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ON A CRITERION OF LOCAL INVERTIBILITY AND CONFORMALITY FOR SLICE REGULAR QUATERNIONIC FUNCTIONS

ANNA GORI[1] AND FABIO VLACCI[2]

[1] *Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italy* (anna.gori@unimi.it)
[2] *Dipartimento di Matematica e Informatica 'U. Dini', Università di Firenze Viale Morgagni, 67/A, 50134 Firenze, Italy* (vlacci@math.unifi.it)

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Abstract A new criterion for local invertibility of slice regular quaternionic functions is obtained. This paper is motivated by the need to find a geometrical interpretation for analytic conditions on the real Jacobian associated with a slice regular function f. The criterion involves spherical and Cullen derivatives of f and gives rise to several geometric implications, including an application to related conformality properties.

Keywords: injective function of a hypercomplex variable; slice regular functions

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1. Preliminaries and introduction

We denote by $\mathbb{H}$ the algebra of quaternions. Let $\mathbb{S}$ be the sphere of imaginary quaternions, i.e. the set of quaternions I such that $I^2 = -1$. Let $\Omega \subseteq \mathbb{H}$ be a domain.

Definition 1.1. We say that Ω is:

- an *axially symmetric domain* if, for all $x + Iy \in \Omega$, with $I \in \mathbb{S}$, the whole sphere $x + \mathbb{S}y$ is contained in Ω;
- a *slice domain* if $\Omega \cap \mathbb{R}$ is non-empty and if, given any $I \in \mathbb{S}$, the complex line $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$ intersecting with Ω is a domain in $\mathbb{C}_I$.

It is possible (see [**2**]) to introduce the notion of regularity for functions defined in any open ball $B(0, r) = \{q \in \mathbb{H} \ : \ |q| < r\}$ (and, more generally, in some axially symmetric slice domains of $\mathbb{H}$) which extends that of holomorphicity in the complex case.

Definition 1.2. If Ω is an axially symmetric slice domain in $\mathbb{H}$, a real differentiable function $f:\Omega\to\mathbb{H}$ is said to be *slice regular* if, for every $I\in\mathbb{S}$, its restriction f_I to the complex line $\mathbb{C}_I=\mathbb{R}+\mathbb{R}I$ passing through the origin and containing 1 and I is holomorphic on $\Omega\cap\mathbb{C}_I$.

We recall that the notion of slice regularity was first introduced in [2]; the theory of slice regular functions has been significantly developed in the past decade by many authors (a short list of contributions can be found in the references of [4]).

Remark 1.3. It can be proved that a function $f:B(0,r)\to\mathbb{H}$ is *slice regular* in $B(0,r)\subset\mathbb{H}$ if and only if there exists a converging power series $\sum_n q^n a_n$ in $B(0,r)$, with $a_n\in\mathbb{H}$ for any $n\in\mathbb{N}$, such that $f(q)=\sum_n q^n a_n$ with $q\in B(0,r)$.

As a direct computation on the real components of a slice regular function, one immediately obtains the following (see [2]).

Lemma 1.4. *If f is a slice regular function on an axially symmetric slice domain $\Omega\subset\mathbb{H}$, then for every $I\in\mathbb{S}$ and any $J\in\mathbb{S}$, $J\perp I$, there exist two holomorphic functions $F_1,F_2:\Omega\cap\mathbb{C}_I\to\mathbb{C}_I$ such that $f_I(z)=F_1(z)+F_2(z)J$ with $z=x+Iy$.*

For the sequel it will be important to recall a natural notion of the product of polynomials (then extended to power series), which turns out to provide a 'regular' multiplication of slice regular functions when represented by converging regular power series.

Definition 1.5. Let $f(q)=\sum_{n=0}^{+\infty}q^n a_n$ and $g(q)=\sum_{n=0}^{+\infty}q^n b_n$ be given power series with coefficients in $\mathbb{H}$ whose radii of convergence are greater than r. We define the *regular product* of f and g as the series $f*g(q)=\sum_{n=0}^{+\infty}q^n c_n$, where $c_n=\sum_{k=0}^{n}a_k b_{n-k}$ for all n, which is convergent in $B(0,r)$.

It is not difficult to see that $f*g$ is a slice regular function defined in the open ball $B(0,r)$. Furthermore, the regular product is extended for slice regular functions defined on a general axially symmetric domain Ω in the following way

$$f*g(q)=\begin{cases}0 & \text{if } f(q)=0,\\ f(q)g(f(q)^{-1}qf(q)) & \text{otherwise.}\end{cases}\tag{1.1}$$

In the spirit of Gateaux, the notion of a derivative is well defined for slice regular functions, namely (see [2]) as follows.

Definition 1.6. Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and let $f:\Omega\to\mathbb{H}$ be a slice regular function. For any $I\in\mathbb{S}$ and any point $q=x+yI$ in Ω (with $x=\Re eq$ and $y=\Im mq$) we define the *Cullen derivative* of f at q as

$$\partial_C f(x+yI)=f'(x+yI):=\frac{1}{2}\left(\frac{\partial}{\partial x}-I\frac{\partial}{\partial y}\right)f_I(x+yI).$$

Since in $\mathbb{H}$ one can choose different imaginary units, it is also worth considering the following.

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Definition 1.7. Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a slice regular function. We define the *spherical derivative* of f at q as

$$\partial_S f(q) := (q - \overline{q})^{-1}[f(q) - f(\overline{q})].$$

It is well known that the possibility of locally inverting a holomorphic function heavily depends on the non-vanishing of the derivative; it is also clear that a holomorphic function which is locally invertible turns out to be conformal. The aim of the present paper is to investigate a generalization of these facts for quaternionic slice regular functions.

2. A local invertibility criterion

Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and $f : \Omega \to \mathbb{H}$ be a slice regular function. If $q_0 \in \Omega$ and $q_0 \notin \mathbb{R}$, take $I \in \mathbb{S}$ so that $q_o \in \mathbb{C}_I$ and $J \in \mathbb{S}$ such that $I \perp J$ as vectors in $\mathbb{R}^3$. According to this choice of local coordinates, consider the corresponding splittings

$$f_I = F_1 + F_2 J \quad \text{and} \quad R_{q_0} f = R_1 + R_2 J,$$

where $R_{q_0} f$ is defined by

$$f(q) - f(q_o) = (q - q_0) * R_{q_0} f(q).$$

We also recall here that

$$R_{q_0} f(q_0) = \partial_C f(q_0) \quad \text{and} \quad R_{q_0} f(\overline{q_0}) = \partial_S f(q_0).$$

Furthermore, from [4, Theorem 8.16] and using the local coordinates as above, the (complex) Jacobian of f at q_0 can be written as

$$df_{q_0} = \begin{pmatrix} R_1(q_0) & -\overline{R_2(\overline{q_0})} \\ R_2(q_0) & \overline{R_1(\overline{q_0})} \end{pmatrix}.$$

We observe first that if f is a slice-preserving function (i.e. if f maps $\mathbb{C}_I \cap \Omega$ into $\mathbb{C}_I$) then, in local coordinates, $f = F_1$ and $R_f = R_1$, and hence

$$df_{q_0} = \begin{pmatrix} R_1(q_o) & 0 \\ 0 & R_1(q_o) \end{pmatrix}$$

which means that the complex Jacobian is invertible if and only if $R_1(q_o) \neq 0$ or, equivalently, if and only if $\partial_C f(q_o) \neq 0$. In general, for a 2×2 quaternionic matrix, its invertibility depends on the non-vanishing of its Dieudonné determinant $\det_{\mathbb{H}}$ which is defined as follows

$$\det{}_{\mathbb{H}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} -bc & \text{if } a = 0, \\ ad - aba^{-1}c & \text{if } a \neq 0. \end{cases}$$

In the case of the Jacobian of f at q_0, we observe that R_1, R_2, $\overline{R_1}$ and $\overline{R_2}$ are all self-maps of $\Omega \cap \mathbb{C}_I$ and hence their products commute; in other words

$$\det{}_{\mathbb{H}} \begin{pmatrix} R_1(q_0) & -\overline{R_2(\overline{q_0})} \\ R_2(q_0) & \overline{R_1(\overline{q_0})} \end{pmatrix} = R_1(q_0)\overline{R_1(\overline{q_0})} + R_2(q_0)\overline{R_2(\overline{q_0})}.$$

Therefore, according to the previous positions, one can write

$$\partial_c f(q_0) = \begin{pmatrix} R_1(q_0) \\ R_2(q_0) \end{pmatrix}, \qquad \partial_S f(q_0) = \begin{pmatrix} R_1(\overline{q_0}) \\ R_2(\overline{q_0}) \end{pmatrix}$$

and hence, using the (standard) Hermitian product $\langle \cdot | \cdot \rangle$ in $\mathbb{C}^2$, one obtains that

$$\det{}_{\mathbb{H}} \begin{pmatrix} R_1(q_0) & -\overline{R_2(\overline{q_0})} \\ R_2(q_0) & \overline{R_1(\overline{q_0})} \end{pmatrix} = \langle \partial_C f(q_0) | \partial_S f(q_0) \rangle.$$

Remark 2.1. The usual quaternionic Hermitian product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ is defined as

$$\langle q, w \rangle_{\mathbb{H}} = q \cdot \overline{w}.$$

If one considers $q = a + bJ$ and $w = c + dJ$, with $a, b, c, d \in \mathbb{C}_I$ and $J \perp I$, then an easy and direct computation shows that $\langle q, w \rangle_{\mathbb{H}}$ splits as the sum of a component along $\mathbb{C}_I$ (namely $a\overline{c} + b\overline{d}$) and another component in $\mathbb{C}_I^{\perp}$. The component along $\mathbb{C}_I$ coincides with the Hermitian product $\langle \cdot | \cdot \rangle$ defined above.

Remark 2.2. We recall that in [5] the same Hermitian product of the Cullen and spherical derivatives of a slice regular function f appears in conditions which guarantee starlikeness for the function f.

We summarize our considerations by stating the following criterion of local invertibility.

Proposition 2.3. *With the above-given notation,*

$$df_{q_0} \text{ is locally invertible} \iff \langle \partial_C f(q_0) | \partial_S f(q_0) \rangle \neq 0.$$

Remark 2.4. The previous proposition can be interpreted in terms of Remark 2.1. Thus, if $q_0 \in \mathbb{C}_I$, with the above notation, df_{q_0} is not invertible if and only if $\langle \partial_C f(q_0) | \partial_S f(q_0) \rangle_{\mathbb{H}}$ belongs to $\mathbb{C}_I^{\perp}$ in accordance with the results in [1] which generalize the ones in [3].

An immediate consequence of the previous proposition is the obvious result that f is not locally invertible if $\partial_C f$ or $\partial_S f$ vanishes. The fact that $\partial_C f(q_0) = 0$ implies non-local invertibility for f is well known and clear as in the holomorphic case.

On the other hand, if $q_0 = x_0 + Iy_0$ and given any $q = x_0 + Jy_0$ with $J \in \mathbb{S}$ (see [4]), it turns out that

$$f(q) = c + Jb$$

with the same b, c for any $q \in \mathbb{S}_{q_0} := \{x_0 + Jy_0 : J \in \mathbb{S}\}$ and $b = \partial_s f(q_0)$. Then it clearly follows that if $\partial_S f(q_0) = 0$, the function f is constant on the sphere $\mathbb{S}_{q_0}$, and so f is not

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invertible. In order to provide an example of a slice regular function whose Cullen and spherical derivatives do not vanish at q_0, but their Hermitian product does, we recall the following fact (see [4]): the Jacobian of f is not invertible at $q_0 = x_0 + Iy_0$ if and only if there exist $\widetilde{q_0} = x_0 + I_1 y_0$ and a slice regular function g such that

$$f(q) - f(q_o) = (q - q_0) * (q - \widetilde{q_0}) * g(q).$$

The previous formula equivalently says that the Jacobian of f is not invertible at q_0 if and only if

$$R_{q_0} f(q) = (q - \widetilde{q_0}) * g(q).$$

Assume now that (with the usual frame associated with the choice of $J \perp I$) we choose the restriction of the slice regular function g along the slice $\mathbb{C}_I$ to be $g_I(q) = q + q^2 J$ and take $\widetilde{q_0} = x_0 + Jy_0$; thus, in this case, the restriction of the slice regular function $R_{q_0} f(q)$ along the slice $\mathbb{C}_I$ is $(q - \widetilde{q_0}) * (q + q^2 J)$ and, in particular, $R_1(q) = q^2 - qx_0 + q^2 y_0$ and $R_2(q) = q^3 - qy_0 - q^2 x_0$. Hence

$$R_1(q_0) = q_0^2 - q_0 x_0 + q_0^2 y_0 = -y_0^2 - y_0^3 + x_0^2 y_0 + (x_0 y_0 + 2x_0 y_0^2) I$$

and

$$R_2(q_0) = q_0^3 - q_0 y_0 - q_0^2 x_0 = -x_0 y_0 - 2x_0 y_0^2 + (-y_0^2 - y_0^3 + x_0^2 y_0) I$$

or

$$R_2(q_0) = -I R_1(q_0).$$

Furthermore, as easily seen from direct computations,

$$R_1(\overline{q_0}) = -y_0^2 - y_0^3 + x_0^2 y_0 - (x_0 y_0 + 2x_0 y_0^2) I = \overline{R_1(q_0)}$$

and

$$R_2(\overline{q_0}) = -x_0 y_0 - 2x_0 y_0^2 + (y_0^2 + y_0^3 - x_0^2 y_0) I = \overline{R_2(q_0)}$$

so that

$$\begin{aligned} \det{}_{\mathbb{H}} \begin{pmatrix} R_1(q_0) & -\overline{R_2(\overline{q_0})} \\ R_2(q_0) & \overline{R_1(\overline{q_0})} \end{pmatrix} &= R_1(q_0) \overline{R_1(\overline{q_0})} + R_2(q_0) \overline{R_2(\overline{q_0})} \\ &= \langle \partial_c f(q_0) | \partial_S f(q_0) \rangle = R_1(q_0)^2 + R_2(q_0)^2 = R_1(q_0)^2 - R_1(q_0)^2 = 0 \end{aligned}$$

even though neither $\partial_c f(q_0)$ nor $\partial_S f(q_0)$ vanishes if q_0 is not real or an imaginary unit.

3. Geometric interpretation of the criterion and characterization of conformality

Let f be a slice regular function on a axially symmetric slice domain Ω. If, for a given $I \in \mathbb{S}$ and $q_o \in \Omega$, one identifies the tangent space $T_{q_0}\Omega$ with $\mathbb{H} = \mathbb{C}_I \oplus \mathbb{C}_I^{\perp}$, then (see [4])

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for all $u \in \mathbb{C}_I$ and $v \in \mathbb{C}_I^{\perp}$,

$$df_{q_o}(u+w) = u\partial_C f(q_0) + w\partial_S f(q_0).$$

We will assume $w = vJ$ with $J \perp I$. Since, using the frame associated with the splitting corresponding to the choice of $J \perp I$, one has

$$\partial_C f(q_0) = R_1(q_0) + R_2(q_0)J,$$
$$\partial_S f(q_0) = R_1(\overline{q_0}) + R_2(\overline{q_0})J,$$

then

$$\begin{aligned} df_{q_o}(u+vJ) &= uR_1(q_0) + vJR_2(\overline{q_o})J + uR_2(q_0)J + vJR_1(\overline{q_o}) \\ &= uR_1(q_0) - v\overline{R_2(\overline{q_o})} + [uR_2(q_0) + v\overline{R_1(\overline{q_o})}]J. \end{aligned}$$

Therefore, after some computations,

$$\begin{aligned} |df_{q_o}(u+vJ)|^2 &= \langle df_{q_o}(u+vJ) | df_{q_o}(u+vJ)\rangle \\ &= |u|^2|R_1(q_0)|^2 + |v|^2|\overline{R_2(\overline{q_o})}|^2 + u[R_2(q_0)R_1(\overline{q_o}) - R_1(q_0)R_2(\overline{q_o})]\overline{v} \\ &\quad + v[\overline{R_1(\overline{q_0})R_2(q_o)} - \overline{R_2(\overline{q_0})R_1(q_o)}]\overline{u} + |u|^2|R_2(q_0)|^2 + |v|^2|\overline{R_1(\overline{q_o})}|^2 \\ &= |u|^2|\partial_C f(q_0)|^2 + |v|^2|\partial_S f(q_0)|^2 \\ &\quad + u[R_2(q_0)R_1(\overline{q_o}) - R_1(q_0)R_2(\overline{q_o})]\overline{v} \\ &\quad + v[\overline{R_1(\overline{q_0})R_2(q_o)} - \overline{R_2(\overline{q_0})R_1(q_o)}]\overline{u}. \end{aligned}$$

In other words, if $\mathcal{A} = u[R_2(q_0)R_1(\overline{q_o}) - R_1(q_0)R_2(\overline{q_o})]\overline{v}$, one gets

$$\begin{aligned} |df_{q_o}(u+vJ)|^2 &= |u|^2|\partial_C f(q_0)|^2 + |v|^2|\partial_S f(q_0)|^2 + \mathcal{A} + \overline{\mathcal{A}} \\ &= |u|^2|\partial_C f(q_0)|^2 + |v|^2|\partial_S f(q_0)|^2 + 2\Re e\mathcal{A}; \end{aligned}$$

therefore, if $\partial_C f(q_0) \neq 0$ and $\partial_S f(q_0) \neq 0$, there exists no pair $(u,v) \neq (0,0)$ such that $df_{q_0}(u+vJ) = 0$ if $\Re e(\mathcal{A}) \geq 0$. In this case, f is then locally invertible. On the other hand, if $\partial_C f(q_0) = \partial_S f(q_0) = 0$, then $\mathcal{A} = 0$. In this case, f is not locally invertible.

Now we want to investigate what happens to $\Re e\mathcal{A}$ when $\langle \partial_C f(q_0) | \partial_S f(q_0)\rangle = 0$ but $\partial_C f(q_0) \neq 0$ and $\partial_S f(q_0) \neq 0$. First of all, one can write

$$\mathcal{A} = u\mathcal{B}\overline{v}$$

where

$$\mathcal{B} := R_2(q_0)R_1(\overline{q_o}) - R_1(q_0)R_2(\overline{q_o}).$$

We observe that

$$\det{}_{\mathbb{H}} \begin{pmatrix} R_1(\overline{q_0}) & R_1(q_0) \\ R_2(\overline{q_0}) & R_2(q_0) \end{pmatrix} = \mathcal{B}.$$

It then turns out that $\mathcal{B} = 0$ if and only if $\partial_C f(q_0)$ and $\partial_S f(q_0)$ are linearly dependent. If one assumes that $\partial_C f(q_0) \neq 0$ and $\partial_S f(q_0) \neq 0$ and $\langle \partial_C f(q_0) | \partial_S f(q_0)\rangle = 0$, then $\partial_C f(q_0)$ and $\partial_S f(q_0)$ are linearly independent, so that $\mathcal{B} \neq 0$. Furthermore, we state the following.

Proposition 3.1. *Let f be a slice regular function on an axially symmetric slice domain Ω and let q_o be in Ω. If $\partial_C f(q_0) \neq 0$ and $\partial_S f(q_0) \neq 0$, then (with the notations introduced so far) the following conditions are equivalent*

- *f is not locally invertible in (a neighbourhood of) q_0;*
- *the matrix associated with df_{q_0} is not invertible or singular;*
- $\langle \partial_C f(q_0) | \partial_S f(q_0) \rangle = 0$
- *the Hermitian matrix*
$$\begin{pmatrix} |\partial_c f(q_0)|^2 & \overline{\mathcal{B}} \\ \mathcal{B} & |\partial_S f(q_0)|^2 \end{pmatrix}$$
is singular or the associated Hermitian product is degenerate.

Proof. The first two conditions are clearly equivalent, and they are both equivalent to the third condition, thanks to Proposition 2.3. Thus, under the assumptions $\partial_C f(q_0) \neq 0$ and $\partial_S f(q_0) \neq 0$, the condition $0 = \langle \partial_C f(q_0) | \partial_S f(q_0) \rangle = R_1(q_0)\overline{R_1(\overline{q_0})} + R_2(q_0)\overline{R_2(\overline{q_0})}$ implies that at least one of the following identities holds

$$R_1(q_0) = \frac{-R_2(q_0)\overline{R_2(\overline{q_0})}}{\overline{R_1(\overline{q_0})}}, \qquad R_2(q_0) = \frac{-R_1(q_0)\overline{R_1(\overline{q_0})}}{\overline{R_2(\overline{q_0})}}.$$

Let us assume that the first one holds, so that, after substitution,

$$\mathcal{B} = \frac{R_2(q_0)}{\overline{R_1(\overline{q_0})}} |\partial_S f(q_0)|^2$$

and hence

$$|df_{q_0}(u + vJ)|^2 = 0 \iff u \frac{R_2(q_0)}{\overline{R_1(\overline{q_0})}} \overline{v} + v \frac{\overline{R_2(q_0)}}{R_1(\overline{q_0})} \overline{u} + |u|^2 \frac{|\partial_C f(q_0)|^2}{|\partial_S f(q_0)|^2} + |v|^2 = 0.$$

This equation can be regarded as the equation which describes $\ker df_{q_0}$. Another way to equivalently write this equation for $\ker df_{q_0}$ is to consider

$$(u, v) \begin{pmatrix} |\partial_c f(q_0)|^2 & \overline{\mathcal{B}} \\ \mathcal{B} & |\partial_S f(q_0)|^2 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} = 0. \tag{3.1}$$

We observe that the matrix in (3.1) is Hermitian, so that it defines a Hermitian product. Therefore, there exists a pair $(u, v) \neq (0, 0)$ such that $df_{q_0}(u + vJ) = 0$ (or, equivalently, df_{q_0} is not invertible) if and only if the Hermitian product introduced in (3.1) is degenerate. Indeed, this is equivalent to saying that

$$\det{}_{\mathbb{H}} \begin{pmatrix} |\partial_c f(q_0)|^2 & \overline{\mathcal{B}} \\ \mathcal{B} & |\partial_S f(q_0)|^2 \end{pmatrix} = |\partial_c f(q_0)|^2 |\partial_S f(q_0)|^2 - |\mathcal{B}|^2 = 0.$$

Now, in general, one has

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$$
\begin{aligned}
|\mathcal{B}|^2 &= [R_2(q_0)R_1(\overline{q_o}) - R_1(q_0)R_2(\overline{q_o})][\overline{R_1(\overline{q_o})R_2(q_0)} - \overline{R_2(\overline{q_o})R_1(q_0)}] \\
&= |R_2(q_0)|^2|R_1(\overline{q_o})|^2 + |R_1(q_0)|^2|R_2(\overline{q_o})|^2 - R_2(q_0)R_1(\overline{q_o})\overline{R_2(\overline{q_o})R_1(q_0)} \\
&\quad - R_1(q_0)R_2(\overline{q_o})\overline{R_1(\overline{q_o})R_1(q_0)}.
\end{aligned}
$$

If $0 = \langle \partial_C f(q_0)|\partial_S f(q_0)\rangle = R_1(q_0)\overline{R_1(\overline{q_0})} + R_2(q_0)\overline{R_2(\overline{q_0})}$ then $R_1(q_0)\overline{R_1(\overline{q_0})} = -R_2(q_0)\overline{R_2(\overline{q_0})}$ and, since $R_1, \overline{R_1}, R_2$ and $\overline{R_2}$ commute, one can equivalently write

$$
|\mathcal{B}|^2 = |R_2(q_0)|^2|R_1(\overline{q_o})|^2 + |R_1(q_0)|^2|R_2(\overline{q_o})|^2 + |R_1(q_0)|^2|R_1(\overline{q_o})|^2 + |R_2(q_0)|^2|R_2(\overline{q_o})|^2
$$

so that

$$
|\mathcal{B}|^2 = |\partial_c f(q_0)|^2|\partial_S f(q_0)|^2
$$

which implies that

$$
|df_{q_0}(u + vJ)|^2 = 0
$$

has a solution $(u, v) \neq (0, 0)$ or df_{q_0} is singular, as desired. □

We conclude this paper by providing an explicit description of $\ker df_{q_0}$ under the assumptions $\partial_C f(q_0) \neq 0$, $\partial_S f(q_0) \neq 0$ and $\langle \partial_C f(q_0)|\partial_S f(q_0)\rangle = 0$. It is known (see [4]) that in general the rank of df_{q_0} (regarded as a 4×4 real matrix) can be 0, 2 or 4. We will show in detail that under the assumptions $\partial_C f(q_0) \neq 0$, $\partial_S f(q_0) \neq 0$ and $\langle \partial_C f(q_0)|\partial_S f(q_0)\rangle = 0$, the rank of df_{q_0} is precisely 2.

First of all, we write $u = t + sI$, $v = x + yI$ and

$$
\mathcal{B} = B_1 + B_2 I, \qquad \overline{\mathcal{B}} = B_1 - B_2 I.
$$

We recall that, under our assumptions,

$$
|\mathcal{B}|^2 = B_1^2 + B_2^2 = |\partial_c f(q_0)|^2|\partial_S f(q_0)|^2.
$$

Hence the equation of $\ker df_{q_0}$ as in (3.1) becomes

$$
(s + tI, x + yI)\begin{pmatrix} |\partial_c f(q_0)|^2 & B_1 - IB_2 \\ B_1 + B_2 I & |\partial_S f(q_0)|^2 \end{pmatrix}\begin{pmatrix} s - tI \\ x - yI \end{pmatrix} = 0.
$$

After some computations, one obtains

$$
(t^2 + s^2)|\partial_c f(q_0)|^2 + (x^2 + y^2)|\partial_S f(q_0)|^2 + 2xB_1 t - 2yB_2 t + 2yB_1 s + 2xB_2 s = 0
$$

or

$$
(t^2 + s^2)\frac{(B_1^2 + B_2^2)}{|\partial_S f(q_0)|^4} + x^2 + y^2 + 2x\frac{(B_1 t + B_2 s)}{|\partial_S f(q_0)|^2} + 2y\frac{(B_1 s - B_2 t)}{|\partial_S f(q_0)|^2} = 0.
$$

This leads us to write

$$
\left[x + \frac{(B_1 t + B_2 s)}{|\partial_S f(q_0)|^2}\right]^2 + \left[y + \frac{(B_1 s - B_2 t)}{|\partial_S f(q_0)|^2}\right]^2 = 0
$$

or

$$x = -\frac{(B_1 t + B_2 s)}{|\partial_S f(q_0)|^2}, \qquad y = -\frac{(B_1 s - B_2 t)}{|\partial_S f(q_0)|^2}.$$

Therefore, the set of pairs (u, v) such that $df_{q_0}(u + vJ) = 0$ is a plane in $\mathbb{R}^4$, and so the rank of the real 4×4 matrix associated with df_{q_0} is 2, as expected.

Remark 3.2. From the above-given calculations, it also follows that a slice regular quaternionic function turns out to be *conformal* at q_0 (in the real sense, as a function from $\Omega \subseteq \mathbb{R}^4 \to \mathbb{R}^4$) if and only if $\mathcal{B} = 0$ and $|\partial_C f(q_0)|^2 = |\partial_S f(q_0)|^2 \neq 0$. This is, for instance, the case for a slice regular function f whose associated (slice regular) function $R_{q_0} f$ is real analytic (i.e. $R_{q_0} f(q) = \sum_n q^n a_n$ with $a_n \in \mathbb{R}$ for any $n \in \mathbb{N}$). The real analyticity of $R_{q_0} f$ is clearly a consequence of the real analyticity of f together with the assumption $q_0 \in \mathbb{R}$ (which implies that f is a slice-preserving and slice regular quaternionic function), but one can consider also other functions such as

$$f(q) = J + (q - I) * \exp(q)$$

where $q_0 = I \in \mathbb{S},\ f(I) = J \neq I,\ J \in \mathbb{S}$ and $\exp(q) = \sum_n (q^n/n!)$. The function f turns out to be not slice-preserving but conformal at $q_0 = I$. On the other hand, if one drops the assumption $q_0 \in \mathbb{R}$, it is not in general true that, for a slice regular and slice-preserving function f, the (associated) slice regular function $R_{q_0} f$ is real analytic, as the function

$$f(q) = q^2 - 2q\Re e q_0 + |q_0|^2 = (q - q_0) * (q - \overline{q_0})$$

clearly demonstrates.

From the previous remark and considerations, we conclude by stating this interesting property on the Cullen and spherical derivatives of a slice regular function which turns out to be also conformal.

Corollary 3.3. *Assume that $f : \Omega \subseteq \mathbb{H} \to \mathbb{H}$ is a slice regular function. If f is conformal at $q_0 \in \Omega$ then there exist two unitary quaternions $U, V \in \mathbb{H}$, with $|U| = |V| = 1$, such that*

$$\partial_C f(q_0) = U \partial_S f(q_0) V.$$

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