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A minimal distal map on the torus with sub-exponential measure complexity

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Abstract. In this paper the notion of sub-exponential measure complexity for an invariant Borel probability measure of a topological dynamical system is introduced. Then a minimal distal skew product map on the torus with sub-exponential measure complexity is constructed.

1. *Introduction*

Let (X, T) be a *topological dynamical system* (t.d.s. for short), that is, X is a compact metric space and $T : X \to X$ is a continuous self map. The distance on X will be denoted by $d(\cdot, \cdot)$ and the set of all T-invariant Borel probability measures on X will be denoted by $\mathcal{M}(X, T)$.

In the measurable dynamics, there are several ways to measure the complexity of a system. Kolmogorov introduced the notion of entropy that measures the average growth rate of the orbits. Positive entropy means the average growth rate of the orbits is exponential. The other well-known definitions of the complexity are due to Katok [**7**] using Bowen balls and to Ferenzi [**1**] using the Hamming distance. To study the Sarnak conjecture, recently Huang, Wang and Ye [**5**] introduced a notion of measure complexity following the idea of Ferenczi in [**1**] using mean distance instead of the Hamming distance, and showed that the Sarnak conjecture holds for all systems with sub-polynomial measure complexity.

In 1968 Parry proved any invariant Borel probability measure of a distal system has zero measure entropy [**11**]. By the Furstenberg structure theorem, any minimal distal system is the inverse limit of equicontinuous extensions [**2**]. It seems that such a system should have lower measure complexity. Surprisingly, this is not the case. Namely, we can construct a minimal distal system with sub-exponential measure complexity for any invariant Borel probability measure of the system.

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We now outline the construction. To do so, first we introduce notions of measure complexity and sub-exponential measure complexity. For any t.d.s. (X, T), $\rho \in \mathcal{M}(X, T)$ and any $n \in \mathbb{N}$, we consider the mean metric $\bar{d}_n$ on X,

$$\bar{d}_n(x, y) = \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y),$$

for any $x, y \in X$. See [**5**, **10**, **14**] for the role of this metric in studying mean dimension and measure complexity.

For any $\epsilon > 0$ and $n \in \mathbb{N}$, let

$$S_n(d, \rho, \epsilon) = \min\left\{m \in \mathbb{N} : \exists x_1, x_2, \ldots, x_m \in X \text{ s.t. } \rho\left(\bigcup_{i=1}^{m} B_{\bar{d}_n}(x_i, \epsilon)\right) > 1 - \epsilon\right\},$$

where $B_{\bar{d}_n}(x, \epsilon) := \{y \in X : \bar{d}_n(x, y) < \epsilon\}$ for any $x \in X$. We remark that $S_n(d, \rho, \epsilon) < \infty$.

We say that the measure-theoretic dynamical system $(X, \mathcal{B}, \rho, T)$ has *sub-exponential measure complexity* if, for any $0 < \tau < 1$,

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty} \frac{\log S_n(d, \rho, \epsilon)}{n^\tau} = +\infty, \quad \text{and} \quad \lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{\log S_n(d, \rho, \epsilon)}{n} = 0.$$

The definition of sub-exponential measure complexity is independent of the metric (see [**5**]). Thus, we can simply say that the measure complexity of (X, T, ρ) is sub-exponential. We note that the above definition is also applied to any measurable system $(X, \mathcal{B}, T, \mu)$ when X is a metrizable space. We also note that in this case it may happen that $S_n(d, \rho, \epsilon) = \infty$. In [**5**] the first and third authors showed that for any ergodic $\rho \in \mathcal{M}(X, T)$, we have

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} \frac{1}{n}\log S_n(d, \rho, \epsilon) = h_\rho(T) = \lim_{\epsilon\to 0}\liminf_{n\to\infty} \frac{1}{n}\log S_n(d, \rho, \epsilon).$$

So when an ergodic system has sub-exponential measure complexity, the entropy $h_\rho(T)$ is zero.

We are now ready to explain the idea of the construction. We construct our example in the following way. First we construct a measurable map $h : \mathbb{R} \to \{0, 1\}$ and obtain a measurable distal system $(X, \mathcal{B}_X, T, \rho)$ on $\mathbb{T}^2$ such that

$$T(x, y) = (x + \alpha, y + \tfrac{1}{2}h(x)), \quad x, y \in \mathbb{T},$$

where $X = \mathbb{T}^2$, α is irrational, $\mathcal{B}_X$ is the Borel σ-algebra and ρ is measure preserving. Using a complicated computation, we show that the measure complexity of the system is sub-exponential. By a result of Lindenstrauss (see [**9**, Theorem 3.1]), there exist a measurable function $p : \mathbb{T} \to \mathbb{T}$ and continuous function $\tilde{h} : \mathbb{T} \to \mathbb{T}$ such that

$$\tilde{h}(x) = \tfrac{1}{2}h(x) + p(x + \alpha) - p(x)$$

for $m_{\mathbb{T}}$-almost every $x \in \mathbb{T}$. We define a distal system $\tilde{T} : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$\tilde{T}(x, y) = (x + \alpha, y + \tilde{h}(x)).$$

Then we prove that the measure complexity of $(\mathbb{T}^2, \tilde{T}, \tilde{\rho})$ is the same as T for any $\tilde{\rho} \in \mathcal{M}(\mathbb{T}^2, \tilde{T})$, and $(\mathbb{T}^2, \tilde{T})$ is minimal, and thus finish the construction. We remark that to show the minimality we use the following proposition (Proposition 4.3): if a skew product map $W : \mathbb{T}^2 \to \mathbb{T}^2$ over an irrational rotation on $\mathbb{T}$ is not minimal then it is equicontinuous.

To conclude the introduction we make the following remarks.

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Remark 1.1. Define a skew product map $T : \mathbb{T}^2 \to \mathbb{T}^2$ with $T(x, y) = (x + \alpha, y + k(x))$, where α is irrational and $k : \mathbb{T} \to \mathbb{R}$ is continuous.

(1) If $k(x) = \beta$, and α and β are rationally independent, then T is minimal and uniquely ergodic. Thus, the measure complexity is bounded for the unique measure (in [**4**] the authors construct a uniquely ergodic, minimal, distal and non-equicontinuous map on $\mathbb{T}^2$ with bounded measure complexity for the unique measure).

(2) If k is a homotopically trivial C^∞-function, Huang, Wang and Ye [**5**] showed that the measure complexity is *sub-polynomial* (i.e. for any $\epsilon > 0$ and any $\tau > 0$, $\liminf_{n\to\infty} (\log S_n(d, \rho, \epsilon)/n^\tau) = 0$) for any $\rho \in \mathcal{M}(\mathbb{T}^2, T)$.

(3) If k has a bounded variation, Qiao [**12**] showed that the measure complexity is *polynomial*.

(4) If $k = \tilde{h}$, then the measure complexity is *sub-exponential*.

This indicates that the simple system $(\mathbb{T}^2, T)$ (depending on k) may have various measure complexities, from the simplest to the most complicated (for a zero-entropy system). Thus, the system is a good touchstone to study the Sarnak conjecture.

Remark 1.2. Let $(\mathbb{T}^2, \tilde{T})$ be the system we defined above and

$$d_n(x, y) = \max_{0\le i\le n-1} d(\tilde{T}^i x, \tilde{T}^i y).$$

It is clear that $\bar{d}_n(x, y) \le d_n(x, y)$ for $x, y \in \mathbb{T}^2$. Thus, $B_{d_n}(x, \epsilon) \subset B_{\bar{d}_n}(x, \epsilon)$. Define

$$r_n^K(d, \rho, \epsilon) = \min\left\{m \in \mathbb{N} : \exists x_1, x_2, \ldots, x_m \in X \text{ s.t. } \rho\left(\bigcup_{i=1}^m B_{d_n}(x_i, \epsilon)\right) > 1 - \epsilon\right\}.$$

It is easy to see that $S_n(d, \rho, \epsilon) \le r_n^K(d, \rho, \epsilon)$ for any $n \in \mathbb{N}$. So the measure complexity of $(\mathbb{T}^2, \tilde{T}, \rho, d)$ is also sub-exponential in Katok's sense if ρ is ergodic (see [**7**]).

Topologically, we may also define the complexity in the same fashion. Namely, let

$$r_n(d, \epsilon) = \min\left\{m \in \mathbb{N} : \exists x_1, x_2, \ldots, x_m \in X \text{ s.t. } \bigcup_{i=1}^m B_{d_n}(x_i, \epsilon) = X\right\}$$

and

$$S_n(d, \epsilon) = \min\left\{m \in \mathbb{N} : \exists x_1, x_2, \ldots, x_m \in X \text{ s.t. } \bigcup_{i=1}^m B_{\bar{d}_n}(x_i, \epsilon) = X\right\}.$$

Then we have

$$r_n^K(d, \rho, \epsilon) \le r_n(d, \epsilon) \text{ and } S_n(d, \rho, \epsilon) \le S_n(d, \epsilon)$$

for any $\rho \in \mathcal{M}(\mathbb{T}^2, \tilde{T})$. Thus, the topological complexity of $(\mathbb{T}^2, \tilde{T}, d)$ in both senses is also sub-exponential, since the topological entropy of $(\mathbb{T}^2, \tilde{T})$ is zero.

This paper is organized as follows. In §2 we construct a measurable map $h : \mathbb{R} \to \{0, 1\}$ with some properties we will need later. In §3 we compute the measure complexity of the measurable distal system $(X, \mathcal{B}_X, T, \rho)$. Then, in the final section, we use Lindenstrauss's result to get a t.d.s. $(\mathbb{T}^2, \tilde{T})$ and show that the measure complexity of $(\mathbb{T}^2, \tilde{T}, \tilde{\rho})$ is sub-exponential for any $\tilde{\rho} \in \mathcal{M}(\mathbb{T}^2, \tilde{T})$.

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2. *The construction of the function h*

In this section we will construct a measurable map $h:\mathbb{R}\to\{0,1\}$ with some properties we will need later. To do so, we fix an irrational number α and an $\eta\in(0,\frac{1}{100})$. Let $R_\alpha:\mathbb{T}\to\mathbb{T}, x\mapsto x+\alpha$ be the rotation on $\mathbb{T}$ by α.

2.1. *Preparation.*

Given an interval $E=[0,a)\subset\mathbb{T}$ with $0<a<1$, define

$$f_1:\mathbb{T}\to\mathbb{R}\quad \text{with } f_1(x)=\chi_E(x)\chi_E(R_\alpha x) \text{ for any } x\in\mathbb{T}$$

and $f_i:\mathbb{T}\to\mathbb{R}$ with

$$f_i(x)=\chi_E(x)\chi_{E^c}(R_\alpha x)\cdots\chi_{E^c}(R_\alpha^{i-1}x)\chi_E(R_\alpha^i x)\quad \text{for any } x\in\mathbb{T},$$

and for any $i=2,3,\ldots$. For given $i\in\mathbb{N}$, $x\in\mathbb{T}$ and $n>i$, set

$$s(i,n,E,x)=\#\{0\le j\le n-i-1: f_i(R_\alpha^j x)=1\}. \tag{2.1}$$

LEMMA 2.1. *For a fixed* $i\in\mathbb{N}$, *the sequence* $\{(1/n)s(i,n,E,x)\}_{n=i+1}^\infty$ *uniformly converges to a constant* $\rho_i(E)$ *for all* $x\in\mathbb{T}$. *Moreover,*

$$\{i\in\mathbb{N}:\rho_i(E)>0\}$$

is a finite set and

$$\sum_{i=1}^{+\infty} i\rho_i(E)=1.$$

Proof. Set $E_1=E\cap R_\alpha^{-1}E$ and

$$E_i=E\cap R_\alpha^{-1}E^c\cap\cdots\cap R_\alpha^{-(i-1)}E^c\cap R_\alpha^{-i}E,$$

for $i=2,3,\ldots$. Clearly, E_i is a finite union of disjoint intervals of $\mathbb{T}$ or $E_i=\emptyset$. Let $m_\mathbb{T}$ be the Lebsegue measure on $\mathbb{T}$. For $i\in\mathbb{N}$, we put $\rho_i(E)=m_\mathbb{T}(E_i)$. Then, for a fixed $i\in\mathbb{N}$, by the unique ergodicity of R_α,

$$\begin{aligned}\frac{1}{n}s(i,n,E,x)&=\frac{1}{n}\sum_{j=0}^{n-i-1} f_i(R_\alpha^j x)\\ &=\frac{1}{n}\sum_{j=0}^{n-i-1}\chi_{E_i}(R_\alpha^j x)\to\int_\mathbb{T}\chi_{E_i}\,dm_\mathbb{T}=m_\mathbb{T}(E_i)=\rho_i(E)\end{aligned}$$

uniformly as n goes to ∞ for all $x\in\mathbb{T}$, as $\chi_{E_i}(R_\alpha^j x)\le 1$ for all $n-i\le j\le n-1$.

In fact, that $\{i\in\mathbb{N}:\rho_i(E)>0\}$ is finite follows from the fact that $E_i=\phi$ for large i. That is, there exists $N\in\mathbb{N}$ large enough so that $\{x,R_\alpha x,\ldots,R_\alpha^{N-1}x\}$ is $a/2$ dense in $\mathbb{T}$ for any $x\in\mathbb{T}$. This means that for any $x\in E$, there is $1\le j=j(x)\le N-1$ such that $R_\alpha^j x\in E$. Hence, $E_i=\phi$ and $f_i(x)\equiv 0$ for any $x\in\mathbb{T}$ and $i>N$, which implies that there are only finitely many indices $i\in\mathbb{N}$ such that $\rho_i(E)>0$.

By the Poincaré recurrence theorem, the map $n_E:E\to\mathbb{N}$,

$$n_E(x)=\inf\{n\ge 1: R_\alpha^n(x)\in E\}$$

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is well defined for $m_{\mathbb{T}}$-almost every $x \in E$. It is well known that $\int_E n_E(x)\, dm_{\mathbb{T}}(x) = 1$ by Kac [**6**]. Thus

$$\sum_{i=1}^{\infty} i\rho_i(E) = \sum_{i=1}^{\infty} i m_{\mathbb{T}}(E_i) = \sum_{i=1}^{\infty} \int_{E_i} n_E(x)\, dm_{\mathbb{T}}(x) = \int_E n_E(x)\, dm_{\mathbb{T}}(x) = 1.$$

We conclude that there exists some index $i \in \mathbb{N}$ such that $\rho_i(E) > 0$. □

Set

$$\mathcal{I}(E) = \{i \in \mathbb{N} : \rho_i(E) > 0\} \quad \text{and} \quad \rho(E) = \tfrac{1}{2} \min\{\rho_i(E) : i \in \mathcal{I}(E)\}.$$

By Lemma 2.1, it is clear that $\mathcal{I}(E)$ is a non-empty finite set and $\rho(E) > 0$.

For $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$, the binomial coefficients are given by the formula

$$C_n^k := \begin{cases} \prod_{i=1}^{k} \dfrac{n+1-i}{i} & \text{if } k \in \mathbb{N}, \\ 1 & \text{if } k = 0. \end{cases}$$

By Stirling's approximation, there exists $C > 0$ such that

$$C_n^{[5\eta n]} \leq C e^{a(\eta)n} \quad \text{with } a(\eta) = -5\eta \log(5\eta) - (1-5\eta)\log(1-5\eta), \tag{2.2}$$

for any $n \in \mathbb{Z}_+$.

LEMMA 2.2. *Let $E = [0, a) \subset \mathbb{T}$ and $0 < t < 1$. Then there exists $N(E, t) \in \mathbb{N}$ such that, for any $n \geq N(E, t)$, $i \in \mathcal{I}(E)$ and $x \in \mathbb{T}$, one has:*

(1) $(1/n) \sum_{i \in \mathcal{I}(E)} i \cdot s(i, n, E, x) > 1 - \eta$;

(2) $(1/n) s(i, n, E, x) > \rho(E)$;

(3) $(1/n^t)(n\rho(E)(\log 2 - a(\eta)) - \log(10\#\mathcal{I}(E)Cn)) > \frac{1}{2}$.

Proof. (1), (2) and (3) follow from Lemma 2.1 and the fact $\lim_{n\to+\infty} (n/n^t) = +\infty$. In fact, since $\mathcal{I}(E)$ is a finite set, when n is large we have

$$\frac{1}{n} \sum_{i \in \mathcal{I}(E)} i \cdot s(i, n, E, x) = \sum_{i \in \mathcal{I}(E)} i \cdot \frac{1}{n} s(i, n, E, x) > (1-\eta) \sum_{i \in \mathcal{I}(E)} i\rho_i(E) = 1 - \eta. \quad \square$$

Let $[a, b) \subseteq \mathbb{R}$. We write

$$\mathcal{P}([a, b)) = \{\Delta : \Delta \text{ is a finite partition } a = a_1 < a_2 < \cdots < a_k = b \text{ of } [a, b)\}.$$

For $\Delta : a = a_1 < a_2 < \cdots < a_k = b \in \mathcal{P}([a, b))$, we define

$$l^*(\Delta) = \max_{1 \leq i \leq k-1} \{a_{i+1} - a_i\} \quad \text{and} \quad l_*(\Delta) = \min_{1 \leq i \leq k-1} \{a_{i+1} - a_i\}.$$

We also consider the function $\xi_\Delta(x)$ on $[a, b)$:

$$\xi_\Delta(x) = \begin{cases} 0 & \text{if } x \in \bigcup_{i=1}^{k-1} \left[a_i, a_i + \dfrac{a_i + a_{i+1}}{2}\right), \\ 1 & \text{if } x \in \bigcup_{i=1}^{k-1} \left[a_i + \dfrac{a_i + a_{i+1}}{2}, a_{i+1}\right). \end{cases}$$

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For $\delta > 0$, let $B \subset [a, b)$ be some disjoint union of intervals with length not less than δ, and set

$$B_\Delta = \bigcup_{1\le i\le k-1 \text{ and } [a_i, a_{i+1})\subset B} [a_i, a_{i+1}). \tag{2.3}$$

Then

$$\frac{m(B_\Delta)}{m(B)} \ge \frac{\delta - 2l^*(\Delta)}{\delta}, \tag{2.4}$$

where m is the Lebesgue measure on $\mathbb{R}$.

For $a < b < c$, $\Delta_1 : a = a_1 < a_2 < \cdots < a_k = b \in \mathcal{P}([a, b))$ and $\Delta_2 : b = b_1 < b_2 < \cdots < b_\ell = c \in \mathcal{P}([b, c))$, we combine Δ_1 and Δ_2 to define a new finite partition

$$\Delta_1 \sqcup \Delta_2 : a = a_1 < a_2 < \cdots < a_k = b_1 < b_2 < \cdots < b_\ell = c$$

of $[a, c)$.

2.2. *The construction.* Set $s_i = 1 - (1/2^i)$ and $E_i = [0, 1/2^i)$ for $i \in \mathbb{N}$. Fix a small positive real number β such that

$$\prod_{l=0}^{\infty}\left(\frac{1}{2}\right)^{\beta^{l+1}} > \frac{9}{10}.$$

As in Lemma 2.2, we let $N_i = N(E_i, s_i)$ for $i \in \mathbb{N}$. Without loss of generality, we assume that $N_{i+1} > N_i$ for $i \in \mathbb{N}$.

We now define a real function $h(x)$ on $(0, 1)$ with range $\{0, 1\}$ by induction for $i \in \mathbb{N}$. To do so, first we choose $K_1 < K_2 < \cdots$ such that, for each $k \in \mathbb{N}$,

$$\frac{N_{k+1}}{2^{K_k - k}} < \frac{\eta}{2} \tag{2.5}$$

and

$$\#\left\{0 \le i \le N_{k+1} : R_\alpha^i(x) \in y + \left[0, \frac{1}{2^{K_k}}\right)\right\} \le 1, \tag{2.6}$$

for any $x, y \in \mathbb{T}$. Recall that η is fixed with $\eta \in (0, \frac{1}{100})$.

We also define a counting function $c(k)$ such that $c(1) = 1$ and, for $k \ge 1$,

$$c(k+1) = c(k) + 2^{K_k - k - 1}.$$

We are now ready to define h using induction.

Step 1. For $i = 1$, we put $\Delta_1 : \frac{1}{2} = a_1 < a_2 = 1 \in \mathcal{P}([\frac{1}{2}, 1))$, $h|_{[\frac{1}{2},1)} = \xi_{\Delta_1}$.

Step k. For $i = k \ge 1$, suppose $h(x)$ has been defined on $[1/2^k, 1)$ with $h|_{[1/2^k,1)} = \xi_{\Delta_k}$ for some defined $\Delta_k \in \mathcal{P}([1/2^k, 1))$.

We divide $E_k = [0, 1/2^k)$ into $2^{K_k - k}$ subintervals

$$E_{k,l} = \left[\frac{1}{2^k} - \frac{l+1}{2^{K_k}}, \frac{1}{2^k} - \frac{l}{2^{K_k}}\right),$$

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where $0 \le l \le 2^{K_k-k}-1$ (see Figures 1 and 2 below). Note that

$$E_{k,2^{K_k-k}-1} = \left[0, \frac{1}{2^{K_k}}\right), \ldots, E_{k,2^{K_k-k-1}-1} = \left[\frac{1}{2^{k+1}}, \frac{1}{2^{k+1}} + \frac{1}{2^{K_k}}\right), \ldots, E_{k,0} = \left[\frac{1}{2^k} - \frac{1}{2^{K_k}}, \frac{1}{2^k}\right).$$

Remark 2.3. Since we divided E_k into finitely many intervals $E_{k,l}$ and Δ_k is a finite partition, we can find $\delta_k > 0$ small enough such that we can find a Borel set $M_k \subseteq \mathbb{T}$ satisfying $m_{\mathbb{T}}(M_k) > 1 - (\eta/2)$, and if $0 \le n \le N_{k+1}$, $x \in M_k$, then:

(1) $R_\alpha^n x \in E_k$ implies $[R_\alpha^n x, R_\alpha^n x + \delta_k) \subset E_{k,\ell}$ for some $0 \le l \le 2^{K_k-k} - 1$;

(2) $R_\alpha^n x \in [1/2^k, 1)$ implies that $[R_\alpha^n x, R_\alpha^n x + \delta_k) \subset [1/2^k, 1)$ and h is constant on $[R_\alpha^n x, R_\alpha^n x + \delta_k)$.

In fact any $\delta_k > 0$ with $\delta_k(2^{K_k-k} + 2\#\Delta_k)N_{k+1} < (\eta/2)$ is the number we want.

Step $k+1$. Now let $\Delta^*_{k+1,0} = \Delta_k$ and $\delta_{k,0} = \frac{1}{2}\min\{\delta_k, l_*(\Delta^*_{k+1,0})\}$. By (2.4), we can find $\Delta_{k,0} \in \mathcal{P}(E_{k,0})$ such that

$$\frac{\delta_{k,0} - 2l^*(\Delta_{k,0})}{\delta_{k,0}} \ge \left(\frac{1}{2}\right)^{\beta^{c(k)+1}}.$$

Suppose, for $\ell \in [0, 2^{K_k-k-1} - 2]$, that we have defined $\Delta_{k,0} \in \mathcal{P}(E_{k,0}), \ldots, \Delta_{k,\ell} \in \mathcal{P}(E_{k,\ell})$ and

$$\Delta^*_{k+1,0} = \Delta_k, \quad \Delta^*_{k+1,j} = \Delta_{k,j-1} \sqcup \Delta^*_{k+1,j-1} \quad \text{for } 1 \le j \le \ell,$$

such that, for each $0 \le j \le \ell$,

$$\frac{\delta_{k,j} - 2l^*(\Delta_{k,j})}{\delta_{k,j}} \ge \left(\frac{1}{2}\right)^{\beta^{j+c(k)+1}},$$

where $\delta_{k,j} = \frac{1}{2}\min\{\delta_k, l_*(\Delta^*_{k+1,j})\}$.

Next let

$$\Delta^*_{k+1,\ell+1} = \Delta_{k,\ell} \sqcup \Delta^*_{k+1,\ell} = \Delta_{k,l} \sqcup \cdots \sqcup \Delta_{k,1} \sqcup \Delta_k$$

and $\delta_{k,\ell+1} = \frac{1}{2}\min\{\delta_k, l_*(\Delta^*_{k+1,\ell+1})\}$. By (2.4), we can find $\Delta_{k,\ell+1} \in \mathcal{P}(E_{k,\ell+1})$ such that

$$\frac{\delta_{k,\ell+1} - 2l^*(\Delta_{k,\ell+1})}{\delta_{k,\ell+1}} \ge \left(\frac{1}{2}\right)^{\beta^{\ell+1+c(k)+1}}.$$

We repeat the above process until $\ell = 2^{K_k-k-1} - 1$. Then we get

$$\Delta_{k,0} \in \mathcal{P}(E_{k,0}), \Delta_{k,1} \in \mathcal{P}(E_{k,1}), \ldots, \Delta_{k,2^{K_k-k-1}-1} \in \mathcal{P}(E_{k,2^{K_k-k-1}-1})$$

and

$$\Delta^*_{k+1,0} = \Delta_k, \quad \Delta^*_{k+1,j} = \Delta_{k,j-1} \sqcup \Delta^*_{k+1,j-1},$$

for $1 \le j \le 2^{K_k-k-1} - 1$, such that, for each $0 \le j \le 2^{K_k-k-1} - 1$,

$$\frac{\delta_{k,j} - 2l^*(\Delta_{k,j})}{\delta_{k,j}} \ge \left(\frac{1}{2}\right)^{\beta^{j+c(k)+1}},$$

where $\delta_{k,j} = \frac{1}{2}\min\{\delta_k, l_*(\Delta^*_{k+1,j})\}$.

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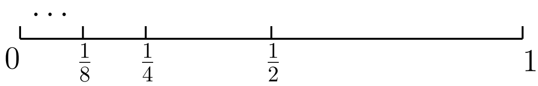

FIGURE 1. $E_k = [0, 1/2^k)$.

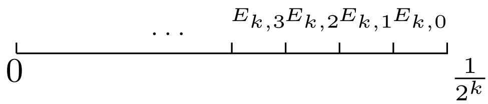

FIGURE 2. $E_{k,i}$.

It is clear that $\Delta^*_{k+1,2^{K_k-k-1}-1} \in \mathcal{P}([1/2^{k+1}, 1))$. Now we put

$$\Delta_{k+1} = \Delta^*_{k+1,2^{K_k-k-1}-1} \quad \text{and} \quad h|_{[\frac{1}{2^{k+1}},1)} = \xi_{\Delta_{k+1}}.$$

Then it is clear that $(h_{\Delta_{k+1}})|_{[1/2^k,1)} = \xi_{\Delta_k}$.

By the induction, we have defined $h(x)$ on $(0, 1)$ as above. Then we set $h(0) = 1$ and by the periodic extension we define

$$h(x) = h(\{x\})$$

for any $x \in \mathbb{R}$, where $\{x\}$ is the decimal part of x.

With the above construction, we now define $T : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$T(x, y) = (x + \alpha, y + \tfrac{1}{2}h(x))$$

for $(x, y) \in \mathbb{T}^2$. It is clear that T is a Borel measurable map from $\mathbb{T}^2$ to $\mathbb{T}^2$. Note that, for any $n \in \mathbb{N}$,

$$T^n(x, y) = \bigg(x + n\alpha, y + \frac{1}{2}\sum_{i=0}^{n-1} h(x + i\alpha)\bigg).$$

In the following remark we extend the definition of $\Delta_{k,j}$ for $2^{K_k-k-1} \le j \le 2^{K_k-k} - 2$.

Remark 2.4. For $k > 0$ and $0 \le j \le 2^{K_k-k} - 2$, there exists a unique partition in $\mathcal{P}(E_{k,j})$ (which will also be denoted by $\Delta_{k,j}$ when $j \ge 2^{K_k-k-1}$) such that $h|_{E_{k,j}}(x) = \xi_{\Delta_{k,j}}$ and if $B \subset E_{k,j}$ is any disjoint union of intervals with length not less than $\frac{1}{2}\min\{\delta_k, l_*(\Delta_k), l_*(\Delta_{k,i}), 0 \le i \le j-1\}$, then

$$\frac{m_{\mathbb{T}}(B_{\Delta_{k,j}})}{m_{\mathbb{T}}(B)} \ge \bigg(\frac{1}{2}\bigg)^{\beta^{j+c(k)+1}}. \tag{2.7}$$

Proof. In fact, given $2^{K_k-k-1} \le j \le 2^{K_k-k} - 2$, let $k(j)$ be the unique integer such that $(1/2^{k(j)+1}) \le (1/2^k) - (j+1)/2^{K_k} < (1/2^{k(j)})$. We have $k+1 \le k(j) < K_k$ since $2^{K_k-k-1} \le j \le 2^{K_k-k} - 2$. Note that $(1/2^{k(j)})$ is an endpoint of $E_{k,2^{K_k-k}-2^{K_k-k(j)}}$. One has

$$\frac{1}{2^{k(j)+1}} \le \frac{1}{2^k} - \frac{j+1}{2^{K_k}} < \frac{1}{2^k} - \frac{j}{2^{K_k}} \le \frac{1}{2^{k(j)}}.$$

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Set

$$j_1 = 2^{K_{k(j)}-k(j)} - 2^{K_{k(j)}-k} + j \cdot 2^{K_{k(j)}-K_k} \quad \text{and} \quad j_2 = j_1 + 2^{K_{k(j)}-K_k}.$$

One has

$$E_{k,j} = \bigsqcup_{j_1 \le l < j_2} E_{k(j),l}.$$

Put

$$\Delta_{k,j} = \Delta_{k(j),j_1} \sqcup \Delta_{k(j),j_1+1} \sqcup \cdots \sqcup \Delta_{k(j),j_2-1}.$$

One has

$$l^*(\Delta_{k,j}) \le l^*(\Delta_{k(j),j_1}) \quad \text{and} \quad \frac{1}{2} \min_{0 \le i \le j-1} \{\delta_k, l_*(\Delta_k), l_*(\Delta_{k,i})\} = \delta_{k(j),j_1}.$$

Hence, if $B \subset E_{k,j}$ is any disjoint union of intervals with length not less than $\frac{1}{2} \min_{0 \le i \le j-1} \{\delta_k, l_*(\Delta_k), l_*(\Delta_{k,i})\} = \delta_{k(j),j_1}$, then by (2.4) and the construction we have

$$\frac{m_{\mathbb{T}}(B_{\Delta_{k,j}})}{m_{\mathbb{T}}(B)} \ge \frac{\delta_{k(j),j_1} - 2l^*(\Delta_{k(j),j_1})}{\delta_{k(j),j_1}} \ge \left(\frac{1}{2}\right)^{\beta^{j_1+c(k(j))+1}} \ge \left(\frac{1}{2}\right)^{\beta^{j+c(k)+1}}.$$

Moreover, it is clear that $h|_{E_{k,j}}(x) = \xi_{\Delta_{k,j}}$. □

3. *The measure complexity*

For a topological space X, let $M(X)$ be the collection of all Borel probability measures on X and (X, T) be the system defined in the previous section. For $\rho \in M(\mathbb{T}^2)$, we say that ρ is T-invariant, if $\rho(T^{-1}A) = \rho(A)$ for any Borel set A of $\mathbb{T}^2$. We denote by $M(\mathbb{T}^2, T)$ the set of all T-invariant measures in $M(\mathbb{T}^2)$. It is clear that the Haar measure $m_{\mathbb{T}^2} \in M(\mathbb{T}^2, T)$.

For any $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$, the metric

$$d((x_1, y_1), (x_2, y_2)) := \max\{\|x_1 - x_2\|, \|y_1 - y_2\|\},$$

where $\|z\| = \min_{k \in \mathbb{Z}} |z - k|$ for $z \in \mathbb{R}$.

In this section we compute the measure complexity of (X, T, ρ) for any $\rho \in M(\mathbb{T}^2, T)$. Since the computation is long we will put the proofs of some technical lemmas in subsections.

3.1. *The computation.*

Before stating the following proposition, let us recall some notation. Let $s_i = 1 - 1/2^i$ and $E_i = [0, 1/2^i)$ for $i \in \mathbb{N}$. Fix a small $\beta > 0$ such that $\prod_{l=0}^{\infty} (\frac{1}{2})^{\beta^{l+1}} > \frac{9}{10}$. For each $k \in \mathbb{N}$, $(N_{k+1}/2^{K_k - k}) < (\eta/2)$ with $0 < \eta < 1/100$.

PROPOSITION 3.1. *For any* $\rho \in M(\mathbb{T}^2, T)$,

$$\lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log S_n(d, \rho, \epsilon)}{n^\tau} = +\infty \tag{3.1}$$

for any $0 < \tau < 1$.

Proof. Let $\rho \in M(\mathbb{T}^2, T)$. Fix $k \in \mathbb{N}$ and $N_k \le n \le N_{k+1}$. By the definition of $S_n(d, \rho, \eta/2)$, there exist $z_1, z_2, \ldots, z_{S_n(d,\rho,\eta/2)} \in \mathbb{T}^2$ such that

$$\rho\left(\bigcup_{i=1}^{S_n(d,\rho,\eta/2)} B_{\bar{d}_n}(z_i, \eta/2)\right) > 1 - \eta/2.$$

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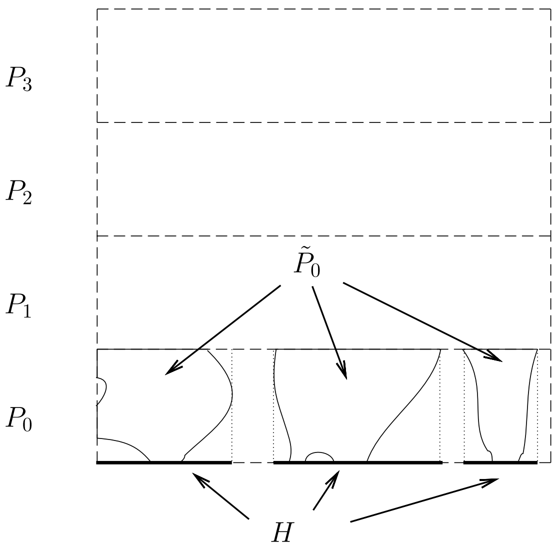

FIGURE 3. $\tilde{P}_0$ and $H = \pi(\tilde{P}_0)$.

Set $P_i = \mathbb{T} \times [i/4, i + 1/4)$, $i = 0, 1, 2, 3$ (see Figure 3). There must be some $0 \le i \le 3$, such that $\rho(P_i) \ge \frac{1}{4}$. Without loss of generality, we suppose $\rho(P_0) \ge \frac{1}{4}$. Write

$$\tilde{P}_0 = P_0 \cap \bigg(\bigcup_{i=1}^{S_n(d,\rho,\eta/2)} B_{\bar{d}_n}(z_i, \eta/2) \bigg).$$

Clearly, $\rho(\tilde{P}_0) > \frac{1}{4} - \eta$. Let $\pi : \mathbb{T}^2 \to \mathbb{T}$ be the projection of the first coordinate. Notice that the marginal of ρ on the first coordinate is the Haar measure $m_{\mathbb{T}}$, and we have $\rho \circ \pi = m_{\mathbb{T}}$. Set

$$H = \pi(\tilde{P}_0).$$

Since $\tilde{P}_0$ is a Borel set of $\mathbb{T}^2$, H is an analytic subset of $\mathbb{T}$ (see [**3**] for the definition) with $m_{\mathbb{T}}(H) > \frac{1}{4} - \eta$. Moreover, for any $x \in H$, we fix some $z(x) \in \tilde{P}_0$ such that $\pi(z(x)) = x$.

At this point let us explain the main idea of the proof. For $x_1, x_2 \in H$ and $j \in \mathbb{N}$, we have

$$d(T^j(z(x_1)), T^j(z(x_2))) \ge \frac{1}{2} - \|y_1 - y_2\| \ge \frac{1}{4}$$

if $z(x_1) = (x_1, y_1)$, $z(x_2) = (x_2, y_2)$ and $\sum_{i=0}^{j-1} h(x_1 + i\alpha) \ne \sum_{i=0}^{j-1} h(x_2 + i\alpha) \pmod 1$, since $y_1, y_2 \in P_0$. This implies that

$$\bar{d}_n(z(x), z(x')) \ge \frac{1}{4n} \# \bigg\{ 0 \le l \le n-2 : \sum_{j=0}^{l} h(R_\alpha^j x) \ne \sum_{j=0}^{l} h(R_\alpha^j x') (\mathrm{mod}\ 1) \bigg\}.$$

So the computation of the measure complexity can be reduced to the study of properties of h. To estimate $S_n(d, \rho, \eta/2)$ we construct a subset $J_{x_0,s} \subset \mathbb{T}$ and consider

$$W = \{1 \le i \le S_n(d, \rho, \eta/2) : \{z(x) : x \in J_{x_0,s} \cap H\} \cap B_{\bar{d}_n}(z_i, \eta/2) \ne \emptyset\}.$$

Using results in Claims 1 and 2 below, we may get a lower bound of W which is the one we need.

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We now begin the proof of the proposition. Setting

$$J = \{x \in \mathbb{T} : m_{\mathbb{T}}([x, x+\delta_k) \cap H) \geq (\tfrac{1}{4} - 2\eta)\delta_k\}, \tag{3.2}$$

we have

$$\begin{aligned}\left(\frac{1}{4} - \eta\right)\delta_k &< \int_0^{\delta_k} \int_{\mathbb{T}} \chi_H(x+y)\, dm_{\mathbb{T}}(x)\, dy \\ &= \int_{\mathbb{T}} \int_0^{\delta_k} \chi_H(x+y)\, dy\, dm_{\mathbb{T}}(x) \\ &\leq m_{\mathbb{T}}(J)\delta_k + \left(\frac{1}{4} - 2\eta\right)\delta_k(1 - m_{\mathbb{T}}(J)) \\ &= m_{\mathbb{T}}(J)\left(\frac{3}{4} + 2\eta\right)\delta_k + \left(\frac{1}{4} - 2\eta\right)\delta_k,\end{aligned}$$

which implies $m_{\mathbb{T}}(J) > (\eta/(3/4) + 2\eta) > \eta$ and $J \cap (M_k \setminus \bigcup_{i=0}^n R_\alpha^{-i}[0, 1/2^{K_k})) \neq \emptyset$ by (2.5) and the fact that $m_{\mathbb{T}}(M_k) > 1 - (\eta/2)$ (see Remark 2.3). Pick $x_0 \in J \cap (M_k \setminus \bigcup_{i=0}^n R_\alpha^{-i}[0, 1/2^{K_k}))$ and set $J_{x_0} = [x_0, x_0 + \delta_k)$.

Let

$$\mathcal{J} = \{0 \leq j \leq n-1 : R_\alpha^j x_0 \in E_k\},$$

and $s = \#\mathcal{J} - 1$. Denote the elements in $\mathcal{J}$ by $j_1, j_2, \ldots, j_{s+1}$ with $0 \leq j_1 < j_2 < \cdots < j_{s+1} \leq n-1$. Let $\mathcal{S} = [1, s] \cap \mathbb{N}$ and

$$\mathcal{S}_i = \{l : 1 \leq l \leq s \text{ and } j_{l+1} - j_l = i\}.$$

We remark that by the definition $s(i, n, E_k, x_0) = \#\{0 \leq j \leq n-i-1 : f_i(R_\alpha^j x_0) = 1\}$. Thus, we have $\#\mathcal{S}_i = s(i, n, E_k, x_0)$ (see (2.1) for the definition).

For a given $x \in J_{x_0}$, and any $x' \in J_{x_0}$, let

$$0 \leq j_1(x, x') < j_2(x, x') < \cdots < j_{s'(x,x')}(x, x') \leq j_s$$

be the collection of $j \in \mathcal{J} \setminus \{j_{s+1}\}$ such that $h(R_\alpha^j x) \neq h(R_\alpha^j x')$. This means that if $j_i(x, x') < p < j_{i+1}(x, x')$ for some i, then $h(R_\alpha^p x) = h(R_\alpha^p x')$.

Set $j_{s'(x,x')+1}(x, x') = j_{s+1}$ and put

$$\mathcal{K}_{x,x'} = \bigcup_{0<2i\leq s'(x,x')+1} \{l : j_l \in [j_{2i-1}(x, x'), j_{2i}(x, x')), 1 \leq l \leq s\}.$$

It is clear that $\mathcal{K}_{x,x'}$ is a subset of $\mathcal{S}$. We remark that if $j_l \in [j_{2i-1}(x, x'), j_{2i}(x, x'))$ then $[j_l, j_{l+1}) \subset [j_{2i-1}(x, x'), j_{2i}(x, x'))$ as $j_{2i}(x, x') \in \mathcal{J}$. Thus

$$\bigcup_{l \in \mathcal{K}_{x,x'}} [j_l, j_{l+1}) \cap \mathbb{N} = \bigcup_{0<2i\leq s'(x,x')+1} \{t \in \mathbb{N} : t \in [j_{2i-1}(x, x'), j_{2i}(x, x'))\} \tag{3.3}$$

is a subset of $0 \leq i \leq n-2$ for which $\sum_{j=0}^i h(R_\alpha^j x) \neq \sum_{j=0}^i h(R_\alpha^j x') \pmod 1$ (see Remark 2.3(2)).

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We have the following claim, whose proof will be given in the next subsection.

CLAIM 1. *If $x \in J_{x_0} \cap H$, then*

$$\#\{\mathcal{K}_{x,x'} \subset \mathcal{S} : x' \in J_{x_0} \cap H \text{ and } \bar{d}_n(z(x), z(x')) < \eta\} \\ \leq \#\mathcal{I}(E_k) C 2^s n e^{-(\log 2 + 5\eta \log(5\eta) + (1-5\eta)\log(1-5\eta))\rho(E_k)n}.$$

Let

$$\mathcal{L}_k := \{\ell \in [0, 2^{K_k - k} - 1] : R_\alpha^{j_i}(x_0) \in E_{k,\ell} \text{ for some } i \in [1, s]\}.$$

By (2.6),

$$\#\{0 \leq i \leq n : R_\alpha^i(x_0) \in E_{k,l}\} \leq 1$$

for any $0 \leq l \leq 2^{K_k - k} - 1$. Hence $\#\mathcal{L}_k = s$ and we rewrite

$$\mathcal{L}_k = \{0 \leq \ell_1 < \ell_2 < \cdots < \ell_s \leq 2^{K_k - k} - 1\}.$$

By the selection of x_0, one has $\ell_s \neq 2^{K_k - k} - 1$ (recall that $E_{k, 2^{K_k-k}-1} = [0, 1/2^{K_k})$). Moreover, for any $l \in \mathcal{L}_k$, we write as $j(l)$ the only element in $\mathcal{J} \setminus \{j_{s+1}\}$ such that $R_\alpha^{j(l)}(x_0) \in E_{k,l}$.

For $i \in [1, s]$, by Remark 2.4 there exists a unique $\Delta_{k,\ell_i} \in \mathcal{P}(E_{k,\ell_i})$ such that $h|_{E_{k,\ell_i}}(x) = \xi_{\Delta_{k,\ell_i}}$ and (2.7) holds. Suppose the partition Δ_{k,l_i} is

$$a_i = a_{i,1} < a_{i,2} < \cdots < a_{i,k_i} = b_i,$$

for $1 \leq i \leq s$. For $i = 1$, we let

$$\mathcal{I}_1 = \{j \in [1, k_1 - 1] : [a_{1,j}, a_{1,j+1}) \subset R_\alpha^{j(l_1)} J_{x_0}\}$$

and

$$J_{x_0,1} = R_\alpha^{-j(l_1)}\bigg(\bigcup_{j \in \mathcal{I}_1} [a_{1,j}, a_{1,j+1})\bigg) = R_\alpha^{-j(l_1)}(R_\alpha^{j(l_1)} J_{x_0})_{\Delta_{k,l_1}}.$$

It is clear that

$$J_{x_0,1} \subset R_\alpha^{-j(l_1)}(R_\alpha^{j(l_1)} J_{x_0}) = J_{x_0}.$$

By induction, for $2 \leq i \leq s$, by (2.3), we put

$$\mathcal{I}_i = \bigg\{ j \in [1, k_i - 1] : \text{ there exists } j' \in \mathcal{I}_{i-1} \text{ such that} \\ [a_{i,j}, a_{i,j+1}) \subset R_\alpha^{j(l_i) - j(l_{i-1})}\bigg(\bigg[a_{i-1,j'}, \frac{a_{i-1,j'} + a_{i-1,j'+1}}{2}\bigg)\bigg) \text{ or} \\ [a_{i,j}, a_{i,j+1}) \subset R_\alpha^{j(l_i) - j(l_{i-1})}\bigg(\bigg[\frac{a_{i-1,j'} + a_{i-1,j'+1}}{2}, a_{i-1,j'+1}\bigg)\bigg)\bigg\}$$

and

$$J_{x_0,i} = R_\alpha^{-j(l_i)}\bigg(\bigcup_{j \in \mathcal{I}_i} [a_{i,j}, a_{i,j+1})\bigg) \\ = R_\alpha^{-j(l_i)}\bigg(\bigcup_{j' \in \mathcal{I}_{i-1}} \bigg(R_\alpha^{j(l_i) - j(l_{i-1})}\bigg(\bigg[a_{i-1,j'}, \frac{a_{i-1,j'} + a_{i-1,j'+1}}{2}\bigg)\bigg)\bigg)_{\Delta_{k,l_i}} \\ \cup \bigg(R_\alpha^{j(l_i) - j(l_{i-1})}\bigg(\bigg[\frac{a_{i-1,j'} + a_{i-1,j'+1}}{2}, a_{i-1,j'+1}\bigg)\bigg)\bigg)_{\Delta_{k,l_i}}\bigg). \tag{3.4}$$

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It is clear that

$$J_{x_0,i} \subset R_\alpha^{-j(l_i)}\bigg(\bigcup_{j'\in\mathcal{I}_{i-1}} R_\alpha^{j(l_i)-j(l_{i-1})}([a_{i-1,j'}, a_{i-1,j'+1}))\bigg)$$
$$= R_\alpha^{-j(l_{i-1})}\bigg(\bigcup_{j'\in\mathcal{I}_{i-1}} [a_{i-1,j'}, a_{i-1,j'+1})\bigg)$$
$$= J_{x_0,i-1}.$$

In this way we get $J_{x_0,s} \subset \cdots \subset J_{x_0,1} \subset J_{x_0}$. It is clear that each $J_{x_0,i}$ is a finite union of subintervals of J_{x_0}.

Next for $t = (t(j))_{j=1}^s \in \{0, 1\}^s$, we define

$$J_{x_0,s}(t) = \{x' \in J_{x_0,s} : h(R_\alpha^{j(\ell_j)} x') = t(j),\ 1 \le j \le s\}.$$

We have the following claim whose proof will be presented in the final subsection.

CLAIM 2. *The following statements hold.*

(1) $m_{\mathbb{T}}(J_{x_0,s} \cap H) > \frac{1}{10}\delta_k$.

(2) $m_{\mathbb{T}}(J_{x_0,s}(t)) \le (1/2^s)\delta_k$ *for any* $t = (t(j))_{j=1}^s \in \{0, 1\}^s$.

Recall that z_i is the point defined at the beginning of the proof. We now let

$$W = \{i \in [1, S_n(d, \rho, \eta/2)] : \{z(x) : x \in J_{x_0,s} \cap H\} \cap B_{\bar{d}_n}(z_i, \eta/2) \neq \emptyset\}.$$

For $i \in W$, we put

$$J_{x_0,s,H}(i) := \{x \in J_{x_0,s} \cap H : z(x) \in B_{\bar{d}_n}(z_i, \eta/2)\}$$

and fix a point $x_i \in J_{x_0,s,H}(i)$. Then let

$$\mathcal{B}_i = \{\mathcal{K}_{x_i,x'} \subset \mathcal{S} : x' \in J_{x_0} \cap H \text{ and } \bar{d}_n(z(x_i), z(x')) < \eta\}.$$

For any $\mathcal{K} \in \mathcal{B}_i$, set

$$J_{x_0,\mathcal{K}}(x_i) = \{x' \in J_{x_0,s} : \mathcal{K}_{x_i,x'} = \mathcal{K}\}.$$

By the definition of h and $x_0 \in M_k$, it is clear that $J_{x_0,\mathcal{K}}(x_i)$ is a union of finite sub-intervals of $J_{x_0,s}$. While $x' \in J_{x_0,\mathcal{K}}(x_i)$, for $1 \le l \le s$, $h(R_\alpha^{j_l} x')$ is decided by x_i and $\mathcal{K}$, which will be denoted by $h_{x_i,\mathcal{K},j_l}$. Hence, by Claim 2(2),

$$\begin{aligned} m_{\mathbb{T}}(J_{x_0,\mathcal{K}}(x_i)) &= m_{\mathbb{T}}(\{x' \in J_{x_0,s} : h(R_\alpha^{j_l} x') = h_{x_i,\mathcal{K},j_l},\ 1 \le l \le s\}) \\ &\le \frac{1}{2^s}\delta_k. \end{aligned} \tag{3.5}$$

Let

$$J^*_{x_0,s,H}(i) := \bigcup_{\mathcal{K}\in\mathcal{B}_i} J_{x_0,\mathcal{K}}(x_i).$$

Then $J^*_{x_0,s,H}(i)$ is also a union of finite sub-intervals of $J_{x_0,s}$ and

$$\begin{aligned} m_{\mathbb{T}}(J^*_{x_0,s,H}(i)) &= \sum_{\mathcal{K}\in\mathcal{B}_i} m_{\mathbb{T}}(J_{x_0,\mathcal{K}}(x_i)) \\ &\le \frac{1}{2^s}\delta_k \#\mathcal{I}(E_k) C 2^s n e^{-(\log 2 + 5\eta\log(5\eta) + (1-5\eta)\log(1-5\eta))\rho(E_k)n} \qquad (3.6) \\ &= \delta_k \#\mathcal{I}(E_k) C n e^{-(\log 2 + 5\eta\log(5\eta) + (1-5\eta)\log(1-5\eta))\rho(E_k)n} \end{aligned}$$

by Claim 1 and (3.5).

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Since $\bar{d}_n(z(x_i), z(x)) < \eta$ for any $x \in J_{x_0,s,H}(i)$, one has

$$J_{x_0,s,H}(i) \subseteq J^*_{x_0,s,H}(i). \tag{3.7}$$

Moreover, as

$$\{z(x) : x \in J_{x_0,s} \cap H\} \subseteq \tilde{P}_0 \subset \bigcup_{i=1}^{S_n(d,\rho,\eta/2)} B_{\bar{d}_n}(z_i, \eta/2),$$

we have

$$J_{x_0,s} \cap H \subset \bigcup_{i \in W} J_{x_0,s,H}(i). \tag{3.8}$$

By (3.7) and (3.8) we obtain

$$J_{x_0,s} \cap H \subset \bigcup_{i \in W} J_{x_0,s,H}(i) \subseteq \bigcup_{i \in W} J^*_{x_0,s,H}(i). \tag{3.9}$$

Combining (3.9) with (3.6) and Claim 2(1), we have

$$\begin{aligned}
&\#W \cdot \delta_k \#\mathcal{I}(E_k) C n e^{-(\log 2 + 5\eta \log(5\eta) + (1-5\eta)\log(1-5\eta))\rho(E_k)n} \\
&\geq \sum_{i \in W} m_{\mathbb{T}}(J^*_{x_0,S,H}(i)) \geq m_{\mathbb{T}}\Big(\bigcup_{i \in W} J^*_{x_0,S,H}(i)\Big) \\
&\geq m_{\mathbb{T}}(J_{x_0,s} \cap H) > \frac{\delta_k}{10},
\end{aligned}$$

which implies

$$\begin{aligned}
S_n(d, \rho, \eta/2) \geq \#W &\geq \frac{\frac{1}{10}\delta_k}{\delta_k \#\mathcal{I}(E_k) C n e^{-(\log 2 + 5\eta \log(5\eta) + (1-5\eta)\log(1-5\eta))\rho(E_k)n}} \\
&= e^{(\log 2 + 5\eta \log(5\eta) + (1-5\eta)\log(1-5\eta))\rho(E_k)n - \log(10\#\mathcal{I}(E_k)Cn)}.
\end{aligned}$$

Thus by Lemma 2.2 (3), we deduce that

$$\frac{\log S_n(d, \rho, \eta/2)}{n^{s_k}} \geq \frac{1}{2} \tag{3.10}$$

for $N_k \leq n \leq N_{k+1}$. Notice that $s_k \nearrow 1$ when k goes to infinity, hence

$$\begin{aligned}
\lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log S_n(d, \rho, \epsilon)}{n^\tau} &\geq \liminf_{n \to \infty} \frac{\log S_n(d, \rho, \eta/2)}{n^\tau} \\
&\geq \liminf_{k \to +\infty} \min_{N_k \leq n \leq N_{k+1}} \frac{\log S_n(d, \rho, \eta/2)}{n^\tau} \\
&= \liminf_{k \to +\infty} \min_{N_k \leq n \leq N_{k+1}} n^{s_k - \tau} \frac{\log S_n(d, \rho, \eta/2)}{n^{s_k}} \\
&\geq \liminf_{k \to +\infty} \min_{N_k \leq n \leq N_{k+1}} n^{s_k - \tau} \frac{1}{2} \quad \text{(by (3.10))} \\
&= +\infty
\end{aligned}$$

for any $0 < \tau < 1$. That is, (3.1) holds. □

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3.2. *Proof of Claim 1.* In this subsection we prove Claim 1.

Proof of Claim 1. Fix $x \in J_{x_0} \cap H$. For any $x' \in J_{x_0} \cap H$, it is clear that

$$\begin{aligned}\bar{d}_n(z(x), z(x')) &\geq \frac{1}{4n}\#\bigg\{0 \leq l \leq n-2 : \sum_{j=0}^{\ell} h(R_\alpha^j x) \neq \sum_{j=0}^{\ell} h(R_\alpha^j x') (\mathrm{mod}\ 1)\bigg\}\\ &\geq \frac{1}{4n}\#\bigg\{l \in \mathbb{N} : l \in \bigcup_{0<2i\leq s'(x,x')+1} [j_{2i-1}(x, x'), j_{2i}(x, x'))\bigg\}\\ &\overset{(3.3)}{=} \frac{1}{4n}\# \bigcup_{l\in\mathcal{K}_{x,x'}} [j_l, j_{l+1}) \cap \mathbb{N}.\end{aligned}$$

Now if, in addition, $\bar{d}_n(z(x), z(x')) < \eta$, then

$$\frac{1}{n}\# \bigcup_{l\in\mathcal{K}_{x,x'}} [j_l, j_{l+1}) \cap \mathbb{N} < 4\eta.$$

This implies

$$4\eta > \frac{1}{n} \sum_{l\in\mathcal{K}_{x,x'}} j_{l+1} - j_l = \sum_{i\in\mathcal{I}(E_k)} i\frac{\#(\mathcal{K}_{x,x'} \cap \mathcal{S}_i)}{\#\mathcal{S}_i}\frac{\#\mathcal{S}_i}{n},$$

where $\#\mathcal{S}_i = s(i, n, E_k, x_0)$. Thus by Lemma 2.2(1),

$$4\eta > (1-\eta) \sum_{i\in\mathcal{I}(E_k)} \frac{\#(\mathcal{K}_{x,x'} \cap \mathcal{S}_i)}{\#\mathcal{S}_i} \frac{i\#\mathcal{S}_i}{\sum_{j\in\mathcal{I}(E_k)} j\#\mathcal{S}_j}.$$

Hence, there exists $i \in \mathcal{I}(E_k)$ such that

$$\frac{\#(\mathcal{K}_{x,x'} \cap \mathcal{S}_i)}{\#\mathcal{S}_i} < \frac{4\eta}{1-\eta} < 5\eta$$

when $\bar{d}_n(z(x), z(x')) < \eta$.

Now, we have the following approximation:

$$\begin{aligned}&\#\{\mathcal{K}_{x,x'} \subset \mathcal{S} : x' \in J_{x_0} \text{ and } \bar{d}_n(z(x), z(x')) < \eta\}\\ &\leq \sum_{i\in\mathcal{I}(E_k)} \#\bigg\{\mathcal{K}_{x,x'} \subset \mathcal{S} : \frac{\#(\mathcal{K}_{x,x'} \cap \mathcal{S}_i)}{\#\mathcal{S}_i} < 5\eta,\ x' \in J_{x_0}\bigg\}\\ &\leq \sum_{i\in\mathcal{I}(E_k)} \sum_{j=0}^{[5\eta s(i,n,E_k,x_0)]} 2^{s-s(i,n,E_k,x_0)} C^j_{s(i,n,E_k,x_0)}\\ &\leq \sum_{i\in\mathcal{I}(E_k)} 2^{s-s(i,n,E_k,x_0)}[5\eta s(i, n, E_k, x_0)] C^{[5\eta s(i,n,E_k,x_0)]}_{s(i,n,E_k,x_0)}\\ &\leq \sum_{i\in\mathcal{I}(E_k)} 2^{s-s(i,n,E_k,x_0)} n C^{[5\eta s(i,n,E_k,x_0)]}_{s(i,n,E_k,x_0)}\\ &\overset{(2.2)}{\leq} \sum_{i\in\mathcal{I}(E_k)} C2^s n e^{-(\log 2+5\eta\log(5\eta)+(1-5\eta)\log(1-5\eta))s(i,n,E_k,x_0)}\\ &\overset{\text{Lem. 2.2(2)}}{\leq} \mathcal{I}(E_k) C2^s n e^{-(\log 2+5\eta\log(5\eta)+(1-5\eta)\log(1-5\eta))\rho(E_k)n}.\end{aligned}$$

This completes the proof of Claim 1. □

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3.3. *Proof of Claim 2.* In this subsection we prove Claim 2.

Proof of Claim 2(1). For $i = 1$, by (2.7) we have

$$\begin{aligned} m_{\mathbb{T}}(J_{x_0,1}) &= m_{\mathbb{T}}(R_\alpha^{-j(l_1)}(R_\alpha^{j(l_1)} J_{x_0})_{\Delta_{k,l_1}}) \\ &= m_{\mathbb{T}}((R_\alpha^{j(l_1)} J_{x_0})_{\Delta_{k,l_1}}) \\ &\geq (\tfrac{1}{2})^{\beta^{l_1+c(k)+1}} m_{\mathbb{T}}(R_\alpha^{j(l_1)} J_{x_0}) = (\tfrac{1}{2})^{\beta^{l_1+c(k)+1}} m_{\mathbb{T}}(J_{x_0}). \end{aligned}$$

For $2 \leq i \leq s$, by (2.7) and (3.4) we have

$$\begin{aligned} m_{\mathbb{T}}(J_{x_0,i}) = &\sum_{j' \in \mathcal{I}_{i-1}} m_{\mathbb{T}}\bigg(\bigg(R_\alpha^{j(l_i)-j(l_{i-1})}\bigg(\bigg[a_{i-1,j'}, \frac{a_{i-1,j'} + a_{i-1,j'+1}}{2}\bigg)\bigg)\bigg)_{\Delta_{k,l_i}}\bigg) \\ &+ \sum_{j' \in \mathcal{I}_{i-1}} m_{\mathbb{T}}\bigg(\bigg(R_\alpha^{j(l_i)-j(l_{i-1})}\bigg(\bigg[\frac{a_{i-1,j'} + a_{i-1,j'+1}}{2}, a_{i-1,j'+1}\bigg)\bigg)\bigg)_{\Delta_{k,l_i}}\bigg) \\ \geq &\left(\frac{1}{2}\right)^{\beta^{l_i+c(k)+1}} \bigg(\sum_{j' \in \mathcal{I}_{i-1}} m_{\mathbb{T}}\bigg(R_\alpha^{j(l_i)-j(l_{i-1})}\bigg(\bigg[a_{i-1,j'}, \frac{a_{i-1,j'} + a_{i-1,j'+1}}{2}\bigg)\bigg)\bigg) \\ &+ \sum_{j' \in \mathcal{I}_{i-1}} m_{\mathbb{T}}\bigg(R_\alpha^{j(l_i)-j(l_{i-1})}\bigg(\bigg[\frac{a_{i-1,j'} + a_{i-1,j'+1}}{2}, a_{i-1,j'+1}\bigg)\bigg)\bigg)\bigg). \end{aligned}$$

It is clear that the right-hand side is equal to

$$\begin{aligned} &\left(\frac{1}{2}\right)^{\beta^{l_i+c(k)+1}} \sum_{j' \in \mathcal{I}_{i-1}} (a_{i-1,j'+1} - a_{i-1,j'}) \\ &= \left(\frac{1}{2}\right)^{\beta^{l_i+c(k)+1}} \sum_{j' \in \mathcal{I}_{i-1}} m_{\mathbb{T}}(R_\alpha^{-j(l_{i-1})}([a_{i-1,j'}, a_{i-1,j'+1}))) \\ &= \left(\frac{1}{2}\right)^{\beta^{l_i+c(k)+1}} m_{\mathbb{T}}(J_{x_0,i-1}). \end{aligned}$$

Thus

$$m_{\mathbb{T}}(J_{x_0,i}) \geq m_{\mathbb{T}}(J_{x_0}) \prod_{j=1}^{i} \left(\frac{1}{2}\right)^{\beta^{l_j+c(k)+1}} \geq \delta_k \prod_{l=0}^{\infty} \left(\frac{1}{2}\right)^{\beta^{l+c(k)+1}} > \frac{9}{10}\delta_k$$

for $1 \leq i \leq s$. Hence

$$m_{\mathbb{T}}([x_0, x_0 + \delta_k) \setminus J_{x_0,s}) = m_{\mathbb{T}}([x_0, x_0 + \delta_k)) - m_{\mathbb{T}}(J_{x_0,s}) < \tfrac{1}{10}\delta_k.$$

Moreover, by (3.2), we have

$$\begin{aligned} m_{\mathbb{T}}(J_{x_0,s} \cap H) &\geq m_{\mathbb{T}}([x_0, x_0 + \delta_k) \cap H) - m_{\mathbb{T}}([x_0, x_0 + \delta_k) \setminus J_{x_0,s}) \\ &\geq \left(\frac{1}{4} - 2\eta\right)\delta_k - \frac{1}{10}\delta_k > \frac{1}{10}\delta_k. \end{aligned} \tag{3.11}$$

□

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Proof of Claim 2(2). Fix $t \in \{0, 1\}^s$. We will prove that $m_{\mathbb{T}}(J_{x_0,s}(t)) \leq (\frac{1}{2})^s \delta_k$. First we let

$$J_{x_0,i}(t) = \{x' \in J_{x_0,i} : h(R_\alpha^{j(\ell_r)} x') = t(i),\ 1 \leq r \leq i\}$$

for $1 \leq i \leq s$.

Next for $i = 1$, we set

$$\mathcal{I}_{t,1} = \mathcal{I}_1 = \{j \in [1, k_1 - 1] : [a_{1,j}, a_{1,j+1}) \subset R_\alpha^{j(l_1)} J_{x_0}\}.$$

By induction, for $2 \leq i \leq s$, we put

$$\mathcal{I}_{t,i} = \begin{cases} \Big\{ j \in [1, k_i - 1] : [a_{i,j}, a_{i,j+1}) \subset R_\alpha^{j(l_i) - j(l_{i-1})} \Big[a_{i-1,j'}, \dfrac{a_{i-1,j'} + a_{i-1,j'+1}}{2} \Big) \\ \qquad \text{for some } j' \in \mathcal{I}_{t,i-1} \Big\} & \text{if } t(i-1) = 0 \\ \Big\{ j \in [1, k_i - 1] : [a_{i,j}, a_{i,j+1}) \subset R_\alpha^{j(l_i) - j(l_{i-1})} \Big[\dfrac{a_{i-1,j'} + a_{i-1,j'+1}}{2}, a_{i-1,j'+1} \Big) \\ \qquad \text{for some } j' \in \mathcal{I}_{t,i-1} \Big\} & \text{if } t(i-1) = 1. \end{cases}$$

It is clear that

$$\mathcal{I}_{t,i} \subset \mathcal{I}_i$$

for $1 \leq i \leq s$.

Then we will show that

$$J_{x_0,i}(t) = R_\alpha^{-j(l_i)} \Big(\bigcup_{j \in \mathcal{I}_{t,i}} I(i, j : t(i)) \Big) \tag{3.12}$$

for $1 \leq i \leq s$, where

$$I(i, j; r) = \begin{cases} \Big[a_{i,j}, \dfrac{a_{i,j} + a_{i,j+1}}{2} \Big) & \text{if } r = 0, \\ \Big[\dfrac{a_{i,j} + a_{i,j+1}}{2}, a_{i,j+1} \Big) & \text{if } r = 1, \end{cases}$$

for $i \in [1, s]$, $j \in [1, k_i - 1]$ and $r \in \{0, 1\}$.

First, for $i = 1$,

$$\begin{aligned} J_{x_0,1}(t) &= \Big\{ x' \in R_\alpha^{-j(l_1)} \Big(\bigcup_{j \in \mathcal{I}_1} [a_{1,j}, a_{1,j+1}) \Big) : h(R_\alpha^{j(l_1)} x') = t(1) \Big\} \\ &= R_\alpha^{-j(l_1)} \Big(\Big\{ x \in \bigcup_{j \in \mathcal{I}_1} [a_{1,j}, a_{1,j+1}) : h(x) = t(1) \Big\} \Big) \\ &= \begin{cases} R_\alpha^{-j(l_1)} \Big(\bigcup_{j \in \mathcal{I}_{t,1}} \Big[a_{1,j}, \dfrac{a_{1,j} + a_{1,j+1}}{2} \Big) \Big) & \text{if } t(1) = 0, \\ R_\alpha^{-j(l_1)} \Big(\bigcup_{j \in \mathcal{I}_{t,1}} \Big[\dfrac{a_{1,j} + a_{1,j+1}}{2}, a_{1,j+1} \Big) \Big) & \text{if } t(1) = 1, \end{cases} \\ &= R_\alpha^{-j(l_1)} \Big(\bigcup_{j \in \mathcal{I}_{t,1}} I(1, j : t(i)) \Big). \end{aligned}$$

That is, (3.12) holds for $i = 1$.

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Assume that (3.12) holds for $i=k\in[1,s-1]$. Then, for $i=k+1$,

$$
\begin{aligned}
&J_{x_0,k+1}(t)\\
&=J_{x_0,k}(t)\cap\{x'\in J_{x_0,k+1}:h(R_\alpha^{j(\ell_{k+1})}x')=t(k+1)\}\\
&=J_{x_0,k}(t)\cap\bigg\{x'\in R_\alpha^{-j(l_{k+1})}\bigg(\bigcup_{j\in\mathcal{I}_{k+1}}[a_{k+1,j},a_{k+1,j+1})\bigg):h(R_\alpha^{j(\ell_{k+1})}x')=t(k+1)\bigg\}\\
&=J_{x_0,k}(t)\cap R_\alpha^{-j(l_{k+1})}\bigg\{x\in\bigcup_{j\in\mathcal{I}_{k+1}}[a_{k+1,j},a_{k+1,j+1}):h(x)=t(k+1)\bigg\}\bigg)\\
&=R_\alpha^{-j(l_k)}\bigg(\bigcup_{j\in\mathcal{I}_{t,k}}I(k,j:t(k))\bigg)\cap R_\alpha^{-j(l_{k+1})}\bigg(\bigcup_{j\in\mathcal{I}_{k+1}}I(k+1,j:t(k+1))\bigg)\\
&=R_\alpha^{-j(l_{k+1})}\bigg(\bigg(\bigcup_{j\in\mathcal{I}_{t,k}}R_\alpha^{j(l_{k+1})-j(l_k)}I(k,j:t(k))\bigg)\cap\bigg(\bigcup_{j\in\mathcal{I}_{k+1}}I(k+1,j:t(k+1))\bigg)\bigg)\\
&=R_\alpha^{-j(l_{k+1})}\bigg(\bigcup_{j\in\mathcal{I}_{t,k+1}}I(k+1,j:t(k+1))\bigg).
\end{aligned}
$$

That is, (3.12) holds for $i=k+1$. Thus by induction, we obtain (3.12) holds for $1\le i\le s$.

Next,

$$m_{\mathbb{T}}(J_{x_0,1}(t))=\tfrac{1}{2}m_{\mathbb{T}}(J_{x_0,1})\le\tfrac{1}{2}m_{\mathbb{T}}(J_{x_0})=\tfrac{1}{2}\delta_k.$$

We suppose

$$m_{\mathbb{T}}(J_{x_0,i}(t))\le(\tfrac{1}{2})^i\delta_k,$$

for some $1\le i\le s-1$. Then

$$
m_{\mathbb{T}}(J_{x_0,i+1}(t))=\begin{cases}
m_{\mathbb{T}}\bigg(R_\alpha^{-j(l_{i+1})}\bigg(\bigcup_{j\in\mathcal{I}_{t,i+1}}\bigg[a_{i+1,j},\dfrac{a_{i+1,j}+a_{i+1,j+1}}{2}\bigg)\bigg)\bigg)\\
\qquad\text{if } t(i+1)=0,\\
m_{\mathbb{T}}\bigg(R_\alpha^{-j(l_{i+1})}\bigg(\bigcup_{j\in\mathcal{I}_{t,i+1}}\bigg[\dfrac{a_{i+1,j}+a_{i+1,j+1}}{2},a_{i+1,j+1}\bigg)\bigg)\bigg)\\
\qquad\text{if } t(i+1)=1.
\end{cases}
$$

By induction, we have

$$
\begin{aligned}
m_{\mathbb{T}}(J_{x_0,i+1}(t))&=\sum_{j\in\mathcal{I}_{t,i+1}}\frac{1}{2}(a_{i+1,j+1}-a_{i+1,j})\le\frac{1}{2}\sum_{j'\in\mathcal{I}_{t,i}}\frac{(a_{i,j'+1}-a_{i,j'})}{2}\\
&=\frac{1}{2}m_{\mathbb{T}}(J_{x_0,i}(t))\le\bigg(\frac{1}{2}\bigg)^{i+1}\delta_k.
\end{aligned}
$$

This implies $m_{\mathbb{T}}(J_{x_0,s}(t))\le(\frac{1}{2})^s\delta_k$.

4. *The final construction: sub-exponential measure complexity*

As in §2, we fix an irrational number α and an $\eta\in(0,\frac{1}{100})$. Let $R_\alpha:\mathbb{T}\to\mathbb{T}$, $z\mapsto z+\alpha$ be the irrational rotation of $\mathbb{T}$ by α. Let h be the function defined in §2.2. Then we set

$$T(x,y)=(x+\alpha,y+\tfrac{1}{2}h(x))$$

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for $(x, y) \in \mathbb{T}^2$. The following result was first proved by Kočergin [**8**] (one can also see Lindenstrauss [**9**, Theorem 3.1] for a more general setting): there exist a measurable function $p : \mathbb{T} \to \mathbb{T}$ and a continuous function $\tilde{h} : \mathbb{T} \to \mathbb{T}$ such that

$$\tilde{h}(x) = \tfrac{1}{2}h(x) + p(x+\alpha) - p(x)$$

for $m_{\mathbb{T}}$-almost every $x \in \mathbb{T}$.

We now define a skew product $\tilde{T} : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\tilde{T}(x, y) = (x + \alpha, y + \tilde{h}(x))$ for $(x, y) \in \mathbb{T}^2$.

THEOREM 4.1. $(\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \tilde{\rho}, \tilde{T})$ *has sub-exponential measure complexity for any* $\tilde{\rho} \in \mathcal{M}(\mathbb{T}^2, \tilde{T})$*,*

Proof. We follow the arguments in the proof of [**5**, Proposition 2.2]. Let $\tilde{\rho} \in M(\mathbb{T}^2, \tilde{T})$. Denote $\phi : (x, y) \to (x, y - p(x))$ and $\rho = \pi^*(\tilde{\rho}) = \tilde{\rho} \circ \phi^{-1}$. It is not too hard to see that $\rho \in M(\mathbb{T}^2, T)$ and $\phi : (\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \tilde{\rho}, \tilde{T}) \to (\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \rho, T)$ is a measure-theoretic isomorphism.

We choose a Borel subset B of $\mathbb{T}$ such that $m_{\mathbb{T}}(B) = 1$, $R_\alpha(B) = B$ and

$$\tilde{h}(x) = \tfrac{1}{2}h(x) + p(x+\alpha) - p(x)$$

for any $x \in B$. Let $\tilde{Z} = Z = B \times \mathbb{T}$. Then $\tilde{T}(\tilde{Z}) = \tilde{Z}$, $T(Z) = Z$, $\phi(\tilde{Z}) = Z$ and $\phi \circ \tilde{T}(\tilde{z}) = T \circ \phi(\tilde{z})$ for all $\tilde{z} \in \tilde{Z}$.

Fix $\epsilon > 0$. By Lusin's theorem there exists a compact subset A of $\tilde{Z}$ such that $\tilde{\rho}(A) > 1 - (\epsilon/4)^2$ and $\phi|_A$ is a continuous function. Choose $\delta \in (0, \epsilon/2)$ such that $\sqrt{2\delta} < (\epsilon/6)$ and

$$d(\phi(\tilde{z}), \phi(\tilde{z}')) < \frac{\epsilon}{3} \quad \text{for any } \tilde{z}, \tilde{z}' \in A \text{ with } d(\tilde{z}, \tilde{z}') < \sqrt{2\delta}. \tag{4.1}$$

Now we show that $S_n(d, \tilde{\rho}, \delta) \geq S_n(d, \rho, \epsilon)$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. For $\tilde{z} \in A$, let $E(\tilde{z}) = \{i \geq 0 : \tilde{T}^i \tilde{z} \in A\}$ and let

$$E_n = \left\{\tilde{z} \in A : \frac{|E(\tilde{z}) \cap [0, n-1]|}{n} \leq 1 - \epsilon/4\right\}.$$

Note that

$$\int_{\mathbb{T}^2} \frac{|E(x) \cap [0, n-1]|}{n}\, d\tilde{\rho}(\tilde{z}) = \int_{\mathbb{T}^2} \frac{1}{n} \sum_{i=0}^{n-1} 1_A(\tilde{T}^i \tilde{z})\, d\tilde{\rho}(\tilde{z}) = \tilde{\rho}(A) > 1 - (\epsilon/4)^2.$$

We have

$$(1 - \epsilon/4)\tilde{\rho}(E_n) + 1 - \tilde{\rho}(E_n) \geq \int_{\mathbb{T}^2} \frac{|E(\tilde{z}) \cap [0, n-1]|}{n}\, d\tilde{\rho}(\tilde{z}) > 1 - (\epsilon/4)^2$$

which implies that $\tilde{\rho}(E_n) < \epsilon/4$.

For $\tilde{z}, \tilde{z}' \in A$, if $\overline{d}_n(\tilde{z}, \tilde{z}') = (1/n) \sum_{i=0}^{n-1} d(\tilde{T}^i \tilde{z}, \tilde{T}^i \tilde{z}') < 2\delta$ then it is easy to see that

$$\frac{1}{n}\#\{i \in [0, n-1] : d(\tilde{T}^i \tilde{z}, \tilde{T}^i \tilde{z}') \geq \sqrt{2\delta}\} < \sqrt{2\delta},$$

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and so, for $\tilde{z}, \tilde{z}' \in A' =: A \setminus E_n$ with $\overline{d}_n(\tilde{z}, \tilde{z}') < 2\delta$ (note that $\rho(A') > 1 - (\epsilon/4)^2 - (\epsilon/4) > 1 - (\epsilon/2)$), one has

$$\begin{aligned}\overline{d}_n(\phi(\tilde{z}), \phi(\tilde{z}')) &= \frac{1}{n}\sum_{i=0}^{n-1} d(T^i\phi(\tilde{z}), T^i\phi(\tilde{z}')) = \frac{1}{n}\sum_{i=0}^{n-1} d(\phi(\tilde{T}^i\tilde{z}), \phi(\tilde{T}^i\tilde{z}'))\\ &\leq \frac{1}{n}(\#\{i \in [0, n-1] : d(T^i\tilde{z}, T^i\tilde{z}') \geq \sqrt{2\delta}\}\\ &\quad + \frac{1}{n}\bigg(\#\{i \in [0, n-1] : \tilde{T}^i\tilde{z} \notin A \text{ or } \tilde{T}^i\tilde{z}' \notin A\}\\ &\quad + \frac{\epsilon}{3}\#\{i \in [0, n-1] : d(\tilde{T}^i\tilde{z}, \tilde{T}^i\tilde{z}') < \sqrt{2\delta}\}\bigg) \text{ (by (4.1))}\\ &\leq \sqrt{2\delta} + \frac{\epsilon}{2} + \frac{\epsilon}{3} < \epsilon.\end{aligned}$$

Pick $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_m \in \mathbb{T}^2$ such that $m = S_n(d, \tilde{\rho}, \delta)$ and

$$\tilde{\rho}\bigg(\bigcup_{i=1}^m B_{\overline{d}_n}(\tilde{z}_i, \delta)\bigg) > 1 - \delta.$$

Let $I_n = \{r \in [1, m] : B_{\overline{d}_n}(\tilde{z}_r, \delta) \cap A' \neq \emptyset\}$. For $r \in I_n$, we choose $\tilde{z}_r^n \in B_{\overline{d}_n}(\tilde{z}_r, \delta) \cap A'$. Then

$$\bigcup_{r\in I_n}(B_{\overline{d}_n}(\tilde{z}_r^n, 2\delta) \cap A') \supseteq \bigcup_{r\in I_n}(B_{\overline{d}_n}(\tilde{z}_r, \delta) \cap A') = \bigg(\bigcup_{i=1}^m B_{\overline{d}_n}(\tilde{z}_i, \delta)\bigg) \cap A'.$$

Thus

$$\tilde{\rho}\bigg(\bigcup_{r\in I_n}(B_{\overline{d}_n}(\tilde{z}_r^n, 2\delta) \cap A')\bigg) \geq \tilde{\rho}\bigg(\bigg(\bigcup_{i=1}^m B_{\overline{d}_n}(\tilde{z}_i, \delta)\bigg) \cap A'\bigg) > 1 - \delta - \bigg(\frac{\epsilon}{4}\bigg)^2 - \frac{\epsilon}{4} > 1 - \epsilon.$$

Since $\overline{d}_n(\phi(\tilde{z}), \phi(\tilde{z}')) < \epsilon$ for $\tilde{z}, \tilde{z}' \in A'$ with $\overline{d}_n(\tilde{z}, \tilde{z}') < 2\delta$, one has

$$\phi(B_{\overline{d}_n}(\tilde{z}_r^n, 2\delta) \cap A') \subseteq B_{\overline{d}_n}(\phi(\tilde{z}_r^n), \epsilon)$$

for $r \in I_n$. Thus

$$\begin{aligned}\rho\bigg(\bigcup_{r\in I_n} B_{\overline{d}_n}(\phi(\tilde{z}_r^n), \epsilon)\bigg) &\geq \rho\bigg(\bigcup_{r\in I_n} \phi(B_{\overline{d}_n}(\tilde{z}_r^n, 2\delta) \cap A')\bigg)\\ &= \rho\bigg(\phi\bigg(\bigcup_{r\in I_n} B_{\overline{d}_n}(\tilde{z}_r^n, 2\delta) \cap A'\bigg)\bigg)\\ &= \tilde{\rho}\bigg(\bigcup_{r\in I_n} B_{\overline{d}_n}(\tilde{z}_r^n, 2\delta) \cap A'\bigg) > 1 - \epsilon.\end{aligned}$$

Hence $S_n(d, \rho, \epsilon) \leq |I_n| \leq m = S_n(d, \tilde{\rho}, \delta)$. This implies

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log S_n(d, \rho, \epsilon)}{n^\tau} \leq \lim_{\delta\to 0}\liminf_{n\to\infty}\frac{\log S_n(d, \tilde{\rho}, \delta)}{n^\tau}$$

for any $0 < \tau < 1$. Moreover, by Proposition 3.1 we have

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log S_n(d, \rho, \epsilon)}{n^\tau} = +\infty,$$

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for any $0 < \tau < 1$. It follows that

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{\log S_n(d, \tilde{\rho}, \delta)}{n^\tau} = +\infty,$$

for any $0 < \tau < 1$.

Finally, since $(\mathbb{T}^2, \tilde{T})$ is distal, the topological entropy of $(\mathbb{T}^2, \tilde{T})$ is zero [**11**]. This implies

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{\log S_n(d, \tilde{\rho}, \delta)}{n} = 0.$$

Hence, $(\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \tilde{\rho}, \tilde{T})$ has sub-exponential measure complexity. □

Remark 4.2. (1) In general, it is true that sub-exponential measure complexity is a measure-theoretic invariant. One can see this by using the methods in [**5**]. (2) The *topological complexity* of the system $(\mathbb{T}^2, \tilde{T}_h)$ is also sub-exponential, since the system has zero topological entropy and the topological complexity is not less than the measure complexity.

We have the following proposition related to skew product maps on $\mathbb{T}^2$.

PROPOSITION 4.3. *If a skew product map $W : \mathbb{T}^2 \to \mathbb{T}^2$ over an irrational rotation on $\mathbb{T}$ is not minimal then it is equicontinuous.*

Proof. Let $W : \mathbb{T}^2 \to \mathbb{T}^2$ be a skew product map on $\mathbb{T}^2$ such that $W(x, y) = (x + \alpha, y + k(x))$ for any $(x, y) \in \mathbb{T}^2$, where α is irrational and $k : \mathbb{T} \to \mathbb{T}$ is continuous.

Let Y be a minimal subset of $(\mathbb{T}^2, W)$. Then $Y \neq \mathbb{T}^2$. First, we consider that $\mathbb{T}$ acts on $\mathbb{T}^2$ by $S_h(x, y) = (x, y + h)$ for any $h \in \mathbb{T}$. It is clear that $S_h \circ W = W \circ S_h$ for any $h \in \mathbb{T}$. Thus if $h \in \mathbb{T}$, then $S_h(Y)$ is a minimal subset of $\mathbb{T}^2$ and $S_h(Y) = Y$ or $S_h(Y) \cap Y = \emptyset$.

Let $H = \{h \in \mathbb{T} : S_h(Y) = Y\}$. Then H is a non-empty closed subgroup of $\mathbb{T}$. Moreover, $H \neq \mathbb{T}$ since $Y \neq \mathbb{T}^2$. This implies that H is a finite subgroup, since a closed subgroup of $\mathbb{T}$ is $\mathbb{T}$ or a finite group.

Next, for $x \in \mathbb{T}$, put $Y(x) = \{y \in \mathbb{T} : (x, y) \in Y\}$. Then $Y(x)$ is a closed subset of $\mathbb{T}$ and

$$Y(x) - Y(x) = H.$$

In fact, for $h \in H$, one has $Y(x) + h = Y(x)$ since $S_h(Y) = Y$. Thus $h \in Y(x) - Y(x)$ when $h \in H$. Conversely, let $h \in Y(x) - Y(x)$. Then $h = y_1 - y_2$ for some $y_1, y_2 \in Y(x)$ and so $(x, y_2) \in S_h(Y) \cap Y$. This implies $S_h(Y) = Y$, that is, $h \in H$.

Combining the fact that $Y(x) - Y(x) = H$ and the fact that H is a finite closed subgroup of $\mathbb{T}$, we know that $Y(x)$ is a finite set and $\#Y(x) = \#H$.

Let $\pi : Y \to \mathbb{T}$ be the projection $\pi(x, y) = x$ for $(x, y) \in Y$. Then $\pi : (Y, W) \to (\mathbb{T}, R_\alpha)$ is a factor map between two minimal systems. Since $(\mathbb{T}, R_\alpha)$ is minimal equicontinuous and π is a $\#H$-to-1 extension, that is, $\#Y(x) = \#H$ for any $x \in \mathbb{T}$, one has that (Y, W) is also a minimal equicontinuous t.d.s. by [**13**, Theorem 2].

We now show that $(\mathbb{T}^2, W)$ is equicontinuous itself. Let $\epsilon > 0$. By the equicontinuity of (Y, W), there is $0 < \delta_1 < \epsilon/4$ such that if $(x_1, y_1), (x_2, y_2) \in Y$ and $d((x_1, y_1), (x_2, y_2)) < \delta_1$ then $d(W^n(x_1, y_1), W^n(x_2, y_2)) < \epsilon/2$ for all $n \in \mathbb{Z}$. It is clear that $\pi' : X \to 2^H$, $x \mapsto \pi^{-1}(x)$ is continuous since π is a distal extension. This means

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that there is $\delta_2 > 0$ such that if $\|x_1 - x_2\| < \delta_2$ then $d_H(\pi'(x_1), \pi'(x_2)) < \delta_1$, where d_H is the Hausdorff distance. Let $\delta = \min\{\delta_1, \delta_2\}$.

Assume that $d((x_1, y_1), (x_2, y_2)) < \delta$. There is $h \in \mathbb{T}$ such that $(x_1, y_1 - h) \in Y$. This implies that there is $y_2^* \in \mathbb{T}$ such that $(x_2, y_2^* - h) \in Y$ and $\|(y_1 - h) - (y_2^* - h)\| < \delta_1$, since $\|x_1 - x_2\| < \delta_2$. Thus, $d((x_1, y_1 - h), (x_2, y_2^* - h)) < \delta_1$. Then, for any $n \in \mathbb{Z}$,

$$\begin{aligned} &d(W^n(x_1, y_1), W^n(x_2, y_2)) \\ &\quad = d(W^n(x_1, y_1 - h), W^n(x_2, y_2 - h)) \\ &\quad \leq d(W^n(x_1, y_1 - h), W^n(x_2, y_2^* - h)) + d(W^n(x_2, y_2^* - h), W^n(x_2, y_2 - h)) \\ &\quad < \epsilon/2 + 2\delta_1 < \epsilon. \end{aligned}$$

We conclude that $(\mathbb{T}^2, W)$ is equicontinuous. □

As a corollary we have the following statement.

COROLLARY 4.4. *$(\mathbb{T}^2, \tilde{T})$ is a minimal distal system with sub-exponential measure complexity for any invariant measure in $\mathcal{M}(\mathbb{T}^2, \tilde{T})$.*

Proof. The corollary follows by the fact that any equicontinuous t.d.s. has bounded complexity. □

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