BULL. AUSTRAL. MATH. SOC. 34K15, 34C10
VOL. 46 (1992) [149–157]

SOME FURTHER RESULTS ON OSCILLATION OF NEUTRAL DIFFERENTIAL EQUATIONS

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We obtain new sufficient conditions for the oscillation of all solutions of the neutral differential equation with variable coefficients

$$\frac{d}{dt}(y(t) - R(t)y(t-r)) + P(t)y(t-\tau) - Q(t)y(t-\sigma) = 0$$

where $P, Q, R \in C([t_0, \infty), R^+)$, $r \in (0, \infty)$ and $\tau, \sigma \in [0, \infty)$. Our results improve several known results in papers by: Chuanxi and Ladas; Lalli and Zhang; Wei; Ruan.

1. INTRODUCTION

Consider the first order neutral delay differential equation with positive and negative coefficients

$$\frac{d}{dt}(y(t) - R(t)y(t-r)) + P(t)y(t-\tau) - Q(t)y(t-\sigma) = 0 \tag{1}$$

where

$$P, Q, R \in C([t_0, \infty), R^+), r \in (0, \infty) \text{ and } \tau, \sigma \in [0, \infty). \tag{2}$$

The oscillation of all solutions of neutral equations with positive and negative coefficients has been investigated recently by several authors; for example, see [**1, 3, 6–9**]. For a survey, one can see [**4**]. The following sufficient conditions for the oscillation of all solutions of Equation (1) were established in [**1, 6, 7, 8**] respectively.

THEOREM A. [**1**] *Assume that (2) holds and that*

$$\tau \geqslant \sigma, \tag{3}$$

$$P(t) \geqslant Q(t+\sigma-\tau) \text{ and } P(t) - Q(t+\sigma-\tau) \not\equiv 0 \text{ for } t \geqslant t_0+\tau-\sigma, \tag{4}$$

$$0 \leqslant R(t) \leqslant r_0 \leqslant 1 \text{ for } t \geqslant t_0 \text{ and some } r_0 \in [0, 1], \tag{5}$$

$$(\tau-\sigma)Q(t) \leqslant 1 - r_0 \text{ for } t \geqslant t_0. \tag{6}$$

Received 16 August 1991
Project supported by the National Natural Science Foundation of The People's Republic of China.

Then each one of the following two conditions

$$\liminf_{t\to\infty}\int_{t-\tau}^{t}(P(s)-Q(s+\sigma-\tau))ds>\frac{1}{e}, \tag{7}$$

$$\limsup_{t\to\infty}\int_{t-\tau}^{t}(P(s)-Q(s+\sigma-\tau))ds>1 \tag{8}$$

implies that every solution of Equation (1) oscillates.

THEOREM B. [6] *Assume that (2) and (7) hold and that*

$$\tau>\sigma>0, \tag{9}$$

$$P(t)-Q(t+\sigma-\tau)\geqslant 0 \text{ and not identically zero}, \tag{10}$$

$$\lim_{t\to\infty}\int_{t-(\tau-\sigma)}^{t}Q(s)ds=0, \tag{11}$$

$$1-R(t)-\int_{t-(\tau-\sigma)}^{t}Q(s)ds\geqslant 0 \text{ for all sufficiently large } t. \tag{12}$$

Then every solution of Equation (1) oscillates.

THEOREM C. [8] *Assume that (2) and (9) hold and that there exist positive constants q and ε such that*

$$Q(t)\equiv q,\quad P(t)\geqslant q+\varepsilon \quad \text{for} \quad t\geqslant t_0, \tag{13}$$

$$0\leqslant R(t)\leqslant 1 \quad \text{and} \quad 1-R(t)-q(\tau-\sigma)>0 \quad \text{for} \quad t\geqslant t_0, \tag{14}$$

and

$$\liminf_{t\to\infty}\int_{t-\tau}^{t}(P(s)-q)ds>\frac{1}{e}. \tag{15}$$

Then every solution of Equation (1) oscillates.

THEOREM D. [7] *Assume that (2) holds and that*

$$R(t)\equiv c<1,\quad \tau>\sigma, \tag{16}$$

$$P(t)\geqslant Q(t+\tau-\sigma) \quad \text{for} \quad t\geqslant t_1\geqslant t_0, \tag{17}$$

$$\liminf_{t\to\infty}\int_{t-\tau}^{t}P(s)ds>\frac{1}{e}, \tag{18}$$

$$\int_{t_0}^{\infty}Q(s)ds<\infty. \tag{19}$$

Then every solution of Equation (1) oscillates.

Our aim in this paper is to improve the above theorems which is possible by using Lemma 1 which is stated and proved in Section 2. Set

$$A = \liminf_{t\to\infty} \int_{t-\tau}^{t} (P(s) - Q(s+\sigma-\tau))\Big(1 + R(s-\tau) + \int_{s-\tau}^{s} Q(u-\tau)du\Big)ds,$$
$$M = \limsup_{t\to\infty} \int_{t-\tau}^{t} (P(s) - Q(s+\sigma-\tau))\Big(1 + R(s-\tau) + \int_{s-\tau}^{s} Q(u-\tau)du\Big)ds.$$

The main result is the following:

THEOREM 1. *Assume that (2), (3), (10) and (12) hold and that either*

(20) $$A > \frac{1}{e}$$

or

(21) $$A \leqslant \frac{1}{e} \text{ and } M > 1 - \frac{1}{2}\Big(1 - A - \sqrt{1-2A-A^2}\Big).$$

Then every solution of Equation (1) oscillates.

The proof of this theorem will be given in Section 3.

REMARK 1. It is easy to see that (5) and (6) imply that (12) holds, and (20) or (21) is an improvement of (7) or (8) respectively. Condition (11) in Theorem B is removed. In Theorem C, Conditions (13) and (15) are stronger than (12) and (20). In Theorem D, (16) and (19) imply (12) holds and (18) implies (20) holds. Thus, Theorem 1 improves four theorems before mentioned.

Let $T = \max\{\tau, \sigma, r\}$. By a solution of Equation (1) we mean a function $y(t) \in C([t_1 - T, \infty), R)$, for some $t_1 \geqslant t_0$, such that $y(t) - R(t)y(t-r)$ is continuously differentiable on $[t_1, \infty)$ and that Equation (1) is satisfied for $t \geqslant t_1$.

Assume that (2) holds and let $\phi \in C(t_0 - T, t_0], R)$ be a given initial function. Then one can easily see by the method of steps that Equation (1) has a unique solution $y(t) \in C([t_0 - T, \infty), R)$ such that

$$y(t) = \phi(t) \quad \text{for} \quad t_0 - T \leqslant t \leqslant t_0.$$

As is customary, a solution of Equation (1) is said to oscillate if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

In the sequel, unless otherwise specified, when we write a functional inequality we shall assume that it holds for all sufficiently large values of t.

2. Important lemmas

In this section we shall first establish a new sufficient condition to guarantee that the first order delay differential inequality

$$x'(t) + H(t)x(t-\tau) \leqslant 0 \tag{22}$$

has no eventually positive solution, where $H \in C([t_0, \infty), R^+)$ and $\tau \in (0, \infty)$. Set

$$a = \liminf_{t\to\infty} \int_{t-\tau}^{t} H(s)ds \quad \text{and} \quad m = \limsup_{t\to\infty} \int_{t-\tau}^{t} H(s)ds.$$

It is well known that (22) has no eventually positive solutions if either $a > 1/e$ or $m > 1$. Erbe and Zhang [**2**] proved that

$$a \leqslant \frac{1}{e} \quad \text{and} \quad m > 1 - \frac{a^2}{4} \tag{23}$$

imply that (22) has no eventually positive solutions. In particular, in the recent paper [**5**] Jianchao proves that (23) can be replaced by the following weaker condition

$$a \leqslant \frac{1}{e} \quad \text{and} \quad m > 1 - \frac{a^2}{2(1-a)}. \tag{24}$$

One of the main results in this section is the following Lemma 1. It is interesting in its own right and improves several known results in the literature. For example, it improves Condition (24).

Lemma 1. *Assume that*

$$a \leqslant \frac{1}{e} \quad \text{and} \quad m > 1 - \frac{1 - a - \sqrt{1 - 2a - a^2}}{2}. \tag{25}$$

Then (22) has no eventually positive solutions.

Proof: If $a = 0$, then (25) yields $m > 1$ which implies that the conclusion of Lemma 1 is true. Next assume that $0 < a \leqslant 1/e$ and let $x(t)$ be an eventually positive solution of (22); we shall derive a contradiction. To this end we define a sequence $\{b_n\}$ of real numbers as follows:

$$\begin{aligned} &b_1 = \frac{1}{4}a^2, \\ &b_n = b_{n-1}^2 + a\,b_{n-1} + \frac{1}{2}a^2, \qquad \text{for } n = 2, 3, \ldots. \end{aligned} \tag{26}$$

We shall finish the following claims:

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CLAIM 1. $x(t) \geqslant b_n x(t-\tau)$, for $n = 1, 2, \ldots$.

Clearly, Claim 1 holds for $n = 1$ by [2]. According to the definition of a, for any $\varepsilon \in (0, a)$ and sufficiently large t, we have

$$\int_{t-\tau}^{t} H(s)ds > a - \varepsilon. \tag{27}$$

Hence, for every t sufficiently large there is $t^* > t$ such that

$$\int_{t}^{t^*} H(s)ds = a - \varepsilon \quad \text{and} \quad \int_{t^*-\tau}^{t^*} H(s)ds > a - \varepsilon \tag{28}$$

which implies that

$$t^* - \tau < t. \tag{29}$$

Integrating both sides of (22) from t to t^*, we have

$$x(t) \geqslant x(t^*) + \int_{t}^{t^*} H(s)x(s-\tau)ds. \tag{30}$$

It is easy to see that

$$t - \tau \leqslant s - \tau \leqslant t^* - \tau < t \quad \text{for} \quad s \in [t, t^*].$$

Again integrating (22) from $s - \tau$ to t, where $s \in [t, t^*]$, and using the fact that $x(t)$ is eventually nonincreasing, we have

$$\begin{aligned} x(s-\tau) &\geqslant x(t) + \int_{s-\tau}^{t} H(u)x(u-\tau)du \\ &\geqslant b_1 x(t-\tau) + x(t-\tau)\int_{s-\tau}^{t} H(u)du \\ &= \left(b_1 + \int_{s-\tau}^{s} H(u)du - \int_{t}^{s} H(u)du\right)x(t-\tau) \\ &\geqslant \left(b_1 + a - \varepsilon - \int_{t}^{s} H(u)du\right)x(t-\tau). \end{aligned}$$

Substituting this into (30), we find

(31)

$$\begin{aligned} x(t) &\geqslant x(t^*) + x(t-\tau)\int_{t}^{t^*} H(s)\left(b_1 + a - \varepsilon - \int_{t}^{s} H(u)du\right)ds \\ &= x(t^*) + x(t-\tau)\left((a-\varepsilon)(b_1 + a - \varepsilon) - \int_{t}^{t^*}\int_{t}^{s} H(s)H(u)duds\right). \end{aligned}$$

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Exchanging the integral order in (31), we have

$$\int_t^{t^*}\int_t^s H(s)H(u)duds = \int_t^{t^*}\int_u^{t^*} H(s)H(u)dsdu$$
$$= \int_t^{t^*}\int_s^{t^*} H(u)H(s)duds$$

and so

$$\int_t^{t^*}\int_t^s H(s)H(u)duds = \frac{1}{2}\left(\int_t^{t^*}\int_t^s H(s)H(u)duds + \int_t^{t^*}\int_s^{t^*} H(u)H(s)duds\right)$$
$$= \frac{1}{2}\left(\int_t^{t^*} H(u)du\right)^2 = \frac{1}{2}(a-\varepsilon)^2.$$

Combining this and (31), we obtain

$$x(t) \geqslant x(t^*) + x(t-\tau)\left((a-\varepsilon)(b_1+a-\varepsilon) - \frac{1}{2}(a-\varepsilon)^2\right). \tag{32}$$

Since $t^* - \tau < t$, it follows that

$$x(t^*) \geqslant b_1 x(t^*-\tau) \geqslant b_1 x(t) \geqslant b_1^2 x(t-\tau).$$

This and (32) yield that

$$x(t) \geqslant \left(b_1^2 + (a-\varepsilon)(b_1+a-\varepsilon) - \frac{1}{2}(a-\varepsilon)^2\right)x(t-\tau).$$

Let $\varepsilon \to 0$; we have

$$x(t) \geqslant \left(b_1^2 + ab_1 + \frac{1}{2}a^2\right)x(t-\tau) = b_2 x(t-\tau).$$

This completes the proof of Claim 1 for $n = 2$. By using a simple induction, we can easily prove in general that

$$x(t) \geqslant b_n x(t-\tau) \quad \text{for } n = 1, 2, \ldots .$$

The proof of Claim 1 is complete.

CLAIM 2. $\lim_{n\to\infty} b_n = \left(1 - a - \sqrt{1 - 2a - a^2}\right)/2$.

We first prove that the sequence $\{b_n\}$ is bounded and strictly increasing. In fact, let

$$c = \frac{1}{2}\left(1 - a - \sqrt{1 - 2a - a^2}\right).$$

Then

$$b_1 = \frac{1}{4}a^2 < c$$

which implies that

$$\begin{aligned} b_2 &= b_1^2 + ab_1 + \frac{1}{2}a^2 \\ &< c^2 + ac + \frac{1}{2}a^2 = c. \end{aligned}$$

Hence, in general we can have

$$b_n < c \quad \text{for } n = 1, 2, \dots$$

and so $\{b_n\}$ is bounded. On the other hand, since

$$b_2 - b_1 = \left(\frac{1}{4}a^2\right)^2 + \frac{1}{4}a^3 + \frac{1}{4}a^2 > 0$$

and $$b_n - b_{n-1} = (b_{n-1} + b_{n-2} + a)(b_{n-1} - b_{n-2}) \quad \text{for } n = 3, 4, \dots,$$

it follows that

$$b_n > b_{n-1} \quad \text{for } n = 2, 3, \dots .$$

That is, $\{b_n\}$ is strictly increasing. Therefore, the limit $\lim_{n\to\infty} b_n = d$ exists and satisfies $d \leqslant c$. Taking limits on both sides of (26), we have

$$d = d^2 + ad + \frac{1}{2}a^2$$

or equivalently $$d^2 + (a-1)d + \frac{1}{2}a^2 = 0.$$

This equation has two positive roots

$$d_1 = \frac{1}{2}\left(1 - a - \sqrt{1 - 2a - a^2}\right) \quad \text{and} \quad d_2 = \frac{1}{2}\left(1 - a + \sqrt{1 - 2a - a^2}\right).$$

Since $d_1 = c < d_2$, it follows that $d = d_1 = c$ and so the proof of Claim 2 is complete.

Finally, integrating (22) from $t - \tau$ to t, we have

$$x(t) - x(t-\tau) + \int_{t-\tau}^{t} H(s)x(s-\tau)ds \leqslant 0.$$

By using Claim 1 and the monotonicity of $x(t)$, we obtain

$$b_n - 1 + \int_{t-\tau}^{t} H(s)ds \leqslant 0 \quad \text{for } n = 1, 2, \ldots$$

which yields, in light of Claim 2, that

$$m = \limsup_{t\to\infty} \int_{t-\tau}^{t} H(s)ds \leqslant 1 - \frac{1}{2}\left(1 - a - \sqrt{1 - 2a - a^2}\right)$$

which contradicts (25) and so the proof of Lemma 1 is finished. ∎

Sinilarly, we have

Lemma 2. *Assume that the hypotheses of Lemma 1 hold. Then the delay differential inequality*

$$x'(t) + H(t)x(t-\tau) \geqslant 0$$

has no eventually negative solutions.

The following lemma was established in [**9**].

Lemma 3. [**9**] *Assume that (2), (3), (10) and (12) hold and let $y(t)$ be an eventually positive solution of Equation (1) and set*

$$x(t) = y(t) - R(t)y(t-\tau) - \int_{t-\tau+\sigma}^{t} Q(s)y(s-\sigma)ds. \tag{33}$$

Then eventually

$$x'(t) \leqslant 0 \quad \textit{and} \quad x(t) > 0.$$

3. Proof of Theorem 1

Proof of Theorem 1: Assume, for the sake of contradiction, that Equation (1) has the eventually positive solution $y(t)$. Set $x(t)$ as in (33). Then by Lemma 3, we have

$$x'(t) \leqslant 0 \quad \text{and} \quad x(t) > 0. \tag{34}$$

From (1) and (33), we see that

$$x'(t) = -(P(t) - Q(t+\sigma-\tau))y(t-\tau). \tag{35}$$

Also from (33) and (34), we have

$$\begin{aligned} y(t) &\geqslant x(t) + R(t)x(t-\tau) + \int_{t-\tau+\sigma}^{t} Q(s)x(s-\sigma)ds \\ &\geqslant \left(1 + R(t) + \int_{t-\tau+\sigma}^{t} Q(s)ds\right)x(t). \end{aligned}$$

Substituting this into (35), we obtain

$$x'(t) + (P(t) - Q(t+\sigma-\tau))\left(1 + R(t-\tau) + \int_{t-\tau+\sigma}^{t} Q(s-\tau)ds\right)x(t-\tau) \leqslant 0.$$

It is well known however, by [4] and Lemma 1, that under condition (20) or (21), the above first order delay differential inequality cannot have an eventually positive solution. This contradicts (34) and so the proof of Theorem 1 is complete. ∎

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