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Abstract

In this paper, we prove that the log minimal model program in dimension $d-1$ implies the existence of log minimal models for effective lc pairs (e.g. of non-negative Kodaira dimension) in dimension d. In fact, we prove that the same conclusion follows from a weaker assumption, namely, the log minimal model program with scaling in dimension $d-1$. This enables us to prove that effective lc pairs in dimension five have log minimal models. We also give new proofs of the existence of log minimal models for effective lc pairs in dimension four and of the Shokurov reduction theorem.

1. Introduction

All the varieties in this paper are assumed to be over an algebraically closed field k of characteristic zero. See § 2 for notation and terminology. For basic notions of the log minimal model program (LMMP) which are not specified below, e.g. singularities of pairs, we follow Kollár and Mori [KM98], although we would need their analogues for $\mathbb{R}$-divisors.

One of the main problems in birational geometry and the classification theory of algebraic varieties in the last three decades or so has been the following conjecture.

Conjecture 1.1 (Minimal model). Any lc pair $(X/Z, B)$ has a log minimal model or a Mori fibre space.

Here B is an $\mathbb{R}$-boundary. Roughly speaking, a model Y birational to X is a log minimal model if $K+B$ is nef on it, and a Mori fibre space if it has a log Fano fibre structure which is negative with respect to $K+B$, where K stands for the canonical divisor.

The two-dimensional case of the above conjecture is considered to be a classical early 20th century result of the Italian algebraic geometry school, at least when X is smooth and $B=0$. The three-dimensional case was proved by the contributions of many people, in particular Mori's theorems on extremal rays [Mor82] and existence of flips [Mor88], Shokurov's results on existence of log flips [Sho93], termination [Sho86, Sho96], and non-vanishing [Sho86], the Kawamata–Viehweg vanishing theorem [Kaw82, Vie82], the Shokurov–Kawamata base point free theorem [Sho86], and Kawamata's termination of log flips [Kaw92]. A much simpler proof of Conjecture 1.1 in dimension three would be a combination of Shokurov's simple proof of existence of log flips in dimension three [Sho03], his termination in the terminal case [Sho86], and his recent method of constructing log minimal models [Sho] (see also [Bir09]). In the case where the pair is effective, i.e. there is an effective $\mathbb{R}$-divisor $M \equiv K_X + B/Z$, Shokurov's existence of log flips in dimension three [Sho03] and this paper give yet another proof.

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In dimension four, Shokurov's existence of log flips [Sho03] and the Kawamata–Matsuda–Matsuki termination theorem [KMM87] prove the conjecture when $B=0$ and X has terminal singularities. The general case in dimension four is a theorem of Shokurov [Sho] with a short proof given in Birkar [Bir09] in the klt case. In dimension five, the conjecture is proved below when $(X/Z, B)$ is effective, in particular if the pair has non-negative Kodaira dimension. In higher dimension, the conjecture is known in the klt case when B is big/Z by Birkar *et al.* [BCHM10], in particular if X/Z is of general type such as a flipping contraction.

To construct a log minimal model or a Mori fibre space one often runs the LMMP. In order to be able to run the LMMP we have to deal with the existence and termination of log flips. As experience teaches us, the best way might be an inductive argument. Existence of log flips in the klt (and $\mathbb{Q}$-factorial dlt) case is treated in [BCHM10, BP], but we will not use their results below since we want to consider existence of log flips as a special case of existence of log minimal models and hence to fit it into the inductive approach below and the results of Shokurov [Sho03] and Hacon and McKernan [HM07]. As for the termination, inductive arguments involve other conjectures: the LMMP in dimension $d-1$ and the ascending chain condition (ACC) for lc thresholds in dimension d imply termination of log flips for effective lc pairs in dimension d by Birkar [Bir07]. Other results in this direction include the reduction of termination of log flips in dimension d to the ACC and semi-continuity for minimal log discrepancies in dimension d by Shokurov [Sho04], and the reduction of termination of log flips for effective lc pairs in dimension d to the LMMP, boundedness of certain Fano varieties, and the ACC for minimal log discrepancies in dimension $d-1$ by Birkar and Shokurov [BS].

In recent years, there have been more attempts in constructing log minimal models and Mori fibre spaces rather than proving a general termination statement. For example, Shokurov [Sho] proved that the LMMP in dimension $d-1$ and termination of terminal log flips in dimension d imply the minimal model conjecture in dimension d. In this paper, using quite different methods, we prove the following inductive theorem.

Theorem 1.2. *Assume the LMMP for $\mathbb{Q}$-factorial dlt pairs of dimension $d-1$. Then, Conjecture 1.1 holds for effective lc pairs in dimension d.*

We must emphasise that unlike in [BCHM10, BP], we do not assume bigness of the boundary B. The abundance conjecture predicts that a lc pair $(X/Z, B)$ has a log minimal model exactly when it is effective. When $(X/Z, B)$ is not effective, existence of a Mori fibre space is predicted.

In Theorem 1.2, the full LMMP in dimension $d-1$ is not necessary; we only need the LMMP with scaling (see Definition 3.2).

Theorem 1.3. *Assume the LMMP with scaling for $\mathbb{Q}$-factorial dlt pairs in dimension $d-1$. Then, Conjecture 1.1 holds for effective lc pairs in dimension d.*

In the previous theorem one can actually weaken the assumption on the LMMP in dimension $d-1$ by replacing it with Conjecture 1.1 in dimension $d-1$, as is shown in [Bir], the sequel to this paper. Furthermore, it is also shown that Conjecture 1.1 in dimension $d-1$ follows from the following weak non-vanishing conjecture in dimension $d-1$.

Conjecture 1.4. If a $\mathbb{Q}$-factorial dlt pair $(X/Z, B)$ is pseudo-effective, then it is effective.

When $d=5$, instead of proving Conjecture 1.4 in dimension four we directly use termination of log flips with scaling in dimension four, which we prove follows from results of Shokurov [Sho] and Alexeev *et al.* [AHK07].

Corollary 1.5. *Log minimal models exist for effective lc pairs in dimension five.*

We also get a new proof of the following result, which was first proved by Shokurov [Sho] (see also [Bir09]).

Corollary 1.6. *Log minimal models exist for effective lc pairs in dimension four.*

Using this corollary, Fujino [Fuj] has proved the finite generation of the lc ring, and hence the existence of lc models, for lc pairs in dimension four.

Our method gives a new proof of the Shokurov reduction theorem, that is, reducing the existence of log flips to the special termination and the existence of pl flips (see Theorem 3.9). A variant of the reduction theorem is an important ingredient in the construction of log flips in [BCHM10, BP].

For other important applications, see [BP, § 2] and [BCHM10, § 5]. To avoid confusion, let us make it clear that this paper is logically independent of [BCHM10, BP].

2. Basics

Let k be an algebraically closed field of characteristic zero fixed throughout the paper.

A *pair* $(X/Z, B)$ consists of normal quasi-projective varieties X, Z over k, an $\mathbb{R}$-divisor B on X with coefficients in $[0, 1]$ such that $K_X + B$ is $\mathbb{R}$-Cartier, and a projective morphism $X \to Z$. For a prime divisor D on some birational model of X with a non-empty centre on X, $a(D, X, B)$ denotes the log discrepancy.

A pair $(X/Z, B)$ is called *pseudo-effective* if $K_X + B$ is pseudo-effective/Z, that is, up to numerical equivalence/Z it is the limit of effective $\mathbb{R}$-divisors. The pair is called *effective* if $K_X + B$ is effective/Z, that is, there is an $\mathbb{R}$-divisor $M \geqslant 0$ such that $K_X + B \equiv M/Z$; in this case, we call $(X/Z, B, M)$ a *triple*. By a log resolution of a triple $(X/Z, B, M)$ we mean a log resolution of $(X, \operatorname{Supp} B + M)$. When we refer to a triple as being lc, dlt, etc, we mean that the underlying pair has such properties.

Let $(X/Z, B)$ be a lc pair. By a *log flip/Z* we mean the flip of a $K_X + B$-negative extremal flipping contraction/Z [Bir07, Definition 2.3], and by a *pl flip/Z* we mean a log flip/Z such that $(X/Z, B)$ is $\mathbb{Q}$-factorial dlt and the log flip is also an S-flip for some component S of $\lfloor B \rfloor$.

A *sequence of log flips/Z starting with* $(X/Z, B)$ is a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ in which $X_i \to Z_i \leftarrow X_{i+1}$ is a $K_{X_i} + B_i$-flip/Z, B_i is the birational transform of B_1 on X_1, and $(X_1/Z, B_1) = (X/Z, B)$.

In this paper, *special termination* means termination near $\lfloor B \rfloor$ of any sequence of log flips/Z starting with a pair $(X/Z, B)$, that is, the log flips do not intersect $\lfloor B \rfloor$ after finitely many of them. There is a more general notion of special termination that claims that the log flips do not intersect any lc centre after finitely many steps, but we do not use it below.

Definition 2.1. For an $\mathbb{R}$-divisor $D = \sum d_i D_i$, let $D^{\leqslant 1} := \sum d_i' D_i$, where $d_i' = \min\{d_i, 1\}$. As usual, D_i are distinct prime divisors.

Definition 2.2. For a triple $(X/Z, B, M)$, define

$$\theta(X/Z, B, M) := \#\{i \mid m_i \neq 0 \text{ and } b_i \neq 1\},$$

where $B = \sum b_i D_i$ and $M = \sum m_i D_i$.

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Construction 2.3. Let $(X/Z, B, M)$ be a lc triple. Let $f\colon W \to X$ be a log resolution of $(X/Z, B, M)$, and let

$$B_W := B^{\sim} + \sum E_j,$$

where $\sim$ stands for the birational transform and E_j are the prime exceptional divisors of f.

Obviously, $(W/Z, B_W)$ is $\mathbb{Q}$-factorial dlt, and it is effective because

$$M_W := K_W + B_W - f^*(K_X + B) + f^*M \equiv K_W + B_W/Z$$

is effective. Note that since $(X/Z, B)$ is lc,

$$K_W + B_W - f^*(K_X + B) \geqslant 0.$$

In addition, each component of M_W is either a component of $M^{\sim}$ or an exceptional divisor E_j. Thus, by construction,

$$\theta(W/Z, B_W, M_W) = \theta(X/Z, B, M).$$

We call $(W/Z, B_W, M_W)$ and $(W/Z, B_W)$ *log smooth models* of $(X/Z, B, M)$ and $(X/Z, B)$, respectively.

Definition 2.4 (cf. [Sho93]). A pair $(Y/Z, B + E)$ is a *log birational model* of $(X/Z, B)$ if we have a birational map $\phi\colon X \dashrightarrow Y/Z$, B on Y which is the birational transform of B on X (for simplicity we use the same notation), and $E = \sum E_j$, where E_j are the exceptional$/X$ prime divisors of Y. $(Y/Z, B + E)$ is a *nef model* of $(X/Z, B)$ if in addition:

(1) $(Y/Z, B + E)$ is $\mathbb{Q}$-factorial dlt; and
(2) $K_Y + B + E$ is nef$/Z$.

And, we call $(Y/Z, B + E)$ a *log minimal model* of $(X/Z, B)$ if in addition:

(3) for any prime divisor D on X which is exceptional$/Y$, we have

$$a(D, X, B) < a(D, Y, B + E).$$

Definition 2.5 (Mori fibre space). A log birational model $(Y/Z, B + E)$ of a lc pair $(X/Z, B)$ is called a Mori fibre space if $(Y/Z, B + E)$ is $\mathbb{Q}$-factorial dlt, there is a $K_Y + B + E$-negative extremal contraction $Y \to T/Z$ with $\dim Y > \dim T$, and

$$a(D, X, B) \leqslant a(D, Y, B + E)$$

for any prime divisor D (on birational models of X) and the strict inequality holds if D is on X and contracted$/Y$.

Our definitions of log minimal models and Mori fibre spaces are slightly different from those in [KM98], the difference being that we do not assume that ϕ^{-1} does not contract divisors. However, in the plt case, our definition of log minimal models and that of [KM98] coincide (see Remark 2.6(iii)).

Remark 2.6. Let $(X/Z, B)$ be a lc pair.

(i) Suppose that $(W/Z, B_W)$ is a log smooth model of $(X/Z, B)$ and $(Y/Z, B_W + E)$ a log minimal model of $(W/Z, B_W)$. We can also write $(Y/Z, B_W + E)$ as $(Y/Z, B + E')$, where B on Y is the birational transform of B on X and E' is the reduced divisor whose components are the exceptional$/X$ divisors on Y. Let D be a prime divisor on X contracted$/Y$. Then,

$$a(D, X, B) = a(D, W, B_W) < a(D, Y, B_W + E) = a(D, Y, B + E'),$$

which implies that $(Y/Z, B + E')$ is a log minimal model of $(X/Z, B)$.

(ii) Let $(Y/Z, B+E)$ be a log minimal model of $(X/Z, B)$ and take a common resolution $f\colon W \to X$ and $g\colon W \to Y$; then,

$$f^*(K_X+B) \geqslant g^*(K_Y+B+E)$$

by applying the negativity lemma [Sho93, 1.1]. This, in particular, means that $a(D, X, B) \leqslant a(D, Y, B+E)$ for any prime divisor D (on birational models of X). Moreover, if $(X/Z, B, M)$ is a triple, then $(Y/Z, B+E, g_*f^*M)$ is also a triple.

(iii) If $(X/Z, B)$ is plt and $(Y/Z, B+E)$ is a log minimal model, for any component D of E on Y, $0 < a(D, X, B) \leqslant a(D, Y, B+E) = 0$, which is not possible, so $E = 0$.

3. Proofs

Before getting into the proofs of our results, we need some preparation.

Lemma 3.1. *Let $(X/Z, B+C)$ be a $\mathbb{Q}$-factorial lc pair, where $B, C \geqslant 0$, K_X+B+C is* nef$/Z$, *and $(X/Z, B)$ is dlt. Then, either K_X+B is also* nef$/Z$ *or there is an extremal ray R/Z such that $(K_X+B)\cdot R<0$, $(K_X+B+\lambda C)\cdot R = 0$, and $K_X+B+\lambda C$ is* nef$/Z$, *where*

$$\lambda := \inf\{t \geqslant 0 \mid K_X+B+tC \text{ is nef}/Z\}.$$

Proof. Suppose that K_X+B is not nef$/Z$ and let $\{R_i\}_{i\in I}$ be the set of (K_X+B)-negative extremal rays$/Z$ and Γ_i an extremal curve of R_i [Sho, Definition 1]. Let $\mu := \sup\{\mu_i\}$, where

$$\mu_i := \frac{-(K_X+B)\cdot\Gamma_i}{C\cdot\Gamma_i}.$$

Obviously, $\lambda = \mu$ and $\mu \in (0, 1]$. It is enough to prove that $\mu = \mu_l$ for some l. By [Sho, Proposition 1], there are positive real numbers $r_1, \dots, r_s$ and a positive integer m (all independent of i) such that

$$(K_X+B)\cdot\Gamma_i = \sum_{j=1}^{s} \frac{r_j n_{i,j}}{m},$$

where $-2(\dim X)m \leqslant n_{i,j} \in \mathbb{Z}$. On the other hand, by [Sho96, First Main Theorem 6.2, Remark 6.4] we can write

$$K_X+B+C = \sum_{k=1}^{t} r'_k(K_X+\Delta_k),$$

where $r'_1, \dots, r'_t$ are positive real numbers such that for any k we have: $(X/Z, \Delta_k)$ is lc with Δ_k being rational, and $(K_X+\Delta_k)\cdot\Gamma_i \geqslant 0$ for any i. Therefore, there is a positive integer m' (independent of i) such that

$$(K_X+B+C)\cdot\Gamma_i = \sum_{k=1}^{t} \frac{r'_k n'_{i,k}}{m'},$$

where $0 \leqslant n'_{i,k} \in \mathbb{Z}$.

The set $\{n_{i,j}\}_{i,j}$ is finite. Moreover,

$$\frac{1}{\mu_i} = \frac{C\cdot\Gamma_i}{-(K_X+B)\cdot\Gamma_i} = \frac{(K_X+B+C)\cdot\Gamma_i}{-(K_X+B)\cdot\Gamma_i} + 1 = -\frac{m\sum_k r'_k n'_{i,k}}{m'\sum_j r_j n_{i,j}} + 1.$$

Thus, $\inf\{1/\mu_i\} = 1/\mu_l$ for some l and so $\mu = \mu_l$. □

Definition 3.2 (LMMP with scaling). Let $(X/Z, B+C)$ be a lc pair such that K_X+B+C is nef$/Z$, $B \geqslant 0$, and $C \geqslant 0$ is $\mathbb{R}$-Cartier. Suppose that either K_X+B is nef$/Z$ or there is an extremal ray R/Z such that $(K_X+B)\cdot R<0$, $(K_X+B+\lambda_1 C)\cdot R=0$, and $K_X+B+\lambda_1 C$ is nef$/Z$, where

$$\lambda_1 := \inf\{t \geqslant 0 \mid K_X+B+tC \text{ is nef}/Z\}.$$

When $(X/Z, B)$ is $\mathbb{Q}$-factorial dlt, the last sentence follows from Lemma 3.1. If R defines a Mori fibre structure, we stop. Otherwise, assume that R gives a divisorial contraction or a log flip $X \dashrightarrow X'$. We can now consider $(X'/Z, B'+\lambda_1 C')$, where $B'+\lambda_1 C'$ is the birational transform of $B+\lambda_1 C$, and continue the argument. That is, suppose that either $K_{X'}+B'$ is nef$/Z$ or there is an extremal ray R'/Z such that $(K_{X'}+B')\cdot R'<0$, $(K_{X'}+B'+\lambda_2 C')\cdot R'=0$, and $K_{X'}+B'+\lambda_2 C'$ is nef$/Z$, where

$$\lambda_2 := \inf\{t \geqslant 0 \mid K_{X'}+B'+tC' \text{ is nef}/Z\}.$$

By continuing this process, we obtain a special kind of LMMP$/Z$ which is called the *LMMP$/Z$ on K_X+B with scaling of C*; note that it is not unique. This kind of LMMP was first used by Shokurov [Sho93]. When we refer to *termination with scaling*, we mean termination of such a LMMP.

Special termination with scaling means termination near $\lfloor B \rfloor$ of any sequence of log flips$/Z$ with scaling of C, i.e. after finitely many steps, the locus of the extremal rays in the process does not intersect $\operatorname{Supp}\lfloor B \rfloor$.

When we have a lc pair $(X/Z, B)$, we can always find an ample$/Z$ $\mathbb{R}$-Cartier divisor $C \geqslant 0$ such that K_X+B+C is lc and nef$/Z$, and so we can run the LMMP$/Z$ with scaling assuming that all the necessary ingredients exist, e.g. extremal rays and log flips.

Lemma 3.3. *Assume the special termination with scaling for $\mathbb{Q}$-factorial dlt pairs in dimension d and the existence of pl flips in dimension d. Let $(X/Z, B)$ be a lc pair of dimension d and let $\{D_i\}_{i\in I}$ be a finite set of exceptional$/X$ prime divisors (on birational models of X) such that $a(D_i, X, B) \leqslant 1$. Then, there is a $\mathbb{Q}$-factorial dlt pair $(Y/X, B_Y)$ such that:*

(1) *Y/X is birational and K_Y+B_Y is the crepant pullback of K_X+B;*

(2) *every exceptional$/X$ prime divisor E of Y is one of the D_i or $a(E, X, B)=0$;*

(3) *the set of exceptional$/X$ prime divisors of Y includes $\{D_i\}_{i\in I}$.*

Proof. Let $f: W \to X$ be a log resolution of $(X/Z, B)$ and let $\{E_j\}_{j\in J}$ be the set of prime exceptional divisors of f. Moreover, we can assume that for some $J' \subseteq J$, $\{E_j\}_{j\in J'} = \{D_i\}_{i\in I}$. Now define

$$K_W+B_W := f^*(K_X+B) + \sum_{j\notin J'} a(E_j, X, B)E_j,$$

which ensures that if $j \notin J'$, then E_j is a component of $\lfloor B_W \rfloor$.

By running the LMMP$/X$ on K_W+B_W with scaling of a suitable ample$/X$ $\mathbb{R}$-divisor, and using the special termination, we get a log minimal model of $(W/X, B_W)$, which we may denote by $(Y/X, B_Y)$. By the negativity lemma, all the E_j are contracted in the process except if $j \in J'$ or if $a(E_j, X, B)=0$. By construction, K_Y+B_Y is the crepant pullback of K_X+B. □

Proposition 3.4. *Assume the special termination with scaling for $\mathbb{Q}$-factorial dlt pairs in dimension d and the existence of pl flips in dimension d. Then, any effective lc pair in dimension d has a log minimal model.*

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Proof.

Step 1. Let $\mathfrak{W}$ be the set of triples $(X/Z, B, M)$ such that:

(1) $(X/Z, B)$ is lc of dimension d;

(2) $(X/Z, B)$ does not have a log minimal model.

It is enough to prove that $\mathfrak{W}$ is empty. Assume otherwise and choose $(X/Z, B, M) \in \mathfrak{W}$ with minimal $\theta(X/Z, B, M)$. Replace $(X/Z, B, M)$ with a log smooth model as in Construction 2.3, which preserves the minimality of $\theta(X/Z, B, M)$.

If $\theta(X/Z, B, M) = 0$, then either $M = 0$, in which case we already have a log minimal model, or by running the LMMP$/Z$ on $K_X + B$ with scaling of a suitable ample$/Z$ $\mathbb{R}$-divisor we get a log minimal model because, by the special termination, flips and divisorial contractions will not intersect $\operatorname{Supp}\lfloor B\rfloor \supseteq \operatorname{Supp} M$ after finitely many steps. This is a contradiction. Note that we only need pl flips here. We may then assume that $\theta(X/Z, B, M) > 0$.

Step 2. Define

$$\alpha := \min\{t > 0 \mid \lfloor (B + tM)^{\leqslant 1}\rfloor \neq \lfloor B\rfloor\}.$$

In particular, $(B + \alpha M)^{\leqslant 1} = B + C$ for some $C \geqslant 0$ supported on $\operatorname{Supp} M$, and $\alpha M = C + M'$, where M' is supported on $\operatorname{Supp}\lfloor B\rfloor$. Thus, outside $\operatorname{Supp}\lfloor B\rfloor$ we have $C = \alpha M$. The pair $(X/Z, B + C)$ is $\mathbb{Q}$-factorial dlt and $(X/Z, B + C, M + C)$ is a triple. By construction,

$$\theta(X/Z, B + C, M + C) < \theta(X/Z, B, M),$$

and so $(X/Z, B + C, M + C) \notin \mathfrak{W}$. Therefore, $(X/Z, B + C)$ has a log minimal model, say $(Y/Z, B + C + E)$. By definition, $K_Y + B + C + E$ is nef$/Z$.

Step 3. Now run the LMMP$/Z$ on $K_Y + B + E$ with scaling of C. Note that we only need pl flips here because every extremal ray contracted would have negative intersection with some component of $\lfloor B\rfloor + E$ by Remark 2.6(ii) and the properties of C mentioned in Step 2. By the special termination, after finitely many steps, $\lfloor B\rfloor + E$ does not intersect the extremal rays contracted by the LMMP and hence we end up with a model Y' on which $K_{Y'} + B + E$ is nef$/Z$. Clearly, $(Y'/Z, B + E)$ is a nef model of $(X/Z, B)$ but may not be a log minimal model because condition (3) of Definition 2.4 may not be satisfied.

Step 4. Let

$$\mathcal{T} = \{t \in [0, 1] \mid (X/Z, B + tC) \text{ has a log minimal model}\}.$$

Since $1 \in \mathcal{T}$, $\mathcal{T} \neq \emptyset$. Let $t \in \mathcal{T} \cap (0, 1]$ and let $(Y_t/Z, B + tC + E)$ be any log minimal model of $(X/Z, B + tC)$. Running the LMMP$/Z$ on $K_{Y_t} + B + E$ with scaling of tC shows that there is $t' \in (0, t)$ such that $[t', t] \subset \mathcal{T}$ because condition (3) of Definition 2.4 is an open condition. The LMMP terminates for the same reasons as in Step 3 and we note again that the log flips required are all pl flips.

Step 5. Let $\tau = \inf \mathcal{T}$. If $\tau \in \mathcal{T}$, then, by Step 4, $\tau = 0$ and so we are done by deriving a contradiction. Thus, we may assume that $\tau \notin \mathcal{T}$. In this case, there is a sequence $t_1 > t_2 > \cdots$ in $\mathcal{T} \cap (\tau, 1]$ such that $\lim t_k = \tau$. For each t_k, let $(Y_{t_k}/Z, B + t_kC + E)$ be any log minimal model of $(X/Z, B + t_kC)$ which exists by definition of $\mathcal{T}$ and from which we get a nef model $(Y'_{t_k}/Z, B + \tau C + E)$ for $(X/Z, B + \tau C)$ by running the LMMP$/Z$ on $K_{Y_{t_k}} + B + E$ with scaling of t_kC. Let $D \subset X$ be a prime divisor contracted$/Y'_{t_k}$. If D is contracted$/Y_{t_k}$, then

$$\begin{aligned} a(D, X, B + t_kC) &< a(D, Y_{t_k}, B + t_kC + E) \\ &\leqslant a(D, Y_{t_k}, B + \tau C + E) \\ &\leqslant a(D, Y'_{t_k}, B + \tau C + E) \end{aligned}$$

but if D is not contracted$/Y_{t_k}$ we have

$$\begin{aligned} a(D, X, B + t_kC) &= a(D, Y_{t_k}, B + t_kC + E) \\ &\leqslant a(D, Y_{t_k}, B + \tau C + E) \\ &< a(D, Y'_{t_k}, B + \tau C + E) \end{aligned}$$

because $(Y_{t_k}/Z, B + t_kC + E)$ is a log minimal model of $(X/Z, B + t_kC)$ and $(Y'_{t_k}/Z, B + \tau C + E)$ is a log minimal model of $(Y_{t_k}/Z, B + \tau C + E)$. Thus, in any case we have

$$a(D, X, B + t_kC) < a(D, Y'_{t_k}, B + \tau C + E).$$

Replacing the sequence $\{t_k\}_{k\in\mathbb{N}}$ with a subsequence, we can assume that all the induced rational maps $X \dashrightarrow Y'_{t_k}$ contract the same components of $B + \tau C$.

Claim 3.5. *$(Y'_{t_k}/Z, B + \tau C + E)$ and $(Y'_{t_{k+1}}/Z, B + \tau C + E)$ have equal log discrepancies for any k.*

Proof. Let $f\colon W \to Y'_{t_k}$ and $g\colon W \to Y'_{t_{k+1}}$ be resolutions with a common W. Put

$$F = f^*(K_{Y'_{t_k}} + B + \tau C + E) - g^*(K_{Y'_{t_{k+1}}} + B + \tau C + E),$$

which is obviously anti-nef$/Y'_{t_k}$. So, by the negativity lemma, $F \geqslant 0$ if and only if $f_*F \geqslant 0$. Suppose that D is a component of f_*F with negative coefficient. By the assumptions, D is not a component of $B + \tau C + E$ on Y'_{t_k} and it must be exceptional$/Y'_{t_{k+1}}$. Moreover, the coefficient of D in f_*F is equal to $a(D, Y'_{t_{k+1}}, B + \tau C + E) - 1$. On the other hand,

$$1 = a(D, X, B + t_{k+1}C) < a(D, Y'_{t_{k+1}}, B + \tau C + E),$$

a contradiction. On the other hand, F is nef$/Y'_{t_{k+1}}$ and a similar argument would imply that $-F \geqslant 0$; hence $F = 0$ and the claim follows. □

Therefore, each $(Y'_{t_k}/Z, B + \tau C + E)$ is a nef model of $(X/Z, B + \tau C)$ such that

$$a(D, X, B + \tau C) = \lim a(D, X, B + t_kC) \leqslant a(D, Y'_{t_k}, B + \tau C + E)$$

for any prime divisor $D \subset X$ contracted$/Y'_{t_k}$.

Step 6. To get a log minimal model of $(X/Z, B + \tau C)$, we just need to extract those prime divisors D on X contracted$/Y'_{t_k}$ for which

$$a(D, X, B + \tau C) = a(D, Y'_{t_k}, B + \tau C + E).$$

This is achieved by applying Lemma 3.3 to construct a suitable crepant model of $(Y'_{t_k}, B + \tau C + E)$, which would then be a log minimal model of $(X/Z, B + \tau C)$. Thus, $\tau \in \mathcal{T}$ and this gives a contradiction. Therefore, $\mathfrak{W} = \emptyset$. □

Lemma 3.6. *The LMMP with scaling in dimension $d - 1$ for $\mathbb{Q}$-factorial dlt pairs implies the special termination with scaling in dimension d for $\mathbb{Q}$-factorial dlt pairs.*

Proof. This follows from the arguments in the proof of [Fuj07, Theorem 4.2.1]. □

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Remark 3.7. Assume the LMMP with scaling for $\mathbb{Q}$-factorial dlt pairs in dimension $d-1$; then, pl flips exist in dimension d by [HM07].

Proof of Theorem 1.3. Immediate by Lemma 3.6, Remark 3.7 and Proposition 3.4. □

Proof of Theorem 1.2. Immediate by Theorem 1.3. □

We now turn to the proofs of the corollaries.

Lemma 3.8. *The LMMP with scaling holds for $\mathbb{Q}$-factorial dlt pairs in dimension four.*

Proof. Log flips exist by [HM07, Sho03], so it is enough to verify the termination. Let $(X/Z, B+C)$ be a $\mathbb{Q}$-factorial dlt pair of dimension four. Suppose that we get a sequence of log flips $X_i \dashrightarrow X_{i+1}/Z_i$, when running the LMMP/$Z$ with scaling of C, which does not terminate. We may assume that $X = X_1$. Let $\lambda = \lim \lambda_i$, where the λ_i are obtained as in Definition 3.2 for this LMMP/Z. By applying [Sho, Corollary 12], we deduce that the λ_i stabilise, that is, $\lambda = \lambda_i$ for $i \gg 0$. We may assume that $\lambda > 0$, otherwise we are done. Therefore, we get an infinite sequence of $K_X + B + \frac{1}{2}\lambda C$-flips such that C is positive on each of these flips. Now, by special termination in dimension four and by [AHK07, Theorem 2.15], we may assume that each of the log flips is of type $(2, 1)$, that is, the flipping locus is of dimension two and the flipped locus is of dimension one. Finally, apply [AHK07, Lemma 3.1]. □

Proof of Corollary 1.5. Immediate by Lemma 3.8 and Theorem 1.3. □

Proof of Corollary 1.6. Immediate by [Sho96] and Theorem 1.3. □

Next we show that the Shokurov reduction theorem [Sho03, Reduction Theorem 1.2] follows easily from Proposition 3.4.

Theorem 3.9 (Shokurov reduction). *Assume the special termination for $\mathbb{Q}$-factorial dlt pairs in dimension d and the existence of pl flips in dimension d. Then, log flips exist for klt (and hence $\mathbb{Q}$-factorial dlt) pairs in dimension d.*

Proof. Let $(X/Z, B)$ be a klt pair of dimension d and $f\colon X \to Z'$ a $(K_X + B)$-flipping contraction/Z. We can apply Proposition 3.4 to construct a log minimal model $(Y/Z', B+E)$ of $(X/Z', B)$. Now, since $(X/Z', B)$ is klt, $E = 0$. So, $(Y/Z', B)$ is also klt and by the base point free theorem [HM07, Theorem 5.2.1] it has a log canonical model which gives the flip of f. □

As mentioned in Remark 3.7, by [HM07], the LMMP with scaling for $\mathbb{Q}$-factorial dlt pairs in dimension $d-1$ implies the existence of pl flips in dimension d. In Lemma 3.6, we proved that the LMMP with scaling for $\mathbb{Q}$-factorial dlt pairs in dimension $d-1$ implies special termination with scaling in dimension d for $\mathbb{Q}$-factorial dlt pairs. Thus, the previous theorem can be restated as saying that the LMMP with scaling for $\mathbb{Q}$-factorial dlt pairs in dimension $d-1$ implies the existence of log flips for klt (and hence $\mathbb{Q}$-factorial dlt) pairs.

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References

AHK07 V. Alexeev, C. Hacon and Y. Kawamata, *Termination of (many) four-dimensional log flips*, Invent. Math. **168** (2007), 433–448.

Bir C. Birkar, *On existence of log minimal models II*, arXiv:0907.4170v1.

Bir07 C. Birkar, *Ascending chain condition for lc thresholds and termination of log flips*, Duke Math. J. **136** (2007), 173–180.

Bir09 C. Birkar, *Log minimal models according to Shokurov*, J. Algebra and Number Theory **3** (2009), 951–958.

BCHM10 C. Birkar, P. Cascini, C. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), 405–468.

BP C. Birkar and M. Păun, *Minimal models, flips and finite generation: a tribute to V. V. Shokurov and Y.-T. Siu*, arXiv:0904.2936v1.

BS C. Birkar and V. V. Shokurov, *Mld's vs thresholds and flips*, arXiv:math/0609539v1.

Fuj O. Fujino, *Finite generation of the log canonical ring in dimension four*, arXiv:0803.1691v2.

Fuj07 O. Fujino, *Special termination and reduction to pl flips*, in *Flips for 3-folds and 4-folds* (Oxford University Press, Oxford, 2007).

HM07 C. Hacon and J. McKernan, *Extension theorems and the existence of flips*, in *Flips for 3-folds and 4-folds* (Oxford University Press, Oxford, 2007).

Kaw82 Y. Kawamata, *A generalization of Kodaira–Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), 43–46.

Kaw92 Y. Kawamata, *Termination of log flips for algebraic 3-folds*, Internat. J. Math. **3** (1992), 653–659.

KMM87 Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, in *Algebraic geometry* (Sendai, 1985), Advanced Studies in Pure Mathematics, vol. 10 (North-Holland, Amsterdam, 1987), 283–360.

KM98 J. Kollár and S. Mori, *Birational geometry of algebraic varieties* (Cambridge University Press, Cambridge, 1998).

Mor82 S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. (2) **116** (1982), 133–176.

Mor88 S. Mori, *Flip theorem and the existence of minimal models for 3-folds*, J. Amer. Math. Soc. **1** (1988), 117–253.

Sho86 V. V. Shokurov, *A nonvanishing theorem*, Math. USSR Izv. **26** (1986), 591–604.

Sho93 V. V. Shokurov, *Three-dimensional log flips*, Russ. Acad. Sci. Izv. Math. **40** (1993), 95–202, with an appendix in English by Yujiro Kawamata.

Sho96 V. V. Shokurov, *3-fold log models*, J. Math. Sci. **81** (1996), 2667–2699.

Sho03 V. V. Shokurov, *Prelimiting flips*, Proc. Steklov Inst. Math. **240** (2003), 75–213.

Sho04 V. V. Shokurov, *Letters of a bi-rationalist V. Mld's and termination of log flips*, Proc. Steklov Inst. Math. **246** (2004), 315–336.

Sho V. V. Shokurov, *Letters of a bi-rationalist VII. Ordered termination*, arXiv:math/0607822v2.

Vie82 E. Viehweg, *Vanishing theorems*, J. Reine Angew. Math. **335** (1982), 1–8.

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