

UNDECIDABILITY OF THE FIRST ORDER THEORIES OF FREE NONCOMMUTATIVE LIE ALGEBRAS

OLGA KHARLAMPOVICH AND ALEXEI MYASNIKOV

Abstract. Let R be a commutative integral unital domain and L a free noncommutative Lie algebra over R . In this article we show that the ring R and its action on L are 0-interpretable in L , viewed as a ring with the standard ring language $+, \cdot, 0$. Furthermore, if R has characteristic zero then we prove that the elementary theory $Th(L)$ of L in the standard ring language is undecidable. To do so we show that the arithmetic $\mathbb{N} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ is 0-interpretable in L . This implies that the theory of $Th(L)$ has the independence property. These results answer some old questions on model theory of free Lie algebras.

§1. Introduction. In this article we continue our program on model theory of groups and algebras outlined at the ICM in Korea in 2014 [7]. Let R be a commutative integral unital domain and L a free noncommutative Lie algebra over R . We show that the ring R and its action on L are 0-interpretable in L , viewed as a ring in the standard ring language $+, \cdot, 0$. Furthermore, if R has characteristic zero then we prove that the arithmetic $\mathbb{N} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ is 0-interpretable in L . Hence the elementary theory $Th(L)$ of L in the standard ring language is undecidable and has the independence property. These answer some old questions on model theory of free Lie algebras. Along the way we further developed the method that uses maximal rings of scalars in Lie rings that gives a general approach to study first order theories of arbitrary noncommutative finitely generated Lie algebras.

The question about decidability of the first-order theory of noncommutative free Lie algebras was well-known in Malcev's school of algebra and logic in Russia. In 1963 Lavrov showed that if the elementary theory $Th(R)$ of the integral domain R is undecidable then the elementary theory $Th(L)$ of L is also undecidable. To this end he interpreted the ring R in L [4].

In 1986 Baudisch proved in [2] that the theory $Th(L)$ is unstable for every such ring of coefficients R . To obtain this result he uniformly interpreted every initial segment of Presburger arithmetic in L . Following Lavrov he also showed that the ring R and its action on L are interpretable (with the use of parameters) in L .

In the same article Baudisch stated the following open problems: Does the theory $Th(L)$ of a free noncommutative Lie algebra L over a commutative integral domain have the independence property? Is $Th(L)$ undecidable? Is it possible to interpret the initial segments of the natural numbers with addition and multiplication in it?

Received June 21, 2017.

2010 *Mathematics Subject Classification.* 03C60.

Key words and phrases. Lie algebra, elementary theory.

© 2018, Association for Symbolic Logic
0022-4812/18/8303-0015
DOI:10.1017/jsl.2017.80

Independently, in the book [3] Bokut' and Kukin asked a similar question: for which integral domains R the theory $Th(L)$ is decidable?

As we have mentioned already, our results completely answer the questions above in the case when the ring R has characteristic zero. It seems plausible that similar results hold for arbitrary infinite integral domains R . However, our techniques do not work if the ring R is finite, so the following question seems to be very interesting. Is the theory $Th(L)$ undecidable when the ring R is a finite field? More precisely, is the arithmetic $\mathbb{N} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ interpretable in a free noncommutative Lie algebra L over a finite field?

We would like to mention that our proofs seem general enough to get similar results for some other Lie algebras, in particular, for various \mathbb{N} -graded Lie algebras where the maximal rings of scalars are integral domains. Actually, we prove that for an arbitrary finitely generated Lie R -algebra L over an arbitrary commutative associative unital ring R the maximal ring of scalars of L and its action on $L/Ann(L)$ and L^2 are 0-interpretable in L . This gives a general approach to study first order theories of finitely generated Lie algebras. To interpret arithmetic in such an algebra L one also needs some weak finiteness divisibility conditions on L , which in the case of a free Lie algebra L come from the fact that L is \mathbb{N} -graded. Note, that the model theory of finite dimensional Lie algebras over fields was studied in [13].

This article is a continuation of the research in [8–10] on model theory of free associative algebras. For some time we thought that model theory of free Lie algebras, though very different from the case of free groups (see [6, 15]), will be somewhat reminiscent of the model theory of free pro- p -groups (see [5, 12]). Now, it looks much more like the model theory of free associative algebras, though the proofs are more technical. The main difference is that in free associative algebras a centralizer of a noninvertible element is isomorphic to the ring of polynomials in one-variable, hence the known results from commutative algebra and number theory can be applied. In free Lie algebras we had to exploit some interesting module structures and unusual divisibility arguments. It seems possible that one can develop current techniques a bit further and study equations in free Lie algebras as well as elementary equivalence of such algebras in the way it was done for free associative ones. There are two interesting open questions here: whether one can interpret the weak second order theory of the ring R in a free noncommutative Lie algebra L with coefficients in R ; and if the Diophantine problem in L is decidable.

§2. Maximal rings of scalars.

2.1. Maximal rings of scalars of bilinear maps. Let R be a commutative associative ring with unity 1, and M_1, M_2, N exact R -modules. Let $f: M_1 \times M_2 \rightarrow N$ be an R -bilinear map. For a subset $E_1 \subseteq M_1$ we define the right annihilator of E_1 (relative to f) by $Ann_r(E_1) = \{y \in M_2 \mid f(E_1, y) = 0\}$. Similarly, for a subset $E_2 \subseteq M_2$ we define the left annihilator of E_2 by $Ann_l(E_2) = \{x \in M_1 \mid f(x, E_2) = 0\}$.

We say that

- 1) f is *nondegenerate* if $Ann_l(M_2) = 0$ and $Ann_r(M_1) = 0$.
- 2) f is *onto* if the submodule (equivalently, the subgroup) $\langle f(M_1, M_2) \rangle$ generated by $f(M_1, M_2)$ is equal to N .

Note that the conditions 1) – 2) do not depend on the ring R , i.e., whether they hold or not in f depends only on the abelian group structure of M and N .

For any nondegenerate onto bilinear map $f: M_1 \times M_2 \rightarrow N$ there is a uniquely defined *maximal ring of scalars* $P(f)$, which is an analogue of the *centroid* of a ring. More precisely, a commutative associative unital ring P is called a “ring of scalars” of f if M_1, M_2 , and N admit the structure of faithful P -modules such that f is P -bilinear. A ring of scalars P of f is called *maximal* if for every ring of scalars P' of f there is a monomorphism $\mu: P' \rightarrow P$ such that for every $\alpha \in P'$ its actions on M_1, M_2 , and N are the same as the actions of $\mu(\alpha)$. It is easy to see that the maximal ring of scalars of f exists, it is unique up to isomorphism, as well as its actions on M_1, M_2 , and N . We denote the unique maximal ring of scalars of f by $P(f)$. In fact, the ring $P(f)$ can be constructed as follows.

Let $\text{End}(M_1), \text{End}(M_2), \text{End}(N)$ be the ring of endomorphisms of M_1, M_2 , and N (here M_1, M_2 , and N are viewed as abelian groups). Below for an endomorphism β and an element x the image of β on x is denoted by βx .

If P is a ring of scalars of f then the actions of P on M_1, M_2 , and N give embeddings $P \rightarrow \text{End}(M_i), P \rightarrow \text{End}(N), i = 1, 2$, which give rise to the diagonal embedding $\Phi: P \rightarrow \text{End}(M_1) \times \text{End}(M_2) \times \text{End}(N)$. Denote the direct product of rings $\text{End}(M_1) \times \text{End}(M_2) \times \text{End}(N)$ by $K(f)$. Let $\tau_i: K(f) \rightarrow M_i, \sigma: K(f) \rightarrow N$ be the canonical projections of $K(f)$ onto its direct factors. Since P is a ring of scalars of f every $\alpha \in \Phi(P) \leq K(f)$ satisfies the following conditions for any $x \in M_1, y \in M_2$:

$$f(\tau_1(\alpha)x, y) = f(x, \tau_2(\alpha)y) = \sigma(\alpha)f(x, y). \quad (1)$$

It is not hard to see that the set $P(f)$ of all elements $\alpha \in K(f)$ which satisfy the condition (1) is a commutative unital subring of $K(f)$. We showed above that every ring of scalars of f embeds into $P(f)$ in such a way that its action on M_1, M_2, N agrees with the action of $P(f)$. Hence $P(f)$ is the maximal ring of scalars of f .

To interpret $P(f)$ in f we need another description of $P(f)$. Let $M(f) = \text{End}(M_1) \times \text{End}(M_2)$ and $\tau = \tau_1 \times \tau_2$ be the canonical projection of $K(f)$ onto $M(f)$. As we mentioned above we may assume that $P(f)$ is a subring of $K(f)$, the restriction of τ on $P(f)$ gives a homomorphism $\tau: P(f) \rightarrow M(f)$. Clearly, $\tau: P(f) \rightarrow M(f)$ is injective and for every $\alpha \in \tau(P(f))$ the following conditions (S) and (W_n) hold for every $n \in \mathbb{N}$:

(S) for every $x \in M_1, y \in M_2$

$$f(\tau_1(\alpha)x, y) = f(x, \tau_2(\alpha)y);$$

(W_n) for every $x_k, x'_k \in M_1, y_k, y'_k \in M_2, k = 1, \dots, n$

$$\sum_{k=1}^n f(x_k, y_k) = \sum_{k=1}^n f(x'_k, y'_k) \rightarrow \sum_{k=1}^n f(\tau_1(\alpha)x_k, y_k) = \sum_{k=1}^n f(\tau_1(\alpha)x'_k, y'_k).$$

We claim that $\tau(P(f))$ consists precisely of those elements $\alpha \in M(f)$ for which the conditions (S) and (W_n) hold for every $n \in \mathbb{N}$. Denote by $\text{Sym}(f)$ the subset of all elements $\alpha \in M(f)$ which satisfy (S) (we call them f -symmetric) and by $W_n(\text{Sym}(f))$ the subset of those $\alpha \in \text{Sym}(f)$ which satisfy (W_n) . Put

$$P_{Sw}(f) = \bigcap_{n=1}^{\infty} W_n(\text{Sym}(f)).$$

Clearly, $\tau(P(f)) \subseteq P_{SW}(f)$. To show the equality it suffices to show that for every $\alpha \in P_{SW}(f)$ there is $\sigma \in \text{End}(N)$ such that (1) holds, i.e., for any $x \in M_1, y \in M_2$

$$f(\tau_1(\alpha)x, y) = f(x, \tau_2(\alpha)y) = \sigma f(x, y).$$

To this end for a given $\alpha \in P_{SW}(f)$ and given $x \in M_1, y \in M_2$ define $\sigma f(x, y) = f(\tau_1(\alpha)x, y)$. Since α satisfies (W_1) this definition is correct, i.e., for any $x' \in M - 1, y' \in M_2$ one has $\sigma f(x, y) = \sigma f(x', y')$. Similarly, since α satisfies all the conditions (W_n) one can correctly extend the definition of σ by linearity on the whole subgroup N_0 generated in N by the set $f(M_1, M_2)$. Since f is onto $N_0 = N$, so $\sigma \in \text{End}(N)$ and (1) holds, as required. This shows that $\tau(P(f)) = P_{SW}(f)$, as claimed.

To study model theoretic properties of $f : M_1 \times M_2 \rightarrow N$ one associates with f a three-sorted structure $\mathcal{A}(f) = \langle M_1, M_2, N; f \rangle$, where M_1, M_2 , and N are abelian groups equipped with the map f (the language of $\mathcal{A}(f)$ consists of additive group languages for M_1, M_2 , and N , and the predicate symbol for the graph of f). Our goal is to show that the ring $P(f)$ as well as its actions on the modules M_1, M_2 , and N , are interpretable in the structure $\mathcal{A}(f)$. For this we need f to satisfy some finiteness conditions.

We say that

- 3) a finite subset $E_1 \subseteq M_1$ is called a *left complete system* for f if $\text{Ann}_r(E_1) = \text{Ann}_r(M_1)$. Similarly, a finite subset $E_2 \subseteq M_2$ is called a *right complete system* for f if $\text{Ann}_l(E_2) = \text{Ann}_l(M_2)$. In this case we say that a pair (E_1, E_2) is a finite complete system for f .
- 4) f has *finite width* if there exists some natural number m , such that for any $z \in N$ there are some $x_i \in M_1, y_i \in M_2, i = 1, \dots, m$ such that $z = \sum_{i=1}^m f(x_i, y_i)$. The least such m is termed the width of f .

THEOREM 2.1. [14] *Let f be an R -bilinear map $M_1 \times M_2 \rightarrow N$ that satisfies 1)–4) above. Then the maximal ring of scalars $P(f)$ for f and its actions on M and N are 0-interpretable in $\mathcal{A}(f)$ uniformly in the size of the finite complete system and the width of f .*

2.2. Maximal rings of scalars of finitely generated Lie algebras. In this section we prove some results on maximal rings of scalars in finitely generated Lie algebras and also in free Lie algebras of arbitrary rank.

Assume that R is an integral domain (commutative associative and unital). Let L be a Lie R -algebra. Denote by L^2 the R -submodule of L generated by all products xy where $x, y \in L$. Then the multiplication map $f_L : L \times L \rightarrow L^2$ is R -bilinear and onto. This map induces a nondegenerate R -bilinear onto map $\bar{f}_L : L/\text{Ann}(L) \times L/\text{Ann}(L) \rightarrow L^2$, where $\text{Ann}(L) = \{x \in L \mid xL = 0\}$.

LEMMA 2.2. *Let L be a finitely generated Lie R -algebra. Then the bilinear map \bar{f}_L satisfies all the conditions 1)–4). In particular, if $\text{Ann}(L) = 0$ then the multiplication f_L satisfies all the conditions 1)–4).*

PROOF. Suppose L is generated (as an algebra) by a finite set X . The map \bar{f}_L satisfies conditions 1) and 2) by construction. To prove 3) it suffices to show that $\text{Ann}(L) = \text{Ann}(X)$. Let $a \in \text{Ann}(X)$ and $b \in L$. To show that $ab = 0$ we may assume by linearity that b is a product of elements from X . If $b \in X$ then $ab = 0$,

otherwise, $b = uv$, where u, v are products of elements of X of shorter length. By induction on length $au = av = 0$. Since L is Lie then $a(uv) = -u(va) - v(au) = u(av) - v(au) = 0$, hence the claim. To show 4) we prove that $L^2 = Lx_1 + \dots + Lx_n$, where $X = \{x_1, \dots, x_n\}$. Clearly, it suffice to show that every product p of elements from X belongs to $M = Lx_1 + \dots + Lx_n$. Note that $p = uv$ for some Lie words u, v in X . We use induction on the length of v (as a Lie word in X) to show that $p \in M$. If v is an element from X then there is nothing to prove. Otherwise, $v = v_1v_2$ where v_1, v_2 are Lie words in X of smaller length. Then $u(v_1v_2) = -v_1(v_2u) - v_2(uv_1) = (v_2u)v_1 + (uv_1)v_2$. Now by induction on the length of the second factors we get that $(v_2u)v_1, (uv_1)v_2$, and hence $(v_2u)v_1 + (uv_1)v_2$, are in M , as required. \dashv

THEOREM 2.3. *Let L be a finitely generated Lie R -algebra. Then the maximal ring of scalars of the bilinear map \bar{f}_L and its action on $L/Ann(L)$ and L^2 are 0-interpretable in L (viewed in the language of rings) uniformly in the size of a finite generating set of L .*

PROOF. Let A be a finite generating set of L . As was shown in Lemma 2.2 the set L^2 is 0-definable in L uniformly in the size of the set A . Hence the bilinear map \bar{f}_L , i.e., the structure $\mathcal{A}(\bar{f}_L)$, is 0-interpretable in L uniformly in the size of A . Now by Theorem 2.1 the maximal ring of scalars of \bar{f}_L and its action on $L/Ann(L)$ and L^2 are 0-interpretable in $\mathcal{A}(\bar{f}_L)$, hence in L , uniformly in the size of a finite complete system of \bar{f}_L and the width of \bar{f}_L , which by Lemma 2.2 are uniform in the size of A . This proves the theorem. \dashv

2.3. Maximal rings of scalars of free Lie algebras. Let L be a free Lie algebra with finite set of free generators X over an integral domain R .

An element $u \in L$ can be uniquely decomposed as a sum of homogeneous elements $u = u_1 + \dots + u_n$ of pair-wise distinct weights (or degrees) with respect to system of free generators X . Notice that $u = 0 \iff u_1 = 0, \dots, u_n = 0$. By \bar{u} we denote the homogeneous component of u of the highest weight. By $wt(u)$ we denote the weight of \bar{u} . Observe, that $wt(\bar{u}\bar{v}) = wt(\bar{u}) + wt(\bar{v})$ provided $\bar{u}\bar{v} \neq 0$.

Denote by \mathcal{H} the set of Hall basis commutators on X (see [11] or [1]), then \mathcal{H} forms an R -basis of L as the R -module. We need the following well-known result, furthermore, since we need the argument used in its proof we provide a short proof as well.

LEMMA 2.4. *Let L be a free noncommutative Lie algebra with system of free generators X over an integral domain R . Then:*

- 1) *for any nonzero $u, v \in L$ if $uv = 0$ then $\alpha u = \beta v$ for some nonzero $\alpha, \beta \in R$.*
- 2) *Let $u \in \mathcal{H}$ be a basic commutator over X . Then for any $v \in L$ if $uv = 0$ then there is $\alpha \in R$ such that $v = \alpha u$.*

PROOF. To show 1) let $u, v \in L$ and $u = \sum u_i, v = \sum v_j$ be their decompositions on homogeneous components. Assume that $u_1 = \bar{u}, v_1 = \bar{v}$. Since $uv = 0$ it follows that $\bar{u}\bar{v} = 0$. Then by Theorem 5.10 from [11] $\alpha\bar{u} = \beta\bar{v}$ for some $\alpha, \beta \in R$. Consider $u' = \alpha u - \beta v$ then $u'v = 0$ and $wt(u') < wt(u)$. The argument above shows that the components of the highest weight in u' and v are linearly dependent, hence either of the same weight, or $u' = 0$. Since $wt(u') < wt(v)$ we get $u' = \alpha u - \beta v = 0$, as claimed.

To prove 2) take $u \in \mathcal{H}$. Suppose $uv = 0$ for some $v \in L$. Consider the decomposition $v = \sum_i v_i$ of v into homogeneous components with respect to X . Then $uv = \sum_i uv_i = 0$ hence $uv_i = 0$ for each such i . It follows from Theorem 5.10 in [11] that u and v_i are linearly dependent over R . Since \mathcal{H} is an R -basis of R it follows that v is homogeneous of the same weight as u and $\alpha v = \beta u$ for some $\alpha, \beta \in R$. Since v is in the same homogeneous component as u it follows that $v = \sum_i \alpha_i u_i$ where u_i are the basic commutators from \mathcal{H} of the same weight as u , so u is one of them, say $u = u_1$. The equality $\alpha v = \sum_i \alpha \alpha_i u_i = \beta u_1$ implies that $\alpha_i = 0$ for $i \geq 2$ and $\alpha \alpha_1 = \beta$. Hence $\alpha v = \alpha \alpha_1 u$, so $v = \alpha_1 u$, as claimed. \dashv

PROPOSITION 2.5. *Let L be a noncommutative free Lie algebra over an integral domain R . Then the maximal ring of scalars $P(f_L)$ of the multiplication bilinear map f_L is isomorphic to the ring R .*

PROOF. Let L be a free Lie algebra over R with system of free generators X . Notice first that $Ann_l(L) = Ann_r(L) = 0$ and f_L is onto (see Lemma 2.2), so the maximal ring of scalars $P = P(f_L)$ exists.

Let \mathcal{H} be a Hall basis of L . By Lemma 2.4 for any $x \in \mathcal{H}$ and $a \in L$ if $ax = 0$ then $a \in Rx$. Let $\alpha \in P$, then the action of α on L gives an R -endomorphism ϕ_α of R -module L such that $\phi_\alpha(xy) = \phi_\alpha(x)y = x\phi_\alpha(y)$. Hence the action by α is completely determined by its action on \mathcal{H} . Take an arbitrary $x \in \mathcal{H}$. One has, $\phi_\alpha(xx) = 0 = (\phi_\alpha(x)x)$, so $\phi_\alpha(x) \in Rx$, say $\phi_\alpha(x) = \alpha_x x$, where $\alpha_x \in R$. Similarly, for $y \in \mathcal{H}$ $\phi_\alpha(y) = \alpha_y y$ for some $\alpha_y \in R$. It follows that $\phi_\alpha(xy) = \alpha_x(xy) = \alpha_y(xy)$, hence $\alpha_x = \alpha_y$ for any $x, y \in \mathcal{H}$. Therefore, ϕ_α acts on L precisely by multiplication of α_x . This shows that $P = R$. \dashv

From Theorem 2.1 and Proposition 2.5 we get the following result.

COROLLARY 2.6. *Let L be a noncommutative free Lie algebra of finite rank over an integral domain R . Then the ring R and its action on L is 0-interpretable in L uniformly in the rank of L .*

Notice that Theorem 2.3 gives the result for any finitely generated noncommutative free Lie algebra. To get an interpretation of R and its action on L for an arbitrary noncommutative free Lie algebra over R one needs to work a bit more.

THEOREM 2.7. *Let L be a noncommutative free Lie algebra over an integral domain R . Then the ring R and its action on L are 0-interpretable in L uniformly on the class of such algebras L .*

PROOF. Let $x \in L, x \neq 0$. The formula

$$\phi(x, z) = (x \in C(z)) \wedge \forall e \exists e' \in C(e)(xe = ze')$$

defines in L the predicate $x \in Rz$ (here by $x \in C(z)$ we denote the formula $xz = 0$). Indeed, if $x = \alpha z$ then $xz = 0$. Take an arbitrary $e \in L$ and put $e' = \alpha e$. Then $e' \in C(e)$ and $xe = \alpha ze = z\alpha e = ze'$, as required. Conversely, suppose $\phi(x, z)$ holds in L on x, z . Then take a basic commutator $e \in L$ that does not appear in the decomposition of x and y into nontrivial linear combinations of basic commutators in A . Since $e' \in C(e)$ it follows from Lemma 2.4 that $e' = \alpha e$ for some $\alpha \in R$. The equality $xe = ze' = z\alpha e$ implies that $(x - \alpha z)e = 0$, so $x - \alpha z \in C(e)$. Because of the choice of e the latter can happen only if $x - \alpha z = 0$, i.e., $x \in C(z)$, as claimed.

Recall that elements of $Sym(f_{x,y})$ are interpreted in $f_{x,y}$ by the values on the complete system x, y , i.e., as elements $(rx, ry), r \in R$. This gives the following interpretation. For a fixed $0 \neq x \in L$ we turn Rx into a ring by interpreting an addition \oplus and a multiplication \otimes as follows. We put $xr \oplus xs$ as the standard addition in L , so $xr \oplus xs = xr + xs = x(r + s)$. To define the multiplication \otimes we need to interpret first the following predicate on $x, x', y, y' \in L$:

$$\exists r \in R(x' = rx \wedge y' = ry).$$

It is easy to see that the condition above holds on elements $x, x', y, y' \in L$ if these elements satisfy the following formula

$$\Phi(x, x', y, y') = (x' \in Rx) \wedge (y' \in Ry) \wedge (x'y = xy').$$

Now we define the multiplication \otimes on Rx : if $x_1, x_2, x_3 \in Rx$ then

$$x_1 \otimes x_2 = x_3 \iff \forall y \neq 0 \exists y' \in L \exists s, t \in R(x_2 = sx \wedge y' = sy \wedge x_3 = tx \wedge x_1 y' = txy).$$

The condition on the right can be written by a formula in the ring language using the formula $\Phi(x, x', y, y')$ above. Observe that the multiplication \otimes corresponds to the multiplication in R . Indeed, since $x_1, x_2, x_3 \in Rx$ then $x_1 = rx, x_2 = sx, x_3 = tx$ for some $r, s, t \in R$. For any $0 \neq y \in L$ there is $y' = sy$, hence $x_1 y' = rs(xy)$, and then $x_3 = rsx$, as required.

The argument above shows that we interpreted the ring R as the structure $R_x = \langle Rx : \oplus_x, \otimes_x \rangle$ in L with the parameter $x \neq 0$ uniformly in x . The formula $\Phi(x, x', y, y')$ defines an isomorphism $R_x \rightarrow R_y$ which maps $x' \rightarrow y'$. Indeed, if $\Phi(x, x', y, y')$ holds in L on elements x, x', y, y' then $x' = rx, y' = ry$ for some (unique) $r \in R$. Thus, for each nonzero $x, y \in L$ we defined an isomorphism $R_x \rightarrow R_y$ uniformly in x, y . Now consider a definable subset in $L \times L$:

$$D = \{(x', x) \mid x \neq 0, x' \in Rx\}.$$

The formula $\Phi(x, x', y, y')$ defines an equivalence relation \sim on D . Moreover, the formulas that interpret operations \oplus_x and \otimes_x on R_x uniformly in $x \neq 0$ allow one to define by formulas operations \oplus and \otimes on the set of equivalence classes D/\sim . Indeed, for $*_x \in \{\oplus_x, \otimes_x\}$ for $(x'_1, x_1), (x'_2, x_2), (x'_3, x_3) \in D$ put

$$[(x'_1, x_1)] *_x [(x'_2, x_2)] = [(x'_3, x_3)] \iff \exists z_1, z_2, z_3, z [z_1 *_x z_2 = z_3 \bigwedge_{i=1}^3 (z_i, z) \sim (x'_i, x_i)].$$

These define operations \oplus and \otimes on D/\sim such that the resulting structure $R_D = \langle D/\sim : \oplus, \otimes \rangle$ is isomorphic to R . Notice that this interpretation does not use any parameters from L . The formula $\Phi(x, x', y, y')$ defines an action of an element $[(x', x)] \in R_D$ on an arbitrary nonzero element $y \in L$, where the result of this action is an element $y' \in L$ such that $(x', x) \sim (y', y)$.

This proves the theorem. –

2.4. Definability of the rank. Now we show that the rank of a free Lie algebra is definable by first-order formulas.

Recall that a Lie ring L has L^2 of *finite width* if there is a number m such that every element $w \in L^2$ is equal to a sum of the type $u_1 v_1 + \dots + u_m v_m$ for some $u_i, v_i \in L$. The minimal such m is called the *width* of L .

We showed in the proof of Lemma 2.2 that every finitely generated Lie algebra has finite width.

LEMMA 2.8. *Let L be a Lie algebra. Then:*

1) *the sentence*

$$\forall u_1, v_1, \dots, u_{m+1}, v_{m+1} \exists u'_1, v'_1, \dots, u'_m, v'_m \left(\sum_{i=1}^{m+1} u_i v_i = \sum_{j=1}^m u'_j v'_j \right)$$

holds in L if and only if the width of L^2 is finite and is less or equal to m .

2) *Consider a formula*

$$\psi_m(a_1, \dots, a_m) = \forall u_1, v_1, \dots, u_{m+1}, v_{m+1} \exists v'_1, \dots, v'_m \left(\sum_{i=1}^{m+1} u_i v_i = \sum_{j=1}^m a_j v'_j \right).$$

Then if L is generated as an algebra by elements u_1, \dots, u_m then $\psi_m(u_1, \dots, u_m)$ holds in L . Furthermore, if $\psi_m(a_1, \dots, a_m)$ holds in an arbitrary algebra Lie L on some elements then L^2 is of width at most m in L and it is defined in L by the following formula

$$S_m(y) = \exists a_1, \dots, a_m \exists v_1, \dots, v_m (\psi_m(a_1, \dots, a_m) \wedge y = \sum_{i=1}^m a_i v_i).$$

PROOF. By a straightforward argument. ⊖

COROLLARY 2.9. *Let R be an integral domain and L a free Lie R -algebra of finite rank. Consider the following formula:*

$$\phi_m(a_1, \dots, a_m) = \forall y \exists \alpha_1, \dots, \alpha_m \in R \exists z_1, \dots, z_m \in L (y = \sum_{i=1}^m \alpha_i a_i + \sum_{i=1}^m \alpha_i z_i),$$

where $\alpha_i \in R$ and $\alpha_i a_i$ mean the corresponding formulas from the interpretation of R and its action on L from Theorem 2.6. Then:

1) *the formula*

$$\Delta_m = \exists a_1, \dots, a_m (\phi_m \wedge \psi_m)$$

(here ψ_m is the formula from Lemma 2.8) holds in L if and only if the rank of L is at most m .

2) *the formula $\Delta_m \wedge \neg \Delta_{m-1}$ holds in L if and only if L has rank m .*

PROOF. To see 1) suppose that Δ_m holds in L , so there are elements $u_1, \dots, u_m \in L$ such that ϕ_m and ψ_m both hold on u_1, \dots, u_m . Then L/L^2 as an R -module is generated by m elements. Conversely, suppose the rank of L is at most m . Then there are elements $u_1, \dots, u_m \in L$ that generate L . Hence by Lemma 2.8 ψ_m holds in L on u_1, \dots, u_m and $L^2 = Lu_1 + \dots + Lu_m$. Note also that L is generated modulo L^2 by u_1, \dots, u_m as an R -module, so the formula Δ_m holds in L . This proves 1) and 2) now follows from 1). ⊖

§3. Interpretability of the arithmetic . Let $A = \{a, b, a_1, \dots, a_n\}$ be a system of free generators of a free Lie algebra L with coefficients in an integral domain R .

If z_1, z_2, \dots, z_n in L , we denote $((z_1, z_2), z_3), \dots, z_n$ by $(z_1, z_2, z_3, \dots, z_n)$. For $u, v \in L$ and $\alpha \in R$ by $u(v + \alpha)$ we denote the element $uv + \alpha u \in L$ and refer to

it as a “product” of u and $v + \alpha$. This is just a notation. One can look at this as follows.

The endomorphism $ad(v) : x \rightarrow [x, v]$ of the R -module L^+ generates in $End(L^+)$ an R -subalgebra $\langle ad(v) \rangle_R$ which is isomorphic to the ring of polynomials $R[v]$, so $R[v]$ acts on L as a subalgebra of $End(L^+)$.

Now we establish some properties of the action above that we will need in the proof of Theorem 3.2:

- a) For any $u, v \in L$ and any $\alpha, \beta \in R$ the following holds:

$$\beta u(v + \alpha) = u(\beta v + \beta \alpha), \quad u(v + \alpha) = u((v + \beta u) + \alpha),$$

- b) For any $u, v \in L$ and any $\alpha \in R$ one has

$$u(v + \alpha) = 0 \iff u = 0.$$

Indeed, if $uv + \alpha u = 0$ then $\bar{u}\bar{v} = 0$, hence by Lemma 2.4 either $\bar{v} = 0$, or $\bar{u} = 0$, or $r\bar{u} = s\bar{v}$ for some nonzero $r, s \in R$. If $\bar{u} = 0$ then $u = 0$, as claimed. If $\bar{v} = 0$ then $v = 0$ hence $0 = u(v + \alpha) = uv + \alpha u = \alpha u$, so $u = 0$. Suppose now that $r\bar{u} = s\bar{v}$ for some nonzero $r, s \in R$. Put $v' = sv - ru$. Then by a)

$$u(v' + s\alpha) = s(u(v + \alpha)) = 0$$

and $\bar{u}\bar{v} \neq 0$ unless $v' = 0$. The argument above shows that $v' = 0$, but then as was mentioned above $u = 0$, as claimed.

- c) For any $u, v \in L$, and any $\alpha_1, \dots, \alpha_n \in R$ if $uv \neq 0$ and $\bar{u}\bar{v} \neq 0$ then

$$wt(u(v + \alpha_1) \dots (v + \alpha_n)) = wt(u) + nwt(v).$$

This property follows by induction on n . In general the following holds:

- d) For any $u, v \in L$, and any $\alpha_1, \dots, \alpha_n \in R$ if $uv \neq 0$ then

$$wt(u(v + \alpha_1) \dots (v + \alpha_n)) = wt(u) + nwt(v'),$$

where $v' = v$ if $\bar{u}\bar{v} \neq 0$, otherwise $v' = \beta v - \alpha u$, where $\alpha, \beta \in R \setminus \{0\}$ are such that $\alpha\bar{u} = \beta\bar{v}$ (such $\alpha, \beta \in R$ always exist if $\bar{u}\bar{v} = 0$).

Indeed, suppose $uv \neq 0$ but $\bar{u}\bar{v} = 0$. Fix any $\alpha, \beta \in R \setminus \{0\}$ such that $\alpha\bar{u} = \beta\bar{v}$. Put $v' = \alpha u - \beta v$. Notice that $uv' \neq 0$ and also $wt(v') < wt(v) = wt(u)$ so $\bar{u}\bar{v}' \neq 0$. Denote

$$w = u(v + \alpha_1) \dots (v + \alpha_n).$$

Then by a)

$$\beta^n w = u(\beta v + \beta \alpha_1) \dots (\beta v + \beta \alpha_n) = u(v' + \beta \alpha_1) \dots (v' + \beta \alpha_n).$$

Notice that $wt(w) = wt(\beta^n w)$. It follows from c) that

$$wt(\beta^n w) = wt(u) + nwt(v'),$$

as claimed.

The following result holds in any Lie R -algebra.

LEMMA 3.1. *Let L be any Lie R -algebra. If $u, v \in L$, and $\alpha_1, \dots, \alpha_n$ are pair-wise distinct elements from R such that*

$$u = u_1(v + \alpha_1), \dots, u = u_n(v + \alpha_n),$$

for some elements $u_1, \dots, u_n \in L$ then

$$\gamma u = w(v + \alpha_1) \dots (v + \alpha_n)$$

for some element $w \in L$ and $0 \neq \gamma \in R$.

PROOF. Case $n = 2$. Let

$$u = u_1(v + \alpha_1) = u_1v + \alpha_1u_1,$$

$$u = u_2(v + \alpha_2) = u_2v + \alpha_2u_2.$$

Then

$$(u_1 - u_2)v + \alpha_1(u_1 - u_2) = (\alpha_2 - \alpha_1)u_2.$$

Notice that $\alpha_2 - \alpha_1 \neq 0$. It follows that

$$(\alpha_2 - \alpha_1)u_2 = (u_1 - u_2)(v + \alpha_1).$$

Hence

$$(\alpha_2 - \alpha_1)u = (\alpha_2 - \alpha_1)u_2(v + \alpha_2) = (u_1 - u_2)(v + \alpha_1)(v + \alpha_2),$$

as required.

Case $n \geq 3$. Let

$$u = u_1(v + \alpha_1), u = u_2(v + \alpha_2), \dots, u = u_n(v + \alpha_n).$$

By induction from the first $n - 1$ equalities one has

$$\gamma_1 u = w_1(v + \alpha_1) \dots (v + \alpha_{n-1}) = w'_1(v + \alpha_{n-1}),$$

where $0 \neq \gamma_1 \in R$, $w'_1 = w_1(v + \alpha_1) \dots (v + \alpha_{n-2})$. Similarly, considering the system obtained from the initial one above by removing the equality $u = u_{n-1}(v + \alpha_{n-1})$ one gets by induction that

$$\gamma_2 u = w_2(v + \alpha_1) \dots (v + \alpha_{n-2})(v + \alpha_n) = w'_2(v + \alpha_n),$$

where $w'_2 = w_2(v + \alpha_1) \dots (v + \alpha_{n-2})$.

Consider a system

$$\gamma_1 u = w'_1(v + \alpha_{n-1}),$$

$$\gamma_2 u = w'_2(v + \alpha_n).$$

Multiplying the first equation by γ_2 and the second—by γ_1 one gets

$$\gamma u = \gamma_2 w'_1(v + \alpha_{n-1}),$$

$$\gamma u = \gamma_1 w'_2(v + \alpha_n),$$

where $\gamma = \gamma_1 \gamma_2 \neq 0$. From the case $n = 2$ one gets

$$(\alpha_n - \alpha_{n-1})\gamma u = (\gamma_2 w'_1 - \gamma_1 w'_2)(v + \alpha_{n-1})(v + \alpha_n).$$

Observe, that

$$\begin{aligned} \gamma_2 w'_1 - \gamma_1 w'_2 &= \gamma_2 w_1(v + \alpha_1) \dots (v + \alpha_{n-2}) - \gamma_1 w_2(v + \alpha_1) \dots (v + \alpha_{n-2}) = \\ &= (\gamma_2 w_1 - \gamma_1 w_2)(v + \alpha_1) \dots (v + \alpha_{n-2}). \end{aligned}$$

Hence

$$\gamma u = ((\alpha_n - \alpha_{n-1})^{-1}(\gamma_2 w_1 - \gamma_1 w_2))(v + \alpha_1) \dots (v + \alpha_{n-2})(v + \alpha_{n-1})(v + \alpha_n),$$

as claimed. ⊖

THEOREM 3.2. *Let R be an integral domain of characteristic 0 and L a free noncommutative Lie algebra over R . Then*

1) *For any $b \in L, b \neq 0$, the formula*

$$\phi(x, b) = (x \in R) \wedge \exists v \neq 0 \exists u \forall k \in R \forall u_1 \exists u_2$$

$$(v = ub \wedge (v = u_1(b + k) \implies (v = u_2(b + k + 1) \vee k = x)))$$

interprets $\mathbb{N} \subseteq R$ in L (in the formula above notation $x \in R$, as well as the action of an $\alpha \in R$ on $u \in L$, means here that x belongs to the interpretation of R in L from Theorem 2.6 and the action by α is also from this interpretation).

2) *The formula $\exists b[(b \neq 0) \wedge \phi(x, b)]$ 0-interprets $\mathbb{N} \subseteq R$ in L .*

PROOF. We prove 1) first. Let $m \in \mathbb{N}$. We need to show that $L \models \phi(m)$. Take any $a \in L, a \neq 0$ and put $v = ab(b + 1) \dots (b + m)$. Then, in the notation above, for any $k \in \mathbb{N}$ for any $i \leq k$ there is $u_i \in L$ such that $ab(b + 1) \dots (b + k) = u_i(b + i)$. Indeed, by the property b) above for any $w \in L$, and for any $i, j \in R$

$$w(b + i)(b + j) = w(b + j)(b + i).$$

This allows one to push $(b + i)$ to the right in the “product” $ab(b + 1) \dots (b + k)$.

Observe that for any $\alpha \in R \setminus \{0, 1, \dots, m\}$ $v \neq u(b + \alpha)$ for any $u \in L$. Indeed, if $v = u(b + \alpha)$ for such an α , then by Lemma 3.1

$$\gamma v = u(b + 0)(b + 1) \dots (b + m)(b + \alpha).$$

In this case by the properties above $wt(v) \geq m + 2$, while by the choice of v we have $wt(v) = wt(a(b + 0)(b + 1) \dots (b + m)) = m + 1$ - contradiction.

This shows that $\phi(m)$ holds in L .

Let now $x \in R \setminus \mathbb{N}$. We need to show that $L \not\models \phi(x)$. Suppose to the contrary that $L \models \phi(x)$ for $x \in R \setminus \mathbb{N}$.

Then there exists $v \in L, v \neq 0$ such that

$$v = u_0 b = u_1(b + 1) = u_2(b + 2) = \dots = u_{n+1}(b + n + 1) = \dots$$

for some $u_i \in L, i \in \mathbb{N}$.

Then by Lemma 3.1 for any $n \in \mathbb{N}$ one has

$$\gamma_n v = w_n(b + 0)(b + 1) \dots (b + n)$$

for some $0 \neq \gamma_n \in R$ and $w_n \in L$. Hence, since $w_n b \neq 0$ (otherwise $v = 0$, but it is not), one has $wt(v) > n$ for every $n \in \mathbb{N}$, but this is impossible since $v \neq 0$. Hence $L \not\models \phi(x)$, as required.

2) follows immediately from 1). This proves the theorem. ⊖

This result answers the question posed by Baudisch in [2] in the case of characteristic zero.

§4. Results. The following theorem answers questions by Baudisch in [2] and by Bokut' and Kukin [3] in the case of characteristic zero.

THEOREM 4.1. *The first order theory in the ring language of a free noncommutative Lie algebra over an integral domain of characteristic zero is undecidable.*

PROOF. By Theorem 3.2 the arithmetic \mathbb{N} is interpretable in L in the ring language. Hence the theory $Th(L)$ is undecidable. \dashv

Let T be a complete theory in a language \mathcal{L} . An \mathcal{L} -formula $\phi(x, y)$ is said to have the independence property (with respect to x, y) if in every model M of T there is, for each $n = \{0, 1, \dots, n-1\} < \omega$, a family of tuples b_0, \dots, b_{n-1} such that for each of the 2^n subsets X of n there is a tuple $a \in M$ for which

$$M \models \varphi(a, b_i) \iff i \in X.$$

The theory T has independence property if some formula does.

Note that the elementary theory of the arithmetic $\mathbb{N} = \langle \mathbb{N}, +, \cdot, 0 \rangle$ is independent. Indeed, the formula “ y divides x ”, i.e., the formula $\exists k(x = ky)$ has the independence property. Clearly the independence property is inherited under interpretations. The following theorem answers the question posed by Baudisch in [2] in the case of characteristic zero.

THEOREM 4.2. *The first order theory of a free noncommutative Lie algebra over an integral domain of characteristic zero has the independence property.*

§5. Acknowledgments. The results of this article were obtained with support of RSF grant (project 14-11-00085) and were inspired by NSF grant proposal. We would also like to thank the referee for giving constructive comments which helped improving the quality of the exposition.

REFERENCES

- [1] Y. BAHTURIN, *Identical Relations in Lie Algebras*, VNU Science Press, Moscow, 1987.
- [2] A. BAUDISCH, *On elementary properties of free Lie algebras. Annals of Pure and Applied Logic*, vol. 30 (1986), pp. 121–136.
- [3] L. A. BOKUT' and G. P. KUKIN, *Algorithmic and Combinatorial Algebra*, Mathematics and Its Applications, vol. 255, Springer, Dordrecht, 1994.
- [4] I. A. LAVROV, *Undecidability of elementary theories of some rings. Algebra i Logika*, vol. 1 (1963), no. 3, pp. 39–45.
- [5] M. JARDEN and A. LUBOTZKY, *Elementary equivalence of profinite groups. Bulletin of the London Mathematical Society*, vol. 40 (2008), no. 5, pp. 887–896.
- [6] O. KHARLAMPOVICH and A. MYASNIKOV, *Elementary theory of free nonabelian groups. Journal of Algebra*, vol. 302 (2006), no. 2, pp. 451–552.
- [7] ———, *Model theory and algebraic geometry in groups, nonstandard actions and algorithmic problems. Proceedings of the International Congress of Mathematicians*, Invited Lectures, vol. 2, Kyung Moon Sa Co., Seoul, 2014, pp. 223–244.
- [8] ———, *Tarski-type problems for free associative algebras. Journal of Algebra*, 2017. Available at <https://doi.org/10.1016/j.jalgebra.2017.10.001>.
- [9] ———, *Equations in algebras*. To appear in the *International Journal of Algebra and Computation*. Preprint, 2016. arXiv:1606.03617.
- [10] ———, *First-order theory of group algebras*. Preprint, 2016. arXiv:1509.04112.
- [11] W. MAGNUS, A. KARRAS, and D. SOLITAR, *Combinatorial Group Theory*, Dover, New York, 1976.
- [12] A. MYASNIKOV, *Recursive p -adic numbers and elementary theories of finitely generated pro- p -groups. Mathematics of the USSR-Izvestiya*, vol. 34 (1988), no. 3, pp. 577–597.

[13] ———, *The structure of models and a criterion for the decidability of complete theories of finite-dimensional algebras*. (Russian) *Izvestiya Akademii Nauk SSSR, Seriya Matematicheskikh*, vol. 53 (1989), no. 2, pp. 379–397; English translation in *Mathematics of the USSR-Izvestiya*, vol. 34 (1990), no. 2, pp. 389–407.

[14] ———, *Definable invariants of bilinear mappings*. *Siberian Mathematical Journal*, vol. 31 (1990), no. 1, pp. 104–115.

[15] Z. SELA, *Diophantine geometry over groups VI: The elementary theory of a free group*. *Geometric and Functional Analysis*, vol. 16 (2006), pp. 707–730.

DEPARTMENT OF MATHEMATICS AND STATISTICS
HUNTER COLLEGE, CUNY
NEW YORK, NY 10065, USA
E-mail: okharlampovich@gmail.com

DEPARTMENT OF MATHEMATICAL SCIENCES
STEVENS INSTITUTE OF TECHNOLOGY
HOBOKEN, NJ 07030, USA
E-mail: amiasnikov@gmail.com