Canad. Math. Bull. Vol. 39 (2), 1996 pp. 241–249

NONNEGATIVE LINEARIZATION AND QUADRATIC TRANSFORMATION OF ASKEY-WILSON POLYNOMIALS

RYSZARD SZWARC

ABSTRACT. Nonnegative product linearization of the Askey-Wilson polynomials is shown for a wide range of parameters. As a corollary we obtain Rahman's result on the continuous q-Jacobi polynomials with $\alpha \geq \beta > -1$ and $\alpha + \beta + 1 \geq 0$.

1. **Introduction.** If $\{p_n\}_{n=0}^{\infty}$ is a system of polynomials such that p_n is a polynomial of degree n, then every polynomial can be expressed as a linear combination of finitely many members of $\{p_n\}_{n=0}^{\infty}$. In particular this applies to all the products p_np_m. In this way we get

$$p_n(x)p_m(x) = \sum c(n, m, k)p_k(x). \tag{1}$$

The numbers $c(n, m, k)$ are called the *linearization coefficients*. Formula (1) is called the *product linearization*.

When $\{p_n\}_{n=0}^{\infty}$ is a system of orthogonal polynomials, then the linearization coefficient $c(n, m, k)$ can be computed from (1) as the integral of the triple product $p_np_mp_k$. Sometimes it can be computed explicitly, like in the case of ultraspherical polynomials (see [2, 4]). Usually, as for the Jacobi polynomials, explicit formulas expressing $c(n, m, k)$ are not available.

It is of great interest to determine whether the linearization coefficients are nonnegative. This property has many important consequences. It gives rise to a convolution structures associated with the polynomials p_n, and opens up the posssibility of applying Banach algebra techniques in the study of orthogonal polynomials. We address the reader to [5, 11] for more details.

In 1970 Richard Askey [1] found a set of conditions that imply nonnegative product linearization. His conditions are given in terms of the coefficients in the three term recurrence formula orthogonal polynomials satisfy. Askey's result could be applied to a wide class of polynomials, including the Jacobi polynomials and their q-analogs. However in the case of the Jacobi polynomials it does not give the whole range of parameters for which nonnegative linearization was known to hold. 22 years later in [11] new conditions were found which applied to the Jacobi polynomials come close to the actual range, found by Gasper [5, 6], where the nonnegative linearization hold. It gives the exact range

Research supported by KBN (Poland) under grant 2 PO3A 030 09.
Received by the editors February 16, 1995.
AMS subject classification: 33D45.
Key words and phrases: orthogonal polynomials, Askey-Wilson polynomials, nonnegative linearization.

for the ultraspherical polynomials. On the way these results imply nonnegative product linearization also for the associated polynomials.

All the results mentioned above assume certain monotonicity conditions of the coefficients in the three term recurrence formula. Sometimes these coefficients are too complicated the expressions to apply the results directly. For example in [11] instead of examining the Jacobi polynomials, we studied the so called generalized Chebyshev polynomials and then used the relation between these polynomials and the Jacobi polynomials. The advantage of doing so is that the recurrence formula for the generalized Chebyshev polynomials is much simpler than the one for the Jacobi polynomials.

The Askey-Wilson polynomials $p_n(x; a, b, c, d \mid q)$ contain the continuous q-Jacobi polynomials as a special case and the Jacobi polynomials as a limit case. We construct the system of polynomials $q_n(x; a, b, c, d \mid q)$ related to the Askey-Wilson in the same way as the generalized Chebyshev polynomials are related to the Jacobi polynomials. The recurrence formula for the polynomials $q_n(x; a, b, c, d \mid q)$ turns up to be much simpler then the one for Askey-Wilson polynomials. Using results of [11] we give conditions on the parameters a, b, c and d which imply nonnegative product linearization of the polynomials $q_n(x; a, b, c, d \mid q)$, and thus of the polynomials $p_n(x; a, b, c, d \mid q)$ taking into account the relationship between these two classes of polynomials.

As a corollary we obtain Rahman's result [10] on the continuous q-Jacobi polynomials with $\alpha \geq \beta > -1$ and $\alpha + \beta + 1 \geq 0$.

Recently Koornwinder obtained nonnegative product linearization for the Askey-Wilson polynomials $p_n(x; q^{\frac{1}{2}}, q^{\frac{1}{2}+\sigma}, q^{\frac{1}{2}}, -q^{\frac{1}{2}-\sigma} \mid q)$. He used quantum groups theoretic methods. The polynomials show up as spherical matrix coefficients of irreducible representations of quantum groups. The linearization coefficients are then positive multiples of the multiplicities of the irreducible representations in the decomposition of tensor product of two such representations. In [9] Koornwinder calls for an analytic proof of his result.

It turns out that the theorems of Section 2 cannot be applied to these polynomials for any nonzero value of σ. That is why, in Section 3, we derive other criteria that give nonnegative product linearization for the above polynomials but with the restriction $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$.

ACKNOWLEDGEMENT. I am grateful to Tom Koornwinder for informing me of his paper and sending me a preprint.

2. **Askey-Wilson polynomials and quadratic transformation.** The Askey-Wilson polynomials are given in terms of the basic hypergeometric series ${}_4\phi_3$. This function is a q-analog of the generalized hypergeometric series ${}_4F_3$ and is defined as

$$ {}_4\phi_3\left[\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix}; q, x\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, a_3, a_4; q)_k}{(b_1, b_2, b_3)_k} \frac{x^k}{(q;q)_k}, $$

where

$$ (a_1, a_2, \ldots, a_n; q)_k = \prod_{i=1}^{n} \prod_{j=0}^{k-1} (1 - a_i q^j). $$

The Askey-Wilson polynomials are defined by

$$p_n(x; a, b, c, d \mid q) = a^{-n}(ab, ac, ad; q)_n \, {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right], \tag{2}$$

where $x = \cos\theta$.

It turns out that the polynomials $p_n(x; a, b, c, d \mid q)$ are invariant for the permutations of the parameters a, b, c, d. These polynomials first appeared in [3], where the recurrence relation and orthogonality measure have been computed explicitly. According to Rahman [10], setting $a = q^{\alpha+\frac{1}{2}}$, $b = -q^{\beta+\frac{1}{2}}$ and $c = -d = q^{\frac{1}{2}}$ yields the continuous q-Jacobi polynomials.

The Askey-Wilson polynomials satisfy the recurrence relation

$$2x\tilde{p}_n(x) = A_n\tilde{p}_{n+1}(x) + [a + a^{-1} - (A_n + C_n)]\tilde{p}_n(x) + C_n\tilde{p}_{n-1}(x), \tag{3}$$

where

$$\tilde{p}_n(x) = \frac{a^n p_n(x; a, b, c, d \mid q)}{(ab, ac, ad; q)_n} \tag{4}$$

and

$$\begin{cases} A_n = \dfrac{(1 - abcdq^{n-1})(1 - abq^n)(1 - acq^n)(1 - adq^n)}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})}, \\ C_n = \dfrac{a(1 - q^n)(1 - bcq^n)(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})}. \end{cases} \tag{5}$$

The polynomials $\tilde{p}_n(x)$ are the Askey-Wilson polynomials normalized at the point $\frac{1}{2}(a + a^{-1})$, *i.e.*,

$$\tilde{p}_n\Big(\frac{1}{2}(a + a^{-1})\Big) = 1.$$

The tilded polynomials are no longer invariant for the permutations of a, b, c, d, unless the parameter a is fixed by the permutation. An advantage of dealing with $\tilde{p}_n(x)$ is the fact that the sum of the coefficients in the recurrence formula (3) is constant and equal to $a + a^{-1}$.

Let $q_n(x; a, b, c, d \mid q)$ be the polynomials defined by the recurrence relation

$$\begin{aligned} 2xq_n(x; a, b, c, d \mid q) = \gamma_n q_{n+1}(x; a, b, c, d \mid q) \\ + \alpha_n q_{n-1}(x; a, b, c, d \mid q), \end{aligned} \tag{6}$$

where

$$\begin{cases} \alpha_{2n} = -ab\dfrac{(1 - q^n)(1 - cdq^{n-1})}{1 - abcdq^{2n-1}}, \\ \gamma_{2n} = \dfrac{(1 - abcdq^{n-1})(1 - abq^n)}{1 - abcdq^{2n-1}}, \\ \alpha_{2n+1} = -\dfrac{a(1 - bcq^n)(1 - bdq^n)}{b(1 - abcdq^{2n})}, \\ \gamma_{2n+1} = \dfrac{(1 - acq^n)(1 - adq^n)}{(1 - abcdq^{2n})}. \end{cases} \tag{7}$$

The polynomials $q_n(x; a, b, c, d \mid q)$ are invariant for the following transformation of the parameters.

$$q_n(x; a, b, c, d \mid q) = q_n(x; -a, -b, -c, -d \mid q) \tag{8}$$

$$q_n(x; a, b, c, d \mid q) = q_n(x; a, b, d, c \mid q). \tag{9}$$

Thus we can always assume that a is positive. Observe that we have

$$\alpha_{2n} + \gamma_{2n} = 1 - ab \text{ and } \alpha_{2n+1} + \gamma_{2n+1} = 1 - ab^{-1}. \tag{10}$$

A key point is a relation between the polynomials $q_{2n}(x; a, b, c, d \mid q)$ and the Askey-Wilson polynomials $\tilde{p}_n(x; a, b, c, d \mid q)$ normalized at $\frac{1}{2}(a + a^{-1})$. Namely we have

$$q_{2n}(x; a, b, c, d \mid q) = \tilde{p}_n\Big(2a^{-1}x^2 + \frac{1}{2}(b + b^{-1}); a, b, c, d \mid q\Big). \tag{11}$$

This is because by using (6) twice we get

$$\begin{aligned} 4x^2 q_{2n}(x) &= \gamma_{2n+1}\gamma_{2n}q_{2n+2}(x) \\ &\quad + (\alpha_{2n+2}\gamma_{2n+1} + \alpha_{2n+1}\gamma_{2n})q_{2n}(x) + \alpha_{2n}\alpha_{2n-1}q_{2n-2}(x), \end{aligned} \tag{12}$$

where $q_n(x) = q_n(x; a, b, c, d \mid q)$. Next, observe that (5), (7) and (10) imply

$$\begin{aligned} aA_n &= \gamma_{2n+1}\gamma_{2n}, \\ aC_n &= \alpha_{2n}\alpha_{2n-1}, \\ (1 - ab)(1 - ab^{-1}) &= \gamma_{2n+1}\gamma_{2n} + (\alpha_{2n+2}\gamma_{2n+1} + \alpha_{2n+1}\gamma_{2n}) + \alpha_{2n}\alpha_{2n-1}. \end{aligned}$$

Therefore (12) can be written as

$$4x^2 q_{2n}(x) = aA_n q_{2n+2}(x) + a[(1 - ab)(a^{-1} - b^{-1}) - (A_n + C_n)]q_{2n}(x) + aC_n q_{2n-2}(x).$$

This together with (3) immediately gives (11)

In the same way one can derive the relation

$$x^{-1}q_{2n+1}(x; a, b, c, d \mid q) = 2(1 - ab)^{-1}\tilde{p}_n\Big(2a^{-1}x^2 + \frac{1}{2}(b + b^{-1}); a, bq, c, d \mid q\Big).$$

In particular combining (4), (6), (11) and (13) gives

$$\begin{aligned} p_n&(x; a, b, c, d \mid q) \\ &= \frac{1 - abcdq^{n-1}}{1 - abcdq^{2n-1}} p_n(x; a, bq, c, d \mid q) \\ &\quad - \frac{b(1 - q^n)(1 - acq^{n-1})(1 - adq^{n-1})(1 - cdq^{n-1})}{1 - abcdq^{2n-1}} p_{n-1}(x; a, bq, c, d \mid q) \end{aligned} \tag{14}$$

and

$$\begin{aligned} 2[x - \frac{1}{2}(b + b^{-1})]&p_n(x; a, bq, c, d \mid q) \\ &= \frac{1}{1 - abcdq^{2n}} p_{n+1}(x; a, b, c, d \mid q) \\ &\quad - \frac{(1 - baq^n)(1 - bcq^n)(1 - bdq^n)}{b(1 - abcdq^{2n})} p_n(x; a, b, c, d \mid q). \end{aligned} \tag{15}$$

https://doi.org/10.4153/CMB-1996-031-x Published online by Cambridge University Press

THEOREM 1. *Let a, b, c, d and q satisfy*
(i) $0 \leq q < 1$,
(ii) $ac < 1, ad < 1, bc < 1, bd < 1$,
(iii) $a > 0, b < 0$ *and* $cd < 0$,
(iv) $a + b \leq 0$ *and* $c + d \leq 0$,
(v) $ab + 1 \geq 0$ *and* $cd + q \geq 0$.
Then the polynomials $q_n(x;\ a,\ b,\ c,\ d \mid q)$ *have nonnegative product linearization.*

Using (3), (11) and the fact that the Askey-Wilson polynomials $p_n(x; a, b, c, d \mid q)$ are invariant for the permutation of a, b, c and d, gives

THEOREM 2. *Let the parameters a, b, c, d and q satisfy*
(i) $0 \leq q < 1$,
(ii) $ac < 1,\ ad < 1,\ bc < 1,\ bd < 1$,
(iii) $ab < 0$ *and* $cd < 0$,
(iv) $a + b \leq 0$ *and* $c + d \leq 0$,
(v) $ab + 1 \geq 0$ *and* $cd + q \geq 0$.
Then the Askey-Wilson polynomials $p_n(x;\ a,\ b,\ c,\ d \mid q)$ *have nonnegative product linearization.*

Observe that when $a = q^{\alpha+\frac{1}{2}}$, $b = -q^{\beta+\frac{1}{2}}$, and $c = -d = q^{\frac{1}{2}}$ the assumptions of Theorem 1 are satisfied if and only if $\alpha \geq \beta > -1$ and $\alpha + \beta + 1 \geq 0$. The polynomials $p_n(x; a, b, c, d \mid q)$ corresponding to this choice of parameters are called the continuous q-Jacobi polynomials and denoted by $P_n^{(\alpha,\beta)}(x; q)$. Thus Theorem 2 yields the following.

COROLLARY 1 (RAHMAN [10]). *Let* $0 \leq q < 1$. *The continuous q-Jacobi polynomials* $P_n^{(\alpha,\beta)}(x;\ q)$ *have nonnegative product linearization if* $\alpha \geq \beta > -1$ *and* $\alpha+\beta+1 \geq 0$.

PROOF OF THEOREM 1. The assumptions (ii) and (iii) of the theorem imply that the coefficients α_n and γ_n defined by (7) are positive. Thus by the Favard theorem the polynomials $q_n(x; a, b, c, d \mid q)$ are orthogonal polynomials. We are going to apply [11, Thm. 1, p. 966]. This theorem states that nonnegativity of the product linearization holds if

$$\begin{cases} \alpha_n \leq \alpha_{n+2}, \\ \alpha_n \leq \gamma_n \\ \alpha_n + \gamma_n \leq \alpha_{n+2} + \gamma_{n+2}, \end{cases}$$

where α_n and γ_n are the coefficients in the recurrence formula (7) satisfied by the polynomials $q_n(x; a, b, c, d \mid q)$.

It can be computed that by setting $z = q^n$ we get

$$\begin{aligned} &\alpha_{2n+2} - \alpha_{2n} \\ &\quad = -ab(1-q)z\frac{abcd(cd+q)z^2 - q(1+q)cd(ab+1)z + q(cd+q)}{(q-abcdz^2)(1-abcdqz^2)}, \end{aligned} \tag{16}$$

$$\begin{aligned} &\alpha_{2n+3} - \alpha_{2n+1} \\ &\quad = -a(1-q)z\frac{abcdq(c+d)z^2 - (1+q)cd(a+b)z + (c+d)}{(1-abcdz^2)(1-abcdq^2z^2)}. \end{aligned} \tag{17}$$

By assumptions we have $0 < -ab \leq 1$ and $0 < -cd \leq q$. Hence $0 < abcd \leq q$. Thus in both formulas the denominators are nonnegative. One can verify that under the assumptions (i)–(iv) also the numerators are nonnegative. Hence we obtain

$$\alpha_{2n} \nearrow -ab, \quad \text{and} \quad \alpha_{2n+1} \nearrow -ab^{-1}.$$

Moreover, in view of (10) we have

$$\gamma_{2n} \searrow 1 \quad \text{and} \quad \gamma_{2n+1} \searrow 1.$$

By (iv) we get $0 < -ab \leq 1$. Dividing (iii) by a^2 gives also $0 < -ab^{-1} \leq 1$. Therefore $\alpha_n \leq \gamma_n$. This completes the proof. ∎

3. **Polynomials considered by Koornwinder.** In [9] Koornwinder showed by quantum group theoretic methods that the Askey-Wilson polynomials $p_n(x;\, q^{\frac{1}{2}},\, -q^{\frac{1}{2}+\sigma},\, q^{\frac{1}{2}},\, -q^{\frac{1}{2}-\sigma} \mid q)$ have nonnegative product linearization for any real value of σ. Unfortunately Theorem 2 can be applied here only in the case $\sigma = 0$. Therefore we have to resort to other methods.

We will give an analytic proof of Koornwinder's result for $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$. Incidentally this is the case when the corresponding orthogonality measure does not admit mass points (see [3, Thm. 2.2]). We were not able to use our methods in the case $|\sigma| > \frac{1}{2}$.

In view of further applications the assumptions in the next proposition are made as weak as possible.

PROPOSITION 1. *Let the numbers q, a, b, c and d satisfy*

(i) $0 \leq q < 1$,

(ii) $ac < 1,\ ad < 1,\ bc < 1,\ bd < 1$,

(iii) $a > 0,\ b < 0,\ cd < 0,\ |b| \leq 1$,

(iv) $abcd(cd+q)z^2 - (1+q)cd(ab+1)z + (cd+q) \geq 0$,
$abcdq(c+d)z^2 - (1+q)cd(a+b)z + (c+d) \geq 0$, *where* $z = q^n$,

(v) $s_1 \leq 0,\ \sqrt{abcd}\,s_1 \leq s_2$, *where*

$$s_1 = (ab+q)(c+d) + (a+b)(cd+q), \tag{18}$$

$$s_2 = (a+b)(ab+q)cd + (c+d)(cd+q)ab. \tag{19}$$

Then the polynomials $q_n(x;\, a, b, c, d \mid q)$ have nonnegative product linearization.

PROOF. We will apply Theorem 3(i) of [12]. To this end we have to consider the orthonormal polynomials.

Since the parameters a and b have opposite signs the polynomials $q_n(x;\, a, b, c, d \mid q)$ are orthogonal polynomials. The corresponding orthonormal polynomials $\hat{q}_n(x;\, a, b, c, d \mid q)$ satisfy the recurrence relation

$$\begin{aligned} x\hat{q}_n(x;\, a, b, c, d \mid q) = {} & \lambda_n \hat{q}_{n+1}(x;\, a, b, c, d \mid q) \\ & + \lambda_{n-1}\hat{q}_{n-1}(x;\, a, b, c, d \mid q), \end{aligned} \tag{20}$$

where

$$\lambda_n^2 = \alpha_{n+1}\gamma_n \tag{21}$$

and α_n, γ_n are defined by (7). We will show that the sequences λ_{2n-1}^2 and $\lambda_{2n-1}^2 + \lambda_{2n}^2$ are nondecreasing, and $\lambda_{2n-1}^2 \le \lambda_{2n}^2$. Then Theorem 3(i) of [12] will give the conclusion.

By (ii) and (iii) the coefficients α_n and γ_n are positive, so are λ_n. Let

$$h(n) = \lambda_{2n-1}^2 + \lambda_{2n}^2. \tag{22}$$

Then it can be computed that

$$h(n) = az\frac{s_1 abcdz^2 - (1+q)s_2 z + qs_1}{(1-abcdz^2)(q^2-abcdz^2)} - a(b+b^{-1}), \tag{23}$$

where $z = q^n$ and s_1, s_2 are given by (18) and (19). Then

$$\begin{aligned} h(n) &= as_1 z\frac{abcdz^2 - (1+q)\sqrt{abcd}z + q}{(1-abcdz^2)(q^2-abcdz^2)} \\ &\quad + \frac{a(1+q)(\sqrt{abcd}s_1 - s_2)z^2}{(1-abcdz^2)(q^2-abcdz^2)} - a(b+b^{-1}) \\ &= \frac{as_1 z}{(1+\sqrt{abcd}z)(q+\sqrt{abcd}z)} \\ &\quad + \frac{a(1+q)(\sqrt{abcd}s_1 - s_2)z^2}{(1-abcdz^2)(q^2-abcdz^2)} - a(b+b^{-1}). \end{aligned}$$

The sequence

$$\frac{z^2}{(1-abcdz^2)(q^2-abcdz^2)}$$

is obviously positive and nonincreasing. So is the sequence

$$k(n) = \frac{z}{(1+\sqrt{abcd}z)(q+\sqrt{abcd}z)},$$

because

$$k(n+1) - k(n) = -\frac{z(1-q)(1-\sqrt{abcd}z)}{(1+\sqrt{abcd}z)(q+\sqrt{abcd}z)(1+q\sqrt{abcd}z)}.$$

Hence the sequence $h(n)$ is nondecreasing beacuse by our assumptions

$$s_1 \le 0 \text{ and } \sqrt{abcd}s_1 - s_2 \le 0.$$

In view of (10) the sequence $\lambda_{2n-1}^2 = \alpha_{2n}\gamma_{2n-1}$ is nondecreasing if the sequence α_{2n} is nondecreasing and α_{2n-1} is nonincreasing. By (16) and (17) this occurs exactly when

$$abcd(cd+q)z^2 - q(1+q)cd(ab+1)z + (cd+q) \ge 0$$
$$abcdq(c+d)z^2 - (1+q)cd(a+b)z + (c+d) \ge 0$$

We also get that $\lambda_{2n}^2 = \alpha_{2n+1}\gamma_{2n}$ is nonincreasing. Thus since $ab < 0$ and $|b| \le 1$ we obtain

$$\lambda_{2n-1}^2 \nearrow -ab \le -\frac{b}{a} \swarrow \lambda_{2n}^2.$$

In this way all the assumptions of [12, Thm. 3(i)] are satisfied. This completes the proof of the theorem. ∎

THEOREM 3. *With the assumptions of Theorem 1 the Askey-Wilson polynomials* $p_n(x;\ a, b, c, d \mid q)$ *have nonnegative product linearization.*

THEOREM 4. *Let the numbers* q, a, b, c *and* d *satisfy*

(i) $0 \le q < 1$,
(ii) $ac < 1$, $ad < 1$, $bc < 1$, $bd < 1$,
(iii) $a > 0$, $b < 0$, $cd < 0$, $|b| \le 1$,
(iv) $cd + q \ge 0$, $ab + 1 \ge 0$, $c + d \ge 0$, $a + b + c + d \le 0$,
$(abcdq + 1)(c + d) \ge (1 + q)cd(a + b)$,
(v) $s_1 \le 0$, $\sqrt{abcd}s_1 \le s_2$,
where s_1 *and* s_2 *are given by (18) and (19).*

Then the polynomials $q_n(x;\ a, b, c, d \mid q)$ *have nonnegative product linearization.*

PROOF. It suffices to show that the assumption (iv) of Proposition 1 is satisfied. Indeed, the first inequality is obvious since all three terms in the expression

$$abcd(cd + q)z^2 - (1 + q)cd(ab + 1)z + (cd + q)$$

are nonnegative.

Let

$$g(n) = abcdq(c + d)z^2 - (1 + q)cd(a + b)z + (c + d).$$

Then using the fact that $abq \ge -1$ and $c + d \ge 0$ we obtain

$$\begin{aligned} g(n+1) - g(n) &= abcdq(c + d)(q^2 - 1)z^2 - (1 + q)cd(a + b)(1 - q)z \\ &= -cd(1 - q^2)[abq(c + d) - (a + b)] \\ &\ge -cd(1 - q^2)[-(c + d) - (a + b)] \ge 0. \end{aligned}$$

Hence the sequence $g(n)$ is nondecreasing. Thus $g(n) \ge 0$ if and only if $g(0) \ge 0$. But

$$g(0) = (abcdq + 1)(c + d) - (1 + q)cd(a + b) \ge 0.$$

Thus all the assumptions of Proposition 1 are satisfied. ∎

COROLLARY 2 ([KOORNWINDER [9]). *The Askey-Wilson polynomials*

$$p_n(x;\ q^{\frac{1}{2}},\ -q^{\frac{1}{2}+\sigma},\ q^{\frac{1}{2}},\ -q^{\frac{1}{2}-\sigma} \mid q)$$

have nonnegative product linearization for $0 < q < 1$ *and any real value of* σ *such that* $-\frac{1}{2} \le \sigma \le \frac{1}{2}$.

PROOF. Set $\beta = q^\sigma$ and

$$a = c = q^{\frac{1}{2}}, \quad b = -q^{\frac{1}{2}}\beta^{-1} \quad d = -q^{\frac{1}{2}}\beta.$$

Since $p_n(x;\ a, b, c, d \mid q)$ does not change when we switch b and d we can restrict ourselves to the case $0 \le \sigma \le \frac{1}{2}$, *i.e.*,

$$q^{\frac{1}{2}} \le \beta \le 1.$$

https://doi.org/10.4153/CMB-1996-031-x Published online by Cambridge University Press

Observe that the assumptions of Theorem 4 are satisfied. Indeed, (ii) is obvious. We have $a > 0$, $b < 0$ and $cd < 0$. Also

$$|b| = q^{\frac{1}{2}}\beta^{-1} < 1.$$

This gives (iii). Concerning (iv) we have

$$\begin{aligned} c+d &= q^{\frac{1}{2}}(1-\beta) \geq 0, \\ cd+q &= q(1-\beta) \geq 0, \\ ab+1 &= 1-q\beta^{-1} \geq 0, \\ a+b+c+d &= q^{\frac{1}{2}}(2-\beta-\beta^{-1}) \leq 0. \end{aligned}$$

Moreover using the notation (18) and (19) we get

$$s_2 = qs_1 = -2q^{\frac{5}{2}}\beta^{-1}(1-\beta)^2.$$

Hence $s_1 \leq 0$ and

$$s_2 - \sqrt{abcd}s_1 = 0.$$

The assumptions of Theorem 4 are thus satisfied. ∎

References

1. R. Askey, *Linearization of the product of orthogonal polynomials*, in Problems in Analysis, R. Gunning, ed., Princeton University Press, Princeton, New Jersey, 1970, 223–228.
2. ———, *Orthogonal polynomials and special functions*, Regional Conference Series in Applied Mathematics 21, Society for Industrial and Applied Mathematics, Philadelphia, Pensylwania, 1975.
3. R. Askey and J. A. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **54**(1985).
4. J. Dougall, *A theorem of Sonine in Bessel functions, with two extensions to spherical harmonics*, Proc. Edinburgh Math. Soc. **37**(1919), 33–47.
5. G. Gasper, *Linearization of the product of Jacobi polynomials*, I, Canad. J. Math. **22**(1970), 171–175.
6. ———, *Linearization of the product of Jacobi polynomials. II, ibid.*, **22**(1970), 582–593.
7. ———, *A convolution structure and positivity of a generalized translation operator for the continuous q-Jacobi polynomials*, in Conference on Harmonic Analysis in Honor of Antoni Zygmund, Wadsworth International Group, Belmont, Calif. (1983), 44–59.
8. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge **35**(1990).
9. T. Koornwinder, *Discrete hypergroups associated with compact quantum Gelfand pairs*, in Applications of Hypergroups and Related Measures Algebras, (eds. W. Connett, M. O. Gebuhrer and A. L. Schwartz), Contemporary Mathematics.
10. M. Rahman, *The linearization of the product of continuous q-Jacobi polynomials*, Can. J. Math. **33**(1981), 961–987.
11. R. Szwarc, *Orthogonal polynomials and a discrete boundary value problem, I, II*, SIAM J. Math. Anal. **23**(1992), 959–964, 965–969.
12. R. Szwarc, *Nonnegative linearization and q-ultraspherical polynomials*, Methods Appl. Anal., to appear.

Institute of Mathematics
Wrocław University
pl. Grunwaldzki 2/4
50–384 Wrocław, Poland