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Algebraic independence of multipliers of periodic orbits in the space of polynomial maps of one variable

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Abstract. We consider the space of complex polynomials of degree $n \geq 3$ with $n - 1$ distinct marked periodic orbits of given periods. We prove that this space is irreducible and the multipliers of the marked periodic orbits, considered as algebraic functions on that space, are algebraically independent over $\mathbb{C}$. Equivalently, this means that at its generic point the moduli space of degree-n polynomial maps can be locally parameterized by the multipliers of $n - 1$ arbitrary distinct periodic orbits. We also prove a similar result for a certain class of affine subspaces of the space of complex polynomials of degree n.

1. *Introduction*

A key point in studying moduli spaces of degree-n rational or polynomial maps of the Riemann sphere $\hat{\mathbb{C}}$ is the choice of a parameterization. The idea of using the multipliers of the fixed points of a map as the parameters of the moduli space appears naturally in many works on the subject. Notably, in [**9**] Milnor used the multipliers of the fixed points to parameterize the moduli space of degree-two rational maps. Using this parameterization, he proved that this moduli space is isomorphic to $\mathbb{C}^2$. In [**8**], he also considered a similar parameterization of the moduli space of cubic polynomials. In the polynomial case, local parameterization by the multipliers of the fixed points was also studied, for instance, in [**13**, **14**].

In the following discussion, let M be the moduli space of either rational or polynomial maps of degree n. As a generalization of the approach described above, instead of the multipliers at the fixed points one can try to use the multipliers of periodic orbits as the local parameters on the moduli space M. It is not hard to see that the map from M to the multipliers of the chosen periodic orbits is defined in a neighborhood of a generic point of M. The main difficulty is to show that this map is a local diffeomorphism when the number of the chosen periodic orbits is equal to the dimension of M. Since multipliers are (multiple-valued) algebraic maps on M, this leads to the question whether there exist 'hidden' algebraic relations between the multipliers of the chosen periodic orbits. In other words, are the chosen multipliers algebraically independent over $\mathbb{C}$, if we view those multipliers as (multiple-valued) functions on M?

In [**7**], McMullen proved that if $n \geq 2$ and M is the moduli space of degree-n rational maps then, except for the flexible Lattès maps, an element of M is determined up to finitely many choices by the multipliers of *all* of its periodic orbits. This implies that one can always choose $2n - 2 = \dim(M)$ distinct periodic orbits whose multipliers, considered as (multiple-valued) functions on M, are algebraically independent over $\mathbb{C}$.

In this paper we focus on the case when M is the moduli space of degree-n polynomial maps. The question about algebraic independence of multipliers of periodic orbits in the polynomial case was asked by Ilyashenko in the context of studying the Kupka–Smale property for volume preserving polynomial automorphisms of $\mathbb{C}^2$ (see [**3**] for a detailed discussion). We notice that the dimension of the moduli space of degree-n polynomials is equal to $n - 1$, so any collection of n or more multipliers will be algebraically dependent. Our main result completely answers the question in the less trivial case, namely, when the number of multipliers is less than n.

THEOREM 1.1. *The multipliers of any $n - 1$ distinct periodic orbits, considered as algebraic (multiple-valued) maps on the space of degree-n polynomials, are algebraically independent over* $\mathbb{C}$.

The best previous result in this direction belongs to Zarhin [**15**], who proved the independence of multipliers under the additional condition that the sum of the periods does not exceed $n - 1$, or the sum of the periods does not exceed n and none of the periodic orbits is a fixed point.

We prove Theorem 1.1 in the following way: we consider the space of complex polynomials of degree $n \geq 2$ with $n - 1$ distinct marked periodic orbits of given periods. This space is a ramified cover over the space of polynomials of degree n. First, in Lemma 1.5, we prove that this space is an irreducible algebraic set. The multipliers that we consider are algebraic functions on this set. Then, in Theorem 1.7, we show that at some point on this set the differentials of the multipliers are linearly independent, which implies the desired algebraic independence of the multipliers.

As a byproduct of this proof, we obtain a similar result about independence of multipliers, when the multipliers are viewed as algebraic functions on certain affine subspaces of the space of polynomials of degree n. In a forthcoming work [**4**], we also extend the methods of the current paper to study independence of multipliers in the space of degree-n rational maps.

Finally, let us mention that a wholly different approach to the subject is given by Epstein's transversality principles in holomorphic dynamics [**2**]. The techniques used in this paper are not directly related, and it is an intriguing question whether our results could be obtained using Epstein's methods.

1.1. *The space of polynomials with k marked periodic orbits.* It is easy to see that even though the change of coordinates $z \mapsto z + c$ in the domain of the polynomial modifies the polynomial, it only shifts all periodic points by c and does not change their multipliers. Thus, we will consider two polynomials to be equivalent if there exists a change of coordinates of the form $z \mapsto z + c$ that brings one polynomial to another. The factor space of all monic polynomials of degree n factored by this equivalence relation can be identified with so-called centered polynomials.

Definition 1.2. By $\mathcal{P}^n \subset \mathbb{C}[z]$ we denote the set of centered monic polynomials of degree n, which are monic polynomials of degree n with the term of degree $n - 1$ being equal to zero.

Definition 1.3. We say that a periodic orbit of a polynomial p is of period m if this periodic orbit consists of m *distinct* points. A point of period m is a point that belongs to a periodic orbit of period m.

Let us fix a positive integer k. By $\mathbf{m}$ we denote the vector of periods $\mathbf{m} = (m_1, \ldots, m_k)$, where $m_1, \ldots, m_k$ are positive integers. Let $N_{\mathbf{m}}^n \subset \mathcal{P}^n \times \mathbb{C}^k$ be an algebraic set that consists of all points $(p, z_1, \ldots, z_k) \in \mathcal{P}^n \times \mathbb{C}^k$, such that $p^{\circ m_j}(z_j) = z_j$, for all $j = 1, 2, \ldots, k$. By $\pi : N_{\mathbf{m}}^n \to \mathcal{P}^n$ we denote the natural projection

$$\pi : (p, z_1, \ldots, z_k) \mapsto p.$$

Consider a polynomial $p \in \mathcal{P}^n$ and its k periodic points $z_1, \ldots, z_k$ belonging to *different* periodic orbits of (minimal) periods $m_1, \ldots, m_k$, respectively. We notice that the chosen periodic points are non-multiple if and only if $(p, z_1, \ldots, z_k) \in N_{\mathbf{m}}^n$ is a regular point of the projection π and any regular point of the projection π belongs to a unique irreducible component of the set $N_{\mathbf{m}}^n$. This means that with any such polynomial $p \in \mathcal{P}^n$ and its non-multiple periodic points $z_1, \ldots, z_n$ belonging to different periodic orbits, one can associate the set $M_{\mathbf{m}}^n$ defined in the following way.

Definition 1.4. The set $M_{\mathbf{m}}^n = M_{\mathbf{m}}^n(p, z_1, \ldots, z_k)$ is the irreducible component of the algebraic set $N_{\mathbf{m}}^n$ that contains the point $(p, z_1, \ldots, z_k)$.

A priori it is not obvious whether the sets $M_{\mathbf{m}}^n$ can be different for different initial choices of $(p, z_1, \ldots, z_k)$. In §5 we will prove the following lemma, which says that all these sets are the same.

LEMMA 1.5. *Given a positive integer $n > 2$ and a vector of periods $\mathbf{m} = (m_1, \ldots, m_k)$, assume that $(p, z_1, \ldots, z_k), (q, w_1, \ldots, w_k) \in N_{\mathbf{m}}^n$ are two regular points of the projection π, such that the points $z_1, \ldots, z_k$ belong to different periodic orbits of p and the points $w_1, \ldots, w_k$ belong to different periodic orbits of q. Then the sets $M_{\mathbf{m}}^n(p, z_1, \ldots, z_k)$ and $M_{\mathbf{m}}^n(q, w_1, \ldots, w_k)$ are equal.*

Because of Lemma 1.5, we are allowed to shorten our notation and write $M_{\mathbf{m}}^n$ instead of $M_{\mathbf{m}}^n(p, z_1, \ldots, z_k)$.

For $k = 1$, the spaces $M_{\mathbf{m}}^n$ and certain subspaces of them were previously studied for instance in [**1**, **5**, **11**, **12**]. Similar spaces for $k = 1$ and rational maps over an arbitrary field instead of polynomials over the field of complex numbers were studied in [**6**].

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Remark 1.6. Lemma 1.5 suggests another way of describing the set $M^n_{\mathbf{m}}$. Namely, the set $M^n_{\mathbf{m}}$ is the closure in $\mathcal{P}^n \times \mathbb{C}^k$ of the set of all points $(p, z_1, \ldots, z_k) \in \mathcal{P}^n \times \mathbb{C}^k$, where $p \in \mathcal{P}^n$ and all z_j are non-multiple periodic points of p belonging to different periodic orbits of corresponding periods m_j.

1.2. *The multiplier map.* We define the map $\Lambda : M^n_{\mathbf{m}} \to \mathbb{C}^k$ that with every point $(p, z_1, \ldots, z_k) \in M^n_{\mathbf{m}}$ associates the vector of multipliers of periodic points $z_1, \ldots, z_k$:

$$\Lambda : (p, z_1, \ldots, z_k) \mapsto ((p^{\circ m_1}(z_1))', (p^{\circ m_2}(z_2))', \ldots, (p^{\circ m_k}(z_k))').$$

If we fix the system of coordinates $\mathbf{a} = (a_0, \ldots, a_{n-2})$ in $\mathcal{P}^n$, naturally related to the coefficients of the polynomials so that a vector $\mathbf{a} = (a_0, \ldots, a_{n-2}) \in \mathbb{C}^{n-1}$ is identified with the polynomial $p_{\mathbf{a}}(z) = z^n + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots + a_1 z + a_0$, then coordinates $\mathbf{a} = (a_0, \ldots, a_{n-2})$ can be viewed as local coordinates on $M^n_{\mathbf{m}}$ around every point $(q, z_1, \ldots, z_k) \in M^n_{\mathbf{m}}$, which is not a critical point of the projection π. In particular, at every such point we can consider the derivative $d\Lambda/d\mathbf{a} = d\Lambda(p_{\mathbf{a}}, z_1(a), \ldots, z_k(a))/d\mathbf{a}$.

Now we are ready to formulate our main theorem.

THEOREM 1.7. *Assume that $n \geq 2$ and $k \leq n-1$. Then, for any vector $\mathbf{m} \in \mathbb{N}^k$ and any k-tuple of indexes $j_1, \ldots, j_k$ satisfying $0 \leq j_1 < j_2 < \cdots < j_k \leq n-2$, the equation*

$$\det\left(\frac{d\Lambda}{d(a_{j_1}, \ldots, a_{j_k})}\right) = 0$$

defines a subset of $M^n_{\mathbf{m}}$ that is contained in an algebraic subset of codimension one in $M^n_{\mathbf{m}}$.

As explained in the Introduction, Theorem 1.1 is an immediate corollary of Theorem 1.7.

In order to formulate another simple corollary, we need the following definition.

Definition 1.8. Assume that $k < n-1$. For $p \in \mathcal{P}^n$ and any k-tuple of indexes $j_1, \ldots, j_k$ satisfying $0 \leq j_1 < j_2 < \cdots < j_k \leq n-2$, denote by $\mathcal{P}^n_{j_1,\ldots,j_k}(p) \subset \mathcal{P}^n$ the affine subspace

$$\mathcal{P}^n_{j_1,\ldots,j_k}(p) = \left\{ p(z) + \sum_{s=1}^{s=k} a_s z^{j_s} \mid a_1, \ldots, a_k \in \mathbb{C} \right\}.$$

COROLLARY 1.9. *For a given $k < n-1$, a vector $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k$ and a k-tuple of indexes $j_1, \ldots, j_k$ satisfying $0 \leq j_1 < j_2 < \cdots < j_k \leq n-2$, there exists a Zariski open subset $S \subset \mathcal{P}^n$, such that for every $p \in S$, the multipliers of any k distinct periodic orbits of corresponding periods $m_1, \ldots, m_k$, viewed as (multiple-valued) functions on $\mathcal{P}^n_{j_1,\ldots,j_k}(p)$, are algebraically independent over $\mathbb{C}$.*

Remark 1.10. Since, according to Definition 1.4, $M^n_{\mathbf{m}}$ is an irreducible algebraic set, in order to prove Theorem 1.7 it is sufficient for every k-tuple of distinct indexes $j_1, \ldots, j_k$ to find a point in $M^n_{\mathbf{m}}$ at which

$$\det\left(\frac{d\Lambda}{d(a_{j_1}, \ldots, a_{j_k})}\right) \neq 0.$$

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2. *Proof of Theorem 1.7 modulo auxiliary results*

The key argument is formulated in the following lemma, which will be proved in §4.

LEMMA 2.1. *Assume that $k \le n-1$. Then, for any k-tuple of distinct indexes $j_1, \dots, j_k$ satisfying $0 \le j_1 < j_2 < \cdots < j_k \le n-2$ and for any k-dimensional vector of periods $\mathbf{m} = (m_1, \dots, m_k)$ there exist corresponding periodic points $z_1, \dots, z_k$ of the polynomial $p_0(z) = z^n$, such that*

$$\det\left(\frac{d\Lambda}{d(a_{j_1}, \dots, a_{j_k})}(p_0, z_1, \dots, z_k)\right) \neq 0. \tag{1}$$

Now we can complete the proof of Theorem 1.7 modulo the auxiliary results formulated earlier.

Proof of Theorem 1.7. Consider the periodic points $z_1, \dots, z_k$ obtained in Lemma 2.1. Since the Jacobian in (1) is non-degenerate, these periodic points belong to different periodic orbits; hence, according to Lemma 1.5, the point $(p_0, z_1, \dots, z_k)$ belongs to $M^n_{\mathbf{m}}$. Now Theorem 1.7 immediately follows from Remark 1.10 and the result of Lemma 2.1. □

3. *Computation of derivatives*

Before we proceed with the proof of Lemma 2.1, we should learn how to compute the Jacobian.

Assume that $w \in \mathbb{C}$ is a non-multiple periodic point of period m for a polynomial $p_{\mathbf{a}_0} \in \mathcal{P}^n$. Then, for all $\mathbf{a}$ in some neighborhood of $\mathbf{a}_0$, we have a periodic point $w(\mathbf{a})$ of the polynomial $p_{\mathbf{a}}$, obtained by analytic continuation of the periodic point w. Thus, for all $\mathbf{a}$ in a neighborhood of $\mathbf{a}_0$ we can consider the corresponding multiplier $\lambda_w(\mathbf{a}) = (p_{\mathbf{a}}^{\circ m})'(w(\mathbf{a}))$ of the periodic point $w(\mathbf{a})$.

LEMMA 3.1. *Let $z_0 \neq 0$ be a periodic point of period m of the polynomial $p_0(z) = z^n$. Then, for any index j satisfying $0 \le j \le n-2$,*

$$\frac{d\lambda_{z_0}(0)}{da_j} = (jn^{m-1} - n^m) \sum_{i=0}^{m-1} z_0^{n^i(j-n)}. \tag{2}$$

Before we give a proof of Lemma 3.1, we can establish the following corollary.

COROLLARY 3.2. *For every positive integers n, m with $n \ge 2$ and every index j satisfying $0 \le j \le n-2$, there exists a non-zero polynomial $P_{n,j,m}(z)$, such that if $z_0 \neq 0$ is a periodic point of period m of the polynomial $p_0(z) = z^n$, then*

$$\frac{d\lambda_{z_0}(0)}{da_j} = \frac{1}{z_0} P_{n,j,m}\left(\frac{1}{z_0}\right).$$

Moreover,

$$\deg P_{n,j,m} = \begin{cases} n^m - jn^{m-1} - 1 & \text{for } 1 \le j \le n-2, \\ n^{m-1} - 1 & \text{for } j = 0. \end{cases} \tag{3}$$

Proof. The existence of such polynomials immediately follows from (2). When j satisfies $1 \le j \le n-2$, then

$$\frac{d\lambda_{z_0}(0)}{da_j} = \frac{jn^{m-1} - n^m}{z_0} \sum_{i=0}^{m-1} \left(\frac{1}{z_0}\right)^{n^i(n-j)-1};$$

thus we can choose $P_{n,j,m}(z) = (jn^{m-1} - n^m) \sum_{i=0}^{m-1} z^{n^i(n-j)-1}$ and then $\deg P_{n,j,m} = n^m - jn^{m-1} - 1$.

When $j = 0$, keeping in mind that $z_0^{n^m} = z_0$, we can rewrite (2) in the following way:

$$\frac{d\lambda_{z_0}(0)}{da_j} = \frac{-n^m}{z_0} \sum_{i=0}^{m-1} \left(\frac{1}{z_0}\right)^{n^i-1}.$$

Then we choose $P_{n,j,m}(z) = -n^m \sum_{i=0}^{m-1} z^{n^i-1}$ and in that case $\deg P_{n,j,m} = n^{m-1} - 1$. □

Remark 3.3. Notice that the (r, s)th entry of the matrix in (1) is equal to the expression (2) with $z_0 = z_r$, $m = m_r$ and $j = j_s$. Hence, according to Corollary 3.2, the (r, s)th entry of the matrix in (1) can be represented as $(1/z_r)P_{n,j_s,m_r}(1/z_r)$.

Proof of Lemma 3.1. As before, given a vector $\mathbf{a} = (a_0, \ldots, a_{n-2}) \in \mathbb{C}^{n-1}$, by $p_{\mathbf{a}}(z)$ we denote the polynomial $p_{\mathbf{a}}(z) = z^n + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots + a_1 z + a_0$. Let $(w_0(\mathbf{a}), w_1(\mathbf{a}), \ldots)$ be a periodic orbit of the polynomial $p_{\mathbf{a}}$, such that $w_i(\mathbf{a}) = w_{i+m}(\mathbf{a})$ for all integers i and $w_0(0) = z_0$. First we will compute the derivatives $(dw_i(\mathbf{a})/da_j)|_{\mathbf{a}=0}$. To simplify the notation, we will write $w_i = w_i(0)$. Since $w_{i+1}(\mathbf{a}) = p_{\mathbf{a}}(w_i(\mathbf{a}))$, we have

$$\left.\frac{dw_{i+1}(\mathbf{a})}{da_j}\right|_{\mathbf{a}=0} = p_0'(w_i)\frac{dw_i}{da_j}(0) + w_i^j.$$

Since $w_i'(\mathbf{a}) = w_{i+m}'(\mathbf{a})$, from the previous identity it follows that

$$\left.\frac{dw_i(\mathbf{a})}{da_j}\right|_{\mathbf{a}=0} = \frac{\sum_{s=0}^{m-1} w_{i+s}^j \prod_{r=s+1}^{m-1} p_0'(w_{i+r})}{1 - \prod_{s=0}^{m-1} p_0'(w_{i+s})}.$$

Now we remember that $w_{i+s} = p_0^{\circ s}(w_i) = w_i^{n^s}$ and w_i is a periodic point of $p_0(z)$ of period m, so $w_i^{n^m} = w_i$. Since $w_0 = z_0 \ne 0$, we know that $w_i \ne 0$ for all $i \in \mathbb{N}$ and hence $w_i^{n^m-1} = 1$. Using this, we get

$$\begin{aligned}\left.\frac{dw_i(\mathbf{a})}{da_j}\right|_{\mathbf{a}=0} &= \frac{\sum_{s=0}^{m-1} w_i^{jn^s} \prod_{r=s+1}^{m-1} nw_i^{n^{r+1}-n^r}}{1-n^m} = \frac{\sum_{s=0}^{m-1} n^{m-s-1} w_i^{n^m-n^{s+1}+jn^s}}{1-n^m} \\ &= \frac{w_i \sum_{s=0}^{m-1} n^{m-s-1} w_i^{n^s(j-n)}}{1-n^m} = \frac{w_0^{n^i} \sum_{s=0}^{m-1} n^{m-s-1} w_0^{n^{s+i}(j-n)}}{1-n^m}.\end{aligned}$$

Now we can compute $d\lambda_{z_0}(0)/da_j$. Since

$$\lambda_{z_0}(\mathbf{a}) = p_{\mathbf{a}}'(w_0(\mathbf{a})) \cdots p_{\mathbf{a}}'(w_{m-1}(\mathbf{a})),$$

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by the formula for the derivative of a product we have

$$\begin{aligned}
&\frac{d\lambda_{z_0}(0)}{da_j}\\
&= \lambda_{z_0}(0)\sum_{i=0}^{m-1}\frac{p_0''(w_i)(dw_i(\mathbf{a})/da_j)|_{\mathbf{a}=0}+jw_i^{j-1}}{p_0'(w_i)}\\
&= n^m\sum_{i=0}^{m-1}\frac{n(n-1)w_i^{n-2}(dw_i(\mathbf{a})/da_j)|_{\mathbf{a}=0}+jw_i^{j-1}}{nw_i^{n-1}}\\
&= n^m\sum_{i=0}^{m-1}\frac{n(n-1)w_0^{n^i(n-2)}\cdot((w_0^{n^i}\sum_{s=0}^{m-1}n^{m-s-1}w_0^{n^{s+i}(j-n)})/(1-n^m))+jw_0^{n^i(j-1)}}{nw_0^{n^i(n-1)}}\\
&= n^{m-1}\bigg(\sum_{i=0}^{m-1}\frac{n(n-1)\sum_{s=0}^{m-1}n^{m-s-1}w_0^{n^{s+i}(j-n)}}{1-n^m}+jw_0^{n^i(j-n)}\bigg)\\
&= jn^{m-1}\sum_{i=0}^{m-1}w_0^{n^i(j-n)}+\frac{n^m(n-1)}{1-n^m}\sum_{s=0}^{m-1}n^{m-s-1}\sum_{i=0}^{m-1}w_0^{n^{s+i}(j-n)}.
\end{aligned}$$

Since $w_0^{n^{s+i}}=w_{s+i}$, and the sequence $w_0, w_1, \ldots$ is periodic of period m, it follows that $\sum_{i=0}^{m-1}w_0^{n^{s+i}(j-n)}=\sum_{i=0}^{m-1}w_0^{n^i(j-n)}$. Hence,

$$\begin{aligned}
\frac{d\lambda_{z_0}(0)}{da_j} &= \bigg(jn^{m-1}+\frac{n^m(n-1)}{1-n^m}\sum_{s=0}^{m-1}n^{m-s-1}\bigg)\sum_{i=0}^{m-1}w_0^{n^i(j-n)}\\
&= (jn^{m-1}-n^m)\sum_{i=0}^{m-1}z_0^{n^i(j-n)}. \qquad \square
\end{aligned}$$

4. *Non-degeneracy of the Jacobian*

In this section we give a proof of Lemma 2.1.

Let $\nu_n(m)$ denote the number of periodic points of the polynomial $p_0(z)=z^n$ with period m. Since this polynomial does not have multiple periodic points, the function $\nu_n(m)$ can be computed inductively by the formula

$$n^m=\sum_{r|m}\nu_n(r) \quad \text{or} \quad \nu_n(m)=\sum_{r|m}\mu(m/r)n^r,$$

where the summation goes over all divisors $r\geq 1$ of m, and $\mu(m/r)\in\{\pm1,0\}$ is the Möbius function.

It is easy to see from these formulas that

$$\begin{gathered}
\nu_n(m)\geq n^m-n^{m-2} \quad \text{for } m\geq 3 \quad \text{and}\\
\nu_n(1)=n, \qquad \nu_n(2)=n^2-n.
\end{gathered}$$

Let $\hat{\nu}_n(m)$ denote the number of non-zero periodic points of the polynomial $p_0(z)=z^n$ with period m. Then, since zero is a fixed point of the polynomial p_0, it follows that

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$\hat{\nu}_n(m) = \nu_n(m)$ for $m > 1$ and $\hat{\nu}_n(1) = \nu(1) - 1$. Thus, from the previous relations on $\nu_n(m)$, we obtain

$$\begin{aligned} \hat{\nu}_n(m) \geq n^m - n^{m-2} \quad \text{for } m \geq 3 \quad \text{and} \\ \hat{\nu}_n(1) = n - 1, \quad \hat{\nu}_n(2) = n^2 - n. \end{aligned} \tag{4}$$

LEMMA 4.1. *For every pair of positive integers n, m with $n \geq 2$ and every index j satisfying $0 \leq j \leq n-2$,*

$$\deg P_{n,j,m} < \hat{\nu}_n(m).$$

Proof. The statement of the lemma immediately follows from (3) and (4). □

Proof of Lemma 2.1. We will give a proof by induction on k.

Case 1: $k = 1$. If $z_1 \neq 0$ is a periodic point of $p_0(z)$ of period m_1, then, according to Corollary 3.2 and Remark 3.3,

$$\det\left(\frac{d\Lambda}{da_{j_1}}(p_0, z_1)\right) = \frac{d\Lambda}{da_{j_1}}(p_0, z_1) = \frac{1}{z_1} P_{n,j_1,m_1}\left(\frac{1}{z_1}\right).$$

Because of Lemma 4.1, it follows that the non-zero periodic point z_1 can be chosen in such a way that $P_{n,j_1,m_1}(1/z_1) \neq 0$, which implies that $d\Lambda/da_{j_1}(p_0, z_1) \neq 0$. This proves Case (1).

Case 2: $k > 1$. Assume that $z_1, \ldots, z_k \in \mathbb{C}$ are non-zero periodic points of corresponding periods $m_1, \ldots, m_k$ for the polynomial $p_0(z) = z^n$. Then, according to Corollary 3.2 and Remark 3.3, the matrix $(d\Lambda/d(a_{j_1}, \ldots, a_{j_k}))(p_0, z_1, \ldots, z_k)$ can be written as

$$\frac{d\Lambda}{d(a_{j_1}, \ldots, a_{j_k})}(p_0, z_1, \ldots, z_k) = \begin{pmatrix} \frac{1}{z_1} P_{n,j_1,m_1}\left(\frac{1}{z_1}\right) & \cdots & \frac{1}{z_1} P_{n,j_k,m_1}\left(\frac{1}{z_1}\right) \\ \cdots & \cdots & \cdots \\ \frac{1}{z_k} P_{n,j_1,m_k}\left(\frac{1}{z_k}\right) & \cdots & \frac{1}{z_k} P_{n,j_k,m_k}\left(\frac{1}{z_k}\right) \end{pmatrix}. \tag{5}$$

Therefore, the determinant of the matrix $(d\Lambda/d(a_{j_1}, \ldots, a_{j_k}))(p_0, z_1, \ldots, z_k)$ can be expressed as

$$\det\left(\frac{d\Lambda}{d(a_{j_1}, \ldots, a_{j_k})}(p_0, z_1, \ldots, z_k)\right) = \frac{1}{z_1} P\left(\frac{1}{z_1}\right), \tag{6}$$

where $P(1/z_1)$ is a polynomial of $1/z_1$ that depends on $z_2, \ldots, z_k$ as parameters. It is easy to see that

$$\deg P \leq \max_{1 \leq i \leq k} \deg P_{n,j_i,m_1};$$

hence, if, for some choice of periodic points $z_2, \ldots, z_k$, the polynomial P is not identically zero, then, according to Lemma 4.1, there exists a non-zero periodic point z_1 for which $P(1/z_1) \neq 0$. Then (6) implies that the Jacobian $(d\Lambda/d(a_{j_1}, \ldots, a_{j_k}))(p_0, z_1, \ldots, z_k)$ is non-degenerate, which is what we had to prove.

Now we use an inductive argument to show that indeed one can choose periodic points $z_2, \ldots, z_k$ so that the polynomial P is not identically zero. Notice that the degrees of polynomials $P_{n,j_1,m_1}, \ldots, P_{n,j_k,m_1}$ are all different, so, for some index i, the degree of P_{n,j_i,m_1} is maximal and greater than zero. Consider the minor of the matrix in (5) obtained by deleting the first row and the ith column. By the inductive assumption, there exist periodic points $z_2, \ldots, z_k$ of corresponding periods $m_2, \ldots, m_k$, such that for these periodic points the considered minor is non-degenerate. This implies that $\deg P = \deg P_{n,j_i,m_1} > 0$ and hence P is not identically zero. This finishes the proof of Lemma 2.1. □

5. *Permutation of periodic orbits*

In this section we will give a proof of Lemma 1.5. We start with two definitions.

Definition 5.1. For $n \geq 2$ we consider the set $\mathcal{P}_0^n \subset \mathcal{P}^n$ of polynomials defined in the following way: $\mathcal{P}_0^n = \{z^n + c \mid c \in \mathbb{C}\}$.

Definition 5.2. For every positive integer m, by $X_m \subset \mathcal{P}^n$ denote the set of all polynomials $q \in \mathcal{P}^n$, such that $q^{\circ m}$ has a multiple fixed point, that is, not a multiple fixed point for any smaller iteration of q. By X'_m we denote the intersection $X'_m = X_m \cap \mathcal{P}_0^n$.

Remark 5.3. It is not hard to show that X_m is an algebraic subset of $\mathcal{P}^n$; however, for our purposes it is enough to notice that X_m is contained in a codimension-one algebraic subset $Y_m \subset \mathcal{P}^n$ of those polynomials $q \in \mathcal{P}^n$ for which $q^{\circ m}$ has a multiple fixed point. Since $Y_m \cap \mathcal{P}_0^n$ is a finite set, X'_m is also finite.

Proof of Lemma 1.5. Consider a polynomial $p \in \mathcal{P}^n$ and its periodic points $z_1, \ldots, z_k$ belonging to different periodic orbits of corresponding periods $m_1, \ldots, m_k$. Consider the vector $\mathbf{m} = (m_1, \ldots, m_k)$ and the corresponding set $M_{\mathbf{m}}^n = M_{\mathbf{m}}^n(p, z_1, \ldots, z_k)$. It follows from Definition 1.4 that in order to prove that the set $M_{\mathbf{m}}^n$ is independent of the initial choice of $(p, z_1, \ldots, z_k)$, it is sufficient to show that for every polynomial $q \in \mathcal{P}^n$ and its non-multiple periodic points $w_1, \ldots, w_k$ belonging to different periodic orbits of corresponding periods $m_1, \ldots, m_k$, the point $(q, w_1, \ldots, w_k)$ belongs to $M_{\mathbf{m}}^n = M_{\mathbf{m}}^n(p, z_1, \ldots, z_k)$. In other words, it is sufficient to show that the point $(q, w_1, \ldots, w_k)$ can be obtained from $(p, z_1, \ldots, z_k)$ by analytic continuation of the periodic points $z_1, \ldots, z_k$ along some curve γ in $\mathcal{P}^n \setminus \bigcup_{j=1}^k X_{m_j}$ connecting the polynomials p and q.

We will show that the curve γ can be constructed from three adjacent pieces γ_1, γ_2 and γ_3. We choose γ_1 to be any curve in $\mathcal{P}^n \setminus \bigcup_{j=1}^k X_{m_j}$ that connects the polynomial p with the polynomial $p_0(z) = z^n$. Similarly, γ_3 is any curve in $\mathcal{P}^n \setminus \bigcup_{j=1}^k X_{m_j}$ that connects the polynomial p_0 with the polynomial q. Analytic continuation of the periodic points $z_1, \ldots, z_k$ along γ_1 produces some periodic points $x_1, \ldots, x_k$ of the polynomial p_0, while analytic continuation of the periodic points $w_1, \ldots, w_k$ along $-\gamma_3$ produces some (possibly different) periodic points $x'_1, \ldots, x'_k$. In order to complete the construction of the curve γ, we have to show that every point $(p_0, x'_1, \ldots, x'_k)$ can be obtained from every point $(p_0, x_1, \ldots, x_k)$ by analytic continuation along some loop $\gamma_2 \subset \mathcal{P}^n \setminus \bigcup_{j=1}^k X_{m_j}$, where $x_1, \ldots, x_k$ are periodic points of p_0 belonging to different periodic orbits of corresponding periods $m_1, \ldots, m_k$ and similarly $x'_1, \ldots, x'_k$ are periodic points of p_0 belonging to different periodic orbits of corresponding periods $m_1, \ldots, m_k$.

We will prove that the loop γ_2 can be chosen inside the set $\mathcal{P}_0^n \setminus \bigcup_{j=1}^k X'_{m_j} \subset \mathcal{P}^n \setminus \bigcup_{j=1}^k X_{m_j}$. First we notice that in the case when $m_1 = m_2 = \cdots = m_k$, the existence of such loop γ_2 follows from the result of Lau and Schleicher [**5**, **11**, **12**], which says that by choosing an appropriate element of the fundamental group $\pi_1(\mathcal{P}_0^n \setminus X_{m_1}, p_0)$ it is possible to realize any permutation of periodic orbits of p_0 of period m_1. Moreover, within each orbit of period m_1 any cyclic permutation of its points can be realized independently from permutations of other periodic points of the same period.

Finally, the case when not all of m_j are equal to each other follows from the same result of Lau and Schleicher and the following proposition.

PROPOSITION 5.4. *For every two distinct positive integers $m_1 \neq m_2$, the sets X'_{m_1} and X'_{m_2} are disjoint.*

Proof. Assume that the set $X'_{m_1} \cap X'_{m_2}$ is non-empty and $p \in X'_{m_1} \cap X'_{m_2}$. Then this means that the polynomial p has two parabolic periodic points z_1 and z_2, such that m_1 and m_2 are the smallest positive integers satisfying the following identities:

$$(p^{\circ m_1}(z_1))' = 1, \quad (p^{\circ m_2}(z_2))' = 1.$$

Since $m_1 \neq m_2$, this means that z_1 and z_2 belong to different parabolic periodic orbits. On the other hand, since the polynomial p is of the form $p(z) = z^n + c$, it has only one critical point; hence, according to [**10**, Corollary 10.11], it cannot have more than one parabolic periodic orbit. This brings us to a contradiction. □

We finish the proof of Lemma 1.5 by splitting the periodic points $x_1, \ldots, x_k$ into classes of points having the same periods. Within each class the necessary permutation of periodic points is realizable according to the above-mentioned theorem of Lau and Schleicher. Since the sets X'_{m_j} are disjoint for different values of m_j, this implies that the necessary permutations of periodic points can be obtained independently within each of the classes. □

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REFERENCES

[**1**] T. Bousch. Sur quelques problèmes de dynamique holomorphe. *PhD Thesis*, Université de Paris-Sud, Centre d'Orsay, 1992.
[**2**] A. Epstein. Transversality in holomorphic dynamics. http://homepages.warwick.ac.uk/ mases/Transversality.pdf.
[**3**] I. Gorbovickis. Some problems from complex dynamical systems and combinatorial geometry. *PhD Thesis*, Cornell University, 2012.
[**4**] I. Gorbovickis. Algebraic independence of multipliers of periodic orbits in the space of rational maps of the Riemann sphere. *Moscow Math. J.* (2014), to appear, arXiv:1401.4713.
[**5**] E. Lau and D. Schleicher. Internal addresses in the Mandelbrot set and irreducibility of polynomials. 1994, http://www.math.sunysb.edu/preprints/ims94-19.pdf.
[**6**] M. Manes. Moduli spaces for families of rational maps on $\mathbb{P}^1$. *J. Number Theory* **129**(7) (2009), 1623–1663.
[**7**] C. McMullen. Families of rational maps and iterative root-finding algorithms. *Ann. of Math. (2)* **125**(3) (1987), 467–493.
[**8**] J. Milnor. Remarks on iterated cubic maps. *Experiment. Math.* **1**(1) (1992), 5–24.

https://doi.org/10.1017/etds.2014.103 Published online by Cambridge University Press

[**9**] J. Milnor. Geometry and dynamics of quadratic rational maps. *Experiment. Math.* **2**(1) (1993), 37–83, with an appendix by the author and Lei Tan.
[**10**] J. Milnor. *Dynamics in One Complex Variable (Annals of Mathematics Studies, 160)*, 3rd edn. Princeton University Press, Princeton, NJ, 2006.
[**11**] P. Morton. Galois groups of periodic points. *J. Algebra* **201**(2) (1998), 401–428.
[**12**] D. Schleicher. Internal addresses in the Mandelbrot set and irreducibility of polynomials. *PhD Thesis*, Cornell University, 1994, ProQuest LLC, Ann Arbor, MI.
[**13**] T. Sugiyama. The moduli space of polynomials of one complex variable. *Preprint*, 2007, arXiv:0708.2512.
[**14**] Yu. G. Zarhin. Polynomials in one variable and ranks of certain tangent maps. *Math. Notes* **91**(4) (2012), 508–516.
[**15**] Yu. G. Zarhin. One dimensional polynomial maps, periodic points and multipliers. *Izv. Math.* **77**(4) (2013), 700–713.