Canad. Math. Bull. 2025, pp. 1–20
http://dx.doi.org/10.4153/S0008439524000432

Generalized torsion orders and Alexander polynomials

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Abstract. A nontrivial element of a group is a generalized torsion element if some products of its conjugates is the identity. The minimum number of such conjugates is called a generalized torsion order. We provide several restrictions for generalized torsion orders by using the Alexander polynomial.

1 Introduction

An element g of a group G is a *generalized torsion element* if there exists a positive integer n and $x_1, \ldots, x_n \in G$ such that they satisfy

$$g^{x_1} g^{x_2} \ldots g^{x_n} = 1. \tag{1.1}$$

Here, we put $g^x := xgx^{-1}$. The *generalized torsion order* $gord(g)$ (often simply called the *order*) is the minimum n such that g satisfies (1.1) for some $x_1, \ldots, x_n \in G$.

The Alexander polynomial is a (multivariable) polynomial invariant of a group G. More precisely, the Alexander polynomial is defined for a group G with surjection $\phi : G \to \mathbb{Z}^s$. Such a surjection ϕ corresponds to a normal subgroup N of G with quotient $G/N = \mathbb{Z}^s$, so we may regard the Alexander polynomial as an invariant of a pair (G, N).

As a slight generalization, we define the Alexander polynomial $\Delta_{\mathcal{A}}(t_1, \ldots, t_s)$ for an *Alexander tuple* $\mathcal{A} = (G; (X, N, H))$ which is a group G and its normal subgroups $X \subset N \subset H$ having several properties (see Definition 3.2 for details).

The aim of this paper is to investigate the relation between generalized torsion elements and Alexander polynomials. For $g \in G$, we define the *generalized torsion equation spectrum* $t(g)$ by

$$t(g) = \{n \in \mathbb{N} \mid g^{x_1} \ldots g^{x_n} = 1 \text{ for some } x_1, \ldots, x_n \in G\}.$$

Namely, $t(g)$ is the set of nonnegative integers n such that equation (1.1) has a solution. We study a relation between Alexander polynomials and generalized torsion equation spectrum.

Received by the editors December 24, 2023; revised July 16, 2024; accepted August 22, 2024.
The author is partially supported by JSPS KAKENHI (Grant Nos. 19K03490, 21H04428, and 23K03110).
AMS subject classification: **20F60**, **20F05**, **20F12**, 57K10.
Keywords: Generalized torsion element, generalized torsion order, Alexander polynomial, G-invariant norm.

For an irreducible polynomial $h(t_1,\dots,t_s) \in \mathbb{Z}[t_1^{\pm 1},\dots,t_s^{\pm 1}]$, we define

$$t(h(t_1,\dots,t_s)) = \{f(1,\dots,1) \mid f \in (h(t_1,\dots,t_s)) \text{ is positive}\} \subset \mathbb{N}.$$

Here, we say that a polynomial f is *positive* if it is nonzero and all the coefficients are nonnegative.

The following main theorem states that $t(h(t_1,\dots,t_s))$ for an irreducible factor h of the Alexander polynomial gives a restriction on the generalized torsion equation spectrum $t(g)$.

Theorem 1.1 *Let $\mathcal{A} = (G;(X,N,H))$ be an Alexander tuple. For an element $g \in N$, if $g \notin X$, then there exists an irreducible factor $h(t_1,\dots,t_s)$ of $\Delta_{\mathcal{A}}(t_1,\dots,t_s)$ such that $t(g) \subset t(h(t_1,\dots,t_s))$.*

One motivation of studying generalized torsion elements comes from orderable groups. A *bi-ordering* of a group G is a total ordering $<$ on G which is invariant under both the left and right multiplications, i.e., $g < h$ implies $agb < ahb$, for all $a, b, g, h \in G$. A group G is *bi-orderable* if G has a bi-ordering. A generalized torsion element serves as a primary obstruction for a group to be bi-orderable.

Recently, the relation between orderable groups and low-dimensional topology is actively studied by many researchers. Therefore, it is also interesting to explore the relation between generalized torsion elements and low-dimensional topology.

As applications of Theorem 1.1, we will discuss generalized torsion elements of knot groups. Our results lead to an interesting connection to homology growth of abelian coverings and the generalized torsion elements.

Theorem 1.2 *Let K be a knot in S^3. Let $G = G(K) = \pi_1(S^3 \backslash K)$ be the knot group, and let $\Sigma_k(K)$ be the k-fold cyclic branched covering of K. Assume that the Alexander polynomial $\Delta_K(t)$ of K is irreducible, $\Sigma_k(K)$ is a rational homology sphere, and $k = p^e$ is a power of a prime p. Then, for $g \notin [[G,G],[G,G]]$ and $n \in t(g)$ either:*

(a) *$n \geq |H_1(\Sigma_k(K);\mathbb{Z})|^{\frac{1}{k-1}}$, or,*
(b) *p divides n*

holds.

The following special case ($k = 2$) of Theorem 1.2 deserves to mention.

Corollary 1.3 (Determinant bound) *Let $G = G(K)$ be the knot group of a knot K. Assume that $\Delta_K(t)$ is irreducible. If $g \notin [[G,G],[G,G]]$ and $n \in t(G(K))$ is odd, then*

$$n \geq \det(K) = |\Delta_K(-1)|.$$

A conjecture of Motegi–Teragaito says that a 3-manifold group has a generalized torsion element if and only if it is not bi-orderable [21]. Although this conjecture predicts the existence of generalized torsion elements for many 3-manifold groups, currently our catalog of generalized torsion elements in 3-manifold groups are quite limited [17].

It is known that for a given $d > 0$, the number of alternating knots K satisfying $\det(K) \le d$ is finite [2]. Thus, Corollary 1.3 says that for each odd k, there are only finitely many alternating knots K having a generalized torsion element $g \notin [[G,G],[G,G]]$ with $gord(g) = k$ and $\Delta_K(t)$ is irreducible. Thus, Corollary 1.3 partially explains why finding a generalized torsion element is difficult.

This observation poses the following finiteness question.

Question 1 For a given integer m, let $M_{\mathrm{alt}}(m)$ be the number of prime alternating knots K other than $(2,k)$-torus knot[1] whose knot group $G(K)$ has a generalized torsion element g with $gord(g) = m$. Is $M_{\mathrm{alt}}(m)$ finite ?

A similar finiteness question makes sense for other appropriate classes of knots. In particular, it is interesting to ask the same finiteness question for hyperbolic knots. On the other hand, as we will see in Proposition 5.4, for the (p,q)-torus knots or (p,q)-cable knots, their knot group have a generalized torsion element of generalized torsion order p, if $p < q$ are primes. This is why we exclude $(2,k)$-torus knots in Question 1.

Theorem 1.2 gives a restriction of $t(g)$ for the case $g \notin [[G,G],[G,G]]$. For general $g \in G$, we have the following.

Theorem 1.4 *Let K be a knot in S^3 and $G = G(K)$ be its knot group. Assume that $[G,G]$ is residually torsion-free nilpotent and that* $\deg \Delta_K(t) = 2g(K)$, *where $g(K)$ is the genus of K. If $\Delta_K(t)$ divides $(t^k - 1)$, where $k = p^a q^b$ for some distinct primes p, q $(p < q)$, then for every $g \in G$,*

$$t(g) \subset p\mathbb{N} \cup \mathbb{N}_{\ge q}.$$

The most fundamental example of knots satisfying the assumption of Theorem 1.4 is a *fibered knot*, a knot whose complement has a structure of a surface bundle over the circle. By applying Theorem 1.4 for torus knots, we get the following.

Corollary 1.5 *Let K be the (p^a, q^b)-torus knot, where $p < q$ are primes. Then, for every $g \in G(K)$, $t(g) \subset p\mathbb{N} \cup \mathbb{N}_{\ge q}$.*

Motivated by these results, we will discuss the *generalized torsion order spectrum* $gord(G)$, the set of generalized torsion orders of a group G in Section 6. We will show that every subset of natural numbers can be realized as the set $gord(G)$ for some countable, torsion-free groups (Corollary 6.4).

2 Generalized torsion order

We summarize the basic facts and definitions on generalized torsion elements and generalized torsion orders.

Definition 2.1 The *generalized torsion equation spectrum* $t(g)$ of an element $g \in G$ is the set

$$t(g) = \{n \in \mathbb{N} \mid g^{x_1} \dots g^{x_n} = 1 \text{ for some } x_1, \dots, x_n \in G\}.$$

[1] This is equivalent to saying that K is a hyperbolic alternating knot [20]

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By definition, $t(g)$ is a sub-semigroup of $\mathbb{N}$: $n, m \in t(g)$ implies $n + m \in t(g)$. Using the set $t(g)$, generalized torsion elements and its generalized torsion orders are defined as follows.

Definition 2.2 An element g is a *generalized torsion element* if $t(g) \neq \emptyset$. The *generalized torsion order* $gord(g)$ is

$$gord(g) = \min t(g).$$

When $t(g) = \emptyset$, we define $gord(g) = \infty$.

A torsion element g is a generalized torsion element. First of all, we discuss several differences between generalized torsion elements and torsion elements.

For a torsion element g of G, clearly

$$t(g) \supset ord(g)\mathbb{N}$$

holds. Here, $ord(g)$ is the order of the torsion element g. In particular,

$$gord(g) \leq ord(g)$$

holds. The next example shows that the difference of $gord(g)$ and $ord(g)$ can be arbitrary large. (See Section 6 for more detailed discussion concerning the difference of orders and generalized torsion orders.)

Example 2.1 For $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, let

$$G_m = \begin{cases} \langle a, b \mid bab^{-1} = a^{-1}, a^m = 1 \rangle & m \in \mathbb{Z}_{\geq 2} \\ \langle a, b \mid bab^{-1} = a^{-1} \rangle & m = \infty. \end{cases}$$

Then, $ord(a) = m$ but $gord(a) = 2$.

The generalized torsion order is often called the *order*. However, since $ord(g) \neq gord(g)$ in general as Example 2.1 shows, it is useful to distinguish the *order* and the *generalized torsion order* when G has a torsion element.

For a subgroup H of G and the inclusion map $i : H \hookrightarrow G$, $h \in H$ is a torsion element of H if and only if $i(h)$ is a torsion element of G. Furthermore, $ord(h) = ord(\iota(h))$. Example 2.1 shows that this is far from true for generalized torsion orders.

We will often write $gord(g)$ as $gord_G(g)$ (and $t(g)$ as $t_G(g)$) to emphasize the group G. For example, when $H \subset G$ is a subgroup of G and $h \in H$, $gord_H(h)$ means a generalized torsion order of h in the group H, whereas $gord_G(h)$ means a generalized torsion order of $\iota(h)$ in the group G, where ι is the inclusion map. as an obstruction for bi-orderability; if G has a bi-ordering $<$ then for every nontrivial element $g \in G$, $1 < g$ or $g < 1$ holds. When $1 < g$ then $1 = xx^{-1} < xgx^{-1}$ for all $x \in G$ hence product of conjugates of x is not trivial. The case $g < 1$ is similar.

To investigate the set $t(g)$, the next simple observation is useful.

Lemma 2.2 (Monotonicity) *Let $f : G \to H$ be a homomorphism. Then, for $g \in G$, $t_G(g) \subset t_H(f(g))$. In particular, $gord_G(g) \geq gord_H(f(g))$.*

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Proof If $g^{x_1} \dots g^{x_n} = 1$, then $f(g)^{f(x_1)} \dots f(g)^{f(x_n)} = 1$. ∎

This leads to the following useful consequences.

Corollary 2.3

(i) *For a prime p and a homomorphism $f : G \to \mathbb{Z}_p$, if $f(g) \neq 1$ then $t(g) \subset p\mathbb{N}$.*

(ii) *If $f : G \to H$ is a homomorphism and $g \in G$ is a generalized torsion element, $f(g)$ is a generalized torsion element unless $f(g) \neq 1$.*

(iii) *If a subgroup H of G is a retract (i.e., there is a map $p : G \to H$ such that the restriction $p|_H : H \to H$ is the identity), then for every $h \in H \subset G$, $t_H(h) = t_G(h)$, and $gord_H(h) = gord_G(h)$.*

(iv) *For every $\phi \in Aut(G)$, $t(g) = t(\phi(g))$ and $gord(\phi(g)) = gord(g)$.*

3 Alexander polynomial criterion

3.1 Alexander polynomial of modules

We quickly review the Alexander polynomial. We refer to [13] for algebraic treatments of Alexander polynomial.

Let $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_s^{\pm 1}]$ be the Laurent polynomial ring of s variables. For $f, g \in \Lambda$, we denote by $f \doteq g$ if $f = ug$, where $u \in \Lambda$ is a unit of Λ. We denote by $\varepsilon : \Lambda \to \mathbb{Z}$, the *augmentation map* $\varepsilon(f(t_1, \dots, t_s)) = f(1, \dots, 1)$.

A Λ-module M is *finitely presented* if there is an exact sequence of Λ-modules

$$\Lambda^m \xrightarrow{A} \Lambda^n \to M \to 0$$

called a *finite presentation* of M. The matrix A is called a *presentation matrix* of M.

Definition 3.1 (Elementary ideal and Alexander polynomial) Let M be a finitely presented Λ-module and A be its presentation matrix. The *kth elementary ideal* $E_k(M)$ is the ideal of Λ generated by $(n-k)$ minors of A (when $k > n$, we define $E_k = \{1\}$). The kth *Alexander polynomial* $\Delta_k(M) \in \Lambda$ is the generator of the smallest principal ideal of R that contains $E_k(M)$.

It is known that the elementary ideal does not depend on a choice of presentation matrix and that $\Delta_k(M)$ is uniquely determined up to multiplication of units of Λ.

Let

$$TM = \{m \in M \mid fm = 0 \text{ for some } 0 \neq f \in \Lambda\}$$

be the torsion submodule of M. The rank of M is defined by

$$\operatorname{rank}(M) = \dim_k k \otimes M$$

where k is the quotient field of Λ. We say that M is a Λ-torsion module if $M = TM$, which is equivalent to $\operatorname{rank}(M) = 0$.

The annihilator ideal of $m \in M$ is an ideal of Λ defined by

$$\operatorname{Ann}(m) = \{f \in \Lambda \mid fm = 0\}.$$

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Similarly, the annihilator ideal of M is defined by

$$\mathrm{Ann}(M) = \bigcap_{m\in M} \mathrm{Ann}(m) = \{f \in \Lambda \mid fm = 0 \text{ for all } m \in M\}.$$

The Alexander polynomial and annihilator ideals are related as follows.

Proposition 3.1

(i) $\Delta_{\mathrm{rank}\,(M)+k}(M) \doteq \Delta_k(TM)$ *[13, Theorem 3.4].*
(ii) $\sqrt{\mathrm{Ann}(M)} = \sqrt{E_0(M)}$ *[13, Theorem 3.1].*
(iii) *If $M = TM$ and M has a square presentation matrix, then* $\mathrm{Ann}(M) = ((\Delta_0(M)/\Delta_1(M))$ *[13, Corollary 3.4.1]*

Here, $\sqrt{I} := \{g \in \Lambda \mid g^n \in \Lambda$ for some $n > 0\}$ is the radical of the ideal I.

We will use the following result later.

Lemma 3.2 [13, Theorem 3.12(3)] *Let $\Phi : M \to N$ be a homomorphism of Λ-modules. If $\Phi|_{TM} : TM \to TN$ is a surjection, then $\Delta_0(TN)$ divides $\Delta_0(TM)$.*

3.2 Alexander tuples

Let N be a normal subgroup of a group G such that its quotient group $G/N = \mathbb{Z}^s$ for some $s \geq 0$. The quotient group $G/N = \mathbb{Z}^s$ acts on the homology group $H_1(N;\mathbb{Z}) = N/[N,N]$ by conjugation. Hence, $N/[N,N]$ has a structure of $\Lambda := \mathbb{Z}[\mathbb{Z}^s]$-module. This Λ-module is called the *Alexander module* and its Alexander polynomial is called the *Alexander polynomial* of a group G. As we already mentioned, we regard them as an invariant of a pair (G,N) of a group G and its normal subgroup N with $G/N = \mathbb{Z}^s$.

We slightly extend this construction.

Definition 3.2 (Alexander tuple) Let G be a group and X, N, H be normal subgroups of the group G. We say that a tuple $\mathcal{A} = (G;(X,N,H))$ is an *Alexander tuple* if they satisfy the following conditions.

(a) $[H,N] \subset X \subset N \subset H$.
(b) The quotient group G/H is the free abelian group $\mathbb{Z}^s$ for $s \geq 0$.

For an Alexander tuple $\mathcal{A} = (G;(X,N,H))$, we put

$$\Lambda = \mathbb{Z}[G/H] = \mathbb{Z}[t_1^{\pm1},\dots,t_s^{\pm1}] \text{ and } M = N/X.$$

Since $[N,N] \subset [H,N] \subset X$, M is an abelian group. The group G acts on M by conjugation because X and N are normal. Furthermore, $[H,N] \subset X$ implies that the conjugation action of H on M is trivial. Thus, the quotient group $G/H = \mathbb{Z}^s$ acts on M hence M is a Λ-module.

Definition 3.3 (Alexander module and polynomials of the Alexander tuple) We call the Λ-module M *the Alexander module* of the Alexander tuple $\mathcal{A} = (G;(X,N,H))$. We call the 0th Alexander polynomial $\Delta_0(TM)$ the *Alexander polynomial* of the Alexander tuple $\mathcal{A}$ and denote by $\Delta_{\mathcal{A}}(t_1,\dots,t_s)$.

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The Alexander polynomial is usually used as an invariant of knots (and links) in the following manner.

Example 3.3 (Alexander polynomial of a knot) Let K be a knot in S^3 and $G = G(K) = \pi_1(S^3 \backslash K)$ be the knot group, the fundamental group of its complement. By Alexander duality $G/[G,G] = H_1(G;\mathbb{Z}) = H_1(S^3\backslash K;\mathbb{Z}) = \mathbb{Z}$. The 0th Alexander polynomial of the Alexander tuple $(G;([[G,G],[G,G]],[G,G],[G,G]))$ is called the Alexander polynomial of a knot K denoted by $\Delta_K(t)$.

3.3 Alexander polynomial and generalized torsion equation spectrum

To state our theorem, we introduce a notion of generalized torsion equation spectrum for an element of Λ-modules.

Definition 3.4 For a Λ-module M and $m \in M$, the *generalized torsion equation spectrum* of m by

$$t(m) = \{\varepsilon(f) = f(1,\dots,1) \mid f \in \mathrm{Ann}(m) \text{ and } f \text{ is a positive element}\}.$$

Here, we say that an element $f \in \Lambda$ is *positive* if $f \neq 0$ and all the coefficients of f are nonnegative.

For the Alexander module of an Alexander tuple, the generalized torsion equation spectrum is nothing but the generalized torsion equation spectrum of suitable quotient group.

Lemma 3.4 *Let $\mathcal{A} = (G;(X,N,H))$ be an Alexander tuple and M be its Alexander module. Then, for $g \in N$, $t(p(g)) = t_{G/X}(q(g))$, where $p : N \to M = N/X$ and $q : G \to G/X$ are the quotient maps.*

Proof By definition, $n \in t_{G/X}(q(g))$ if and only if there exists $x_1,\dots,x_n \in G$ such that

$$g^{x_1}\dots g^{x_n} \in X.$$

Since $g \in N$, by taking the projection map $p : N \to M$ it is equivalent to

$$p(g^{x_1})\dots p(g^{x_n}) = \left(\sum_{i=1}^{n} \phi(x_i)\right) p(g) = 0 \in M = N/X,$$

where $\phi : G \to G/H \subset \Lambda$ is the projection map. Therefore, $n \in t_{G/X}(q(g))$ if and only if $n \in t(p(g))$. ∎

Definition 3.5 For an irreducible element $h \in \Lambda$, we define

$$t(h) = \{\varepsilon(f) \mid f \in (h(t_1,\dots,t_s)), f \text{ is a positive element}\}.$$

The set $t(h)$ for the case $s = 1$ (i.e., the case $G/H = \mathbb{Z}$) will be studied in the next section. Now we are ready to prove the main theorem stated in Sectio.

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Theorem 1.1 *Let $\mathcal{A} = (G; (X, N, H))$ be an Alexander tuple. For an element $g \in N$, if $g \notin X$, then there exists an irreducible factor $h(t_1, \dots, t_s)$ of $\Delta_{\mathcal{A}}(t_1, \dots, t_s)$ such that $t(g) \subset t(h(t_1, \dots, t_s))$.*

Proof If $t(g) = \varnothing$, we have nothing to prove so we assume that $t(g) \neq \varnothing$. We put $m = p(g)$, where $p : N \to M = N/X$ is the quotient map. By Lemma 2.2, $t(g) \subset t(m)$ hence $t(m) \neq \varnothing$. In particular, $m \in TM$.

Let Λm be the sub Λ-module of TM generated by m. By Proposition 3.1,

$$\begin{array}{ccccccccc}
\mathrm{Ann}(m) = \mathrm{Ann}(\Lambda m) & \subset & \sqrt{\mathrm{Ann}(\Lambda m)} & = & \sqrt{E_0(\Lambda m)} & \subset & \sqrt{(\Delta_0(\Lambda m))} \\
\cup & & \cup & & \cup & & \cup \\
\mathrm{Ann}(TM) & \subset & \sqrt{\mathrm{Ann}(TM)} & = & \sqrt{E_0(TM)} & \subset & \sqrt{(\Delta_0(TM))} \\
 & & & & \cup & & \cup \\
 & & & & E_0(TM) & \subset & (\Delta_0(TM)).
\end{array}$$

Thus, $\sqrt{(\Delta_0(\Lambda m))}$ is a principal ideal that contains $(\Delta_0(TM))$. Since we are assuming $g \notin X$, $m = p(g) \neq 0$. Thus, $\sqrt{(\Delta_0(\Lambda m))}$ is not the whole Λ. Therefore, there exists a nontrivial irreducible factor $h(t_1, \dots, t_s)$ of $\Delta_0(TM) = \Delta_{\mathcal{A}}(t_1, \dots, t_s)$ such that

$$\mathrm{Ann}(m) \subset \sqrt{(\Delta_0(\Lambda m))} \subset (h(t_1, \dots, t_s))$$

hence,

$$t(g) \subset t(m) \subset t(h(t_1, \dots, t_s)).$$ ∎

Remark 3.10 Although in Theorem 1.1 we used the Alexander polynomial $\Delta_{\mathcal{A}}(t_1, \dots, t_s) = \Delta_0(TM)$, if we know the structure of the annihilator ideals we can often improve the theorem. For example, if $M = TM$ and M has a square presentation matrix

$$t(g) \subset t(h(t_1, \dots, t_s))$$

for some irreducible factor $h(t_1, \dots, t_s)$ of $\Delta_0(M)/\Delta_1(M)$, because $\mathrm{Ann}(M) = (\Delta_0(M)/\Delta_1(M))$ by Proposition 3.1(iii).

For normal subgroups N and H of G such that $N \subset H$, the *H-lower central series* of N

$$\gamma_0^H N \supset \gamma_1^H N \supset \cdots \supset \gamma_k^H N \supset \gamma_{k+1}^H N \supset \dots$$

is defined by $\gamma_0^H N = N$ and $\gamma_{k+1}^H N = [H, \gamma_k^H N]$. When $N = H$, this is the usual lower central series of H. We put $\gamma_\infty^H N = \bigcap_{k \geq 0} \gamma_k^H N$. Then, iterated use of Theorem 1.1 for the Alexander tuple $(G; (\gamma_{k+1}^H N, \gamma_k^H N, H))$ gives the following.

Corollary 3.11 *Let N and H be normal subgroups of a group G that satisfy the conditions:*

(a) *$N \subset H$.*
(b) *The quotient group G/H is the free abelian group $\mathbb{Z}^s$ for $s \geq 0$.*

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For $k > 0$, let $\mathcal{A}_k = (G; (\gamma^H_{k+1}N, \gamma^H_k N, H))$ be an Alexander tuple. If $g \in \Gamma^H_k N$ and $g \notin \Gamma^H_{k+1}N$, then there exists an irreducible factor $h(t_1, \dots, t_s)$ of $\Delta_{\mathcal{A}_k}(t_1, \dots, t_s)$ such that

$$t(g) \subset t(h(t_1, \dots, t_s)).$$

We give the simplest application, the case $H = G$. For a prime number p, a group G is *residually finite* p if for every nontrivial $g \in G$, there exists a surjection $f : G \to Q$ to a finite p-group Q such that $f(g) \neq 1$.

Corollary 3.12 *If G is a residually finite p-group, then for every nontrivial $g \in G$, $t(g) \subset p\mathbb{N}$.*

Proof By the monotonicity (Lemma 2.2), it is sufficient to show the assertion for finite p-groups. Let G be a finite p-group. We apply Corollary 3.11 for $N = H = G$ (thus $\Lambda = \mathbb{Z}$). Since a finite p-group is nilpotent, there exists $k \geq 0$ such that $g \in \gamma_k G$ but $g \notin \gamma_{k+1}G$. Since G is a finite p-group, $\gamma_k G/\gamma_{k+1}G$ is an abelian p-group hence $t(g) \subset p\mathbb{N}$. ■

Finally, we give useful variant of Corollary 3.11 that only uses one Alexander module M for an Alexander tuple $\mathcal{A} = (G; ([N, H], N, H))$.

For a Λ-module M, let $M^{\otimes k}$ be the tensor product of the $\mathbb{Z}$-module (i.e., abelian group) M. We view $M^{\otimes k}$ as a Λ-module by the diagonal action; for $t \in \mathbb{Z}^s \subset \lambda = \mathbb{Z}[\mathbb{Z}^s]$ and $m_1, \dots, m_k \in M$, we define

$$t(m_1 \otimes m_2 \otimes \cdots \otimes m_k) = tm_1 \otimes tm_2 \otimes \cdots \otimes tm_k).$$

Corollary 3.13 *Let $\mathcal{A} = (G; ([N, H], N, H))$ be an Alexander tuple and $M = N/[N, H]$ be its Alexander module. If M is a Λ-torsion module, then for every $g \in G$, if $g \notin \gamma^H_\infty N$ then $t(g) \subset t(h(t_1, \dots, t_s))$ for some irreducible factor $h(t_1, \dots, t_s)$ of $\Delta_0(M^{\otimes k})$.*

Proof Since $g \notin \gamma^H_\infty N$, there exists $k > 0$ such that $g \in \gamma^H_k N$ and that $g \notin \gamma^H_{k+1}N$. Let M_k be the Alexander module of the Alexander tuple $\mathcal{A}_k = (G; (\gamma^H_{k+1}N, \gamma^H_k N, H))$ and $f : M^{\otimes(k+1)} \to M_k$ be the map defined by

$$f(a_1 \otimes a_2 \otimes \cdots \otimes a_{k+1}) = [a_1, [a_2, [\dots, [a_k, a_{k+1}] \dots]]] \qquad (a_i \in N).$$

The map f is a surjective Λ-module homomorphism (see [24, 5.2.5]). Since we are assuming M is a Λ-torsion module, so is $M^{\otimes k}$. Therefore, by Lemma 3.2, $\Delta_{\mathcal{A}_k}(t_1, \dots, t_s) = \Delta_0(TM_k)$ divides $\Delta_0(T(M^{\otimes k})) = \Delta_0(M^{\otimes k})$ so the assertion follows from Corollary 3.11. ■

Remark 3.14 (Noncommutative settings) Throughout this section, we assume that G/H is a finitely generated free abelian group (assumption (b) of the Alexander tuple). However, the arguments presented in this section works without this assumption.

Let X, N, H be normal subgroups of G such that $[N, H] \subset X \subset N \subset H$. Let $\Lambda := \mathbb{Z}[G/H]$ be the group ring of the (possibly noncommutative) quotient group G/H.

Then, the conjugation of G induces a structure of right Λ-module for the quotient group $M = N/X$.

By the same argument, we see that if $g \in N$, then

$$t(g) \subset t_{G/X}(q(g)) = \{\varepsilon(f) \mid f \in \mathrm{Ann}(p(g)) \text{ and } f \text{ is a positive element}\}$$

where $p : N \to N/X$ and $q : G \to G/X$ are the projection maps.

Unfortunately, it is not easy to use this noncommutative version.

4 The set $t(h)$

To utilize the results in the previous section, we need to know $t(h(t_1, \dots, t_s))$ for irreducible $h(t_1, \dots, t_s)$. In this section, we discuss the structure of the set $t(h)$ for irreducible one-variable Laurent polynomial $h \in \Lambda = \mathbb{Z}[t^{\pm 1}]$.

To begin with, we observe the following simple properties. For a positive integer $k > 0$, let $\Phi_k(t)$ be the kth cyclotomic polynomial and P_k be the set of roots of Φ_k, the set of primitive kth root of unities.

Lemma 4.1 *Assume that $h(t) = a_m t^m + \cdots + a_1 t + a_0$ $(a_0, a_m \neq 0,\ m \geq 1)$ is irreducible.*

(i) *$t(h) \neq \varnothing$ if and only if h has no positive real root.*
(ii) *$t(h) \subset |h(1)|\mathbb{Z}$.*
(iii) *If $n \in t(h(t))$ then $n \geq |a_m| + |a_0|$.*
(iv) *$2 \in t(h(t))$ if and only if $h(t) = \Phi_{2s}$ for some $s > 0$.*

Proof (i) This is proven in [10] (see also [4]).

(ii) If $n \in t(h(t))$, then there exists $g(t) = b_k t^k + \cdots + b_0$ $(b_k, b_0 \neq 0)$ such that $f(t) = g(t)h(t)$ is positive and that $n = f(1)$. Then, $n = g(1)h(1) = |g(1)||h(1)|$ so $|h(1)|$ always divides n.

(iii) Since $f(t) = g(t)h(t)$ is positive,

$$n = a_m b_k + \cdots + a_0 b_0 \geq |a_m b_k| + |a_0 b_0| \geq |a_m| + |a_0|.$$

(iv) If $2 \in t(h(t))$ there exists $g(t) \in \Lambda$ such that $f(t) = g(t)h(t) = 1 + t^d$ for some d or $g(t)h(t) = 2$. The latter case does not happen since we are assuming that h is not a constant. Thus, $h(t)$ divides $1 + t^d$ which implies that $h(t) = \Phi_{2s}$ for some divisor s of d. ∎

To get more constraints, we use the following quantity.

Definition 4.1 For $h(t) \in \Lambda$, we define

$$R_k(h) = \prod_{\zeta \in P_k} |h(\zeta)| \in \mathbb{Z}_{\geq 0}$$

where P_k is the set of primitive kth root of unities.

This is the absolute value of the resultant of $h(t)$ and the k-th cyclotomic polynomial $\Phi_k(t)$.

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Proposition 4.2 *Let $h \in \Lambda$ be an irreducible polynomial and $k = p^e$ be a power of a prime p. Then, if $n \in t(h)$, then either*

(a) *p divides n. Furthermore, if $h \neq \Phi_k(t)$, then $p|h(1)|$ divides n, or,*
(b) *$n^{\phi(k)} \geq R_k(h)$.*

holds. Here, $\phi(k) := \#P_k$ is the Euler's totient function.

Proof Assume that $f(t) = g(t)h(t)$ is positive and that $n = f(1) = g(1)h(1)$. Since $f(t)$ is positive, $n = |f(1)| \geq |f(\omega)|$ for all $\omega \in \{z \in \mathbb{C} \mid |z| = 1\}$. In particular, $n = f(1) \geq |f(\zeta)|$ for every root of unity ζ. Therefore,

$$n^{\phi(k)} = |f(1)|^{\phi(k)} \geq R_k(f) = R_k(g)R_k(h)$$

holds for all $k > 0$.

If $R_k(g) = 0$, then we may write $f(t) = \Phi_k(t)f^*(t)$ for some $f^*(t)$. Since $\Phi_k(1) = p$ if $k = p^e$,

$$n = f(1) = |\Phi_k(1)||f^*(1)| = p|f^*(1)|.$$

Furthermore, if $h \neq \Phi_k(t)$, then $f(t) = \Phi_k(t)g^*(t)h(t)$ for some $g^*(t)$, hence,

$$n = f(1) = |\Phi_k(1)||g^*(1)||h(1)| = p|h(1)||g^*(1)|.$$

Thus, in this case (a) holds.

If $R_k(g) \neq 0$ then $R_k(f) = R_k(g)R_k(h) \geq R_k(h)$ so (b) holds. ∎

The *Mahler measure* $M(f)$ of a polynomial $f(t) = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_0 \in \mathbb{Z}[t^{\pm 1}]$ is defined by

$$M(f) = |a_d| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$$

where $\alpha_1, \ldots, \alpha_d$ are zeros of $f(t)$. It is known that [12, 23, 25]

$$\lim_{k \to \infty} \left(\prod_{\zeta^k = 1} h(\zeta) \right)^{\frac{1}{k}} = \left(\prod_{d|k} R_d(h) \right)^{\frac{1}{k}} = M(h).$$

Thus by Proposition 4.2, we get the following interesting connection to Mahler measure.

Corollary 4.3 (Mahler measure bound) *If $h(t)$ is irreducible, then $n \geq M(h)$ for all $n \in t(h)$.*

We give some simple calculations which will be used later.

Example 4.4 Let $k = p^a$ be a power of a prime p. Since $\Phi_k(1) = p$, by Lemma 4.1 $t(\Phi_k) \subset p\mathbb{N}$. Indeed, $\Phi_k = t^{(p-1)p^{a-1}} + t^{(p-2)p^{(a-1)}} + \cdots + 1$ is positive so $p \in t(\Phi_k)$. Thus,

$$t(\Phi_{p^a}) = p\mathbb{N}.$$

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Let $k = p^a q^b$ where $p < q$ be primes and $a, b > 0$. Assume that $n \in t(\Phi_k)$. Since $R_p(\Phi_k) = q$ (see [1]), by Proposition 4.2, p divides n, or, $n \geq q$. Similarly, since $R_q(\Phi_k) = p$, either q divides n, or, $n \geq p$. Since $p < q$, we conclude that

$$t(\Phi_{p^a q^b}) \subset p\mathbb{N} \cup \mathbb{N}_{\geq q}.$$

5 Application: Generalized torsion elements of knot groups

In this section, we apply our arguments for knot groups. We refer to [9] as a reference for the knot theory and its relation to orderable group theory. Actually, our arguments can be applied for an *augmented group*, a pair (G, χ) consisting of a finitely generated group G and epimorphism $\chi : G \to \mathbb{Z}$ [25].

Let K be a knot in S^3, and $G = G(K) := \pi_1(S^3 \backslash K)$ be the knot group, the fundamental group of the knot complement. As we have mentioned in Example 3.3, the Alexander polynomial $\Delta_K(t)$ of the knot K in knot theory is the Alexander polynomial of the Alexander tuple $(G; ([[G,G],[G,G]],[G,G],[G,G]))$.

Proof of Theorem 1.2 Let $\Sigma_k(K)$ be the k-fold cyclic branched covering of K. If $\Sigma_k(K)$ is a rational homology sphere (that is, equivalent to saying that $\Delta_K(\zeta) \neq 0$ for every (not necessarily primitive) kth root of unities), then the order of homology is given by

$$|H_1(\Sigma_k(K);\mathbb{Z})| = \prod_{i=1}^{k} |\Delta_K(\zeta^i)| = \prod_{d|k} R_d(\Delta_K)$$

[26]. Since we are assuming that $\Delta_K(t)$ is irreducible, if $g \neq [[G,G],[G,G]]$ by Proposition 4.2, we have either (a) or (b). ■

Proof of Theorem 1.4 Since $\Delta_K(t)$ divides $(t^k - 1)$, $\Delta_K(t)$ is monic. A knot having the properties that $\deg \Delta_K(t) = 2g(K)$ and that $\Delta_K(t)$ is monic is called (integrally) *homologically fibered knot*. For such a knot, the Alexander module M of $G(K)$ has a square presentation matrix of the form $A = tI_{2g} - S$, where S is certain $2g \times 2g$ integer matrix [11].

By the definition of the tensor product module $M^{\otimes m}$, $M^{\otimes m}$ has a presentation matrix $A_m = tI_{(2g)^m} - S^{\otimes m}$ where $S^{\otimes m} : (\mathbb{Z}^{2g})^{\otimes m} \to (\mathbb{Z}^{2g})^{\otimes m}$ is the tensor product of S.

Let $\alpha_1, \ldots, \alpha_{2g} \in \mathbb{C}$ be the roots of the Alexander polynomial $\Delta_K(t)$. Then, for $m \geq 1$

$$\Delta_0(M^{\otimes m}) = \det(tI_{(2g)^m} - S^{\otimes m}) = \prod_{i_1=1}^{2g} \prod_{i_2=1}^{2g} \cdots \prod_{i_m=1}^{2g} (t - \alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_m}).$$

Since $\Delta_K(t)$ divides $t^k - 1$, $\alpha_1, \ldots, \alpha_{2g}$ are (not necessarily primitive) k-th root of unities. Thus their products $\alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_m}$ are also the kth root of unities. Since $k = p^a q^b$, every irreducible factor of $\Delta_0(M^{\otimes m})$ is a cyclotomic polynomial $\Phi_{p^{a'} q^{b'}}$ $(0 \leq a' \leq a, 0 \leq b' \leq b)$. Thus, by Corollary 3.13

$$t(g) \subset t(\Phi_{p^{a'} q^{b'}})$$

for some a', b' hence by Example 4.4, $t(g) \subset p\mathbb{N} \cup \mathbb{N}_{\geq q}$. ■

Example 5.1 Let K be the $(2,5)$ torus knot $K = T_{2,5}$ or the knot 10_{132}. They are fibered knots with the Alexander polynomial $\Delta_{T_{2,5}}(t) = t^4 - t^3 + t^2 - t + 1 = \Phi_{10}(t)$. By Theorem 1.4, $t(g) \subset 2\mathbb{N} \cup \mathbb{N}_{\geq 5}$ for every $g \in G(K)$. In particular, $G(K)$ has no generalized torsion element of generalized torsion order 3.

On the other hand, the knot $T_{2,5}$ has a generalized torsion element of generalized torsion order 2, whereas the knot 10_{132} has no generalized torsion element of generalized torsion order 2 (because it is hyperbolic).

Our result can be used to determine the generalized torsion order.

Example 5.2 (Himeno's generalized torsion element [14]) Let $K = T_{2,k}$ be the $(2,k)$-torus knot with $k > 3$. Himeno showed for $n > 0$, the element $E_n \in G(K) = \langle a, b \mid a^2 = b^k \rangle$ given by

$$E_n = [a,b]^n [a,b^2][a,b]^{n+1}[a,b^{k-1}],$$

satisfies $4 \in t(E_n)$. He verified $gord(E_1) = 4$ for small k by using computer calculations of the stable commutator length and the lower bound of generalized torsion order given in [16].

It is easy to check that $2 \notin t(E_n)$ because E_n and E_n^{-1} are not conjugate. If $k = q^e$ is a power of a prime $q > 3$, $3 \notin t(E_n)$ by Theorem 1.4. Therefore, $gord(E_n) = 4$ as expected.

Our result gives a following partial answer to [22, Question 6.6] which asks the existence of generalized torsion of $G(K)$ when $\Delta_K(t)$ is nontrivial and has no positive real roots.

Proposition 5.3 *Let K be a knot in S^3 and $G = G(K)$ be the knot group. If $\Delta_K(t)$ has no positive real root and is nontrivial then $G/[[G,G],[G,G]]$ has a generalized torsion element.*

Proof This is an immediate consequence of Lemma 3.4 and Lemma 4.1(i). ■

A knot satisfying the condition $\deg \Delta_K(t) = 2g(K)$ is called *rationally homologically fibered knot*. In [15] we showed that for a rationally homologically fibered knot K, if $\Delta_K(t)$ has no positive real root then $G = G(K)$ is not bi-orderable. This result and Proposition 5.3 raise the following question.

Question 2 If K is rationally homologically fibered and $\Delta_K(t)$ has no positive real root, is there a generalized torsion element $g \in G = G(K)$ such that $g \notin [[G,G],[G,G]]$?

Finally, we observe the following existence result of generalized torsion elements.

Proposition 5.4 *Let K be a (p^a, q^b)-torus knot T_{p^a,q^b} or a (p^a, q^b)-cable knot where $p < q$ are primes. Then, its knot group $G(K)$ admits a generalized torsion elements with $gord(g) = p$ such that $g \notin [[G(K),G(K)],[G(K),G(K)]]$.*

Proof Assume that K is a (p^a, q^b)-cable of a knot C where we allow C to be the trivial knot (in such case, K is just the (p^a, q^b)-torus knot). Then, its knot group is the amalgamated free product $G(K) = G(C) *_{\mu^{q^b}\lambda^{p^a}=y^{p^a}} \langle y \rangle$, where μ and λ are the meridian and the longitude of P. Since $[\mu, y^{p^{a-1}}] \neq 1$ but $[\mu, y^{p^a}] = [\mu, \mu^{q^b}\lambda^{p^a}] = 1$, $p \in t_{G(K)}([\mu, y^{p^{a-1}}])$ [22]. Furthermore, $[\mu, y^{p^{a-1}}] \notin [[G(K), G(K)], [G(K), G(K)]]$.

Let $\pi : G(C) \to \mathbb{Z}$ be the projection map, $x = \pi(\mu)$ and let $G = G(T_{a,b})$ be the torus knot group. The projection induces the epimorphism

$$\pi : G(K) = G(C) *_{\mu^{q^b}\lambda^{p^a}=y^{p^a}} \mathbb{Z} \to \mathbb{Z} *_{x^{q^b}=y^{p^a}} \mathbb{Z} = G(T_{p^a,q^b}) = G.$$

Since $\Delta_{T_{p^a,q^b}}(t) = \frac{(t^{p^a q^b}-1)(t-1)}{(t^{p^a}-1)(t^{q^b}-1)}$, the irreducible factors of $\Delta_{T_{p^a,q^b}}(t)$ are cyclotomic polynomial $\Phi_{p^{a'}q^{b'}}(t)$ for some $0 \leq a' \leq a, 0 \leq b' \leq b$. Since $f(\pi([\mu, y^{p^{a-1}}])) \notin [[G,G],[G,G]]$, by Theorem 1.1, Example 4.4, and the monotonicity

$$t_{G(K)}([\mu, y^{p^{a-1}}]) \subset t_G(\pi([\mu, y^{p^{a-1}}])) \subset p\mathbb{N} \cup \mathbb{N}_{\geq q}.$$

Therefore, $gord_{G(K)}([\mu, y^{p^{a-1}}]) = p$. ∎

6 Generalized torsion order spectrum

For a group G, the *torsion order spectrum* $ord(G)$ is the set defined by

$$ord(G) = \{ord(g) \mid g \text{ is a torsion element of } G\}.$$

It is easy to see that $ord(G)$ cannot be arbitrary because $ord(G)$ is *factor-complete*, which means that if $pq \in ord(G)$ with $p, q \neq 1$ then $p, q \in ord(G)$.

The torsion order spectrum is characterized as follows [8].

Theorem 6.1

(i) *For a factor-complete subset A of $\mathbb{N}_{\geq 2}$, there exists a finitely generated group G such that $ord(G) = A$.*

(ii) *For a factor-complete subset A of $\mathbb{N}_{\geq 2}$, there exists a finitely presented group G such that $ord(G) = A$ if and only if A is a Σ^0_2-set ([8, Theorem 6.3]).*

Here, Σ^0_2-set is a set appeared in a theory of arithmetical hierarchy, and is larger than Σ^0_1-set, the recursively enumerable sets.

As a natural generalization of torsion order spectrum, it is natural to investigate the following set.

Definition 6.1 Let G be a group. The *generalized torsion order spectrum* $gord(G)$ of G is

$$gord(G) = \{gord(g) \mid g \text{ is a generalized torsion element of } G\}.$$

The *strict generalized torsion order spectrum* $gord^*(G)$ of G is

$$gord^*(G) = \{gord(g) \mid g \text{ is a non-torsion, generalized torsion element of } G\}.$$

https://doi.org/10.4153/S0008439524000432 Published online by Cambridge University Press

By definition, $gord^*(G) \subset gord(G) \subset \bigcup_{g \in G} t(g)$. Unlike torsion order spectrum, generalized torsion order spectrum is not necessarily factor-complete as the next lemma shows.

Lemma 6.2 *For every $n \in \mathbb{Z}_{\geq 2}$, there exists a finitely presented torsion-free group G_n such that $gord^*(G_n) = gord(G_n) = \{n\}$.*

Proof When $n = 2$, let $G_2 = \langle a, t \mid tat^{-1} = a^{-1} \rangle$ be the infinite diherdal group. If $g \in G_2$ is a generalized torsion element, then $g \in \langle a \rangle$. However, every element in $\langle a \rangle$ is a generalized torsion element of generalized torsion order two because $ta^k t^{-1} \cdot a^k = 1$ for every k. Thus, $gord^*(G_2) = \{2\}$.

For $n \geq 3$ and $n \neq 4$, let A be the free abelian group of rank two generated by a, b and let

$$G_n = \langle t, a, b \mid tat^{-1} = a^{-n+2}b, tbt^{-1} = a^{-1}, [a, b] = 1 \rangle.$$

G_N is the HNN extension $1 \to A \to G \to \mathbb{Z} = \langle t \rangle$. If $g \in G_n$ is a generalized torsion element, then $g \in \langle a, b \rangle$. The Alexander polynomial of G is an irreducible polynomial $t^2 + (n-2)t + 1$. If $g \in G_n$ is a generalized torsion element, then $g \in [G_n, G_n] = A$. For every $1 \neq g \in A$, $t^2 g t^{-2} (tgt^{-1})^{n-2} g = 1$. Furthermore, by Theorem 1.1 and Lemma 4.1, $t(g) \subset n\mathbb{N}$. Thus, $gord^*(G_n) = \{n\}$.

The group G_4 is constructed in a similar manner. Let A be the free abelian group generated by a, b, c. We define

$$G_4 = \left\langle t, a, b, c \,\middle|\, \begin{array}{c} tat^{-1} = b, tbt^{-1} = c, tct^{-1} = a^{-1}b^{-2} \\ [a, b] = [a, c] = [b, c] = 1 \end{array} \right\rangle.$$

G_4 is the HNN extension $1 \to A \to G \to \mathbb{Z} = \langle t \rangle \to 1$. Its Alexander polynomial is $t^3 + 2t + 1$. Since it is irreducible, we conclude $gord^*(G_4) = \{4\}$ by the same argument. ∎

The strict generalized torsion order spectrum behaves nicely with respect to the free product.

Theorem 6.3 *If G and H are torsion-free, then $gord^*(G * H) = gord^*(G) \cup gord^*(H)$.*

Proof By [16, Theorem 1.5], a generalized torsion element x of $G * H$ is conjugate to a generalized torsion element of G or H if G and H are torsion-free. Since generalized torsion order is invariant under conjugation, we assume that $x \in G \subset G * H$ (or $x \in H$). Since the inclusion map $G \hookrightarrow G * H$ is a retract, $gord_G(x) = gord_{G*H}(x)$ by Corollary 2.3. Thus, $gord^*(G * H) = gord^*(G) \cup gord^*(H)$. ∎

These two results give the following realization result.

Corollary 6.4 *For every subset $A \subset \mathbb{N}_{\geq 2}$, there exists a countable, torsion-free group G such that $gord^*(G) = gord(G) = A$.*

It is interesting to ask when we can take such a group G finitely generated (or, finitely presented, with suitable complexity assumption on A). For a torsion spectrum case, Higman–Neumann–Neumann embedding theorem allows us to embed countable groups to finitely generated groups so that the set of torsion elements are the same. For generalized torsion case, we do not know similar embedding is possible or not.

Question 3 (Higman–Neumann–Neumann embedding preserving generalized torsion equation spectrum/generalized torsion orders) Let G be a countable group. Is there an embedding of G into a finitely generated group H such that $t_G(g) = t_H(g)$ (or, $gord_G(g) = gord_H(g)$) for all $g \in G$?

A Appendix. Generalized torsion elements and G-invariant norms

In this appendix, we show that G-invariant norms can be used to evaluate the generalized torsion orders. This is an extension of our previous observation that the stable commutator length gives a lower bound of the generalized torsion order [16].

Definition A.2 Let N be a normal subgroup of a group G. We say that a function $v : N \to \mathbb{R}_{\geq 0}$ is

- *G-invariant* if $v(gag^{-1}) = v(a)$ for all $a \in N$ and $g \in G$.
- *Symmetric* if $v(a^{-1}) = v(a)$ for all a.
- *Homogeneous* if $v(a^n) = nv(a)$ for all a and $n \in \mathbb{N}$.
- A *norm* if $v(ab) \leq v(a) + v(b)$ for all $a, b \in N$.
- A *quasi-norm* if there exists a constant $D_v \geq 0$ such that $v(ab) \leq v(a) + v(b) + D_v$ for all $ab \in N$. We call D_v the *defect* of v.

We often allow v to take the value ∞. When $N = G$, a G-invariant norm is usually called a *conjugation-invariant norm* of the group G. Such a norm has been studied in several places. See [5] for the relation to geometry.

For $v : N \to \mathbb{R}$, we define its *symmetrization* $v^s : N \to \mathbb{R}$ by $v^s(a) = \frac{v(a)+v(a^{-1})}{2}$. Symmetrization preserves the property that v is G-invariant, homogeneous, norm, quasi-norm.

Remark A.5

(i) Although we call v a norm, it is actually a *semi-norm* since we do not require $v(g) = 0$ iff $g = 1$. Indeed, we even do not assume $v(1) = 0$.
(ii) If v is a quasi-norm, then $v + D_v : G \to \mathbb{R}_{\geq 0}$ given by $(v + D_v)(g) = v(g) + D_v$ is a norm. In particular, if v is a G-invariant quasi-norm then $v + D_v$ is G-invariant norm.

Example A.6 (G-invariant quasimorphism) A map $\phi : N \to \mathbb{R}$ is a *quasimorphism* if

$$D_\phi = \sup_{a,b \in N} |\phi(ab) - \phi(a) - \phi(b)| < \infty.$$

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D_ϕ is called the *defect* of ϕ. A quasimorphism ϕ is G-invariant if $\phi(gag^{-1}) = \phi(a)$. The absolute value $|\phi|$ of a G-invariant quasimorphism ϕ gives a G-invariant quasi-norm with defect D_ϕ. Thus, $|\phi| + D_\phi : N \to \mathbb{R}$ given by $(|\phi| + D_\phi)(g) = |\phi(g)| + D_\phi$ is a G-invariant norm.

Example A.7 (Mixed commutator length) The *mixed commutator length* $cl_{G,N}(g)$ of an element $g \in [G, N]$ is the minimum number of commutators of the form $[x, a]$ or $[a, x]$ ($x \in G, a \in N$) whose product is equal to g. Clearly, the mixed commutator length $cl_{G,N}$ is a G-invariant symmetric norm.

For a G-invariant quasi-norm ν, its *stabilization* (or, *homogenization*) $\overline{\nu} : G \to \mathbb{R}_{\geq 0}$ is defined by

$$\overline{\nu}(g) = \lim_{n\to\infty} \frac{\nu(g^n)}{n}.$$

Then, $\overline{\nu}$ is homogeneous and G-invariant. The next lemma gives a sufficient condition for $\overline{\nu}$ to be a quasi-norm.

Lemma A.8 *For a G-invariant quasi-norm ν of N,*

$$\overline{\nu}(gh) \leq \overline{\nu}(g) + \overline{\nu}(h) + \frac{1}{2} \sup_{s,t\in N} \nu([s,t]).$$

To prove the lemma, we use the following.

Lemma A.9 *For $g_1, \dots, g_n \in G$ and $k > 0$,*

$$(g_1 g_2 \dots g_n)^{2k} = g_1^{2k} g_2^{2k} \dots g_n^{2k} \cdot ((n-1)k \textit{ commutators}).$$

Similarly,

$$(g_1 g_2 \dots g_n)^{2k-1} = g_1^{2k-1} g_2^{2k-1} \dots g_n^{2k-1} \cdot ((n-1)k \textit{ commutators}).$$

Proof We prove the lemma by induction on n. The case $n = 2$ is well-known; see [6, Lemma 2.24] for example. For $n > 1$, by induction

$$\begin{aligned}(g_1 g_2 \dots g_n)^{2k} &= (g_1(g_2 \dots g_n))^{2k} \\ &= g_1^{2k}(g_2 \dots g_n)^{2k} \cdot (k \text{ commutators}) \\ &= \dots \\ &= g_1^{2k} g_2^{2k} \dots g_n^{2k} \cdot ((n-1)k \text{ commutators}).\end{aligned}$$

The $2k-1$ case is similar. ∎

Proof of Lemma A.8 By Lemma A.9, $(gh)^{2n} = g^{2n}h^{2n}(n \text{ commutators})$ hence

$$\nu((gh)^{2n}) \leq \nu(g^{2n}) + \nu(h^{2n}) + n \sup_{t,s\in N} \nu([t,s]).$$ ∎

Then, we have the following estimate of generalized torsion order.

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Theorem A.10 *Let N be a normal subgroup of a group G. Assume that $g \in N$ and $n \in t_G(g)$. Then, for a G-invariant norm ν of N, if $\sup\{\nu([x,t]) \mid x \in G, t \in N\} < \infty$, then*

$$\overline{\nu}(g) \leq \frac{n-2}{2n} \sup\{\nu([t,s]) \mid t,s \in N\}.$$

Proof Since $g \in N$ satisfies the order n generalized torsion equation, there exists $x_1, \ldots, x_{n-1} \in G$ such that

$$g^{-1} = g^{x_1} \ldots g^{x_{n-1}}.$$

Therefore, by taking $2k$th powers for $k > 0$, by Lemma A.9

$$g^{-2k} = (g^{x_1})^{2k}(g^{x_2})^{2k} \ldots (g^{x_{n-1}})^{2k} \cdot ((n-2)k \text{ commutators})$$

so

$$g^{-2nk} = [x_1, g^{2k}]^*[x_2, g^{2k}]^* \ldots [x_{n-1}, g^{2k}]^* \cdot ((n-2)k \text{ commutators}).$$

Here, $[x,y]^*$ means a conjugate of $[x,y]$. Furthermore, the $(n-2)k$ commutators are actually commutators of elements of N (by Lemma A.9 applied to $G = N$). Therefore,

$$\frac{\nu(g^{2nk})}{2nk} \leq \frac{n-1}{2nk} \sup_{x \in G, t \in N} \nu([x,t]) + \frac{(n-2)}{2n} \sup_{s,t \in N} \nu([s,t]).$$

By taking $k \to \infty$, we get the desired inequality. ■

Theorem A.10 makes sense only if $\sup_{s,t \in N} \nu([s,t]) < \infty$. In this case, the assumption $\sup\{\nu([x,t]) \mid x \in G, t \in N\} < \infty$ is automatically satisfied.

Example A.11 (Stable mixed commutator length [18, 19]) The *stable mixed commutator length* $scl_{G,N}(g)$ is the stabilization of the mixed commutator length $cl_{G,N}$. When $N = G$, the (stable) mixed commutator length is called the *(stable) commutator length* of G denoted by cl_G and scl_G, respectively.

For an element $g \in G$ such that $g^\ell \in [G,N]$ for some $\ell > 0$ one can define the stable mixed commutator length $scl_{G,N}(g)$ by $scl_{G,N}(g) = \frac{scl_{G,N}(g^\ell)}{\ell}$.

Applying Theorem A.10 for mixed commutator length or G-invariant homogeneous quasimorphisms, we get the following.

Corollary A.12 *If $g \in N$ and $n \in t_G(g)$*

$$scl_{G,N}(g) \leq \frac{n-2}{2n} \left(< \frac{1}{2}\right) \tag{A.1}$$

and

$$|\phi(g)| \leq \frac{n-2}{2n}\left(\sup_{s,t \in N} |\phi([s,t])| + D_\phi\right) = \frac{n-2}{n} D_\phi{}^{2} \tag{A.2}$$

for every G-invariant homogeneous quasimorphism $\phi : N \to \mathbb{R}$.

[2] Here, we use the equality $\sup_{s,t \in N} |\phi([s,t])| = D_\phi$ [3, 6].

https://doi.org/10.4153/S0008439524000432 Published online by Cambridge University Press

When $N = G$, (A.1) is nothing but [16, Theorem 2]. Since $scl_G(g) \leq scl_{G,N}(g)$, Corollary A.12 gives stronger restriction.

Remark A.13 It is known that $\sup_\phi \frac{|\phi(g)|}{2D_\phi} = scl_{G,N}(g)$ where ϕ runs all G-invariant homogeneous quasimorphism which is not a homomorphism (Bavard's duality; [3, 18]) so (A.2) follows from (A.1) and vice versa. We remark that our argument does not use these results whose proof uses Hahn–Banach theorem, although it requires that g is a generalized torsion element.

In a similar vein, we have the following variant of Theorem A.10.

Proposition A.14 *Let N be a normal subgroup of a group G. Assume that $g \in N$ and $n \in t_G(g)$. Then, for a symmetric G-invariant norm ν of N, $\nu(g^n) \leq (n-1)\sup\{\nu([x,g]) \mid x \in G\}$ holds. In particular,*

$$\overline{\nu}(g) \leq \frac{n-1}{n}\sup\{\nu([x,g]) \mid x \in G\}.$$

Proof Assume that $g^{-1} = g^{x_1} \dots g^{x_{n-1}}$ for some $x_1, \dots, x_{n-1} \in G$. Then,

$$g^{-n} = [x_1, g]^* \dots [x_{n-1}, g]^*$$

where $[x_i, g]^*$ means a suitable conjugate of $[x_i, g]$. ∎

Example A.15 ((Stable) γ_k-length [7]) The γ_k*-length* ℓ_{γ_k} of a group G is the minimum number the kth commutator $[g_1,[g_2,[\dots,[g_{k-1},g_k]]\dots]](g_i \in G)$ and its inverses that is needed to express g. The γ_k-length is a G-invariant norm on the $(k-1)$th lower central subgroup $\Gamma_{k-1}G = [G,[G,[\dots,[G,G]]\dots]]$.

Assume that $g \in \Gamma_k G$ and $n \in t_G(g)$. If $\ell_{\gamma_{k-1}}(g) = 1$, then $\ell_{\gamma_k}([x,g]) = 1$ for all $x \in G$ so Proposition A.14 shows that the stable γ_k-length satisfies

$$\overline{\ell_{\gamma_k}}(g) \leq \frac{n-1}{n}.$$

Acknowledgements The author is partially supported by JSPS KAKENHI Grant Numbers 19K03490, 21H04428, 23K03110. We thank the anonymous referees for helpful suggestions to improve the exposition.

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