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# Essentially exact asymptotic solutions for Asian derivatives

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In this paper, we will show how to obtain asymptotic solutions for the problem of pricing Asian options. Under the assumption that the underlying follows geometric Brownian motion, we will derive Taylor expansion series for the fixed and floating strike Asian options. While there will be no analytical formulae for calculating expansion coefficients, we will provide relatively simple algorithms for calculating them. The methodology is particularly effective for the case of continuously sampled fixed-strike Asian calls where it takes only seconds to obtain constants for the Taylor expansion series that can converge beyond 10 significant digits. It is needless to say that we need to calculate Taylor expansion constants only once and the option price would be an analytical expression constructed from a cumulative normal distribution function, an exponential function and finite sums.

**Key words:** Asian options, option pricing, matched asymptotic expansions, financial derivatives.

## 1 Introduction

The Asian options have been around for decades. In plain terms, these are similar to standard vanilla options except that the payout depends on the value of the underlying over the life of the contract rather than just on its value at maturity. For example, the payout may depend on an average price of the underlying calculated over the time of the contract. This could be preferable for a buyer because average price is less susceptible to short-term fluctuations caused by shocks to supply or demand. While these financial instruments make sense from a business perspective, the problem of pricing them is not trivial at all.

If we assume that the underlying follows geometric Brownian Motion, the price for a standard vanilla option can be calculated as an expectation with respect to a normally distributed random variable. In the case of Asian options, the payout will depend on the full path of the Brownian Motion and we need to take expectations with respect to a random variable that is not normally distributed. For example, if we have a continuously sampled fixed-strike call option, the payout will depend on an integral of the geometric Brownian Motion. The full probabilistic analysis of this case is well described by H. Geman and M. Yor [2]. As a matter of fact, the analytics has been developed to the extent that we can write down Laplace transforms of the solutions or even have integral representations for the distribution functions [13]. Unfortunately, this still leaves us with the problem of calculating integrals for the expectations or inverse Laplace transforms.

Alternatively, we can reduce the problem to the solution of a parabolic partial differential equation (PDE). This gives us a whole new range of options because there are many numerical methods developed for solving PDEs. Perhaps the most known algorithm, for the case of Asian options, has been introduced by J. Vecer [9].

Other algorithms include the matched asymptotic expansions method. It has been first introduced into the field of financial mathematics by S. Howison [4].

He has successfully applied it to a number of problems including discretely sampled Barrier options [5]. Specifically, for Asian options, the method has been developed by L. Rogers and Z. Shi [7] and then applied by J. Zhang in [11]. In his paper, J. Zhang is using first-order approximation which is similar to our zero order for the case of the fixed-strike Asian call. The remaining orders are solved numerically. He has further developed on this by replacing the numeric algorithm with a perturbation method [12]. This resulted in several orders of approximation that have very similar form to what we present here. The other developments along this root include a paper by J. Dewynne and W. Shaw [1] which derives Black–Scholes-like asymptotic solutions for the "Asian" PDE. In their paper, we can find a Taylor series expansion algorithm for the PDE solution with explicit analytical expressions for the first several orders. For all practical purposes, the algorithm gives good approximation with simple analytical expressions.

Inspired by these results, we have derived a relatively easy algorithm for calculating *all* Taylor expansion coefficients. In addition, we have given full consideration to the floating-strike case which is often not covered as well in the literature as the fixed-strike case.

We will start with formulating the problem of pricing Asian options and then move on to describe analytics for expanding the 'Asian' PDE solution in Taylor series. We will consider two expansions. One is for the fixed-strike Asian options and the other is for the floating-strike Asian options. While most of the analytics developed here will apply to the case of continuously sampled options, a similar approach should work for other cases as well. We will comment on how to extend algorithms beyond continuously sampled case when appropriate. In the final section, we will validate numeric results for the same test cases as in [1] and show how many expansion terms are required to obtain each result.

## 2 Reducing pricing problem to a parabolic PDE

For the purpose of this analysis, we will consider two types of Asians – fixed-strike calls and floating-strike puts. The remaining two options (fixed-strike puts and floating-strike calls) are trivially derived from the first ones (e.g. see [1]). Mathematical treatment of these problems is almost the same. In contrast, fixed-strike calls and floating-strike puts are fundamentally different problems. If we are to solve them probabilistically, for the first problem we will need distribution of an integral of the geometric Brownian Motion and, for the floating-strike put, we will need joint distribution functions between the integral and a geometric Brownian Motion. It is clear that we should expect the second problem to be more complex.

We can write the price of options as discounted expectation of the final pay-off in a risk-neutral measure. For the fixed-strike Asian call, we have

$$C_x(S_t, K, r, \delta, t, T) = e^{-r(T-t)} E_t \left[ \left( \frac{1}{T} \int_0^T \omega(u) S_u du - K \right)^+ \right], \tag{2.1}$$

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and for the floating-strike Asian put

$$P_f(S_t, \lambda, r, \delta, t, T) = e^{-r(T-t)} E_t \left[ \left( \frac{1}{T} \int_0^T \omega(u) S_u du - \lambda S_T \right)^+ \right]. \tag{2.2}$$

Options are written at time zero and valued at time $t$. We are using the usual notation: $T$ denotes time at maturity, $S_t$ – stock price at time $t \in [0, T]$, $K$ – strike, $r$ is the risk-free rate of interest, $\delta$ is the continuous dividend yield on the stock and $\lambda$ is a constant that is usually set to 1. Both $r$ and $\delta$ are assumed to be constant. $\omega(u)$ is a weight function which is non-negative, its integral over time interval $[0, T]$ is equal to $T$ and $\int_0^t \omega(u) du$ is piecewise continuous on $[0, T]$. There are two special cases of interest. When $\omega(u)$ is equal to 1, we have continuously sampled options and, if $\omega(u)$ is a sum of delta functions multiplied by $T/N$ ($N$ is the number of delta functions), we have discreetly sampled Asian options. When $S_t$ follows geometric Brownian Motion and $t = 0$, the problems are mathematically equivalent

$$C_x(S_0, K, r, \delta, 0, T) = P_f\left( S_0, \frac{K}{S_0}, \delta, r, 0, T \right). \tag{2.3}$$

The identity comes from V Henderson and R Wojakowski [3]. Unfortunately, for $t > 0$, the relationship does not hold. Nevertheless, we can reduce these problems to the same parabolic PDE but with two different initial conditions. From [1], we know that this is the PDE we need to solve

$$\frac{\partial \phi(\eta, \tau)}{\partial \tau} = \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2 \phi(\eta, \tau)}{\partial \eta^2} + \left( \frac{W(\tau)}{T} - (r - \delta)\eta \right) \frac{\partial \phi(\eta, \tau)}{\partial \eta} \tag{2.4}$$

with the initial condition of $\phi(\eta, 0)$ equal to $\max(\eta, 0)$ for the fixed-strike call and $\max(\eta - \lambda, 0)$ for the floating-strike put. By $\tau$ we define time to maturity $(T - t)$ and $W(\tau)$ is simply $\omega(T - t)$. The following equalities will hold

$$C_x(S_t, K, r, \delta, t, T) = S_t e^{-q(T-t)} \phi\left( \frac{I_t - KT}{S_t T}, T - t \right), \tag{2.5}$$

$$P_f(S_t, \lambda, \delta, r, t, T) = S_t e^{-q(T-t)} \phi\left( \frac{I_t}{S_t T}, T - t \right), \tag{2.6}$$

where $I_t = \int_0^t \omega(u) S_u du$.

Let us now see how we can simplify equation (2.4) further. First, we can make times $\tau$ and $T$ dimensionless by scaling them with volatility: $\tau_\sigma = \sigma^2 \tau$ and $T_\sigma = \sigma^2 T$. The volatility scaled times would normally be less than 0.01 which makes them good candidates for expanding solutions in Taylor series. The physical times will not appear in the context of this problem anymore. Scaling times with volatility leads us to redefine the risk-free rate of interest and dividends as follows:

$$\rho = \frac{r - \delta}{\sigma^2}. \tag{2.7}$$

Here, $\rho$ has no dimension but its value would not necessarily be small because we obtain

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it by dividing the difference between the interest rate and the dividend yield by the square of the volatility and these are usually of the same order of magnitude. And, finally, we substitute $W(u)$ with

$$W_\sigma(u) = W\left(\frac{u}{\sigma^2}\right). \tag{2.8}$$

Now we are ready to simplify equation (2.4) by introducing the following change of variables

$$\phi(\eta,\tau) = \psi\left(\eta e^{-\rho\tau_\sigma} + \frac{1}{T_\sigma}\int_0^{\tau_\sigma} W_\sigma(u)e^{-\rho u}du, \tau_\sigma\right) = \psi(\xi,\tau_\sigma). \tag{2.9}$$

Substituting the above expression (2.9) into our original PDE (2.4), we can see that the first derivative with respect to $\xi$ disappears and we are left with the following expression:

$$\frac{\partial\psi(\xi,\tau_\sigma)}{\partial\tau_\sigma} = \frac{1}{2}e^{-2\rho\tau_\sigma}\eta^2\frac{\partial^2\psi(\xi,\tau_\sigma)}{\partial\xi^2}. \tag{2.10}$$

Noting that $\xi = \eta e^{-\rho\tau_\sigma} + \frac{1}{T_\sigma}\int_0^{\tau_\sigma} W_\sigma(u)e^{-\rho u}du$, we immediately get

$$\frac{\partial\psi(\xi,\tau_\sigma)}{\partial\tau_\sigma} = \frac{1}{2}(\xi - \eta^*(\tau_\sigma))^2\frac{\partial^2\psi(\xi,\tau_\sigma)}{\partial\xi^2}, \tag{2.11}$$

where

$$\eta^*(\tau_\sigma) = \frac{1}{T_\sigma}\int_0^{\tau_\sigma} W_\sigma(u)e^{-\rho u}du. \tag{2.12}$$

The initial conditions are the same as before

$$\psi(\xi,0) = \max(\xi,0) \tag{2.13}$$

for the fixed-strike call and

$$\psi(\xi,0) = \max(\xi-\lambda,0) \tag{2.14}$$

for the floating-strike put.

We now have a PDE (2.11) that will be the starting point for our expansion in Taylor series. The equation is slightly different from the one used by J. Vecer [9] or W. Shaw [1]. This is because we would like to introduce two Taylor expansions and, as we will see, this form of PDE is more "natural" to work with. The problems however are equivalent and we will obtain the same (up to a simple change of variables) first orders of the expansion as in [1].

## 3 Expanding PDE solutions in Taylor series

In the previous section, we have introduced a parabolic PDE (2.11) with dimensionless parameters which solves our original problem of pricing Asian options. No useful analytic solutions are known for equations (2.11)–(2.14) and one of the popular numerical methods would be the Finite Difference method. We will attempt to get full expansion for the solution in Taylor series.

First, let us note that $\psi$ is a function of two variables $\tau_\sigma$, $\xi$ and it is also a functional of $\eta^*(\tau_\sigma)$. While we would love to have direct Taylor expansion in $\tau_\sigma$ and $\xi$, there is no obvious way of getting it from the PDE (2.11) itself. If we attempt to search for solutions as Taylor series, function $\eta^*(\tau_\sigma)$ would stop us from getting simple identities between expansion coefficients and it is not clear how to make sure that $\psi$ satisfies the initial conditions.

Since we do not have any other variables, the prospect of solving this problem gets obscure. Luckily, there is nothing to stop us from adding more variables. Then, we can obtain the solution to our original problem by setting made-up variables to a constant (e.g. let us say 1).

There seems to be two variables that work best. We will call them $\alpha$ and $\beta$. Let us now modify our PDE as follows:

$$\frac{\partial \psi(\xi, \tau_\sigma, \alpha, \beta)}{\partial \tau_\sigma} = \frac{1}{2}(\alpha\xi - \beta\eta^*(\tau_\sigma))^2 \frac{\partial^2 \psi(\xi, \tau_\sigma, \alpha, \beta)}{\partial \xi^2}, \tag{3.1}$$

where we have the fixed-strike call initial condition $\max(\xi, 0)$ for the $\alpha$ expansion and the floating-strike put initial condition $\max(\xi - \lambda, 0)$ for the $\beta$ expansion.

The choice for $\alpha$, $\beta$ and initial conditions makes sense in the following context. We want to expand $\psi(\xi, \tau_\sigma, \alpha, \beta)$ in $\alpha$ and $\beta$ around $\alpha = 0$, $\beta = 0$ and it is preferable that zero expansion orders have simple analytical form and satisfy the initial conditions. The latter will make sure that all the following orders $(1, 2, 3, \ldots)$ are zero at time 0. It is clear that PDEs with zero initial conditions could be much easier to solve.

By setting $\alpha$ or $\beta$ to zero in equation (3.1), we can see that zero-order solution for the $\beta$ expansion is equal exactly to the solution of the Black–Scholes PDE and in the $\alpha$ expansion case it has an analytical form as well.

For the special case of our original PDE, $\alpha$ and $\beta$ are equal to 1 which is relatively far from the Taylor expansion point zero. This means that we should not expect our algorithms to perform well and we therefore need to understand under what conditions each expansion works best. We can get some insights from the formula (3.1). In particular, we can see that $\alpha$ and $\beta$ expansions are in some way complementary. When $|\alpha\xi| \gg |\beta\eta^*(\tau_\sigma)|$, small changes in $\beta$ will have little impact on how the solution of equation (3.1) develops across the time dimension and, conversely, when $|\alpha\xi| \ll |\beta\eta^*(\tau_\sigma)|$, the solution should have little sensitivity to changes in $\alpha$. Therefore, we should expect the $\alpha$ expansion to work better for low values of $|\xi|$ and the $\beta$ expansion should perform better for high values of $|\xi|$. It is also clear that we should expect both $\alpha$ and $\beta$ expansions to perform well for low values of $\sigma^2(T - t)$ simply because this is our volatility-scaled time $\tau_\sigma$ and by moving away from the initial condition at time zero we can only expect our approximation to get worse.

The choice of initial conditions naturally allocates $\alpha$ expansion to the problem of the fixed-strike Asian call and $\beta$ expansion nicely fits the problem for the floating-strike Asian put. If we value options at time zero, the symmetry relationship (2.3) will make these algorithms interchangeable – we can use either of them to price any type of the Asian option.

### 3.1 Fixed-strike Asian call

In this section, we will show how to get full Taylor expansion for the price of a fixed-strike Asian call. The expansion will be in $\alpha$ around zero and we therefore start by differentiating

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both sides of equation (3.1) with $\alpha$ and setting $\alpha$ to zero. This will give us a system of differential equations for $\alpha$ expansion orders. We need to differentiate $n$ times to get $n's$ expansion order. We will also set $\beta$ to 1. For better readability of formulas, let us remove $\sigma$ subscript from times $\tau$ and $T$. From now on, $\tau$ and $T$ will denote volatility-scaled times. Let us also denote partial derivatives with subscripts and $\frac{\partial^n \psi(\xi,\tau_\sigma,\alpha,1)}{\partial \alpha^n}$ evaluated at $\alpha = 0$ with $\psi^{(n)}$ $(n = 0, 1, \ldots)$. This gives us

$$
\begin{aligned}
&\psi^{(0)}_\tau - \frac{1}{2}\eta^*(\tau)^2\psi^{(0)}_{\xi\xi} = 0,\\
&\psi^{(1)}_\tau - \frac{1}{2}\eta^*(\tau)^2\psi^{(1)}_{\xi\xi} = -\eta^*(\tau)\xi\psi^{(0)}_{\xi\xi},\\
&\psi^{(2)}_\tau - \frac{1}{2}\eta^*(\tau)^2\psi^{(2)}_{\xi\xi} = -2\eta^*(\tau)\xi\psi^{(1)}_{\xi\xi} + \xi^2\psi^{(0)}_{\xi\xi}\\
&\qquad\qquad\qquad\ldots\\
&\psi^{(n)}_\tau - \frac{1}{2}\eta^*(\tau)^2\psi^{(n)}_{\xi\xi} = -n\eta^*(\tau)\xi\psi^{(n-1)}_{\xi\xi} + \frac{1}{2}n(n-1)\xi^2\psi^{(n-2)}_{\xi\xi}\\
&\qquad\qquad\qquad\ldots
\end{aligned}
\tag{3.2}
$$

with the initial condition of $\max(\xi, 0)$ for $\psi^{(0)}$ and 0 for $(\psi^{(1)}, \psi^{(2)}, \ldots)$. Then, we would hope that the solution to our original PDE (2.11) is

$$
\psi(\xi,\tau) = \sum_{n=0}^{+\infty} \frac{1}{n!}\psi^{(n)}(\xi,\tau). \tag{3.3}
$$

Strictly speaking, the above sum may not converge at all or it may not converge to the solution of our original PDE. The maximum we can do is to test convergence empirically and see if our calculations agree with other algorithms accepted for solving this particular problem. We will do this in the Validation of results section. Also, it is interesting to note that $n!$ in the above expression is in some way redundant. We could have removed it (e.g. as it has been done in [1]) and got rid of $n$ and $n(n-1)$ in equations (3.2) as well.

Let us now start with solving our system of differential equations (3.2) for $\psi^{(0)}$

$$
\psi^{(0)}(\xi,\tau) = \xi N\left(\frac{\xi}{\sqrt{\hat\eta(\tau)}}\right) + \sqrt{\frac{\hat\eta(\tau)}{2\pi}}e^{-\frac{\xi^2}{2\hat\eta(\tau)}}, \tag{3.4}
$$

where $N$ is the normal cumulative density function and $\hat\eta(\tau)$ is the square of $\eta^*(u)$ function integrated over $[0, \tau]$

$$
\hat\eta(\tau) = \int_0^\tau \eta^*(u)^2 du. \tag{3.5}
$$

It is clear that $\hat\eta(\tau)$ is a strictly increasing function of its argument and it is zero at time zero. We can therefore view $\hat\eta$ as a 'natural' clock for our problem. In particular, it defines how $\psi^{(0)}$ develops over time. It is relatively easy to guess $\psi^{(1)}$ as

$$
\psi^{(1)}(\xi,\tau) = -\frac{\xi}{\sqrt{2\pi\hat\eta(\tau)^3}}e^{-\frac{\xi^2}{2\hat\eta(\tau)}}\int_0^\tau \hat\eta(u)\eta^*(u)du. \tag{3.6}
$$

It is interesting to note that instead of guessing solutions we could have rewritten the problems using $\hat\eta$ as an independent time-like variable and derive equations (3.4) and (3.6) in a very straightforward way similar to how it has been done in [1]. This method however has very limited utility beyond order 1. The solutions for higher orders are not easy to obtain and we have not seen them presented for a general form of $\eta^*(\tau)$. Everything now depends on how successfully we can solve PDE of the following type:

$$\frac{\partial\psi(\xi,\tau)}{\partial\tau} - \frac{1}{2}\eta^*(\tau)^2\frac{\partial\psi(\xi,\tau)}{\partial\xi^2} = \phi(\xi,\tau), \tag{3.7}$$

where the initial condition is $\psi(\xi,0) = 0$ and $\phi(\xi,\tau)$ is given. In the context of our problem, $\psi(\xi,\tau)$ would be $\psi^{(n)}$ $(n = 2,3,\ldots)$ with $\phi(\xi,\tau)$ calculated from two preceding orders. We can write the solution to the above PDE (3.7) as some transformation $E_{x,\hat{\eta}}[.]$ applied to the function $\phi(\xi,\tau)$

$$\psi(\xi,\tau) = E_{x,\hat{\eta}}\left[\phi(\xi,\tau)\right] = E\left[\int_0^\tau \phi(\xi + x\sqrt{\hat{\eta}(\tau)-\hat{\eta}(u)}, u)du\right], \tag{3.8}$$

where $x$ is a normally distributed random variable with variance 1. We can prove this relationship as follows. First, let us introduce a standard Brownian motion in the above equation

$$\psi(\xi,\tau) = E\left[\int_0^\tau \phi(\xi + B_{\hat{\eta}(\tau)-\hat{\eta}(u)}, u)du\right]. \tag{3.9}$$

From Itô's lemma, we know that

$$\phi(\xi + B_v, u) = \phi(\xi,u) + \int_0^v \phi_\xi(\xi + B_v, u)dB_v + \int_0^v \frac{1}{2}\phi_{\xi\xi}(\xi + B_v, u)dv. \tag{3.10}$$

Therefore, we can write equation (3.9) as

$$\psi(\xi,\tau) = \int_0^\tau \phi(\xi,u)du + \frac{1}{2}\int_0^\tau\int_0^{\hat{\eta}(\tau)-\hat{\eta}(u)} E[\phi_{\xi\xi}(\xi + B_v, u)]dvdu. \tag{3.11}$$

We can then differentiate both sides with $\tau$ and replace the remaining expectation with $\psi$ using identity (3.9). This will give us our result (3.7). Throughout the proof, we assumed that $\phi$ is in $C^{2,1}(R,[0,T])$ and $L^1$. This is sufficient for Itô's lemma to work and all of the above expectations, integrals and derivatives will exist.

Straightforward application of this result (3.8) to our system of differential equations (3.2) immediately yields all orders greater then 1

$$\psi^{(n)}(\xi,\tau) = E_{x,\hat{\eta}}\left[-n\eta^*(\tau)\xi\psi^{(n-1)}_{\xi\xi} + \frac{1}{2}n(n-1)\xi^2\psi^{(n-2)}_{\xi\xi}\right]. \tag{3.12}$$

Therefore, we have a viable way of obtaining all orders of the $\alpha$ expansion. For example, this is how we can get $\psi^{(2)}$. First, we note that $\psi^{(0)}$ and $\psi^{(1)}$ are known from equations (3.4) and (3.6). We then substitute them into the above identity (3.12) and calculate the argument for the transformation $E_{x,\hat{\eta}}[.]$. Finally, we apply the transformation. For that we need to replace $\tau$ with $u$ and $\xi$ with $\xi + x\sqrt{\hat{\eta}(\tau)-\hat{\eta}(u)}$. Then, we take expectation with respect to $x$ (it is normally distributed with variance 1) and integrate over $u$ from 0 to $\tau$.

This would give us the result

$$\psi^{(2)}(\xi,\tau) = e^{-\frac{\xi^2}{2\hat{\eta}(\tau)}}(2\xi^4\eta 2^{(1,1,1)}(\tau) + \xi^2(\hat{\eta}(\tau)^2\eta 2^{(0,2,0)}(\tau) - 12\hat{\eta}(\tau)\eta 2^{(1,1,1)}(\tau) + 6\hat{\eta}(\tau)^2\eta 2^{(1,0,1)}(\tau)) + (-\hat{\eta}(\tau)^3\eta 2^{(0,2,0)}(\tau) + \hat{\eta}(\tau)^4\eta 2^{(0,1,0)}(\tau) + 6\hat{\eta}(\tau)^2\eta 2^{(1,1,1)}(\tau) - 6\hat{\eta}(\tau)^3\eta 2^{(1,0,1)}(\tau)))/\sqrt{2\pi\hat{\eta}(\tau)^9}, \tag{3.13}$$

where

$$\eta 2^{(k,l,m)}(\tau) = \int_0^\tau \eta^*(u)^k\hat{\eta}(u)^l\left(\int_0^u \eta^*(u1)\hat{\eta}(u1)du1\right)^m du. \tag{3.14}$$

Immediately, we can see that the functional form for $\psi^{(2)}$ is not as simple as we have seen for the previous orders. In order to obtain $\psi^{(3)}$, we need to follow equation (3.12) again. However, differentiating by $\xi$ in equation (3.12) will result in even more complex analytical expressions. We need to optimise our algorithm further. For that, we note that the expansion orders 1 (3.6) and 2 (3.13) have similar forms with respect to $\xi$. They are both $\xi$ polynomials multiplied by the exponential function. Substituting this type of expressions into equation (3.12) will yield the argument for $E_{x,\hat{\eta}}[.]$ transformation which again is a $\xi$ polynomial multiplied by the exponential function. We will see next what happens when we apply $E_{x,\hat{\eta}}[.]$ transformation to this class of functions. Let us start with the following identity:

$$E\left[(\xi+ax)^n e^{-\frac{1}{2}\left(\frac{\xi+ax}{b}\right)^2}\right] = e^{-\frac{\xi^2}{2(a^2+b^2)}}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!\xi^{n-2k}a^{2k}}{2^k(n-2k)!k!}\left(\frac{b^2}{a^2+b^2}\right)^{\frac{1}{2}+n-k}, \tag{3.15}$$

where, as before, $x$ is a normally distributed random variable with variance one, $a$, $b$ are real constants, $n$ is an integer and by $\lfloor n/2\rfloor$ we denote the greatest integer less than or equal to $n/2$. We can see that this identity holds by re-writing the expectation as an integral w.r.t $x$; then, through a simple change of variables, we can remove the first power of $x$ in the exponential, re-write multiplier in front of the exponential as $x$ polynomial and the result will follow from the known formulae for moments of a normally distributed random variable. For our problem, this means that

$$E_{x,\hat{\eta}}\left[\xi^n e^{-\frac{\xi^2}{2\hat{\eta}(\tau)}}f(\tau)\right] = e^{-\frac{\xi^2}{2\hat{\eta}(\tau)}}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!\xi^{n-2k}}{2^k(n-2k)!k!}\int_0^\tau f(u)g_{n,k}(\tau,u)du, \tag{3.16}$$

where

$$g_{n,k}(\tau,u) = (\hat{\eta}(\tau)-\hat{\eta}(u))^k\left(\frac{\hat{\eta}(u)}{\hat{\eta}(\tau)}\right)^{\frac{1}{2}+n-k}. \tag{3.17}$$

It is clear from the above identity (3.16) that applying transformation $E_{x,\hat{\eta}}[.]$ to a $\xi$ polynomial multiplied by the exponential function gives us $\xi$ polynomials multiplied by the same exponential function and even the maximum power of $\xi$ stays the same. This means that all $\alpha$ expansion orders will be very predictable functions of $\xi$. As a matter of fact, if trying to derive $\psi^{(3)}$ and above manually, we would see even more patterns emerging. In particular, we would be tempted to search for solutions in the following form:

$$\psi^{(n)}(\xi,\tau) = \frac{\tau^{5n}}{T^{3n}\sqrt{2\pi\hat{\eta}(\tau)^{3n-1}}} e^{-\frac{\xi^2}{2\hat{\eta}(\tau)}} \sum_{m=0}^{3n-2} A_{n,m}(\tau)\hat{\eta}(\tau)^{-\frac{m}{2}}\xi^m \tag{3.18}$$

where $n \geqslant 1$. We can substitute the above formula into equation (3.12) and, with the help of identity (3.16), obtain the following relationship for $A_{n,m}(\tau)$ :

$$A_{n,m}(\tau) = \int_0^\tau \sum_{k=m}^{3n-2} \frac{(1+(-1)^{m+k})k!}{2m!((k-m)/2)!} \frac{u^{5n-1}}{\tau^{5n}} \left(\frac{\hat{\eta}(\tau)-\hat{\eta}(u)}{2\hat{\eta}(\tau)}\right)^{\frac{k-m}{2}} \left(\frac{\hat{\eta}(\tau)}{\hat{\eta}(u)}\right)^{\frac{3n-m-2}{2}}$$
$$(-nT^3u^{-4}\eta^*(u)\hat{\eta}(u)\hat{A}_{n-1,k-1}(u) + \frac{1}{2}n(n-1)T^6u^{-9}\hat{\eta}(u)^3\hat{A}_{n-2,k-2}(u))du, \tag{3.19}$$

where by $\hat{A}_{n,k}(u)$ we denote

$$\hat{A}_{n,k}(u) = A_{n,k-2} - (1+2k)A_{n,k}\,\mathbb{I}_{\{n>0\}} + (k+1)(k+2)A_{n,k+2}, \tag{3.20}$$

with $A_{0,-2} = 1$ and $A_{n,m} = 0$ for all other negative $m$'s, $n < 1$ and for $m > 3n-2$. This is all we need to obtain any $\alpha$ expansion order. We have analytical expression for $\psi^{(0)}$ (3.4) and equation (3.18) gives us all orders above zero. Let us now look at the above algorithm more closely.

Firstly, we notice that $A_{n,m}$ is calculated from two preceding orders of the $\alpha$ expansion multiplied by $n$ and $n(n-1)$. This introduces $n!$ type of growth and we therefore need to be careful not to assume that the $\alpha$ expansion converges unconditionally. This is simply not true.

Secondly, we can see that $\hat{\eta}(u)$ appears in the denominator and, since it is zero at the start of our integration interval, the integral may not exist. In particular, for discreetly sampled options, if the last sample is not taken at maturity, the $\hat{\eta}(u)$ and all its derivatives will be zero around a neighbourhood of zero. The integral will not exist and we will therefore need to enforce that the last sample is taken at maturity. But this is fine because, if interest rates are constant, the final payout is already known if we know the last sample.

Let us now see how we can apply this algorithm for the special case of continuously sampled fixed-strike Asian call. For continuously sampled options, the $\eta^*(\tau)$ function takes the following form:

$$\eta^*(\tau) = \frac{1}{\rho T}(1-e^{-\rho\tau}). \tag{3.21}$$

We would like to remind that $T$ and $\tau$ are volatility-scaled times and should not be confused with physical times.

Then, we just need to implement the algorithm (3.19). If we substitute the above equation for $\eta^*(\tau)$ and $\hat{\eta}(\tau)$ (which is just an integral of $\eta^*(u)^2$ over $[0,\tau]$) into our algorithm (3.19), we will find that $A_{n,m}(\tau)$'s will only depend on $\rho\tau$. This should not be a surprise because, it is due to our choice of the functional form for the solution (3.18), we have things working out nicely for this special case.

It should be noted that the expression (3.19) includes integration over time which creates some problems. Firstly, we would want to have analytical expressions for all expansion orders which means that we cannot do integration numerically. We need to use tools which can take integrals analytically, e.g. Wolfram Mathematica [10]. Secondly, looking at the form of the integrand, it is easy to see that there is no guarantee that the integrals can be taken analytically. Our algorithm would then clearly fail. And, finally, taking integrals analytically takes time. The time it takes to obtain the solution will increase with each order and for the 8th order it took us around 24 h with an average modern personal computer.

Having gone through the process, we would obtain quite complex analytical expressions with 8th order taking more then 20 screens of a Wolfram Mathematica [10] notebook. The expressions will not be numerically stable for high orders and we will need to expand $A_{n,m}(\tau)$ in Taylor series with respect to $\tau$. This means that, although it is nice to have analytical expressions for the $\alpha$ expansion orders, they are really of little practical use. We will only need Taylor expansions in $\tau$ for all $\alpha$ orders.

$A_{m,n}$'s are functions of $\rho\tau$ and we obtain them through multiple integration of analytical functions of the known form. If we remember what we already know about our solution (3.18), we can easily derive the following representation for $\psi^{(n)}$'s:

$$\psi^{(n)}(\xi,\tau) = \frac{\tau^{5n}}{\sqrt{2\pi}\hat{\eta}(\tau)^{\frac{(3n-1)}{2}}\,T^{3n}} e^{-\frac{\xi^2}{2\hat{\eta}(\tau)}} \sum_{i=0}^{+\infty}\sum_{j=0}^{3n-2} a_{n,i,j}\hat{\eta}(\tau)^{-\frac{j}{2}}\xi^j(\rho\tau)^i, \tag{3.22}$$

where $a_{n,i,j}$'s are real constants. One way of calculating them is to directly expand $A_{m,n}(\tau)$'s that we had obtained earlier. There is however a simpler way. It should be noted, that, in order to calculate $a_{n,i,j}$'s, we do not need any(!) of the analytics we have developed so far. We can just use a brute force approach. We take the above representation (3.22) as given and substitute it directly into the system of differential equations at the start of this section (3.2). At first sight, it may look complicated because we have to multiply a Taylor expansion with unknown coefficients by functions $\hat{\eta}(u)$ and $\eta^*(u)$ and then we need to solve for $a_{n,i,j}$'s which, for a given $n$, is a matrix with one finite and one infinite dimension. Such problems could be notoriously hard to solve. However, for our special case, it all works out very nicely. The products give us convolution sums and the resulting system of linear equations is surprisingly easy to solve.

This will give us the algorithm we were looking for

$$\begin{aligned} a_{n,i,j} = -3\Bigg(&\sum_{k=1}^{i}((2+j-3n)(k+3)+2(i-k+5n))\Psi_{1,k}a_{n,i-k,j} \\ &-\sum_{k=0}^{i}((k+3)(2+3j+j^2)\Psi_{1,k}a_{n,i-k,j+2} - 2n\Psi_{2,k}\hat{a}_{n-1,i-k,j-1} \\ &+n(n-1)\Psi_{3,k}\hat{a}_{n-2,i-k,j-2})\Bigg)\Bigg/(3(2+j-3n)+2(i+5n)), \end{aligned}$$

where, as before, by the hat over $a$ we mean the following identity:

$$\hat{a}_{n,i,j}(u) = a_{n,i,j-2} - (1+2j)a_{n,i,j}\mathbb{I}_{\{n>0\}} + (j+1)(j+2)a_{n,i,j+2}, \tag{3.23}$$

and by $\Psi_{m,k}$'s we denote Taylor expansion coefficients for the functions in $\{.\}_k$ brackets

$$\Psi_{1,k} = \{u^{-3}T^2\rho^3\hat{\eta}(u)\}_k = (-1)^k(2^{k+2}-2)/(k+3)!,$$

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$$\Psi_{2,k} = \{u^{-7}\rho T\eta^*(u)(\rho^3 T^2\hat{\eta}(u))^2\}_k = (-1)^k(397 - 2^{9+k}29 + 3^{6+k}118 - 4^{7+k}9 + 5^{7+k} + (80 - 2^{8+k}5 + 3^{6+k}4)k + 4k^2)/4/(k+7)!,$$

$$\Psi_{3,k} = \{u^{-12}(\rho^3 T^2\hat{\eta}(u))^4\}_k = (-1)^k(-15987 + 2^{10+k}7971 + 2^{32+3k} - 3^{12+k}523 + 2^{10+k}3^{12+k}31 + 4^{11+k}613 - 5^{11+k}197 - 7^{12+k} + 2(-1889 + 2^{11+k}323 - 3^{11+k}80 + 2^{9+k}3^{11+k} + 4^{9+k}297 - 5^{11+k}3)k + 12(-25 + 2^{9+k}11 + 2^{17+2k} - 3^{10+k})k^2 + 8(-1 + 2^{7+k})k^3)/(k+12)!.$$

We start with $a_{0,0,-2} = 1$ and we set $a_{n,i,j}$'s to zero for all other negative $j$'s, for $n < 1$, $i < 0$ and for $j > 3n - 2$. For given $n$ and $i$, the index $j$ should run in reverse order from $3n - 2$ to 0. $n$ runs from 1 and $i$ from 0 up. The maximum $n$ will determine the number of expansion orders required in $\alpha$ and the maximum $i$ determines the number of expansion terms in $\tau$. If we want to keep error for each $n$ of the same order of magnitude with respect to $\tau$, we would need to increase maximum $i$ as $n$ goes up. This is because our representation (3.22) explicitly includes $\tau$, $T$ and our new clock $\hat{\eta}$ which are of the same order of magnitude as $\tau$. It is clear that the maximum $i$ will need to go up by 1 for each extra $\alpha$ order.

When interest rates are equal to the dividend yield on the stock, we have $\rho = 0$ and from the representation (3.22) we can see that the summation in $i$ will disappear. Our algorithm (3.22) will then simplify significantly. First sum will disappear all together and, in the second sum, we will only have to keep the term with $k$ equal to zero. $\Psi_{m,k}$ will become real numbers. The resulting algorithm (in C++ or Wolfram Mathematica [10]) will then consist of only 2 'for' loops around 1 relatively short line of code.

The algorithm we have presented here is invariant to changes in any of our parameters. We calculate Taylor expansion constants $a_{n,i,j}$'s and these are just real numbers independent of our inputs. It is sufficient to calculate them only once and have them stored for future re-use through the formula (3.22). This completes specifications for the algorithm.

We have kept things general up to a point where we started to search for solutions in the form (3.18). This choice has been made to ensure that, for continuously sampled options, our $A_{n,m}(\tau)$'s functions only depend on $\rho\tau$. For other cases, it may not be possible to find a solution that would make our algorithm invariant to changes in $\rho$. It is however clear that the algorithm for calculating $A_{n,m}$'s (3.18) holds for a general case which means that, if everything else fails, we can still try to get explicit expressions for the $\alpha$ expansion orders by integrating expressions, constructed from two previous orders, over volatility-scaled time.

### 3.2 Floating-strike Asian put

In this section, we will show how to get full Taylor expansion for the price of the floating-strike Asian put. This time, we need to expand PDE solutions in $\beta$ and, analogously to the $\alpha$ case, we can get a system of equations for all $\beta$ expansion orders from equation (3.1) by differentiating both sides with $\beta$ and setting $\beta$ to zero. We need to differentiate $n$ times to get $n's$ expansion order. We will also set $\alpha$ to 1. Then, we have

$$\psi_\tau^{(0)} - \frac{1}{2}\xi^2\psi_{\xi\xi}^{(0)} = 0,$$

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$$
\begin{aligned}
&\psi^{(1)}_\tau - \frac{1}{2}\xi^2\psi^{(1)}_{\xi\xi} = -\eta^*(\tau)\xi\psi^{(0)}_{\xi\xi},\\
&\psi^{(2)}_\tau - \frac{1}{2}\xi^2\psi^{(2)}_{\xi\xi} = -2\eta^*(\tau)\xi\psi^{(1)} + \eta^*(\tau)^2\psi^{(0)}_{\xi\xi}\\
&\qquad\qquad\qquad\dots\\
&\psi^{(n)}_\tau - \frac{1}{2}\xi^2\psi^{(n)}_{\xi\xi} = -n\eta^*(\tau)\xi\psi^{(n-1)}_{\xi\xi} + \frac{1}{2}n(n-1)\eta^*(\tau)^2\psi^{(n-2)}_{\xi\xi}\\
&\qquad\qquad\qquad\dots
\end{aligned} \tag{3.24}
$$

with the initial condition of $\max(\xi-\lambda, 0)$ for $\psi^{(0)}$ and 0 for $(\psi^{(1)},\psi^{(2)},\dots)$. Firstly, we need to remove $\xi^2$ in front of the second derivative through the following change of variables:

$$
\zeta_- = Log\left[\frac{\xi}{\lambda}\right] - \frac{1}{2}\tau. \tag{3.25}
$$

This will transform the above system of differential equations into

$$
\begin{aligned}
\psi^{(0)}_\tau - \frac{1}{2}\psi^{(0)}_{\zeta_-\zeta_-} &= 0,\\
\psi^{(1)}_\tau - \frac{1}{2}\psi^{(1)}_{\zeta_-\zeta_-} &= -\eta^*(\tau)\lambda^{-1}e^{-\zeta_- - \frac{1}{2}\tau}\left(\psi^{(0)}_{\zeta_-\zeta_-} - \psi^{(0)}_{\zeta_-}\right),\\
\psi^{(2)}_\tau - \frac{1}{2}\psi^{(2)}_{\zeta_-\zeta_-} &= -2\eta^*(\tau)\lambda^{-1}e^{-\zeta_- - \frac{1}{2}\tau}\left(\psi^{(1)}_{\zeta_-\zeta_-} - \psi^{(1)}_{\zeta_-}\right)\\
&\quad + \eta^*(\tau)^2\lambda^{-2}e^{-2\zeta_- - \tau}\left(\psi^{(0)}_{\zeta_-\zeta_-} - \psi^{(0)}_{\zeta_-}\right)\\
&\quad \dots\\
\psi^{(n)}_\tau - \frac{1}{2}\psi^{(n)}_{\zeta_-\zeta_-} &= -n\eta^*(\tau)\lambda^{-1}e^{-\zeta_- - \frac{1}{2}\tau}\left(\psi^{(n-1)}_{\zeta_-\zeta_-} - \psi^{(n-1)}_{\zeta_-}\right)\\
&\quad + \frac{1}{2}n(n-1)\eta^*(\tau)^2\lambda^{-2}e^{-2\zeta_- - \tau}\left(\psi^{(n-2)}_{\zeta_-\zeta_-} - \psi^{(n-2)}_{\zeta_-}\right)\\
&\quad \dots
\end{aligned} \tag{3.26}
$$

with the initial conditions of $\lambda\max(e^{\zeta_-} - 1, 0)$ for $\psi^{(0)}$ and 0 for the higher orders. Mathematically, the problem is still analogous to the $\alpha$ case and we will therefore proceed in the same way. First, let us guess $\psi^{(0)}$, i.e.

$$
\psi^{(0)}(\zeta_\pm, \tau) = \lambda e^{\zeta_+ - \frac{1}{2}\tau}N\left(\frac{\zeta_+}{\sqrt{\tau}}\right) - \lambda N\left(\frac{\zeta_-}{\sqrt{\tau}}\right). \tag{3.27}
$$

Here, we define $\zeta_+$ as $\zeta_-$ (3.25) but with the plus sign in front of the half $\tau$. The two $\zeta$'s are similar to the two inputs into the Black–Scholes formula for the vanilla call. These are sometimes denoted by $h^+$, $h^-$ or $d1$, $d2$. For orders $(1, 2, 3, \dots)$, similarly to the $\alpha$, case, we can write

$$
\begin{aligned}
\psi^{(n)}(\zeta_\pm, \tau) = E_{x,\tau}\big[&-n\eta^*(\tau)\lambda^{-1}e^{-\zeta_- - \frac{1}{2}\tau}(\psi^{(n-1)}_{\zeta_-\zeta_-} - \psi^{(n-1)}_{\zeta_-})\\
&+ \frac{1}{2}n(n-1)\eta^*(\tau)^2\lambda^{-2}e^{-2\zeta_- - \tau}(\psi^{(n-2)}_{\zeta_-\zeta_-} - \psi^{(n-2)}_{\zeta_-})\big],
\end{aligned} \tag{3.28}
$$

where we define $\psi^{(n)} = 0$ for negative $n$ (to allow for the case of $n$=1) and the transformation $E_{x,\tau}[.]$ is the same as before except that we replace $\hat{\eta}(\tau)$ with $\tau$

$$E_{x,\tau}[\phi(\zeta_\pm, \tau)] = E\left[\int_0^\tau \phi(\zeta_- + x\sqrt{\tau - u}, u)du\right], \tag{3.29}$$

where $x$ is a normally distributed random variable with variance 1. It is interesting that, for the $\beta$ case, our PDEs (3.26) are nothing but canonical Heat Equations with the Source. Their solutions would usually be expressed as a convolution integral of their fundamental solution with the Source function which is equivalent to the above transformation. We can now substitute equation (3.27) into equation (3.28) and obtain $\psi^{(1)}$

$$\psi^{(1)}(\zeta_\pm, \tau) = -\frac{1}{\sqrt{2\pi\tau}} \int_0^\tau e^{-\frac{(\zeta_- + u)^2}{2\tau}} \eta^*(u)du. \tag{3.30}$$

$\zeta_-$ cannot be taken outside integration which means that we cannot split $\tau$ and $\zeta_-$ in a way we did for the $\alpha$ case. It is therefore not clear if we can improve on our algorithm (3.28) further. The $\beta$ expansion appears to be more difficult to handle for a general case.

Let us now look at the special case of continuously sampled floating-strike Asian put. The $\eta^*(\tau)$ function is given by equation (3.21). We have analytical expression for $\psi^{(0)}$ (3.27) and we can obtain $\psi^{(1)}$ from equation (3.30)

$$\begin{aligned}\psi^{(1)}(\zeta_\pm, \tau) = &\frac{1}{\rho T} e^{\rho\zeta_- + \frac{1}{2}\rho^2\tau} \left(N\left(\frac{\zeta_+ + \rho\tau}{\sqrt{\tau}}\right) - N\left(\frac{\zeta_- + \rho\tau}{\sqrt{\tau}}\right)\right) \\ &+ \frac{1}{\rho T}\left(N\left(\frac{\zeta_-}{\sqrt{\tau}}\right) - N\left(\frac{\zeta_+}{\sqrt{\tau}}\right)\right).\end{aligned} \tag{3.31}$$

Our zero order (3.27) is exactly equal to the solution of the Black–Scholes PDE and the first order above shows what happens if we deviate slightly from the 'Black–Scholes world'.

For the following orders $(2, 3, \ldots)$, we need to use transformation (3.28) again. This is not as complicated as it looks thanks to the following identity:

$$\begin{aligned}&E_{x,\tau}\left[e^{a\zeta_- + \frac{a^2\tau}{2} + b\tau} N\left(\frac{\zeta_- + (a+c)\tau}{\sqrt{\tau}}\right)\right] \\ &= \frac{1}{b} e^{\frac{a^2\tau}{2} + a\zeta_-}\left(e^{bt} N\left(\frac{\zeta_- + (c+a)\tau}{\sqrt{\tau}}\right) - N\left(\frac{\zeta_- + a\tau}{\sqrt{\tau}}\right)\right) \\ &\quad + \frac{1}{b} e^{(a - \frac{b}{c})\zeta_- + \frac{(a-\frac{b}{c})^2\tau}{2}}\left(N\left(\frac{\zeta_- + (a - \frac{b}{c})\tau}{\sqrt{\tau}}\right) - N\left(\frac{\zeta_- + (c + a - \frac{b}{c})\tau}{\sqrt{\tau}}\right)\right),\end{aligned} \tag{3.32}$$

where $a, b, c$ are real constants. We can prove it by substituting its right-hand side into $\psi_\tau - \frac{1}{2}\psi_{\zeta_-\zeta_-}$ and making sure that we get the argument of the $E_{x,\tau}[.]$ transformation. This is simply because we have introduced $E_{x,\tau}[.]$ as a way to solve the Heat Equation with the Source.

The above identity (3.32) combined with our algorithm (3.28) and the functional forms of $\psi^{(0)}$ (3.27) and $\psi^{(1)}$ (3.31) should make it clear that all $\beta$ expansion terms greater then

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zero will have the following representation:

$$\psi^{(n)}(\zeta_{\pm},\tau)=\sum_{k=0}^{n-1}\sum_{m=0}^{M_k}\frac{\partial^k}{\partial\zeta_-^k}\left(e^{a_{k,m}\zeta_-+b_{k,m}\tau}N\left(\frac{\zeta_-+c_{k,m}\tau}{\sqrt{\tau}}\right)\right), \tag{3.33}$$

where $a_{k,m}$'s, $b_{k,m}$'s and $c_{k,m}$'s are real constants One way to show this is to substitute the above equation into our algorithm (3.28) and see what happens. It helps to know that, because our functions are from $C^{+\infty,1}(R,(0,T])$ and $L^1$, $\frac{\partial^k}{\partial\zeta_-^k}$ and $E_{x,\tau}[.]$ will commute.

While we can continue beyond order 1, we have found that the algorithm is not as easy to implement with Wolfram Mathematica [10] as in the $\alpha$ case. The main reason is that the identity (3.32) does not seem to be recognised by the software and we have to apply it ourselves. In any case, we will get complex analytical expressions with 8th order taking more than 30 screens of a Wolfram Mathematica [10] notebook. Similar to the $\alpha$ case, high orders will not be numerically stable and we will need to have them expanded in Taylor series with respect to $\tau$ and $\zeta_-$ anyway.

Perhaps we should take a different approach. Let us re-write our system of differential equation (3.26) in the following form:

$$\psi^{(n)}_\tau-\frac{1}{2}\left(\psi^{(n)}_{\zeta_+\zeta_+}-2\psi^{(n)}_{\zeta_+}\right)=\phi^{(n)}, \tag{3.34}$$

$$\begin{aligned}\phi^{(n)}=&-n\eta^*(\tau)\lambda^{-1}e^{-\zeta_++\frac{1}{2}\tau}\left(\psi^{(n-1)}_{\zeta_+\zeta_+}-\psi^{(n-1)}_{\zeta_+}\right)\\&+\frac{1}{2}n(n-1)\eta^*(\tau)^2\lambda^{-2}e^{-2\zeta_++\tau}\left(\psi^{(n-2)}_{\zeta_+\zeta_+}-\psi^{(n-2)}_{\zeta_+}\right),\end{aligned} \tag{3.35}$$

with $\psi(\zeta_+,0)=0$ and $\psi^{(n)}=0$ for negative $n$. This is for $n\geqslant 1$ and $\psi^{(0)}$ is defined by equation (3.31). It happens that for our special case of $\eta^*(\tau)$ (3.21), we can write $\phi^{(n)}$ as

$$\phi^{(n)}(\zeta_+,\tau)=\frac{1}{(\lambda T)^{n-1}}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{\zeta_+^2}{2\tau}}\left(\frac{1}{T}\right)\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{j-1}c_{n,i,j,k}\zeta_+^i\tau^j\rho^k \tag{3.36}$$

and look for solutions in the form

$$\psi^{(n)}(\zeta_+,\tau)=\frac{1}{(K\lambda)^{n-1}}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{\zeta_+^2}{2\tau}}\left(\frac{\tau}{T}\right)\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{j-1}b_{n,i,j,k}\zeta_+^i\tau^j\rho^k, \tag{3.37}$$

where for $k\geqslant j$, $n=0$ and for negative indexes we define $\{b_{n,i,j,k}\}$ and $\{c_{n,i,j,k}\}$ as equal to $\{0\}$. By notation $\{.\}$, we mean a collection of $b_{n,i,j,k}$'s or $c_{n,i,j,k}$'s with all indexes running from $-\infty$ to $+\infty$ unless we specify otherwise. Substituting analytical expression for $\phi^{(0)}$ (3.31) and $\psi^{(0)}$ (3.37) into equation (3.35), we get a very simple algorithm for calculating $\{c_{1,i,j,k}\}$

$$c_{1,0,j,j-1}=\frac{(-1)^j}{j!} \tag{3.38}$$

and $c_{1,i,j,k}=0$ for $i>0$, $k\neq j-1$ and for negative indexes. This will be our starting point. The next step is to substitute expressions for $\psi^{(n)}$ (3.37) and $\phi^{(n)}$ (3.36) into equation (3.34)

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and get an algorithm for calculating $\{b_{n,i,j,k}\}$

$$b_{n,i,j,j-l} = \frac{b_{n,i-1,j,j-l}}{j+i+1} + \frac{0.5(1+i)(2+i)b_{n,i+2,j-1,j-l}}{j+i+1} - \frac{(1+i)b_{n,i+1,j-1,j-l}}{j+i+1} + \frac{c_{n,i,j,j-l}}{j+i+1}. \tag{3.39}$$

For a given $n$, we will know $\{c_{n,i,j,k}\}$ and this will be enough to calculate $\{b_{n,i,j,j-1}\}$ then $\{b_{n,i,j,j-2}\}$ and so on (it helps if we remember that $\{b_{n,i,j,k}\} = \{0\}$ for $k = j$, $n = 0$ and for negative indexes). The above equation (3.39) therefore maps $\{c_{n,i,j,k}\}$ to $\{b_{n,i,j,k}\}$ for a given $n$. The map is linear and we can therefore write it as

$$b_{n,i,j,k} = B^{i,j,k} c_{n,i,j,k}, \tag{3.40}$$

where $B$ is a tensor defined by the relationship (3.39) and we using the usual shorthand notation for tensors which excludes $\sum$'s. Now the only remaining piece is to explain how to calculate $\{c_{n,i,j,k}\}$ from $\psi^{(n-1)}$ and $\psi^{(n-2)}$. The formula for $\phi^{(n)}$ (3.36) and equation (3.35) give us the required relationship

$$\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{j-1} c_{n,i,j,k}\zeta_+^i \tau^j \rho^k = -n\left(\frac{(1-e^{-\rho\tau})}{\rho\tau}e^{-\zeta_+ + \frac{1}{2}\tau}\right)\Psi^{(n-1)} + \frac{1}{2}n(n-1)\tau\left(\frac{(1-e^{-\rho\tau})}{\rho\tau}e^{-\zeta_+ + \frac{1}{2}\tau}\right)^2 \Psi^{(n-2)}, \tag{3.41}$$

where $\Psi^{(n-1)}$ and $\Psi^{(n-2)}$ are calculated from sums in the expression for $\psi^{(n)}$ (3.37)

$$\Psi^{(n)} = \left(\tau^2\frac{\partial^2}{\partial\zeta_+^2} - (2\zeta_+\tau + \tau^2)\frac{\partial}{\partial\zeta_+} + (\zeta_+^2 - \tau + \zeta_+\tau)\right)\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}\sum_{k=0}^{j-1} b_{n,i,j,k}\zeta_+^i\tau^j\rho^k. \tag{3.42}$$

We need to get rid of sums and for that we need to know what happens to $\{b_{n,i,j,k}\}$ when we apply three basic transformations. Namely, when we calculate $\Psi^{(n)}$ from equation (3.42), we get

$$b_{n,i,j,k} \to b_{n,i-2,j,k} + b_{n,i-1,j-1,k} - (1+2i)b_{n,i,j-1,k} - (i+1)b_{n,i+1,j-2,k} + (i+2)(i+1)b_{n,i+2,j-2,k}. \tag{3.43}$$

When we multiply by $\frac{(1-e^{-\rho\tau})}{\rho\tau}e^{-\zeta_+ + \frac{1}{2}\tau}$

$$b_{n,i,j,k} \to \sum_{l=0}^{i}\sum_{m=k+1}^{j}\sum_{n=0}^{k}\frac{(-1)^{i-l+k-n}2^{-j+m}}{(i-l)!(j-m)!(k-n+1)!}b_{n,l,n+m-k,n}, \tag{3.44}$$

and, trivially, when we multiply by $\tau$

$$b_{n,i,j,k} \to b_{n,i,j-1,k}. \tag{3.45}$$

All the three transformations (3.43)–(3.45) map $\{b_{n,i,j,k}\}$ to $\{b_{n,i,j,k}\}$. The maps, again, are linear and we can define them with tensors $Z^{i,j,k}$, $H^{i,j,k}$, $T^{i,j,k}$. Then, we can write our algorithm in full

$$b_{n,i,j,k} = B^{i,j,k} H^{i,j,k} \left( -nZ^{i,j,k} b_{n-1,i,j,k} + \frac{1}{2} n(n-1) T^{i,j,k} H^{i,j,k} Z^{i,j,k} b_{n-2,i,j,k} \right). \quad (3.46)$$

This is for $n > 2$. For $n = 2$, we have

$$b_{2,i,j,k} = B^{i,j,k} H^{i,j,k} (-2Z^{i,j,k} b_{1,i,j,k} - T^{i,j,k} c_{1,i,j,k}), \quad (3.47)$$

and, finally, for $n = 1$

$$b_{1,i,j,k} = B^{i,j,k} c_{1,i,j,k}, \quad (3.48)$$

where $\{c_{1,i,j,k}\}$ is given by (3.38). Tensors in the above equations do not necessarily commute which means that the order they are applied in is important. Also, when implementing tensor $H^{i,j,k}$, there is no need to do three-dimensional summation in equation (3.44). It is more efficient to calculate first sum (counting from the right) for indexes $(l, m, n)$ then use results to calculate the second sum for indexes $(l, m, k)$ and so on.

This gives us the algorithm for calculating $\{b_{n,i,j,k}\}$ except that we have to start with $\{c_{1,i,j,k}\}$ defined by equation (3.38) and it can only have a finite number of elements (let us say $i \in [0, i_0]$ and $j \in [0, j_0]$). This will result in $\{b_{n,i,j,k}\}$ having a finite number of elements as well (let us say $i \in [0, i_n]$ and $j \in [0, j_n]$). This means that instead of exact solution, we will have its approximation such that

$$\psi^{(n)}(\zeta_+, \tau) = \frac{1}{(\lambda T)^{n-1}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{\zeta_+^2}{2\tau}} \left( \frac{\tau}{T} \right) \left( \sum_{i=0}^{i_n} \sum_{j=0}^{j_n} \sum_{k=0}^{j-1} b_{n,i,j,k} \zeta_+^i \tau^j \rho^k + O(\tau^{j_n+1}, \zeta_+^{i_n+1}) \right). \quad (3.49)$$

While we are fine with the $O(\tau^{j_n+1})$ ($\tau$ is usually small), $O(\zeta_+^{i_n+1})$ may present a problem. After all, $\zeta_+$ is not necessarily small. However, looking at the form of expansion (3.37) we can see that only when $\zeta_+ = O(\tau^{\frac{1}{2}})$ we would be concerned about the multiplier in front of $e^{-\frac{\zeta_+^2}{2\tau}}$. For all other cases, we can expect the exponential to be sufficiently small so that it dampens the effect of whatever $\zeta_+$ polynomial multiplies it. We can therefore assume that $\zeta_+$ is of the same order as $\tau^{\frac{1}{2}}$ which results in the following approximation for $\psi^{(n)}$

$$\psi^{(n)}(\zeta_+, \tau) \approx \frac{1}{(\lambda T)^{n-1}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{\zeta_+^2}{2\tau}} \left( \frac{\tau}{T} \right) \left( \sum_{i=0}^{2j_n} \sum_{j=0}^{j_n} \sum_{k=0}^{j-1} b_{n,i,j,k} \zeta_+^i \tau^j \rho^k + O(\tau^{j_n+1}) \right). \quad (3.50)$$

This keeps expansion in $\zeta_+$ under control and slightly simplifies the problem – we only need to know $i_0$ and $j_0$ for a given $j_n$. It is easy to see that $j_n = j_0$. This is because none of the algorithms (3.39), (3.43)–(3.45) require elements with time indexes higher then $j$. For $i_0$, this is not the case. From equation (3.39), it follows that for calculating $b_{n,i_n,j,j-l}$, we need two extra elements in $i$ from a previous step - $b_{n,i_n+1,j-1,j-l}$, $b_{n,i_n+2,j-1,j-l}$. For a given $n$, we have $j$ steps ($l = 1, 2, \ldots j$) to complete the algorithm and we will therefore lose $2j_n$ expansion orders in $\zeta_+$ by moving from $n$ to $n+1$. As a matter of fact, for $n > 0$, two extra orders in $\zeta_+$ will be lost through differentiation by $\zeta_+$ (3.43) as well. Our total loss by moving from 0 to $n$ can be accounted for as $2(j_n + 1)n - 2$. We should therefore have

Table 1. *Test cases*

| No. | $S$ | $K$ | $r$ | $\delta$ | $\sigma$ | T | No | $S$ | $K$ | $r$ | $\delta$ | $\sigma$ | T |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.9 | 2 | 0.05 | 0 | 0.5 | 1 | $1^*$ | 1.9 | 2 | 0.05 | 0.1 | 0.5 | 1 |
| 2 | 2.0 | 2 | 0.05 | 0 | 0.5 | 1 | $2^*$ | 2.0 | 2 | 0.05 | 0.1 | 0.5 | 1 |
| 3 | 2.1 | 2 | 0.05 | 0 | 0.5 | 1 | $3^*$ | 2.1 | 2 | 0.05 | 0.1 | 0.5 | 1 |
| 4 | 2.0 | 2 | 0.02 | 0 | 0.1 | 1 | $4^*$ | 2.0 | 2 | 0.02 | 0.04 | 0.1 | 1 |
| 5 | 2.0 | 2 | 0.18 | 0 | 0.3 | 1 | $5^*$ | 2.0 | 2 | 0.18 | 0.36 | 0.3 | 1 |
| 6 | 2.0 | 2 | 0.0125 | 0 | 0.25 | 2 | $6^*$ | 2.0 | 2 | 0.0125 | 0.025 | 0.25 | 2 |
| 7 | 2.0 | 2 | 0.05 | 0 | 0.5 | 2 | $7^*$ | 2.0 | 2 | 0.05 | 0.1 | 0.5 | 2 |
| $4A$ | 2.0 | 2 | 0.02 | 0 | 0.05 | 1 | $4A^*$ | 2.0 | 2 | 0.02 | 0.1 | 0.05 | 1 |
| $4B$ | 2.0 | 2 | 0.02 | 0 | 0.01 | 1 | $4B^*$ | 2.0 | 2 | 0.02 | 0.05 | 0.01 | 1 |
| $4C$ | 2.0 | 2 | 0.02 | 0 | 0.005 | 1 | $4C^*$ | 2.0 | 2 | 0.02 | 0.01 | 0.005 | 1 |
| $4D$ | 2.0 | 2 | 0.02 | 0 | 0.001 | 1 | $4D^*$ | 2.0 | 2 | 0.02 | 0.005 | 0.001 | 1 |

Test Cases with two stars ($1^{**}, 2^{**}, \ldots$) are the same as above except that $r = \delta = 0$.

$i_0 = 2j_n + 2(j_n + 1)n - 2$. Since we know that $j_0 = j_n$, this completes specifications for the algorithm.

The algorithm is easy to implement in low-level programming languages like C++. This clearly gives an advantage of speed. However, we will still encounter performance issues for higher orders. The problem is that the maximum dimension for the tensors is defined by $i_0 = 2j_n + 2(j_n + 1)n - 2$, where $j_n$ is our required accuracy for expansion in $\tau$ and, if multiplied by $n$, it can easily yield dimensions above 100. In addition, $T^{n-1}$ in the denominator of equation (3.50) will compound the problem because, as $n$ increases, we will have to increase $j_n$ as well. We should remind though – it is sufficient to run this algorithm once. This will give us expansion constants $b_{n,i,j,k}$'s which we can store for repeated use through the formula (3.50). The algorithm for calculating option prices would be just an analytical expression constructed from the Black–Scholes formula $\psi^{(0)}$, exponential functions and finite sums.

### 3.3 Validation of results

Let us now see how our algorithms perform for real-life pricing problems. For benchmarking purpose, we will use the same test cases as in [1]. We present them in Table 1. We should note that $T$ denotes time to maturity and it is not volatility-scaled as before. The results are given for the fixed-strike continuously sampled Asian call valued at time zero and we will calculate them directly with the $\alpha$ expansion algorithm. For the $\beta$ case, we will have to use symmetry relationship (2.3) and we therefore need to make sure that the test cases are relevant for the $\beta$ algorithm. When we apply equation (2.3), there are two things that happen. First, we have to replace $\lambda$ of the floating-strike option with $K/S$ and, for the given test cases, $K/S$ will be close to 1 which makes perfect sense. This parameter is usually chosen to be 1 anyway. The other thing is that we have to swap interest rate with dividends yield but, again, this should not be a problem because they are of the same magnitude and we are testing for them being greater as well as less than one another.

Table 2. *Test results*

| No. | $\alpha_{10}$ $\beta_6$ | Price | $\alpha_{10}$ $\beta_6$ | Price* | $\alpha_{10}$ $\beta_6$ | Price** |
|---|---|---|---|---|---|---|
| 1 | 08 14 | 0.1931737903 | 09 15 | 0.1475616974 | 08 15 | 0.1692015821 |
| 2 | 08 16 | 0.2464156905 | 08 16 | 0.1917469269 | 10 16 | 0.2178147105 |
| 3 | 09 20 | 0.3062203648 | 08 16 | 0.2423155009 | 10 19 | 0.2729242550 |
| 4 | 05 11 | 0.0559860415 | 04 12 | 0.0357853435 | 04 12 | 0.0451430808 |
| 5 | 07 14 | 0.2183875466 | 08 14 | 0.0522606611 | 06 16 | 0.1151881714 |
| 6 | 07 15 | 0.1722687410 | 07 16 | 0.1453078196 | 08 06 | 0.1583800234 |
| 7 | 12 ** | 0.3500952190 | 12 ** | 0.2404951680 | 10 ** | 0.2913151865 |
| 4*A* | 04 11 | 0.0339411768 | 04 11 | 0.0140247437 | 04 11 | 0.0225755301 |
| 4*B* | 03 10 | 0.0199277917 | 03 10 | 0.0001902534 | 02 06 | 0.0045153614 |
| 4*C* | 02 09 | 0.0197357059 | 02 10 | 0.0000003799 | 02 09 | 0.0022576847 |
| 4*D* | 00 00 | 0.0197353227 | 00 00 | 0.0000000000 | 00 07 | 0.0004515372 |
| No. | $\alpha_{10}$ $\beta_6$ | Delta | $\alpha_{10}$ $\beta_6$ | Delta* | $\alpha_{10}$ $\beta_6$ | Delta** |
| 1 | 09 21 | 0.4980939002 | 10 20 | 0.4091308357 | 10 19 | 0.4524927373 |
| 2 | 10 18 | 0.5660494294 | 09 19 | 0.4742428248 | 09 24 | 0.5192587592 |
| 3 | 09 ** | 0.6291244896 | 09 ** | 0.5365266568 | 10 18 | 0.5821693826 |
| 4 | 05 18 | 0.5721077914 | 05 18 | 0.4273173018 | 05 18 | 0.4991286298 |
| 5 | 08 17 | 0.6615412349 | 08 19 | 0.2491029903 | 07 16 | 0.4406882508 |
| 6 | 09 18 | 0.5499955934 | 09 15 | 0.4893201669 | 07 12 | 0.5193605520 |
| 7 | 12 ** | 0.5834996605 | 11 ** | 0.4436099158 | 10 ** | 0.5108971711 |
| 4*A* | 04 ** | 0.6334899176 | 04 ** | 0.3579777783 | 05 ** | 0.4946145273 |
| 4*B* | 04 ** | 0.9490766340 | 04 ** | 0.0406269249 | 03 ** | 0.4910024096 |
| 4*C* | 03 ** | 0.9898046675 | 03 ** | 0.0002598102 | 03 ** | 0.4905508737 |
| 4*D* | 00 ** | 0.9900663347 | 00 ** | 0.0000000000 | 01 ** | 0.4901896441 |

Price* and Price** are results for the Test Cases with stars - ($1^* - 4D^*$) and ($1^{**} - 4D^{**}$).
$\alpha_{10}$ and $\beta_6$ columns show the minimum number of $\alpha$ and $\beta$ expansion terms required to calculate 10 and six significant digits, respectively.

For the $\alpha$ algorithm, test cases $1-7$ cover parameter values commonly used in practice and $4A - 4D$ give us better view of the low-volatility scenarios. Then, we want to test for dividends yield being higher than interest rate and we introduce test cases $1^* - 4D^*$. In contrast, for the $\beta$ algorithm, it all works the other way. Test cases $1^* - 7^*$ will cover parameter values commonly used in practice (for test purposes, these would be equivalent to the cases with no dividends and interest rate being equal to $\delta - r$). While test cases $1 - 4D$, on the other hand, are equivalent to having dividends yield higher then interest rate. Finally, two star test cases have the same interpretation for both algorithms – they are for the case of interest rate being equal to dividends yield.

We have results presented in Table 2. For all data points, we show the first 10 digits after the comma and, in addition, we can see how many $\alpha$ and $\beta$ orders it takes to calculate each result with 10 and six significant digits, respectively. The series converges and the results are precise for all visible digits. The price values exactly match outputs from all non-asymptotic algorithms presented in [1] including V. Linetsky's [6] algorithm which is known for its high accuracy. We have also used a contour integral Mathematica

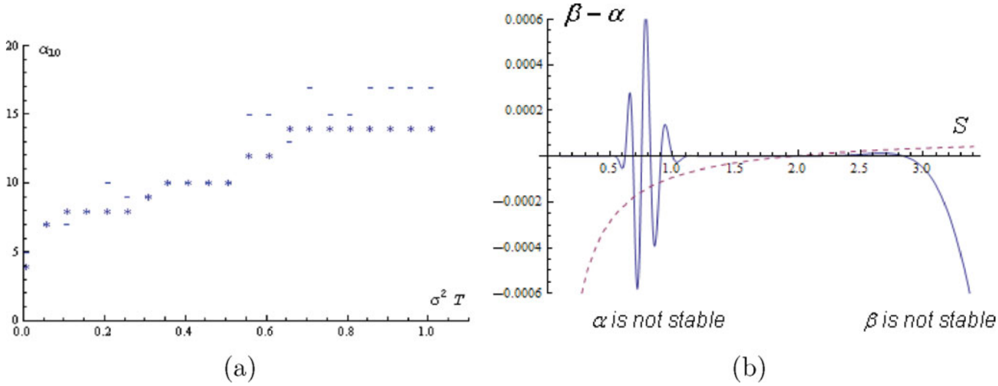


FIGURE 1. (Colour online) Test Case 2: (a) number of $\alpha$ expansion terms required to achieve 10 significant digits of accuracy, '*' is for the price and '-' is for the delta; (b) $\beta$ minus $\alpha$ expansion, the dotted line is $\xi/10000$.

algorithm from [8] to validate both the price and the delta values. Looking at the number of expansion orders, we can see that the $\alpha$ algorithm requires at most 12 expansion orders and this is for the case of high volatility ($\sigma = 0.5$) and long maturity ($T = 2$). For low-volatility cases $4A - 4D$, we can see that it takes only four $\alpha$ orders to converge. Our algorithms therefore seem to work better when volatility is low and when we are closer to maturity. This was expected because we had PDEs evolving along volatility-scaled time $\tau$ which means that low $\sigma$ and $T$ would reduce approximation error by taking us closer to the initial conditions. The convergence does not seem to be impacted much if we vary $r$ or/and $\delta$. Finally, for the $\beta$ case, we can see similar patterns except that we only manage to achieve six significant digits and, for the high-volatility Test Cases 7 and $7^{**}$, we could only get four. Also, the $\beta$ algorithm failed to achieve six digits accuracy on many test cases when calculating option's delta.

While algorithms perform well for the Test Cases, we know that they would not be stable for extreme values of our parameters. For example, let us see what happens if we dramatically change the ratio $S/K$. For the fixed-strike Asian call, this is equivalent to moving deep into or out of the money region. Figure 1(b) shows what happens. Looking at the difference between $\alpha$ and $\beta$ algorithms, we can see two regions of instability. For the deep out of the money case, the $\alpha$ expansion fails. We can see an oscillating error approximately when our spatial parameter $\xi$ (dotted line) drops below –1. Increasing number of expansion orders will make the frequency and amplitude of oscillations grow. The series simply does not converge. This makes sense. We have originally introduced $\alpha$ parameter in equation (3.1) as a multiplier in the form of $\xi$ and it is natural to expect that, for high $|\xi|$, the evolution of our solution across time coordinate gets very sensitive to even small changes in $\alpha$. This would imply that high orders of $\alpha$ expansion might start contributing more to the dynamics of the solution leading to the explosion in the series. The region of instability does not go all the way to zero and this is because we have the exponential function which, for very high $|\xi|$, seems to dominate everything that multiplies it. We have not detected any instabilities for deep in the money case and since this is the region of low $|\xi|$'s there is no reason to expect any.

Let us now look at the region of $\beta$ instability. The ratio of $K/S$ is substituted in the $\beta$ algorithm instead of its parameter $\lambda$ which is a multiplier in front of the floating strike $S_T$. The algorithm fails when $\lambda$ is small. Looking at the form of our expansion orders in $\beta$ (3.49), we can see that $\lambda$ impacts our solution in a very trivial way. Its $n-1$ power is included in the denominator of each $\beta$ expansion order. Smaller $\lambda$ therefore induces growth of exponential type and the series explodes. This actually suggests that the convergence of the $\beta$ algorithm is of a geometric progression type. We have found no instabilities for low values of $S$ and, since this is the region of high $\lambda$'s, there is no reason to expect any.

If we dramatically increase $\sigma$ and/or $r$, both regions of instabilities will expand and the length of the overlapping convergence region will (visibly) tend to zero.

### 3.4 Conclusion

We have derived Taylor expansion algorithms for the fixed-strike Asian call and floating-strike Asian put. The expansions are by made-up variables $\alpha$ and $\beta$ introduced in a parabolic PDE with dimensionless parameters. For both cases, we have a generic algorithm computing an expansion order from two preceding ones. It requires taking expectations with respect to a normally distributed random variable and integrating over time. We have simplified it further for the fixed-strike Asian call case, where we have an explicit functional form with respect to our spatial coordinate and the algorithm only requires integration over time.

For the special case of continuously sampled options, we have numeric algorithms computing Taylor expansion constants and there is no need for any analytical transformations. The algorithm is particularly effective for the case of the fixed-strike Asian call where it is feasible to obtain expansion orders sufficient to satisfy any realistic requirement for accuracy. For the floating-strike Asian put, accuracies up to six significant digits can easily be obtained as well. In any case, we need to compute Taylor expansion constants only once and the option price is then essentially given by analytical expressions constructed from a cumulative distribution function, an exponential function and finite sums.

When we value options at time zero, we have some empirical evidence to suggest that the algorithms are complementary. If price of the underlying is extremely high or low, the solutions may not be stable. Nevertheless, if we are sufficiently close to maturity, it seems that the algorithms will complement each other in such a way that if one fails the other will provide even better approximation.

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