before us two papers which suggest improvements and alternatives. One way of comparing models is to see how well they fit actual market prices. I have taken FT-SE option prices from the *Financial Times*, and calculated how well they fit the theories. I have not tried to estimate the parameters from fundamentals; instead, I have taken the best fit choice of parameters implied by current market prices alone. Of course, given any single option price, it is easy to choose parameters which fit it exactly. What I seek is an integrated model which can simultaneously explain option prices, both calls and puts, for a range of strikes.

The models that I have fitted are:

(1) simple Black-Scholes with the Garman-Kohlhagen dividend adjustment, as Mr Kemp gives in his Section 7.5;
(2) the cost of capital model described in Appendix B of Mr Kemp's paper, which I helped him develop in 1995; and
(3) the net call and put prices for Professor Clarkson's equilibrium model, following the net pricing methodology in his Section 6.2.

As I expected, the cost of capital model provided a substantially better empirical fit than Black-Scholes. For traded FT-SE options, the sum of squares typically reduces by a factor of between 5 and 10. The results of the dynamic equilibrium model were rather more surprising. I was unable to find any improvement over Black-Scholes. The best fit model was precisely the Black-Scholes special case. Professor Clarkson's three other parameters collapsed to trivial values. I tried different option maturities; different equity markets; I even tried long-dated over-the-counter put option prices from an investment bank. I always obtained the same conclusion: the best fit of the dynamic equilibrium model to option prices is the Black-Scholes special case.

The reasons for this are subtle. The first point relates to put-call parity. This is not just a theoretical construct; it can also be confirmed empirically. On the FT-SE this typically works to within one index point. Given transaction costs, this is sufficient to preclude arbitrage. We can use put-call parity to construct bonds and forwards out of options. If options are priced by discounted expected cash flow, then the synthetic bonds will yield the discount rate, and the forward price will be the mean of the distribution. Professor Clarkson's suggested generalisation, in ¶7.1.7, allowing the user to specify the mean and discount rate exogenously, does not help us in explaining option prices. These parameters inevitably collapsed to the forward price and bond yield respectively in the fitting process.

It now remains to explain why the log-harmonic, or more conventionally beta, distribution does not help us to fit option prices. Both authors cite the extensive evidence which has built up that, empirically, stockmarket returns have had sharper peaks and fatter tails than a fitted normal or log-normal distribution. This positive kurtosis is widely held to be a major factor driving the observed smile effect in option prices.

Professor Clarkson claims, in ¶8.2.1, that his own proposed convolution of normal and beta distributions generates the required peak and fat tails, and thus produces a smile effect. However, the reverse is the case. The assertion is apparently based on a two-point approximation to the compound distribution in ¶4.3.15. Such approximations can, in general, only fit the first three moments of a distribution, so could hardly be adequate for his application of comparing kurtosis. The claimed smile and skew in Professor Clarkson's ¶6.2.5 now deserve some explanation. We have to bear in mind that, in real life, options do satisfy put-call parity. The discount rate and forward prices used by traders in the implied volatility calculation will be those implied by the market. However, he has not used the values implied by his option prices, but has chosen some other, apparently arbitrary, inputs. He has devised his own private definition of implied volatility which, by coincidence, has a downward slope vaguely like implied volatilities calculated in the market. This is an artefact of inconsistent forward prices within the calculations, and has nothing to do with the log-harmonic distribution. The same result would have appeared with a log-normal distribution. However, using the accepted market definition of implied volatility, I find that the option prices in his Tables 2 and 4 generate a slight frown effect, which is almost never seen in market prices.

As Mr Kemp has recognised, alternative option pricing theories abound. These vary from complex mathematical constructions to empirical bodges invented by traders. Almost all generalise the Black-Scholes formula to incorporate market smile effects. We may reasonably expect these models to provide a better fit to observed prices than Black-Scholes does, simply because there are additional parameters to play with. This will happen irrespective of the merits or otherwise of the underlying theories. However, Professor Clarkson's theory, although allowing three extra parameters, does not improve the fit of Black-Scholes to market prices. It could be argued that such a comparison does an injustice to Professor Clarkson's approach. Presumably he believes that his method provides a rational assessment of option value. If markets are profoundly inefficient, as he claims, we would hardly expect them to reproduce his option prices. Paradoxically, the best empirical support he could expect would be for the predictions of his theory to be completely at variance with observed market prices. Curiously, that is exactly what I found.

Dr A. J. G. Cairns, F.F.A.: We have one paper which gives a balanced and well-informed view of derivatives, their uses and the mathematics underlying their pricing, and we have another paper which rejects everything that financial economics has to offer.

I turn first to Mr Kemp's paper. It is a paper which I think all actuaries should read thoroughly. I stress this point, because there are many points made in this paper which dispel many of the myths put about by Professor Clarkson and the supporters of his 'Actuaries are Best' philosophy. The main points which I draw out of Mr Kemp's paper are as follows:

(1) The market price of a derivative is the price at which market makers are prepared both to buy and to sell, subject to a bid offer spread. The formula in Professor Clarkson's paper only gives prices above which he is prepared to sell. More generally, Mr Kemp notes that utility theory can be used by individual investors to assess how they can maximise their expected utility by buying, selling or holding.

(2) In the real world the individual assumptions of the Black-Scholes model break down, as has already been pointed out. However, the underlying theory of financial economics can still be used to construct hedging strategies which reduce significantly the level of risk relative to a static strategy.

(3) In this world of imperfect hedging, it is necessary for banks and other issuers of derivatives to hold additional reserves or contingency margins against the risk of insolvency. Such reserves can be calculated by using the methods of value at risk described in the paper. The size of these reserves can be reduced significantly by the use of dynamic hedging strategies.

(4) Financial economists understand that the Black-Scholes model is not perfect, and they have developed models to take account of this, contrary to what Professor Clarkson would have us believe. For example, there are models in financial economics which explain the smile effect observed in share options.

I turn now to Professor Clarkson's paper.

The first error occurs in ¶1.1.2, where there seems to be some confusion between the definition of a premium and a reserve. A premium is made up of expected claims (suitably discounted) plus expenses plus a margin for profit. The margin for profit is not the same as the contingency margin against the risk of insolvency, which is what the author uses. The margin for profit ensures that, in the long run, the line of business will be profitable for shareholders. A contingency margin for the risk of insolvency is the extra money put in by shareholders — not the policyholder — with the aim that it will obviate any necessity for future injections of shareholders' funds. Furthermore, the profit margin in the premium will, in the long run, provide an adequate return to shareholders for having put up the contingency margin in the first place.

In the world of derivatives trading, sensible banks are the majority which hedge derivatives or find counter-parties for their risks. Such strategies give pay offs at maturity which are similar to Mr Kemp's Tables 1-5. The key point is that dynamic hedging significantly reduces a bank's risks. If a small margin for profits is added to an option price, then this ensures that the bank has a low risk and profitable business (even though these profits are unspectacular). It is an important point that we do not see derivatives operations going bankrupt month after month. The only banks which have failed, such as Barings, were those which did not employ appropriate hedging strategies.

In ¶1.1.3 the author defines his formula for option prices. This defines the price at which Professor Clarkson is prepared to sell an option. He is apparently not willing to buy options. If he is not willing to buy, then he is not a market-maker, and therefore his formula says nothing about what market prices should be. It only defines at what point he is prepared to enter the market. It is just as well for

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him that he is not prepared to make a market at these prices, since they do not satisfy put-call parity, as I will mention later.

There are two significant problems in the numerical example given in ¶1.1.6. The numbers just do not add up!

First, let us consider the expected pay offs at time 1:

$$\begin{aligned} q_c &= \text{expected payoff from a call} \\ &= \mathrm{E}[\max(S_1-X, 0)] \\ q_p &= \text{expected payoff from a put} \\ &= \mathrm{E}[\max(X-S_1, 0)] \\ q_c-q_p &= \mathrm{E}[S_1-X] \end{aligned}$$

but;

$$\begin{aligned} q_c-q_p &= 8.42 - 4.49 = 3.93 \\ \mathrm{E}[S_1-X] &= 95 \times 1.03 \times 1.05 - 100 = 2.74. \end{aligned}$$

Perhaps the author would like to clarify this discrepancy.

Second, let us compare the prices of the call and put options:

$$\begin{aligned} P_c &= \text{price of the call} \\ &= 8.68 \\ P_p &= \text{price of the put} \\ &= 4.63. \end{aligned}$$

According to put-call parity we should have:

$$P_c = S_0 - Xe^{-r^*} + P_p$$

where r^* is the risk-free rate of interest over the next year. r^* is the only unknown quantity, and it is simple to show that to have this equality we must have $r^* = 9.49\%$. This is a rate of interest which is not just in excess of the expected rate of growth for the index, which is 3%, but it is also significantly higher than the expected return on that marvellous, but risky asset, which is going to pay 5% more than the index.

Much more likely is that the risk-free rate is 3%. It cannot be less than 3%, because otherwise Professor Clarkson should not have been investing in the risky index at all. Suppose, then, that we buy one put option, go short in cash and buy one share. This costs us just 2.54. If we just hold this portfolio until time 1, which is a static hedge, then we will precisely replicate the pay-off on a call option. So why should we be prepared to buy a call option at 8.68?

These problems, I believe, arise because of the monumental leaps of good old actuarial judgement used in this paper. Such subjectivity has left us with a model for option prices which is internally inconsistent.

In Section 1 the author criticises the Black-Scholes model for being too simple. In ¶7.1.10 he states that the methodology is 'complex in the extreme'. Which is it to be? These do not sound like the words of someone who has taken the time to examine financial economics properly. I therefore urge actuaries to make up their own minds by acquainting themselves with the Black-Scholes model. They will find that it is not the slightest bit complex. In fact, it is very easy to understand if it is taught in the correct way — for example as the limit of a binomial lattice, as described in Mr Kemp's paper. They will also find that, when presented in this way, it makes a lot of common sense — something which I find sadly lacking in Professor Clarkson's model.

Mr C. J. Exley, F.I.A.: We need to establish the purpose of actuarial involvement in derivatives. If Professor Clarkson wants to take proprietary positions in the derivatives market on his own account,

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or if he can persuade some fund management firm or investment bank to back him, then the proof of the pudding will be in the eating, and I wish him every success in trying to beat the markets, and especially in trying to beat put-call parity consistently. On the other hand, most actuaries will probably have more limited ambitions — for example, they may have a liability with characteristics very similar to traded derivatives. If we seek to price this liability, then it is not really very helpful to start by arguing that the prices of all similar traded instruments are all wrong and proceed to construct a completely different framework for pricing both the market and the liability. If this actuarial answer is not tested in the market, then I am afraid that it will appear indistinguishable from a pure guess.

If the liability is traded, then sowing intellectual seeds of doubt about market prices is no comfort if the actuary subsequently reaps the whirlwind of arbitrage. Whether or not the market prices are right, these are the ruling prices at which arbitrageurs can replicate the liability. The simplest example of this is a trivial derivative known as a gilt. This is a forward contract delivering cash at some future date. If a portfolio of these derivatives replicates an annuity certain, then any price other than the price of the portfolio is open to arbitrage. This applies whether or not Professor Clarkson thinks that the prices of these derivatives are right or wrong.

The financial economists' approach to derivative pricing is described in Mr Kemp's paper. It is firmly in the positivist camp. It probably understates the beauty of the mathematics to say that, at the very least, it provides a framework for calibrating pricing models against traded instruments. For example, a term structure model used to price limited price indexation could be calibrated against swaptions and caps in the market. However, the value of such a framework, when compared with Professor Clarkson's approach, should not be underestimated. Crucially, the approach described by Mr Kemp ties derivative pricing very firmly to an associated hedge portfolio. As a member of the Institute, I cannot think of a more perfect example of our motto *Certum ex Incertis* than that provided by the Black-Scholes hedging methodology. Incidentally, I think that the basic hedging idea is so simple that a child could understand it, especially when described by a binomial tree.

It is a matter of great sadness to me that the profession seems rather churlish in not universally acknowledging that the followers of Black-Scholes have stolen a march on us intellectually. I think that Mr Kemp's paper goes a long way towards such unreserved acknowledgement, and I would urge members of the profession to follow this lead rather than being distracted by Professor Clarkson's alternative theory.

Mr J. M. Pemberton, F.I.A.: Let me contrast two much simpler investments than were used by Mr. Kemp in his introductory remarks. This example was used by Professor Wilkie in the discussion of the paper 'Allowing for Asset, Liability and Business Risk in the Valuation of a Life Office', by S. J. B. Mehta (*J.I.A.* **119**, 449).

The first investment comprises two entitlements: an entitlement to one if a coin is tossed and it lands up heads; and a further entitlement to one if a second coin is tossed and it lands up tails. The second investment is the same, except that the coin toss is the same for both entitlements — now you receive one whether the coin lands up heads or tails. You can see that in the first investment you might get zero, one or two as a pay-off, with probabilities $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$ respectively. Under the second, you always get one.

The two investments look very similar because, in both cases, you have an entitlement to two unit cash flows, each with a probability of a half, but the only difference is that in the second case they are causally linked. There is a difference between these two types of investments. Indeed, the work that Kahneman & Tversky (1979) have done demonstrates that, in practice, people do respond differently to those propositions. They value the one which is risky as less than the one which is not risky.

When Mr Kemp, in his introductory example, has brought his partial option cash flows together to synthesise a risk-free bond, what he has done is to put a causal link between these cash flows in a way which insists that the cash flow always relates to the same outcome event. Without that causal link we have an entitlement to between zero and N with a rather skewed distribution, and, in this case, we may have additivity. If you bring them together in a way which always gives you one, the risk-free bond which Mr Kemp was talking about, you do not have additivity, in my view, for the same reason that you do not value my two investments as the same. We need to go back and question the original proposition that the sum of the value of two cash flows is always the same. We need to look at the question of the no arbitrage proposition in more detail.

Mr Kemp seems to imply that you do not need strong constraints on assumptions about the beliefs of investors to underpin the mathematical ideas in his paper; but you do need the homothetic assumptions. You do need investors to agree on what are the probabilities of the outcomes of the prices of all the investments in the market. I do not believe that we should brush these aside as irrelevant.

The two papers reflect very well the underlying traditions from which they come. Mr Kemp's represents idealisation mathematics, which is characterised by universal assumptions and abstract mathematics. Professor Clarkson's paper characterises an empirical approach. It is characterised by a concern for the facts and models based on the facts. I see that as the traditional actuarial approach.

I am concerned that Mr Kemp's paper does not manage to translate between financial economic terms and actuarial terms, but rather is a presentation of financial economic methods. I am concerned with the lack of focus on facts that is entailed within the idealisation mathematics approach adopted.

In Professor Clarkson's paper, I thought there was a helpful and thorough review of many of the problems of the idealisation mathematics approach. One powerful part of this paper is the parallel with mortality and the history of the use of mortality statistics by this profession. In the early days we relied on crude models, and crude models played an important role in allowing us to develop an understanding of mortality. As we developed, and as the subject grew, what became increasingly important was to develop those models in an empirical fashion so that they developed a respect for the facts of mortality. I see a close parallel between that position and the position which we are in now, in developing the valuation of options. Mathematical models have played an important part and continue to do so. They are simple to use, they are crude, but they are often a good rule of thumb. We now need to move towards refining them with the use of empirical facts.

Within the actuarial profession there is a realisation beginning to dawn that the empiricist approach is an important one, and we need to start taking seriously again the need to bring facts within our models much more carefully. The pendulum started to swing towards idealisation mathematics with innovations in economics in the 1960s and the 1970s. This profession caught up with those moves to some extent in the 1980s and the early 1990s. I think that the pendulum has now swung too far. In my view we are now set to see the re-emergence of empirical models, and the paper from Professor Clarkson confirms that we are starting to travel in that direction. As we see the pendulum swing back towards empiricism, I think that we should anticipate an exciting time for the actuarial profession. Whereas we have followed behind in idealisation mathematics, in the rise of empiricism we have the opportunity to lead. We have precisely the skills and the methods required to make a major contribution to developments in the wider fields.

When future historians look back, I believe that they will see Mr Kemp's paper as marking the high point of confidence in idealisation mathematics, and Professor Clarkson's paper as one of the opening developments in a re-birth of confidence in the actuarial profession in actuarial science and empirical methods. I have no doubt that this change in direction will be marked by more than usually lively debate. It is essential that as broad a group as possible within the profession take part in it, and it is important that this debate continues. I share the sentiments of Mr Jones that it is equally important that we begin to explore constructively the differences thrown up by this debate in a way that will help us to resolve them.

Reference

KAHNEMAN, D. & TVERSKY, A. (1979). Prospect theory: an analysis of decision-making. *Econometrica*, **47**, 263–291.

Mr S. J. B. Mehta, F.I.A.: Mr Kemp should be congratulated on pulling together so many diverse

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strands of thinking into such a practical single reference document. Some of the numerical examples give special insight into the ways in which the assumptions underlying the Black-Scholes method can be relaxed. Having been involved with investment markets for many years and having studied the available evidence, I would strongly agree with Mr Kemp that substantial arbitrage opportunities are few and far between.

I would like to comment on Professor Clarkson's observation that markets are inefficient, because of some adverse market reaction to recent comments from Mr Alan Greenspan. I would argue that the degree and speed of response to the comments of, perhaps, the most influential institution in the world could be taken as an indication of a high degree of efficiency.

As someone who now works for an investment bank, I make a plea for the profession to distance itself from actuarial theories of option pricing. I suggest that it serves no purpose whatever to put forward new theories so much at odds with market realities and available evidence. Rather, we should recognise the brilliance of the concepts underlying the Black-Scholes framework and recognise that all subsequent credible work has built on this framework, rather than found it wanting. Fundamental to modern financial theory and Black-Scholes is the law of one price, that two similar cash flows have the same values and will trade at the same price. Black & Scholes then showed that arbitrage arguments drive option prices towards the Black-Scholes values, given assumptions that are approximately, but not exactly, valid. Subsequent work has relaxed these assumptions one by one, and still finds the Black-Scholes formula to give a useful benchmark, not just for equity options, but also in the commodities, foreign exchange, interest rate and other markets. Recent years have seen the theory extended to applications in, for example, long-term capital projects, the value of flexibility in decision making and in the life and pensions industries. I suggest that there is no alternative robust approach. One of the main reasons why financial economists are held in such high regard is the extensive empirical work and testing of their theories. There are literally thousands of research programmes seeking to test empirically some of these theories. Sadly, this rigorous approach is not applied to the so-called actuarial approach of option pricing, as Mr Smith demonstrated.

In defending the Black-Scholes framework, I do not believe that I am standing against scientific progress — rather, those who attempt to discredit theories so clearly approximately true risk relegating the profession to the sidelines of modern financial thinking and development.

Dr M. W. Baxter (a visitor): Despite being one of the technicians whose name has been taken in vain or held up as part of the problem of this issue, I will try to keep myself non-technical in these remarks.

Firstly, I agree with Professor Clarkson. I agree that you should price an option by taking an expectation and working out its expected value. I also agree that you should remember to discount for the time value of money, as he does. To receive £1 later is not as good as receiving £1 now. I also agree that you should be careful to make sure that you charge enough to cover your profits, your expenses and contingencies and have a little bit for caution in case something goes wrong with all of the above. All that is correct, and in all that I agree with him.

The problem arises, as some speakers have pointed out, when some of the prices he gets do not work in the sense that some things are too expensive, others are too cheap. Other speakers brought out the point that the price of forwards is wrong and put-call parity is wrong. Selling a forward is remarkably easy, because, if you promise to deliver a share later, instead of working out its expected worth later and getting in the right amount of money and the reserves ready in time to buy it later, you can just buy it now. Buy it now, put it in a drawer and wait. At the end of the term take it out of the drawer; it is there; you promised to deliver it; you hand it over. You are exposed to no risk. That is what we call a static hedge. You have managed to avoid market risk completely in the forward share transaction just by buying now at the current market price. So long as you make a note of how much you pay for it and how much that debit is going to mount up to, you have the price that you should sell the forward for. That has to be right.

So where has Professor Clarkson gone wrong? I am sure many actuaries are wondering, having heard about these two approaches, what the essential difference is. The difference arises because of a fundamental difference in the way expectations are used in mathematical finance and investment banking, and how they are used in the actuarial profession. There is a clue to this in almost the first line of Professor Clarkson's example on calls in ¶1.1.3, where he considers writing *N* identical options. This is very much in tune with actuarial theory, where you consider *N* identical lives which behave independently and write assurances or annuities on them. What you are going to do is average out the behaviour over all those separate lives, over all those separate options. You are assuming that, at the end of the day, the average value when you divide by *N* will be the expected value. This is the strong law of large numbers. If you take enough independent and identical things and average them out, you find the mean. If you charge the mean plus profit, you are going to be all right. That is the only way that you can do it. However, there are problems with finance taking this approach across to option pricing. Firstly, you are not writing *N* identical options. You may be writing one. If you average out one, you do not get anything other than the original random number. Nothing is going to be made any better. Much options trading is in over-the-counter specialised products. You sell only one and you sell it only once. The law of averages is not going to help one bit. Further, even if you do write many options on the same market, they will certainly not be independent. This method of averaging out independent risks is inferior to the method of hedging which Professor Clarkson has avoided almost all mention of except occasionally to bring it up as a problem. I beg you to see hedging not as a problem, but as an opportunity. With the simple forward case the hedge was simply to put the stock in a drawer. It was a simple hedge; it was a static hedge; it could not go wrong. So long as your drawer is secure, then it will still be there, which introduces the different matter of risk control. In more complicated situations the static hedge is not enough and you need to start changing your hedge with time and create a dynamic hedge. That is where all the technicalities and mathematicians like me have to come in.

I now describe another case, slightly more complicated than the forward, to show why hedging is something you should try to do. Say you sell a very simple European call, the right to buy (but not the obligation) a share. Say that it is struck at the current price. If you do not hedge at all; if you just take the money, cross your fingers and hope and watch the screen, you might win. However, you might lose, and that could be a bit of a problem. Say, instead of hoping, you put on a very simple static hedge, buying just half a share in the stock. If the option is for 100 units, you buy 50 units of the stock. If you work it out, this does not change your expectation at all, but it reduces your risk by a factor of 2. There is no transaction cost, only the actual stock purchase, but you have halved your risk. If you believe that that works, you should start thinking whether you can get the risk down a little further. Maybe you might change the hedge half way through the contract. Maybe you might change it twice throughout the life of the contract. You could work out what was best for you through your trading system and see what the transaction costs were. I would certainly prefer to hedge than not to hedge. The expectation is the same, but every time you make your hedge a little bit more sophisticated, you remove some risk. At the end of the day, you hope to be approaching the idealised Black-Scholes model, where you are completely free of market risk.

In practice, these are sophisticated markets and complicated products. You are never going to be fully protected, which is why, as Professor Clarkson wisely suggests, you should charge extra. The more uncertain you are about model specification or about model parameters, the more insurance premium you should charge in case things go slightly wrong. However, you should certainly try to hedge. The price that you should charge is roughly the likely price the hedge will be plus some more. The extra should be more the more uncertain you are about whether it is going to work. This is what investment banks do in practice. If the product they are selling is very close to existing traded market products and the hedge is almost exact, then they add very little. If it is something that has never been traded before, they calculate the price and then add £½m. Over a 5-year contract a bit of that will run down, but it should still get you through to positive profit at the end of the day. That is why the expectation that Mr Kemp brings out, which, in essence, is the expectation to allow you to hedge, is the one that I recommend you look for.

The actuarial profession is left with a choice of which model to believe in. Here I believe there is a more genuine choice. There are opportunities for actuaries in a variety of areas, one of them, moving in Mr Kemp's direction, would be to start to get involved in the risk control side of investment banks and the managing of their books. They need much good advice about how to run a

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book with proper margins and with due diligence, care and a questioning of assumptions. These are things that the traders do not do. Traders are just looking for the next deal. There is a great opportunity for the profession to come in and help investment banks with risk control. When you do that, you have to be arbitrage pricing; you have to be hedging; and you have to be following admittedly complex mathematics.

Most actuaries will not end up in a niche like that. Most actuaries who encounter derivatives will be managing a fund or looking after a company that owns derivatives. That is, they meet derivatives as a user rather than as a supplier. Here the calculations are different. As a user, which I think is the direction that Professor Clarkson is really coming from, you do not need to quote two-way prices. You do not need to trade if you do not want to. You just look at the price in the market, look at the value you think the instrument has, and decide whether you want to buy or not. In that sense you are in a position where you have to think more about why you want this product and whether it is going to off-set a natural hedge that you have or a natural exposure that has occurred in your business. You have to start thinking more in value terms.

In risk control you have to think about price, because money is what matters. In investment you have to think about value, because that is what you are trying to maximise. There are questions for the actuarial profession to answer that only you can make up your minds about, but there are certainly exciting opportunities for those who want to get involved.

Dr A. S. Macdonald, F.F.A.: The background to this discussion can be simply stated: the actuarial profession has traditionally approached risk from one particular point of view, a reserving paradigm in which capital is allocated to collectives of independent risks to cover adverse contingencies with some level of probability. Our education system equips actuaries to handle this paradigm, and the approach set out in ¶1.1.3 of Professor Clarkson's paper is a typical and perfectly good example of this line of thought. More recently a different paradigm has appeared, a hedging paradigm, in which risky assets are combined into less risky portfolios with consequences for pricing and reserving. Our education system does not yet equip actuaries to handle this paradigm, and thus we have a problem to which these papers offer two very different solutions.

Mr Kemp invites us, not to choose between them, but to educate ourselves in the hedging paradigm and add it to our tool kit. The problem with this is that we would need to introduce some more advanced mathematics; not more advanced perhaps, in the light of all the examples that Professor Clarkson gave us, than, say, probability theory would have looked to the actuaries of 150 years ago, but more advanced than anything we currently require. The easy option, therefore, would be to find reasons not to do this, which is Professor Clarkson's purpose. His paper can be summed up in one line: 'New Maths, New Danger'.

Reasons to follow Professor Clarkson and to turn aside from modern financial mathematics spring from two sources. One is all too human; we do not like to be told that our expertise is less than complete in a subject so close to home. Financial economists are trespassers, pure and simple. We must boot them off our patch. Thus their work is false, dangerous, unsound, and, to use the most pejorative word any actuary knows, academic. In the U.K. this chimes with the habit, not confined to this profession, of regarding theory and practice as adversaries rather than allies. Professor Clarkson's paper is full of these sentiments, and they do him no credit. The second possible reason to agree with Professor Clarkson is that he might be making some valid points. This deserves to be taken more seriously. Two factors need to be considered: the nature of a mathematical model and the role of judgement in the application of a mathematical model, which I know is a subject of great interest to Professor Clarkson and to Mr Pemberton.

Any mathematical model rests on assumptions, and we draw conclusions within the model by working out the mathematics. At all stages we must be reasonably fluent in the language of mathematics. It is precise, unambiguous and, most important, it lays bare any simplifications which are being made where waffly prose will serve to obscure them. Anyone who says that a mathematical formulation obscures the assumptions is talking nonsense. That is the exact opposite of the truth. Our mathematical conclusions are not statements about reality. They relate to the model world. To interpret them and to use them we need to apply judgement, and it is at this stage that statistical procedures are most helpful. Again, anyone who says that the use of statistics negates judgement can only be talking out of a profound ignorance of statistics.

As with any language, be it mathematics, statistics or English, it is not possible to assess works written in that language without first learning it to some degree. This, I agree, presents a genuine problem to the uncommitted actuary; how can one decide if it is worth the considerable effort of going back and dragging out the old text books (and even some new text books) without having gone to that effort in the first place. It is a Catch-22 situation. One thing is sure; the usefulness of a piece of mathematics has absolutely nothing to do with how complex it appears to those who have not learned it. The use of stochastic calculus cannot be judged by whether or not actuaries or, as Professor Clarkson says at one point, typical business school graduates will understand it. If it is, indeed, the case that stochastic calculus is too hard for actuaries, then there is only one thing for actuaries to do: get out of the way. There is no law of nature that says that actuaries must always and forever be more expert in financial matters than anyone else; but there is such a thing as the Memorandum on Professional Conduct.

The effect of new ideas, new mathematics, new models, is to shift the area in which it is most necessary to exercise judgement. Take, for example, a forward contract. Without any knowledge of arbitrage-free models, the problem might quite reasonably be formulated as a discounted expected value based on the true probabilities, and judgement will be needed in assessing the distribution of the underlying rate and in choosing a discount rate. With knowledge of arbitrage-free models, the problem shifts to estimating the risk-free rate of return or some approximation to it. Judgement is still needed, but within a different paradigm. It is easy to imagine the distress that this could cause to members of a profession whose expertise, education, history and even self-esteem lay in the application of judgement within the old paradigm.

Professor Clarkson is chiefly critical of the assumptions underlying the original Black-Scholes model. He says they are unrealistic. Let us agree with him about that, but let us be cautious about dismissing a model for its assumptions alone. All models are false. As an example of a model based on assumptions at least as far removed from reality, let us take the net premium valuation of a life office. This is part of a mathematical model whose assumptions include constant and known interest; no lapses; no new business; no expenses; no bonuses; and people dying in fractions as dictated by the life table. Here is a question: how is it that actuaries have been able to use such an absurd model with sufficient success to steer the life insurance industry for 200 years? It has had its critics, but why do we not hear cries of 'unsound' and 'dangerous'? Why are non-actuaries not writing papers saying that actuaries believe in unrealistic models and should not be allowed to manage life offices?

It is a pity that Mr Kemp was not allowed to present his excellent paper unhindered. It meets, very well, the needs of the actuary seeking a way into this subject. It shows clearly, especially in Sections 9 onwards, where the interface between the hedging paradigm and the reserving paradigm lies. It gives the lie to some of the myths put about by Professor Clarkson, and, most important, it shows us how our education system needs to change, and the sooner the better.

Professor A. D. Wilkie, F.F.A., F.I.A.: Mr Jones mentioned that utility functions were not very useful. However, many of the results of financial economists are based simply on the assumption that the utility function is a risk-averse one, and they do not say anything about the actual numerical values or the particular function involved. One can obtain many useful results from the concept that people have utility functions and are risk averse.

There is going to be a very large experiment taking place in the British gilts market later this year when many gilts are sub-divided into strips. I can see three things that could happen. One is that nobody converts coupon gilts into strips at all because they like coupon gilts; the other extreme is that all the coupon gilts are converted wholly into strips because nobody likes coupon gilts. The other option is that some of the issues are converted into strips and some remain as coupon gilts, in which case you can be sure that the sum of the value of the strips will very closely equal the value of the gilts plus or minus transaction costs.

Dr Macdonald made some comments about actuaries not being familiar with hedging. However immunisation is precisely to do with hedging, and actuaries have been quite familiar with that. Life

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offices selling non-profit annuities do look closely at how they are going to hedge by looking at the prices of gilts in the market. Annuity rates change frequently in accordance with gilt prices. We are entirely familiar with quoting life assurance premiums, not on the basis of the interest rates that we think ought to be in the market, but actually looking at the market and seeing what is there.

I am interested in time series modelling and I find Professor Clarkson's mathematical model potentially interesting, but, when I look at it closely, it is totally incoherent. It cannot possibly represent real share price movements at all. Either share prices change by plus and minus 15% every second or every δt, or they change with some sort of frequency which you know something about. The most generous way that I can make sense of it, that one would like to see a second order time series model which is the equivalent of his harmonic motion, which if you remember Newton's dot notation is driven by $\ddot{x} = -ax$. The acceleration is towards the centre each time, causing things to tick backwards and forwards. Put an error term into it, and you have something that might possibly be a coherent time series. However, the idea that there is periodic motion in the share market is not true. There is no empirical evidence for it whatsoever. So, apart from all the other comments, I do not find the mathematical contributions in Professor Clarkson's paper of the slightest help to me.

Professor R. S. Clarkson, F.F.A., (replying): Mr Kemp hopes that mathematics will continue to be used by actuaries. I am not against relevant mathematics. However, I believe that the pendulum has swung much too far towards elaborate mathematics and it is likely to swing back. Another actuary who has come to similar conclusions is Mr Booth, but I shall not pre-empt his written reply to this discussion.

One quick answer to Dr Cairns: is Black–Scholes simple or complex? The standard formula involving normal distributions and natural logarithms looks very complicated. However, in functional terms it is similar to Gompertz Law, a one-parameter law of mortality. That is why I refer to it as simple.

Mr Exley suggested that the Black–Scholes supporters have stolen a march on the actuarial profession. At the seminar that I gave earlier today I suggested that, if you took Black–Scholes as a narrow mathematical model, it was a great step forward at the time, but my main complaint is that people have forgotten to look at the simplifying assumptions. The market practitioners have moved on and make a lot of *ad hoc* adjustments, but there is no coherent new theory that people can develop. I am hoping that we can find an actuarially-based framework that will help us move forward.

I was interested in the comments of Dr Baxter about the importance of hedging. I have covered these points in Section 5.4. I was delighted to hear him suggest that, perhaps, we should move back towards value.

My main disagreement with Dr Macdonald is that he says that mathematical models lay bare the assumptions. In the textbook, *Introduction to Stochastic Calculus Applied to Finance* (Lamberton & Lapeyre, 1996), there is no discussion whatsoever on risk or uncertainty. On page 1 there is reference to the 'riskless asset'; a 'risky asset' is then defined, unhelpfully, as one which is not the 'riskless asset'.

Dr Macdonald said that, as actuaries, we may have to get out of the way of stochastic calculus. I think that it is the other way round, in that it is stochastic calculus that should be thrown out by actuaries. I came into the actuarial profession as one where we are seen by the public at large as experts in managing financial risk in important real world applications. To use esoteric mathematics where you do not even define risk or uncertainty strikes me as unsound in the extreme, and unworthy of the actuarial profession.

The President (Mr P. H. Grace): We have had a very lively discussion, mainly on pricing issues, for which I should like to thank all who participated. There would be no discussion without the efforts of the authors of the two papers. Mr Kemp's paper gave us an excellent overview of derivatives, the markets and their uses. Professor Clarkson's paper dealt with the issue of pricing on which there are clearly differing views. Whether this is the last time we hear the subject debated at the Faculty remains to be seen. I ask you to join me in thanking both Mr Kemp and Professor Clarkson for their stimulating papers.

WRITTEN CONTRIBUTIONS

Mr P. M. Booth, F.I.A.: I welcome Mr Kemp's paper for presenting such a clear exposition of option pricing theory and, most importantly, for discussing the uses to which options can be put by institutional investors. It is often said that actuaries are not sufficiently familiar with option pricing mathematics. This may be true. However, there is also an unfamiliarity with market terminology and with the uses to which options can be put. Options can play a major part in financial risk management for institutions with actuarial liabilities. It is very difficult to find anywhere in the financial or actuarial literature a coherent analysis of the uses to which options can be put. Mr Kemp is to be congratulated for providing that.

At the meeting, there was more controversy about Professor Clarkson's paper. However, I believe that it also provides an important contribution. I come to actuarial science from the direction of economics. I sometimes get worried about people in the actuarial profession (which provides its members with little formal training in economics) who speak in dogmatic terms about financial modelling. The philosophy of economic modelling has come on a long way in the last 45 years (for example, econometrics has been created and, to a large extent, discredited in that time), yet many of these developments seem to be ignored by some members of the profession. So my first point is that we should not make dogmatic critical statements about the principles on which the Clarkson paper is based, because, as a profession, we do not specialise in the field of economic and financial model building and we have no special expertise in the subject area of economics in general.

I do not wish to enter the debate started by Mr Pemberton, at a Staple Inn Actuarial Society meeting, as to whether the Black–Scholes formula has any use. I am prepared to accept that it has. However, consider an institution with a compex risk profile buying long-term options. How might it assess the price it is willing to pay for (as opposed to the market value of) the options, if the assumptions underlying the Black–Scholes formula do not hold? One approach is to say that we must adapt and relax the assumptions behind option pricing mathematics. This may be possible. However, it is wrong to say that this must be the only approach. It seems to be the approach of (as one actuary put it to me) somebody who only has a hammer and believes that every problem is a nail. People who have mathematics as their only tool for analysing financial problems can usefully investigate how mathematical pricing formulae can be developed. However, it is not mistaken nor mathematically ignorant to say that a whole different approach might be necessary, and that we might need to look again at the financial and economic framework and come up with a solution for valuation which is not inconsistent with the economic realities.

It is easy, as a mathematician, to forget the true nature of the underlying economic processes when developing mathematical tools. It is particularly easy to make that mistake with problems where complex mathematical methods have been developed, as is the case with option pricing. The development of the mathematical tool (the Black & Scholes approach) has the tendency of moving the problem out of its legitimate paradigm (economics) into another paradigm (mathematics). The whole point of the analysis can then be lost. There is then a concentration of pursuing mathematical analysis by relaxing the trivial assumptions, and making the mathematics more complex, whilst ignoring the issues which really matter because relaxation of assumptions pertaining to these issues would render the problem mathematically intractable. This takes me back to hammers and nails. If an economic problem has features which make it not amenable to mathematical analysis, it is just hard luck. It is a serious academic error to shape the problem so that mathematical analysis is possible (unless the necessary simplifying assumptions do not matter too much). It is vital to remember that economic problems are not simply more complex versions of other types of problems which are analysed using mathematics; this was the disastrous mistake of the great communist economists, such as Oskar Lange, who was honoured by mathematicians and statisticians in the west for his grave errors in economics. Economic problems are different types of problems.

Of course, it is not possible for actuaries to be experts in everything. Some will be very good at developing mathematical tools to solve complex pricing problems, given a particular financial model. Others may have a better understanding of the underlying economics of the problem, but not be well equipped to develop the mathematical tools (if indeed they can be developed). However, those who specialise in the mathematics will not fall into any traps if they remember just one thing. In a discussion of the philosophy of jurisprudence, I have seen it remarked that we "develop abstractions as a way of coping with our ignorance". The word abstract has a different meaning in jurisprudence from that in financial modelling. However, the same profound observation is important. The abstract financial model and the related mathematical tools have been developed because we do not know how to take all factors into account. Indeed, the philosophy of economics tells us that the only thing we do know is that we will never know how to take all factors into account. It is most useful that we have found a tool (Black–Scholes) which helps us to take many factors into account. However, it is an imperfect tool, although it may be the best that we have at the moment. It is reasonable for some people to invest effort in not just developing elaborations of the existing tool, but in developing different types of models which may help us cope with our ignorance better. If we do not do that, others will. If we take other approaches to analysis, we might find, that, one day, a spontaneous breakthrough (in economics, finance or in mathematics) will make the problem amenable to the use of mathematical tools within the new framework. Then the neat analytical processes, required by mathematicians, could be used in the new framework. However, experience tells us that we should not look in the most obvious place for that breakthrough. In the meantime these other approaches may have to employ approximate methods which do not have the analytical rigour of Black & Scholes, but which capture the underlying economic realities better. This is not to say that anybody has yet found a better framework: once again, I do not wish to comment on the details of Professor Clarkson's paper. However, we should welcome research which considers the option pricing problem in a different paradigm and not be instinctively hostile to it.

Professor P. P. Boyle, F.I.A.: The modern theory of derivative pricing started with the seminal Black & Scholes paper. To many observers this theory represents the greatest intellectual achievement in investment science in the 20th century. As Robert Merton has noted, this theory is both beautiful and practical. It provides the foundations for most of the trading and hedging models of large financial institutions all over the world. Some of the most advanced research in this area is now conducted within the banks, blurring the distinction between theoretical and practical. This is an area that should be of interest to actuaries: it is encouraging to see the profession becoming more involved in the whole area.

The two papers take very different perspectives. Mr Kemp's paper provides a survey of existing models. Professor Clarkson challenges the orthodox view and presents an alternative approach to the Black–Scholes model. This is termed an actuarial approach, since it is patterned after the traditional approach to the pricing of life insurance products. Of the two papers, Professor Clarkson's is the more controversial, and the purpose of my remarks is to provide a perspective for his approach.

There are three differences between the Clarkson model and the Black–Scholes model, but only one of them is fundamental. These differences are:

(1) *The basic valuation paradigm.* The Black–Scholes model is based on the no arbitrage principle, which is discussed in Mr Kemp's paper. The Clarkson model is based on discounting the expected value of the option payout. He uses real world probabilities to compute the expected value.

(2) *Inclusion of transactions costs and credit risk.* The Black–Scholes model does not include these frictions in its pristine form, but they can be, and have been, incorporated with some additional complexity.

(3) *The distribution of returns.* The original Black–Scholes model was based on log-normally distributed returns, but the approach can be used with other assumptions. Professor Clarkson uses a specific assumption for stock return that has more degrees of freedom than the log-normal.

Of these three, the only important difference is the first — the basic valuation paradigm. The Black–Scholes approach has already been extended to incorporate the frictions and flexibility mentioned as the second and third items.

Professor Clarkson's idea of computing the expected payout under the real world probabilities is

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not unreasonable. In fact this is precisely how academic researchers started to tackle the option pricing problem in the 1960s. Papers from this era by Sprenkle, Boness and Samuelson reflect this approach. These authors priced the option by discounting its expected payout based on realistic probabilities. The issue they struggled with was how to determine the appropriate rate to use to discount the expected payout. There was no theoretical framework to help them with this task. If one uses the expected rate of return on the underlying asset, the resulting option formulae violate put-call parity. Professor Clarkson's suggestion to use r, the return on the assets of the financial institution, does not solve the problem; in fact it creates more problems. He assumes that the financial institution writing the option collects the premiums and invests them in the assets of the company. This does not allow for the way that options are handled and traded in the real world.

The way out of the theoretical impasse was provided by Black & Scholes in the early 1970s. They showed that the option payout could be replicated with a portfolio of the underlying asset and a riskless asset. The investment strategy, together with the no arbitrage principle, furnished the pricing model. The relevant interest rate is the risk-free rate, and uses the risk-neutral distribution to compute the expected value.

In this contribution I have attempted to discuss the key issues which separate the Black–Scholes approach from Professor Clarkson's. In this respect I have tried to follow the advice contained in the quotation at the start of the paper.

> "Genuine progress never consists in a purely formal exposition, but always in the discovery of the guiding ideas which underlie any proof."

In summary, the main idea in Professor Clarkson's model has already been introduced in several papers in the 1960s. The Black–Scholes approach, in one brilliant stroke, showed us how to overcome the fundamental difficulty with those papers.

Dr A. J. G. Cairns, F.F.A.: On reading the paper by Professor Clarkson one might believe that the world of financial economics has not progressed since the early work of Markovitz and Black & Scholes.

In Section 8.3 of Mr Kemp's paper he lists how the assumptions contained in the Black–Scholes model might break down; something which is central to Professor Clarkson's rejection of the subject:

— *Markets are not arbitrage free.* This is very much open to argument, but it is *prudent* to assume that they are arbitrage free.

— *Markets can jump.* Financial economics has taken account of this: for example Smith (1995, 1996), and in work by Geman and others on the pricing of catastrophe futures. Furthermore Kemp points out, in ¶6.5.8 of his paper, that the best hedgers were able to reduce substantially the effects of the October 1987 crash.

— *Future volatility is uncertain.* Again there are papers in the financial economics literature which allow for this: for example, Longstaff & Schwartz (1992) use stochastic volatility in the pricing of bond options.

— *The market is not frictionless.* I refer the reader to the works of Davis and others, for instance Davis, Panas & Zariphopoulou (1993), which consider the optimal pricing and hedging of derivatives in the presence of transaction costs.

Professor Clarkson either ignores or is unaware of these developments which have arisen over the last 20 years. Instead, he uses his superior actuarial judgement to come up with a formula which appears to be based more on hand waving than on solid mathematics.

In ¶¶1.3.5 and 1.3.7 he quotes Föllmer (1991) and Geman & Ané (1996), who have questioned the validity of the diffusion process assumption contained in the Black–Scholes model. Föllmer and Geman are two of the foremost proponents of financial economics and, as far as I am aware, they are still so (despite their amazing revelations)! The result of their work is that many financial economists are considering alternatives which do not throw away the work of Black & Scholes, but build on it.

In ¶1.5.1 there is a misleading error. I quote:

"In the Black–Scholes world it is assumed, not only that risk is equivalent to the variance of return (which is furthermore assumed to be constant over time), but also that in some miraculous way, it can be 'diversified away' to create a 'risk-free' asset."

Now let me state some facts about the Black–Scholes model:
— The model is an approximation to reality; all models are (including Professor Clarkson's), and financial economists know this and understand this.
— *If* the model is true, then:
  — there is a hedging strategy which uses the underlying asset and the risk-free money market, and which produces, without risk of any sort, the correct payoff; and
  — at no point does the Black–Scholes model use any definition of risk, variance of return or otherwise (contrary to Professor Clarkson's claim).

It is only when we go beyond Black & Scholes and introduce jumps and friction that measures of risk become relevant, and I am quite happy to consider downside risk as a good measure. Within such a model the measure of risk can be used to construct a hedging strategy to minimise the expected risk; whatever measure is used. The author, on the other hand, clearly does not believe in hedging, and so his strategy will:
(a) be much more risky (even on his own measure); and
(b) as a consequence will have to charge rather higher prices than we see in the market to make up for this extra risk.

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Mr S. J. Green, F.I.A.: My views on Mr Kemp's comprehensive paper appear in *B.A.J.* **2**, I, 165–167. I add that I have revisited my sources and would not want to retract a single word. I wish, however, to repeat one of the remarks which I made at the previous discussion: "In the short term the unrealistic assumptions do not matter too greatly. ... In the longer term its inadequacies are shown up." I would, therefore, repeat for the benefit of those who misunderstood me: as an approximation, Black–Scholes can be made to work for short-term derivatives, but for longer-term derivatives the formulae's inadequacies are a matter of record, as noted, for example, by du Payrat in his paper to AFIR 1.

I now concentrate on Professor Clarkson's paper and the actuarial model for option pricing which he has described. As he writes, there are a number of reasons why it is to be preferred to the Black–Scholes model, based, as the latter is, on a flawed concept of markets and inappropriate mathematics. In common with many other economic fallacies derived from modern portfolio theory, Black–Scholes confuses market price with value. In practice there is no reason why Black–Scholes cannot be applied to any price of the underlying, but the theoretical basis of the mathematics loses its validity when that price is not the market price. For this reason disciples of MPT refused to recognise that the value of, say, a share may differ from its market price.

But let us follow Professor Clarkson's example, and apply our attention to the analogy of life assurance. Consider a single endowment policy, of a number of years standing, but still with several years to run to maturity. Does any actuary dispute that such a policy has a number of values? There is one value for the life assured who wishes to surrender it; another value, viewed as a single policy, for the office which issued it; yet another for the same office when it is considered as a component of a portfolio of similar policies; another for the specialist investment trust which wishes to add it to its portfolio of policies; and yet other values, as those who have attended any auctions will testify, for the various individuals who might bid for it. What, then, is the price of that policy? If a derivative is secured on that policy, on what price should that derivative be based? Can the price only be settled when the policy is surrendered or traded?

I turn now to more usual underlyings. In single trading sessions, on five separate days, in the week immediately prior to my writing this, the Dow–Jones fell by 2%, the Nikkei fell by $3\frac{1}{2}$%, a former FT-SE 100 share fell by 18%, Brent crude fell by 10% and spot copper was fluctuating by as much as 5% over periods of less then five minutes.

I have never understood how some people fail to appreciate that market price does not represent value. After all, the laws of supply and demand were the first that they learned in their student days. If they had any practical experience, then they would realise that, forced selling apart, shares, or any other underlyings, are only traded when a buyer thinks that their value is greater than the market price or a seller thinks that their value is less. Even if a share is not traded, it does not mean that all buyers and sellers agree that the price represents true value — only that the price is close enough to their estimates of value, to ensure that the risk of loss or prospect of gain, after taking into account expenses, does not justify the trade.

Professor Clarkson, because he bases his theory on actuarial principles, as opposed to modern portfolio mythology, recognises the difference between price and value, and his method can be applied, with validity, to any pricing model of the underlying. Black-Scholes and its derivatives cannot. Were it for this reason alone, Clarkson is to be preferred to Black–Scholes; but, when taken together with the comparative reality of Clarkson's assumptions and the transparency of his mathematics, there can be no comparison; the superiority of Clarkson to Black–Scholes is beyond doubt.

As Professor Clarkson recognises, much work will be needed to fine tune his ideas and to apply them to American options, futures, forwards and other derivatives, but such work will be rewarding for actuaries, traders, regulators and, academics alike.

Dr P. McCloud: This is a cautionary note on the use of the stochastic calculus in financial engineering.

The stochastic calculus, more accurately the gaussian stochastic calculus, deals only with the dynamics of stochastic processes with gaussian gradient. This is both its strength and its weakness. The simplifications afforded by the gaussian assumption make the gaussian stochastic calculus the ideal tool for dealing with dynamic uncertainty. On the downside, the results obtained can only ever be regarded as approximate, as the higher moments of the gradient, such as skew and kurtosis, are neglected.

These objections to the gaussian stochastic calculus are of little concern to the physicist. Effectively, the calculus tracks both the mean and the variance of the dynamic process. As such, it provides an estimate of the process and some measure of the uncertainty in the estimate. It is unlikely that the physicist will be concerned with tracking the higher moments of the process.

This contrasts with the application of the gaussian stochastic calculus in finance. The beauty of the Black–Scholes approach lies in the observation that the risk in a hedged portfolio can be removed via dynamic hedging. This is undoubtedly a significant achievement. A result that is exact in the gaussian world is, however, only approximate in the real world.

Dynamic hedging only removes the variance-dependent term in a power series for the risk in the hedged portfolio, leaving higher order terms dependent on the higher moments. In a very real sense, dynamic hedging invalidates the assumptions that underly the use of the gaussian stochastic calculus in finance. In the absence of the variance term, it is precisely the higher moment terms that dominate the expression for residual risk, and these are not accommodated in the gaussian stochastic calculus.

The gaussian hedging strategy of the Black–Scholes approach is designed to make the hedged portfolio resilient to movements in the market. The apparent residual risk in the portfolio occurs as a result of market movements that are not easily accommodated into the gaussian model.

Whilst the gaussian hedging strategy may be the optimal risk reduction strategy in the gaussian world, this may not be the case in the real world. Indeed, it is conceivable that there are situations where the gaussian hedging strategy actually exposes the hedged portfolio to greater risk than the 'do nothing' strategy. In general, the optimal risk reduction strategy must take into account the higher moments, to make the hedged portfolio more resilient to real market movements. This will clearly have consequences for the pricing of derivatives.

Mr E. F. Smith, F.F.A.: In Section 10 of his paper Mr Kemp draws an important distinction between the different uses of derivatives within a life office or a pensions fund:

— Model B type activities relate to the use of derivatives for what has become known as EPM. I believe that such activities are now fairly well established.
— Model A type activities relate to the management of a trading book of derivatives, i.e. where the life office is acting as principal. A good example here would be if a life office offering one of the common 5-year growth bonds containing guarantees decided to provide the guarantees itself rather than by matching the risk via the purchase of a tailor-made derivative from a bank. I believe that such activities are far less prevalent, despite the large volume of such products being sold and the profit margin that the banks are pricing into these products, profit which could potentially, in theory, but perhaps not, in practice, remain with the life office.

I agree with much of what Mr Kemp says about model B activities, although I would make a few comments:

— I believe that it is fundamental that the use of derivatives should be an integral part of the fund management activities. It is when a derivatives team becomes a separate profit centre within an investment department that trouble may begin. This principle is embedded within EPM, but I think that it is worth emphasising.
— I may be mis-interpreting what is being implied in ¶3.2.10, but I would be more than a little concerned about the use of derivatives to avoid regulatory requirements. Regulations are there for a good reason.
— I am surprised at the example given of the use of options within a portfolio in ¶3.4.1. For me a more natural use of options would be to fine tune the fluctuations of stock prices by selling into short-term excessive strength and buying into short-term excessive weakness. This can be achieved by selling calls against stocks approaching the top of their trading range and puts against stocks approaching the bottom of theirs. This can never be more than a marginal activity within a portfolio, but the income produced can prove a useful contribution to performance.
— The suggestion, in ¶3.5.3, of using options to maintain exposure to equity markets, whilst at the same time protecting the solvency position of the company, is indeed valid; but there is a cost — the option premium which wastes over the lifetime of the option. The anticipated enhanced return of equities over matching assets will have to at least meet this cost — of course this is a matter of professional judgement, but the cost invariably looks expensive to me.

Model A type activities are more controversial. The paper gives a good intuitive explanation of the Black–Scholes pricing model. The Clarkson paper describes a very different approach and, at the same time, is critical of Black–Scholes or, to be more accurate, of the underlying assumptions.

The investigation carried out in Section 9 of Mr Kemp's paper is important:

— Table 2 shows that market jumps are a significant problem.

The paper suggests that the control process can be improved by separately insuring the cost associated with such jumps. I agree, but I am not at all sure how practical or costly this would be:

— Table 5 shows a very substantial improvement in the control process. I would like further explanation of what is being done here, although the text suggests that the price charged for the put will be adjusted to reflect the eventual out-turn of the control process. This must be true, but does not seem to be directly relevant to what we are trying to test.

I am not sure what Mr Kemp's overall conclusion is — but, from the tone of what is written and the comments in ¶6.5.8, I suspect that Mr Kemp believes that Black–Scholes, with some refinements, does have a viable role to play:

— My understanding of the findings of the Brady Commission Report do seem to be at odds with what is suggested in ¶6.5.8.

This conclusion is also at odds with many of the comments in Professor Clarkson's paper. For my own part, I would like to see more testing of the type carried out by Mr Kemp.

Professor Clarkson's paper was a very enjoyable read, but I would like to see more detail supporting the claims made, as I do not find some of the conclusions of the paper as intuitive as the author suggests they should be.

It seems reasonable to me that, if a life office is writing a series of options associated with various tranches of a 5-year guaranteed bond, then the lack of independence between the outcome of each guarantee will mean that high capital reserves would be required to ensure that the probability of ruin is satisfactorily low. Assuming the cost of capital to be greater than the earned rate of return, this will tend to make the price the office should charge for the guarantees expensive, yet Professor Clarkson suggests that his methodology leads to lower option prices than under a Black–Scholes environment.

To me, the lack of independence suggests that the construction of hedging techniques should be the basic building block to both the option pricing and the management of a trading book. It may be that the Black–Scholes approach, which is based on such an approach, has practical difficulties, but it does seem a reasonable starting point.

A couple of further examples where I find Professor Clarkson's paper counter intuitive are:

(1) In a situation where the trading book was long a call and short a put on the same stock with same strike price, then, as the position is identical to owning the underlying stock, the trading book would be unwise to do anything other than shorting the stock to hedge the position. It is not clear to me that this would be a trivial consequence of Professor Clarkson's approach.

(2) A very similar feature is present in ¶7.1.6, where the author suggests that, if the trading book becomes more bullish for a stock, it should increase the price of calls and reduce the price of puts relative to the current position. This confuses me, because holding stock plus put is equivalent to owning a call option once the cost of carry is adjusted for. This must mean that, in the example given, if call prices are marked up then put prices must also be increased, or else the trading book will be effectively offering an arbitrage opportunity to the market.

I would suggest that, no matter which approach — Black–Scholes, Clarkson's or whatever, an office is using when pricing and reserving for options, a margin should be included to allow for what has been described as model risk, i.e. that the underlying model is wrong.

Mr C. A. Speed, F.F.A.: I would like to relate the discussion about the papers to the ongoing debate about the future of the profession.

I have become a Fellow comparatively recently, and can distinctly recall why I joined the profession and how the profession presented itself to me. I think that the reasons are important and relevant. Having completed a mathematical degree, I wanted to apply the knowledge I had gained in my career and expand this knowledge. Along with an interest in finance, the actuarial profession, with the respect it commanded and the high levels of responsibility, was an obvious route. So I joined the profession that proudly promoted itself as applying mathematical and statistical techniques to financial risk management.

Now, consider the high quality mathematical graduates of today and future years that the profession surely must wish to attract. Many of these graduates will be aware of the advances in financial economics which have led to the boom in global derivative products. This is arguably the most important development in the financial world in recent years. Surely graduates entering the actuarial profession would expect to become acquainted with such major developments in the financial world and to develop an understanding of the mathematics underlying these developments. Indeed, many of today's graduates will have studied stochastic processes and may have been introduced to stochastic calculus, which underlies the mathematics of derivative pricing. Thus, the absence of these topics from the actuarial education syllabus will appear strange to a recent graduate.

Now consider Fellows just after qualification. Once the pleasure at passing the exams has cooled,

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they may well ask just how much have their abilities to tackle financial problems advanced and prepared them for a career that may span many decades. In particular, new Fellows may question why they have not been encouraged to develop a knowledge and understanding of the powerful tools that are now being used to create derivative products, especially given their importance to financial institutions and the suspicion that many people have of them. I believe that they would be right in thinking that useful tools have not been provided.

For these reasons I strongly support the proposals to modernise the actuarial education syllabus, and congratulate Mr Kemp for putting many of the ideas that are familiar to others before the profession in such a clear and accessible manner.

I am not advocating that the profession blindly adopts all the work that has been done by the financial economists, but we must, first of all, develop an understanding of the methodology and the principles. I would then hope that the profession can critically assess its importance from a position of strength and, where necessary, adapt the work to meet the problems faced by actuaries. This is exactly what we see in Section 9 of Mr Kemp's paper, where some of the criticisms often levelled at the Black–Scholes model are investigated. What Mr Kemp shows is that the hedging approach cannot provide the perfect security of the idealised world of Black–Scholes, but this is what we expect, the Black–Scholes formulation is, after all, only a model. It is not reality. However, the usefulness of the hedging approach is clearly demonstrated on actual data. The results give useful pointers as to how actuaries can use the models in practice, and contribute to the prudent use of derivatives to achieve investment and risk management objectives. I hope that this is the way in which the profession will develop, taking on board the progress in other fields and using the knowledge and experience of actuaries to apply these 'new' techniques.

I find Professor Clarkson's aversion to financial economics regrettable. It is dangerous to dismiss or denigrate a large, and highly reputable, branch of mathematics on account of supposedly unrealistic assumptions unless you have a full understanding of the relevance of the assumptions. I feel this is particularly the case with a comparatively new area which has such an obvious practical relevance. Before we criticise the hard work of others (in an area where most actuaries are definitely not experts), let us gain a firm understanding. Only then will we be able to see clearly the advantages and the problems with these tools.

The actuarial profession contains many very able people, so let us use the ability within the profession and make the effort, as Mr Kemp and others are doing, to come to terms with the work being done elsewhere. Once we have a thorough grounding, let us use the skills and experience of the profession to adapt these powerful tools to the problems faced.

I see two major consequences if the profession does not embrace and adapt to the advances that have been made:

(1) the profession will shut the door on an area where actuarial employment could expand; and
(2) the profession's ability to continue to attract top quality mathematical graduates will be seriously undermined. These must be the people that the profession needs for a robust and healthy future.

Dr J. T. S van Bezooyen: As a financial economist, I am extremely surprised that the theory and methodology presented by Mr Kemp is subject to criticism by part of the actuarial profession.

It is generally agreed in financial economic literature that the Black–Scholes formula is not the ultimate truth, and a series of improvements and adjustments have been suggested since the publication of the original paper in 1973. Financial economists do, however, agree that the valuation methodology initiated by Black & Scholes is superior to discounted cash flow methods. The application of the latter techniques to the pricing of options has been analysed by, amongst others, Paul Samuelson and Robert C. Merton in the 1960s. The main problem that they could not solve was the determination of an appropriate discount rate. When Black & Scholes published their paper, Samuelson and Merton were the first to admit that this new valuation methodology was superior to theirs, and Merton went on to contribute his part of the development of the risk-neutral pricing methods.

The method suggested by Professor Clarkson is basically a return to the discounted cash flow methods. In my view as a financial economist, his approach is a major step back and does not offer the "theoretical advantages over the Black–Scholes and related methodologies of modern finance theory" that are claimed by Professor Clarkson.

From the discussion, I get the idea that a large part of the opposition is caused by the unfamiliarity of many actuaries with the mathematics used in derivative pricing theory. In my opinion, the point of view of a profession which has built its reputation on the application of various mathematical techniques should not be driven by such fright for the unknown.

The claim that the mathematics needed for derivative pricing methods are too difficult and are only understood by so-called rocket scientists is not true, and is unworthy of a mathematical profession. Option pricing theories are currently taught to business students all around the world and are an integral part of most standard financial economic textbooks. At the very least, actuaries should, therefore, be familiar with such basic theory if they are not to be left in a position of woeful ignorance and foolishness on elementary issues such as put-call parity and risk-neutral laws.

Professor A. D. Wilkie, F.F.A., F.I.A.: This is an extension of my contribution at the discussion, and explains certain things.

Mr Pemberton misquoted my remarks at Mehta's paper (*J.I.A.*, **119**, 449). It is best if I reproduce these:

> "For a simple example of timing risk, assume that at the end of year 1 a claim is payable which will either be £0 or £1 million, with equal probabilities. At the end of year 2 another claim is payable, again of amount £0 or £1 million, with equal probabilities. The total of claims may be £0, £1 million or £2 million, with appropriate probabilities. Contrast this with the situation where £1 million is certainly payable, but at the end of year 1 or year 2, with equal probabilities. The total claim is £1 million with certainty, but there is uncertainty about the date of payment. Looking at the years individually, the situation seems to be the same as in the first case, but this ignores the dependence between years. The second case is much less risky than the first."

A number of authors of otherwise good textbooks make the mistake of assuming that it is reasonable to ignore dependence between years, and that one can calculate expected utility for each year and then discount, instead of discounting first and then taking expected utilities.

I also comment on Professor Clarkson's mathematical model. Simple harmonic motion is generated by the differential equation:

$$\frac{d^2x}{dt^2} = -a^2.x$$

for which the solution is:

$$x = b.\sin(a.t + \omega)$$

where a gives the frequency, b is the amplitude and ω is the phase (i.e. where you are at time $t = 0$). Thus x is a regular periodic function.

There is no evidence in any share price series that I have examined of regular periodicity with any frequency.

I think that Professor Clarkson agrees with this, so he suggests using a distribution related to simple harmonic motion. He calls this a harmonic distribution. The distribution function for x in the range $-b$ to $+b$ is:

$$F(x) = \frac{1}{\pi}.\sin^{-1}\left(\frac{x}{b}\right) + \frac{1}{2}$$

and the density function is:

$$f(x) = \frac{1}{\pi.\sqrt{x^2 + b^2}}.$$

As Mr A. Smith observed, this is the same as a beta distribution with parameters $(\frac{1}{2}, \frac{1}{2})$ over the range $-b$ to $+b$.

Professor Clarkson then chooses to pick at random from this distribution to find the share price at time t. Here comes the problem. If the share price at time t is distributed in this way (with other adjustments), then how is it distributed at time $t + \delta t$? Either the price at $t + \delta t$ is independent of the price at time t, which means that prices move over extremely short periods over the whole range from $-b$ to $+b$, so that a graph of the share price would look like a solid band; or else the prices at neighbouring times are correlated, in which case we are back at a periodic contribution to the share price. But if there is a periodic contribution, then we should be able to estimate the frequency and the phase from past data, and we would also know where we are in the cycle, so there is no need to pick from this (log)harmonic distribution. This is why I said that the mathematics was incoherent.

A possible way forward is to suggest the model:

$$dx = v.dt$$

$$dv = -a^2.dt + \sigma.dW$$

where dW is the derivative of a Wiener process W. In such a model the acceleration is still towards the centre, but is subject to small random shocks at every instant. I have not developed this idea further.

Professor R. S. Clarkson, F.F.A. subsequently wrote: The two most serious criticisms running through the contributions of those who were hostile to my suggested new framework for option pricing were that I had failed to recognise the practical relevance of 'modern' concepts and methodologies such as put-call parity and stochastic calculus, and that I was thereby impeding the wider acceptance within the U.K. actuarial profession of these new mathematical tools.

To put the practical relevance issue into a proper perspective, consider the following quotation:

> "The Black–Scholes formula is still around, even although it depends on at least ten unrealistic assumptions. Making the assumptions more realistic hasn't produced a formula that works better across a wide range of circumstances. In special cases, though, we can improve the formula. If you think investors are making an unrealistic assumption like one of those used in deriving the formula, there is a strategy you may want to follow that focuses on that assumption."

These words were written in 1989, not by someone who believed the Black–Scholes approach to be intrinsically unsound, but by the late Fischer Black himself, in an article in the *Journal of Applied Corporate Finance* entitled 'How to use the holes in Black–Scholes'. Black then took each of the 'ten unrealistic assumptions' in turn, and derived strategies "that make sense if investors continue to make unrealistic assumptions".

My suggested new framework, which makes explicit allowance for important behavioural aspects of the real world, such as the possibility of significant outperformance or underperformance of the underlying security, offers a detailed and transparent structure for exploiting, precisely as Black suggested, divergences between the simplifying assumptions of modern finance theory and a realistic view of future financial behaviour. Mr A. Smith missed the point completely in this regard by suggesting that my model is unnecessary in that my new parameters 'collapse to trivial values'. The precise opposite is the case, in that market prices do not yet appear to reflect the additional explanatory variables that I have incorporated, thereby creating potentially profitable opportunities for those prepared to use my new approach.

There was a predictable polarisation in the general manner in which speakers viewed my more pragmatic and more detailed model. Mr Jones, as a practitioner, was prepared to use it if it helped to make money. Others, such as Dr Cairns and Dr Macdonald, whose main interest is in teaching the advanced mathematics that Fischer Black and Myron Scholes first used to translate their 'ten unreasonable assumptions' into a very useful one-parameter graduation formula, viewed any attempt to move beyond the Black–Scholes paradigm as sacrilege. Also, Mr Jones's practical experience had led him to conclude, as I do, that utility theory, while very elegant in mathematical terms, can lead to unsatisfactory results in attempted real world applications. Professor Wilkie, on the other hand, saw utility theory as a very important mathematical tool of modern finance theory.

A few weeks after the presentation of my paper there was yet another well-publicised derivatives fiasco, this time involving options trading at a major U.K. retail bank. Regardless of what specific conclusions are drawn in this case after the appropriate investigations by the company and the regulators, one general conclusion seems inescapable. Until the current highly mathematical theories of option pricing are replaced by a much more transparent theoretical framework, possibly along the lines of the one that I have suggested, similar or even more serious financial accidents will continue to occur without warning.

Mr M. A. D. Kemp, F.I.A. subsequently wrote: The main criticisms of my paper were made by Professor Clarkson and Mr Pemberton. I believe that Mr Pemberton's comments help to clarify some of the key issues, and so I will concentrate on them.

Mr Pemberton argues that everyone needs to agree the same probabilities of events happening for there to be uniquely agreed prices. I disagree. If we go back to the strips market, there will almost certainly be, as Professor Wilkie has pointed out, equivalence between the price of a gilt unstripped and the price of the corresponding instruments generated by stripping the gilt.

Mr Pemberton also argues that it is not valid to combine values together in an additive fashion. Looking closely at the above, we can see that what we need for prices/values to be additive is for the cash flow packages to be *freely traded.* Suppose that we have two separate representations of the same cash flows, both of which are freely traded and each of which can be reconstituted as the other. The laws of economics from Adam Smith onwards require freely traded cash flow packages to be priced at a level at which supply and demand are balanced. In effect, they imply a 'law of one price' — two equal packages of cash flows will trade at the same price. If the relationship between the prices permits arbitrage, then arbitrageurs will operate to bring supply and demand for the two separate representations of the same cash flows back into balance, if necessary converting one representation to the other, benefiting from a 'free lunch' in the process. I believe that this is what Professor Clarkson is getting at when he refers to the need for markets to be 'complete'.

What happens if different investors place different intrinsic 'values' on the same (freely tradeable) cash flows or have different views as to how likely are events on which the cash flows depend? This will affect the overall balance between supply and demand, but it will *not* affect the equivalence in price between separate representations of the same cash flows. Any investor who offers to buy and sell on different terms the two equivalent cash flow packages will merely be writing a cheque to some arbitrageur. Indeed, we can go further. In the above situation the investor himself can freely convert one representation of the cash flow package to the other. Therefore the intrinsic 'value' he places on each must also be the same, since, if there is any difference, he will himself reconstitute any of the version he considers is less valuable into the version he considers is more valuable.

So, the key distinction that we should be focusing on is whether the cash flows involved are reasonably freely tradeable or not. If the cash flows are freely tradeable, then the principle of no arbitrage is almost certain to apply. If the cash flows are not, then a wider range of models could be justified. This probably explains why Mr A. Smith's analysis finds Professor Clarkson's model so unsatifactory. Mr Smith has concentrated on market prices, and thus on options that are at least reasonably marketable, which is precisely when Professor Clarkson's model can be expected to perform worst, because it does not, in general, satisfy no arbitrage.

Do we need one set of models for cash flows that are freely traded and another set for those that are not? I think not. I believe that we should, instead, be aiming for a generalised framework that handles both situations simultaneously, by incorporating parameters linked to the degree to which the underlying cash flows are freely tradeable.

We can, for example, measure the degree to which a cash flow package is freely tradeable by its bid/offer spread. A zero bid/offer spread corresponds to a completely freely tradeable instrument, whereas an infinite bid/offer spread corresponds to an instrument that does not trade at all. In practice all instruments will lie somewhere in between these two extremes. As explained in Appendix B of my paper, it is possible to construct generalisations of the Black–Scholes model that take such transactions costs specifically into account. A consequence is that there is no longer a single price at which options might trade. They are, themselves, potentially subject to a bid/offer spread, but, as the transactions costs tend to zero, they have the intuitively attractive characteristic that they tend back to the Black–Scholes model.

In a sense this confirms the view that Black–Scholes needs modification, but, as several speakers have already commented, no reputable financial economist would now wish to claim that Black–Scholes is a completely accurate representation of how the world actually operates.

Other weaknesses in the Black–Scholes formulation (i.e. the possibility that markets might jump or that volatility will be uncertain) can also be explicitly accommodated in a similar fashion — see again Appendix B of my paper.

I think that focusing on liquidity can also provide the bridge that some (like Mr Green) argue exists between the 'short term' and the 'long term'. It has always seemed to me impossible to identify when the short term ends and the long term starts. The long term is, after all, merely a collection of short terms. However, there is a general falling off of liquidity with length of contract. Thus we can expect a difference between the short term and the long term, but the driving force is the change in liquidity, rather than the longer timespan *per se.*

There is a corollary. As derivatives markets develop and liquidity deepens, the range of circumstances in which the sorts of models proposed by Professor Clarkson might have some validity will steadily shrink. I prefer the 'win-win' approach of accommodating all time frames/liquidity depths simultaneously. However, I accept that it does make the mathematics more complex.