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# Discrepancy of rational points in simple algebraic groups

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## ABSTRACT

The aim of the present paper is to derive effective discrepancy estimates for the distribution of rational points on general semisimple algebraic group varieties, in general families of subsets and at arbitrarily small scales. We establish mean-square, almost sure and uniform estimates for the discrepancy with explicit error bounds. We also prove an analogue of W. Schmidt’s theorem, which establishes effective almost sure asymptotic counting of rational solutions to Diophantine inequalities in the Euclidean space. We formulate and prove a version of it for rational points on the group variety, with an effective bound which in some instances can be expected to be the best possible.

## 1. Introduction

Our goal is to analyze the discrepancy of distribution for rational points on certain homogeneous algebraic varieties; that is, the behavior of the counting function for the number of rational solutions of Diophantine inequalities. The most classical setting for this problem is that of rational points in Euclidean spaces, and let us begin by recalling some of the most basic results. For vectors  $x \in \mathbb{R}^d$ , one considers the inequality

$$\|x - p/q\|_\infty < \psi(q), \quad \text{with } (p, q) \in \mathbb{Z}^d \times \mathbb{N}, \quad (1.1)$$

where  $\|\cdot\|_\infty$  denotes the maximum norm on  $\mathbb{R}^d$ , and  $\psi : (0, \infty) \rightarrow (0, 1)$  is a non-increasing function. According to Khinchin’s theorem, the inequality (1.1) has infinitely many solutions for almost all  $x \in \mathbb{R}^d$  if and only if

$$\sum_{q \geq 1} q^d \psi(q)^d = \infty.$$

This result raises the problem of estimating the number of solutions for the inequality (1.1) satisfying a specified bound on the denominators, namely analyzing the counting function

$$N_T(x) := |\{(p, q) \in \mathbb{Z}^d \times \mathbb{N} : 1 \leq q \leq T \text{ and (1.1) holds}\}|.$$

The study of this question led to a major quantitative refinement of Khinchin’s theorem that was established in full generality by Schmidt [Sch60] (see also [Erd59, LeV59] for previous results). It is natural to embed the set  $\mathbb{Q}^d$  as a lattice subgroup in the space  $\mathbb{A}^d$ , where  $\mathbb{A}$  denotes the rational adèles. Then  $N_T(x)$  can be interpreted as the number of lattice points contained in

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the corresponding domains of  $\mathbb{A}^d$ . Therefore, one expects that  $N_T(x)$  is approximated by the following volume sum

$$V_T := \sum_{1 \leq q \leq T} \text{vol}(B(x, \psi(q)))q^d = \sum_{1 \leq q \leq T} (2\psi(q))^d q^d,$$

where  $B(x, \epsilon) := \{y \in \mathbb{R}^d : \|x - y\|_\infty < \epsilon\}$ . Indeed, Schmidt [Sch60] proved that when  $V_T \rightarrow \infty$ , for every  $\theta > 1/2$ ,

$$N_T(x) = V_T + O_{x,\theta}(V_T^\theta) \quad \text{for a.e. } x \in \mathbb{R}^d. \tag{1.2}$$

The natural problem of proving analogues of Khinchin’s and Schmidt’s theorems for rational points on homogeneous algebraic varieties, and in particular on algebraic groups, was raised by Lang [Lan65, p. 189].

In previous work [GGN14], an analogue of Khinchin’s theorem for rational points on semisimple group varieties was established. The first main goal of the present paper is to establish an analogue of Schmidt’s asymptotic formula (1.2) in this setting. The second main goal is to study systematically the discrepancy of distribution for rational points on general semisimple group varieties.

We now turn to presenting the results in the simplest case: for simply connected groups defined and almost-simple over  $\mathbb{Q}$ .

### 1.1 An analogue of Schmidt’s theorem for group varieties

Let  $G \subset GL_n$  be a linear algebraic group defined over  $\mathbb{Q}$ . For a set of primes  $S$ , we denote by  $\mathbb{Z}_S$  the ring of rational numbers which are integral for every  $p \notin S$  (also called  $S$ -integers), namely whose reduced denominator is divisible only by  $p^k$ ,  $k \geq 0$ ,  $p \in S$ . We consider the group  $G(\mathbb{Z}_S)$  consisting of matrices whose entries belong to the ring  $\mathbb{Z}_S$ . The compact open ring of  $p$ -adic integers in  $\mathbb{Q}_p$  will be denoted  $\widehat{\mathbb{Z}}_p$ , and  $G(\widehat{\mathbb{Z}}_p)$  denotes the group of  $\widehat{\mathbb{Z}}_p$ -points in  $G(\mathbb{Q}_p)$ .

It is natural to order the rational points in  $G(\mathbb{Z}_S)$  with respect to the height function

$$H_f(r) := \prod_{p\text{-prime}} \max(1, \|r\|_p) = \prod_{p \in S} \max(1, \|r\|_p) \quad \text{for } r \in G(\mathbb{Z}_S), \tag{1.3}$$

where  $\|\cdot\|_p$  denotes the  $p$ -adic norm on the matrix space  $\text{Mat}_n(\mathbb{Q}_p)$ . For  $x \in G(\mathbb{R})$  and a parameter  $b > 0$ , we consider the inequality

$$\|x - r\|_\infty < H_f(r)^{-b}, \quad \text{with } r \in G(\mathbb{Z}_S). \tag{1.4}$$

In previous works [GGN13, GGN14], the existence of solutions of (1.4) when  $G$  is a connected semisimple algebraic group was investigated. In particular, when  $G$  is almost-simple, simply connected and isotropic over  $S$ , namely  $G(\mathbb{Q}_p)$  is non-compact for some  $p \in S$ , the following was proved. There exist explicit positive exponents  $b_1(S) > b_2(S)$  such that for  $b < b_1(S)$ , the inequality (1.4) has infinitely many solutions for almost all  $x \in G(\mathbb{R})$ , and for  $b < b_2(S)$ , the inequality (1.4) has infinitely many solutions for all  $x \in G(\mathbb{R})$ . In [GGN22] an asymptotic formula for the number of solutions of the inequality (1.4) was established, for all  $x \in G(\mathbb{R})$  and every  $b < b_2(S)$ .

We now turn to stating (an instance of) our first main result, namely an asymptotic formula analogous to the classical estimate (1.2). Instead of working with (1.4), it will be more convenient to work with an equivalent inequality defined in terms of a fixed right-invariant Riemannian metric  $\rho$  on  $G(\mathbb{R})$ . For a parameter  $b > 0$ , we consider the inequality

$$\rho(x, r) \leq H_f(r)^{-b}, \quad \text{with } r \in G(\mathbb{Z}_S). \tag{1.5}$$

As will be shown in Lemma 2.9 below, in a ball of fixed radius  $R$  centered at a point  $x$ , the distances in (1.4) and (1.5) are comparable up to a multiplicative constants (depending on  $R$ ).

We define

$$N_T(x) := |\{r \in G(\mathbb{Z}_S) : 1 \leq H_f(r) \leq T \text{ and (1.5) holds}\}|. \tag{1.6}$$

Note that this counting function is analogous to that appearing in (1.2), when the gauge function used is given by  $\psi_b(q) = 1/q^b$ .

We will show that as in (1.2) this counting function can be approximated by a suitable adelic volume. We denote by  $G_S$  the *restricted direct product* of the groups  $G(\mathbb{Q}_p)$ ,  $p \in S$ , with respect to the compact open subgroups  $G(\widehat{\mathbb{Z}}_p)$ .

The definition of the height function (1.3) extends to the group  $G_S$  as

$$H_f(g) := \prod_{p \in S} \max(1, \|g_p\|_p) \quad \text{for } g = (g_p)_{p \in S} \in G_S. \tag{1.7}$$

The diagonal embedding  $G(\mathbb{Z}_S) \hookrightarrow G(\mathbb{R}) \times G_S$  realizes  $G(\mathbb{Z}_S)$  as lattice subgroup in the product  $G(\mathbb{R}) \times G_S$ . We fix Haar measures  $m_\infty$  and  $m_S$  on the groups  $G(\mathbb{R})$  and  $G_S$ , respectively, such that the subgroup  $G(\mathbb{Z}_S)$  has covolume one in  $G(\mathbb{R}) \times G_S$  with respect to  $m_\infty \times m_S$ . We consider the volume sum:

$$V_T := \sum_{1 \leq h \leq T} m_\infty(B(x, h^{-b}))m_S(\Sigma_S(h)),$$

where

$$B(x, \epsilon) := \{y \in G(\mathbb{R}) : \rho(y, x) < \epsilon\} \quad \text{and} \quad \Sigma_S(h) := \{g \in G_S : H_f(g) = h\}.$$

We note that because of invariance of the distance  $\rho$  and Haar measure  $m_\infty$ , this sum is independent of  $x$ . The sets  $\Sigma_S(h)$  are the *height spheres* on the group  $G_S$ , and they constitute compact open subsets of  $G_S$  (and can be empty, a possibility that can certainly occur for certain values of  $h$ ).

With this notation, we prove the following analogue of Schmidt’s result (1.2).

**THEOREM 1.1.** *Let  $G$  be a connected simply connected  $\mathbb{Q}$ -almost simple linear algebraic group defined over  $\mathbb{Q}$ , and let  $S$  be a finite set of primes such that  $G$  is isotropic for all  $p \in S$ . Then there exists explicit  $b_0 = b_0(S) > 0$  such that for every parameter  $b \in (0, b_0)$ ,*

$$\|N_T - V_T\|_{L^2(Q)} \ll_{S,Q} V_T^\theta$$

with explicit  $\theta = \theta(S, b) \in (0, 1)$  and an arbitrary bounded measurable subset  $Q$  of  $G(\mathbb{R})$ . Moreover, for every  $\theta' \in (\theta, 1)$ ,

$$N_T(x) = V_T + O_{S,x,\theta'}(V_T^{\theta'}) \quad \text{for almost every } x \in G(\mathbb{R}).$$

We note that in Theorem 1.1 we require the assumption that  $G$  is isotropic (namely  $G(\mathbb{Q}_p)$  is non-compact) for every  $p \in S$ . Our other results below will be proved under the weaker condition that  $G$  is isotropic for at least one  $p \in S$ .

Theorem 1.1 will be proved in § 3 (see Theorem 3.2 and Corollary 3.3). We show that the parameters  $b_0$  and  $\theta$  can be estimated explicitly in terms of the integrability exponents (see (2.1)–(2.2) and (2.3) below) of the relevant automorphic representations. Moreover, when the automorphic representations are known to be tempered and  $G$  is unramified over  $S$ , Theorem 1.1 holds with  $b_0$  being the divergence exponent of the sum  $V_T$  and  $\theta = 1/2 + \eta$  for every  $\eta > 0$

(see Corollary 3.4). Hence, in this situation, we obtain the estimate: for every  $\eta > 0$ ,

$$N_T(x) = V_T + O_{S,x,\eta}(V_T^{1/2+\eta}) \quad \text{for a.e. } x \in G(\mathbb{R}).$$

We note that in this case, the exponent of the error estimate of  $N_T$  is the square root of the main term. Thus, it is of the same quality as Schmidt’s theorem stated in (1.2) above, and can be expected to be best possible. We refer to §3.2 for further discussion.

**1.2 Strong approximation and discrepancy bounds**

Let  $G \subset GL_n$  be a connected  $\mathbb{Q}$ -almost-simple linear algebraic group defined over  $\mathbb{Q}$ . In order to render our introduction more transparent, in the present section we assume that  $G$  is simply connected, and defer for later a discussion of the general case. Then it is known that  $G$  satisfies the strong approximation property [PR94, §7.4]. This property implies that for every set  $S$  of primes such that  $G$  is isotropic over  $\mathbb{Q}_p$  for some  $p \in S$ , the diagonal embedding

$$G(\mathbb{Z}_S) \hookrightarrow G(\mathbb{R}) \times \mathcal{I}^S, \quad \text{where } \mathcal{I}^S := \prod_{p \notin S} G(\widehat{\mathbb{Z}}_p),$$

is dense.

For a measurable subset  $E$  of  $G(\mathbb{R})$  and  $x \in G(\mathbb{R})$ , we set

$$E(x) := Ex.$$

We will analyze the distribution of the rational points  $G(\mathbb{Z}_S)$  in the subsets  $E(x) \times W$ , where  $W$  are compact open subsets of  $\mathcal{I}^S$ . We fix the invariant probability measure  $m^S$  on  $\mathcal{I}^S$ . In addition, we recall that  $m_S$  denotes the Haar measure on  $G_S$ , and  $m_\infty$  denotes the Haar measure on  $G(\mathbb{R})$ , which are normalized, so that  $G(\mathbb{Z}_S)$  has covolume one in  $G(\mathbb{R}) \times G_S$ . We set

$$R_S(h) := \{\gamma \in G(\mathbb{Z}_S) : H_f(\gamma) \leq h\},$$

$$B_S(h) := \{g \in G_S : H_f(g) \leq h\} \quad \text{and} \quad v_S(h) := m_S(B_S(h)).$$

We will be interested in estimating the cardinality  $|R_S(h) \cap (E(x) \times W)|$ . However, this cardinality might be infinite (for instance, when  $E$  contains a coset of the group  $G(\mathbb{Z})$ ). To address this issue, we define

$$\mathcal{N}(E) := |\{\gamma \in G(\mathbb{Z}) : m_\infty(E \cap \gamma E) > 0\}|,$$

and assume that  $\mathcal{N}(E) < \infty$ . For example, if  $E$  is bounded, this is always the case. We note that the assumption  $\mathcal{N}(E) < \infty$  does not imply that  $|R_S(h) \cap (E(x) \times W)| < \infty$  for all  $x$ . For example, if  $m_\infty(E) = 0$  and  $E$  contains  $G(\mathbb{Z})$ , then  $\mathcal{N}(E) = 0$  and the intersection is infinite for  $x = e$ , but nevertheless is empty for almost every  $x$ . In general, finiteness of  $\mathcal{N}(E)$  will allow us to control the  $L^2$ -norm of this cardinality, as a function of (almost every)  $x$ .

We define the discrepancy of the rational points  $G(\mathbb{Z}_S)$  as

$$\mathcal{D}(R_S(h), E(x) \times W) := \left| \frac{|R_S(h) \cap (E(x) \times W)|}{v_S(h)} - m_\infty(E)m^S(W) \right|. \tag{1.8}$$

Remarkably, we show that the discrepancy can be estimated for general measurable domains  $E$  of finite measure, as follows.

**THEOREM 1.2.** *Let  $G$  be a connected simply connected  $\mathbb{Q}$ -almost-simple linear algebraic group defined over  $\mathbb{Q}$ , and let  $S$  be a set of primes such that  $G$  is isotropic for some  $p \in S$ . Then there exists  $\mathfrak{k}_S(G) > 0$  such that for every measurable subset  $E$  of  $G(\mathbb{R})$  with finite measure satisfying*

$\mathcal{N}(E) < \infty$ , for every compact open subset  $W$  of  $\mathcal{I}^S$ , and for every  $\eta > 0$ ,

$$\|\mathcal{D}(R_S(h), E(\cdot) \times W)\|_{L^2(Q)} \ll_{S,Q,\eta} \mathcal{N}(E)^{1/2} m_\infty(E)^{1/2} m^S(W)^{1/2} v_S(h)^{-\mathfrak{k}_S(\mathbb{G})+\eta}$$

for every bounded measurable subset  $Q$  of  $\mathbb{G}(\mathbb{R})$ .

*Remark 1.3.* – The degenerate case of our theorem with  $m_\infty(E) = 0$  says that under this assumption  $|\mathbb{G}(\mathbb{Z}_S) \cap (E(x) \times W)| = 0$  for almost all  $x$ . This is also easy to check directly. When  $\mathcal{N}(E) = 0$ , we also have  $m_\infty(E) = 0$ .

- The discrepancy  $\mathcal{D}(R_S(h), E(x) \times W)$  might be infinite for some  $x$ . The theorem only gives an  $L^2$ -bound on the discrepancy.
- The spectral exponent  $\mathfrak{k}_S(\mathbb{G})$  will be given an explicit form in our discussion of the proof of Theorem 1.2 in § 4.

Using the  $L^2$ -bound established in Theorem 1.2, we also deduce an almost sure estimate on the discrepancy.

**THEOREM 1.4.** *With notation as in Theorem 1.2, for every  $0 < \mathfrak{k} < \mathfrak{k}_S(\mathbb{G})$ , and for almost every  $x \in \mathbb{G}(\mathbb{R})$ , and for every  $\eta > 0$ ,*

$$\mathcal{D}(R_S(h), E(x) \times W) \ll_{S,E,W,x,\mathfrak{k},\eta} (\log v_S(h))^{3/2+\eta} v_S(h)^{-\mathfrak{k}}.$$

*Remark 1.5.* We note that when  $m_\infty(E) = 0$ , Theorem 1.4 provides the almost sure upper bound

$$|R_S(h) \cap (E(x) \times W)| \ll_{S,E,W,x,\mathfrak{k},\eta} (\log v_S(h))^{3/2+\eta} v_S(h)^{1-\mathfrak{k}}$$

for the number of rational points of height bounded by  $h$  in  $E$  satisfying the congruence constraints defined by  $W$ . Therefore, it amounts to a general non-concentration phenomenon for rational points. As an example,  $E \subset G_\infty$  can be any smoothly embedded submanifold of positive co-dimension satisfying  $\mathcal{N}(E) < \infty$ .

Clearly, one has to impose additional assumptions on the sets  $E$  to expect an estimate for the discrepancy  $\mathcal{D}(R_S(h), E(x) \times W)$  that is valid for all  $x$ . Indeed, when the sets  $E$  satisfy a suitable regularity property, we establish such a pointwise bound, as follows.

We say that a measurable subset  $E$  of  $\mathbb{G}(\mathbb{R})$  is *right-stable* if for some  $\epsilon_0 > 0$

$$m_\infty(E_\epsilon^+ \setminus E_\epsilon^-) \ll \epsilon \quad \text{for every } \epsilon \in (0, \epsilon_0), \tag{RS}$$

where

$$E_\epsilon^+ := EB(e, \epsilon) \quad \text{and} \quad E_\epsilon^- := \{x \in \mathbb{G}(\mathbb{R}) : xB(e, \epsilon) \subset E\}.$$

*Remark 1.6.* The definition of right-stability is motivated by the notion of well-roundedness from [GN12], but is considerably more general. Note that many sets of measure zero (including finite sets) are right-stable. These include, for example, the intersection of smoothly embedded positive-codimension submanifolds of  $G$  with a norm ball.

Our pointwise error estimate will now depend on the dimension  $d := \dim_{\mathbb{R}}(\mathbb{G}(\mathbb{R}))$ .

**THEOREM 1.7.** *Let  $\mathbb{G}$  be a connected simply connected  $\mathbb{Q}$ -almost-simple linear algebraic group defined over  $\mathbb{Q}$ , and let  $S$  be a set of primes such that  $\mathbb{G}$  is isotropic for some  $p \in S$ . Let  $E$  be a right-stable finite-measure subset of  $\mathbb{G}(\mathbb{R})$  satisfying  $\mathcal{N}(E_{\epsilon_0}^+) < \infty$ , and  $W$  compact open subset of  $\mathcal{I}^S$ . Then for every  $x \in \mathbb{G}(\mathbb{R})$  and  $0 < \mathfrak{k} < \mathfrak{k}_S(\mathbb{G})$*

$$\begin{aligned} |R_S(h) \cap (E(x) \times W)| &= m_\infty(E)m^S(W)v_S(h) \\ &\quad + O_{S,E,x,\mathfrak{k}}(m^S(W)^{(d+1)/(d+2)} v_S(h)^{1-2\mathfrak{k}/(d+2)}), \end{aligned}$$

provided that  $m^S(W) \gg_{S, \mathfrak{k}} v_S(h)^{-2\mathfrak{k}}$ . When  $m_\infty(E) > 0$ , this condition is equivalent to the error term in the estimate being bounded by the main term.

Explicitly, if the volume growth satisfies  $v_S(h) \gg_{S, \mathfrak{a}} h^\mathfrak{a}$ , with some  $\mathfrak{a} > 0$ , then the estimate holds provided that the height  $h$  satisfies  $h \gg_{S, \mathfrak{a}, \mathfrak{k}} (m^S(W))^{-1/2\mathfrak{a}\mathfrak{k}}$ . Moreover, the above estimate is uniform for  $x$  ranging in compact subsets of  $G(\mathbb{R})$ .

In particular, when  $m_\infty(E) > 0$ , it follows that for such  $h$ ,  $R_S(h) \cap (E(x) \times W) \neq \emptyset$ , namely the domain  $E(x) \times W$  contains a point in  $G(\mathbb{Z}_S)$  (namely an  $S$ -integral point) with height at most  $h$ .

We note that Theorem 1.7 and its more general version stated in § 4 generalize several earlier results, including [Clo02], [Duk03] and [BO12, Theorem 1.8].

The method of the proof of Theorem 1.7 can be used to establish bounds on discrepancy which are uniform over variable families of sets. To demonstrate the utility of this fact, we analyze the discrepancy with respect to the entire family of Riemannian balls  $B(x, l)$  in  $G(\mathbb{R})$ , with  $x \in G$  and  $0 < l < l_0$ , and establish explicit discrepancy estimates for all points in the group, at arbitrary small scales.

**THEOREM 1.8.** *Let  $G$  be a connected simply connected  $\mathbb{Q}$ -almost-simple linear algebraic group defined over  $\mathbb{Q}$ , and let  $S$  be a set of primes such that  $G$  is isotropic for some  $p \in S$ . Fix a compact open subset  $W$  of  $\mathcal{I}^S$ , and  $0 < \mathfrak{k} < \mathfrak{k}_S(G)$ . For a suitable  $\ell_0 > 0$ , we set, for  $0 < \ell < \ell_0$ ,*

$$\mathcal{E}_{\ell, W}(h) := m_\infty(B(e, \ell))^{d/d+2} m^S(W)^{(d+1)/(d+2)} v_S(h)^{1-2\mathfrak{k}/(d+2)},$$

Then, for every  $x \in G(\mathbb{R})$ ,

$$|R_S(h) \cap (B(x, \ell) \times W)| = m_\infty(B(e, \ell)) m^S(W) v_S(h) + O_{S, x, \mathfrak{k}}(\mathcal{E}_{\ell, W}(h)),$$

provided that  $m_\infty(B(e, \ell))^2 m^S(W) \gg_{S, \mathfrak{k}} v_S(h)^{-2\mathfrak{k}}$ . We note that this condition is equivalent to the error term in the estimate being bounded by the main term.

Explicitly, if the volume growth satisfies  $v_S(h) \gg_{S, \mathfrak{a}} h^\mathfrak{a}$ , with some  $\mathfrak{a} > 0$ , then the estimate holds provided that the height  $h$  satisfies

$$h \gg_{S, \mathfrak{a}, \mathfrak{k}} \ell^{-d/\mathfrak{k}\mathfrak{a}} m^S(W)^{-(d+2)/2\mathfrak{k}\mathfrak{a}}.$$

Moreover, this estimate is uniform for  $x$  ranging in compact subsets of  $G(\mathbb{R})$ .

The results stated in this section will be proven in § 4.

We note that the effective estimates on discrepancy stated in Theorems 1.7 and 1.8 can also be viewed as establishing an effective count for the number of rational solutions of intrinsic Diophantine inequalities. This latter problem can be reduced to a lattice counting problem, and was given a short and simple solution in [GGN22], under more restrictive hypotheses than those of Theorems 1.7 and 1.8. Indeed  $G(\mathbb{Z}_S)$  can be viewed as a lattice subgroup in  $G(\mathbb{R}) \times G_S$ , and when a (variable) family of balls  $D(x, \varepsilon) \times B_S(h)$  is suitably well-rounded (as a function of  $\varepsilon$ ), the effective solution of the lattice point counting problem in [GN12] can be applied.

Theorems 1.7 and 1.8 generalize the results in [GGN22] in several respects, as follows.

- The set  $E$  in Theorem 1.7 need only be a measurable set of finite measure satisfying right-stability, a considerably weaker condition than well-roundedness.
- A stronger error bound for the discrepancy  $\mathcal{D}(R_S(h), B(x, l) \times \mathcal{I}^S)$  is established in Theorem 1.8. For comparison, in the case of Riemannian balls  $B(x, \ell)$ , the method of the present paper gives the bound, for any  $\eta > 0$

$$\ll_{S, x, \eta} m_\infty(B(e, \ell))^{d/(d+2)} v_S(h)^{(-2\mathfrak{k}/(d+2))+\eta},$$

whereas the bound established in [GGN22, Theorem 1.3] gives, for a set  $S$  of unramified primes (see the discussion following Theorem 2.6 below), and for the choice  $W = \mathcal{I}^S$  (namely in the absence of congruence conditions),

$$\ll_{S,x,\eta} m_\infty(B(e, \ell))^{d/(d+1)} v_S(h)^{(-\mathfrak{k}/(d+1))+\eta}.$$

- An arbitrary congruence constraint is allowed on the rational points in  $G(\mathbb{Z}_S)$  involved in the approximation process, given by an arbitrary compact open subset  $W$  of  $\mathcal{I}^S$ .

Furthermore, we will generalize Theorems 1.2–1.8 in two additional important respects, namely we will consider *every* connected almost  $\mathbb{Q}$ -simple algebraic group, not only simply connected groups, and we will consider *every* non-empty subset  $S \subset P$  of primes over which the  $\mathbb{Q}$ -group  $G$  is isotropic, including subsets which contain ramified primes. Each of these extensions requires elaborate arguments using structure theory of adèle groups and spectral results in their automorphic representations. Now let us turn to formulating the corresponding results.

### 1.3 Discrepancy bounds for general $\mathbb{Q}$ -groups

We now allow the group  $G$  to be a general  $\mathbb{Q}$ -almost simple group, not necessarily simply connected. In this case, the strong approximation property fails, and, in particular, the embedding of  $G(\mathbb{Z}_S)$  in  $G(\mathbb{R})$  is not dense, typically. Nonetheless, one can show (cf. Corollary 2.2) that the closure of  $G(\mathbb{Z}_S)$  in  $G(\mathbb{R})$  is a finite index subgroup of  $G(\mathbb{R})$ . Since  $\overline{G(\mathbb{Z}_S)}$  decomposes as a finite union of cosets of  $G(\mathbb{R})^0$ , the connected component of identity in  $G(\mathbb{R})$ , it is sufficient to analyze the distribution of  $G(\mathbb{Z}_S)$ -points in  $G(\mathbb{R})^0$ .

To state our results precisely, we need to take into account the contribution of automorphic characters, as follows. Let  $\mathcal{X}(G, \mathcal{I}_f)$  denote the set of continuous unitary characters  $\chi$  on the adèle group  $G(\mathbb{A}_{\mathbb{Q}})$  such that  $\chi(G(\mathbb{Q})) = 1$  and  $\chi(G(\widehat{\mathbb{Z}}_p)) = 1$  for all  $p$ . We denote by  $G^{\text{ker}}$  the joint kernel of this (finite) set of characters of  $G(\mathbb{A}_{\mathbb{Q}})$ , and note that  $G^{\text{ker}}$  is a finite index subgroup of  $G(\mathbb{A}_{\mathbb{Q}})$  (see, for instance, [GGN13, Lemma 4.4]).

We set

$$G_S^{\text{ker}} := G_S \cap G^{\text{ker}} \quad \text{and} \quad v_S^{\text{ker}}(h) := m_S(\{g \in G_S^{\text{ker}} : H_f(g) \leq h\}).$$

Let  $m_\infty^0$  be the Haar measure on  $G(\mathbb{R})^0$  normalized so that the intersection of  $G(\mathbb{Z}_S)$  with  $G(\mathbb{R})^0 \times G_S^{\text{ker}}$  has covolume one in  $G(\mathbb{R})^0 \times G_S^{\text{ker}}$  with respect to the measure  $m_\infty^0 \times m_S$ . For  $E \subset G(\mathbb{R})^0$ , we consider the discrepancy

$$\mathcal{D}(R_S(h), E) := \left| \frac{|R_S(h) \cap E|}{v_S^{\text{ker}}(h)} - m_\infty^0(E) \right|. \tag{1.9}$$

Note that in the foregoing expression we did not impose any congruence conditions on the rational points involved, as we did in the simply connected case, although our methods certainly allow for this possibility. However, since in the non-simply connected case certain congruence obstructions typically do arise, a complete analysis of the discrepancy of rational points subject to congruence conditions requires considerably more notation and further discussion, which we will avoid in order to keep the exposition more accessible.

We shall show that, with this setup, an analogue of Theorem 1.2 holds.

**THEOREM 1.9.** *Let  $G$  be a connected  $\mathbb{Q}$ -almost-simple linear algebraic group defined over  $\mathbb{Q}$ , and let  $S$  be a set of primes such that  $G$  is isotropic for some  $p \in S$ . Then there exists  $\mathfrak{k}_S(G) > 0$  such that for every measurable subset  $E$  of  $G(\mathbb{R})^0$  of finite measure satisfying  $\mathcal{N}(E) < \infty$ , and*



any  $\eta > 0$

$$\|\mathcal{D}(R_S(h), E(\cdot))\|_{L^2(Q)} \ll_{S,Q,\eta} \mathcal{N}(E)^{1/2} m_\infty^0(E)^{1/2} m^S(W)^{1/2} v_S^{\ker}(h)^{-\ell_S(\mathbf{G})+\eta}$$

for every bounded measurable subset  $Q$  of  $\mathbf{G}(\mathbb{R})^0$ .

Theorem 1.9 will be proved in §5.

*Remark 1.10.* The estimate of Theorem 1.9 is similar in form to Theorem 1.2, but there is an important, if subtle, difference between them. The normalization of Haar measure  $m_\infty^0$  is typically different than that of  $m_\infty$ , and the volume functions  $v_S(h)$  and  $v_S^{\ker}(h)$ , while comparable, are typically also different. Therefore, the existence of automorphic characters influences the size of the main term in the asymptotic formulas associated with (1.8) and (1.9), replacing  $v_S(h)m_\infty(E)$  which arise in the simply connected case by  $v_S^{\ker}(h)m_\infty^0(E)$ .

*Remark 1.11.* We note that the closure  $\overline{\mathbf{G}(\mathbb{Z}_S)}$  in  $\mathbf{G}(\mathbb{R})$  is a union of finitely many cosets  $\gamma_i \mathbf{G}(\mathbb{R})^0$  with  $\gamma_i \in \mathbf{G}(\mathbb{Z}_S)$ . Given a subset  $E$  of  $\overline{\mathbf{G}(\mathbb{Z}_S)}$ , we may decompose it as  $E = \bigsqcup_i E_i$  with  $E_i := E \cap \gamma_i \mathbf{G}(\mathbb{R})^0$  and estimate the discrepancy separately for each of the subset  $E_i$ . Hence, our method can be used more generally to analyze discrepancy for subsets  $E \subset \overline{\mathbf{G}(\mathbb{Z}_S)}$ .

Once the mean-square bound for  $\mathcal{D}(R_S(h), E(x))$  has been established, we can also prove generalizations of Theorems 1.2–1.8 with obvious modifications, which will be explained further in §5.

In summary, in the present section we gave an account of our results for almost-simple linear algebraic groups defined over  $\mathbb{Q}$ , to simplify the presentation. Our methods, however, are completely general, and from now on we will turn to developing discrepancy estimates when:

- the ground field  $K$  is an arbitrary algebraic number field, namely a finite-dimensional extension of  $\mathbb{Q}$ ;
- the group  $\mathbf{G}$  is an arbitrary connected semisimple linear algebraic group defined and simple over  $K$ , not necessarily simply connected;
- the set of places  $S$  of  $K$  may contain ramified places;
- the approximating rational elements are subject to an arbitrary congruence constraint when the group is simply connected.

## 2. Notation and preliminary results

### 2.1 Algebraic groups over number fields

We start by reviewing basic properties of semisimple algebraic groups over number fields (cf. [PR94, Ch. 3–5]). Throughout the paper,  $K$  denotes an algebraic number field (namely, a finite extension of the field of rationals  $\mathbb{Q}$ ). Let  $V_K$  be the set of normalized absolute values  $|\cdot|_v$  of the field  $K$ . The set of non-trivial absolute values decomposes as

$$V_K = V_K^\infty \sqcup V_K^f,$$

where  $V_K^\infty$  is the finite set of Archimedean absolute values and  $V_K^f$  the set of non-Archimedean absolute values. For  $v \in V_K$ , we write  $K_v$  for the corresponding completion of  $K$ , and when  $v \in V_K^f$ , we denote by

$$O_v := \{x \in K_v : |x|_v \leq 1\}$$

the ring of integers in  $K_v$ . For  $S \subset V_K^f$ , we write

$$O_S := \{x \in K : |x|_v \leq 1 \text{ for } v \in V_K^f \setminus S\},$$

for the ring of  $S$ -integers,<sup>1</sup> namely elements in  $K$  that are integral with respect to every completion except possibly those in  $S$ . In particular,  $O = O_\emptyset$  denotes the ring of integers in  $K$ .

We denote by

$$\mathbb{A}_K := \{(x_v)_{v \in V_K} : |x_v|_v \leq 1 \text{ for almost all } v\}$$

the ring of adèles of  $K$ , which is the restricted direct product of  $K_v$ ,  $v \in V_K$ , with respect to compact open subrings  $O_v \subset K_v$ ,  $v \in V_K^f$ .

Let  $G \subset \text{SL}_n$  be a connected semisimple linear algebraic group defined over the field  $K$ . The focus of our investigation is the distribution of the set of  $S$ -integral points

$$\Gamma_S := G(O_S).$$

We denote by

$$G_v := G(K_v) \quad \text{for } v \in V_K,$$

the locally compact groups of  $K_v$ -points of  $G$  equipped with the topology defined by the corresponding absolute values  $|\cdot|_v$ . We say that  $G$  is *isotropic* over  $v$  if  $G_v$  is non-compact, and  $G$  is *isotropic* over  $S \subset V_K$  if  $G_v$  is non-compact for at least one  $v \in S$ .

For  $v \in V_K^f$ , we consider

$$G(O_v) := \{g \in G_v : \|g\|_v \leq 1\},$$

which is a compact open subgroup of  $G_v$ . We introduce compact groups

$$\mathcal{I}_S := \prod_{v \in S} G(O_v) \quad \text{and} \quad \mathcal{I}^S := \prod_{v \in V_K^f \setminus S} G(O_v).$$

For  $S \subset V_K$ , we write

$$G_S := \prod_{v \in S}^I G_v = \{(g_v)_{v \in S} : g_v \in G_v, \|g_v\|_v \leq 1 \text{ for almost all } v\}$$

for the restricted direct product of the groups  $G_v$  with respect to the compact open subgroups  $G(O_v)$ ,  $v \in S \cap V_K^f$ . Then  $G_S$  is a locally compact group. For instance,  $G(\mathbb{A}_K) = G_{V_K}$  is the adèle group associated to  $G$ . To simplify notation, we also write

$$G_\infty := G_{V_K^\infty} \quad \text{and} \quad G_f := G_{V_K^f}.$$

We recall that when  $G$  is simply connected, the group  $G_\infty$  is connected with respect to the Euclidean topology (cf. [PR94, Ch. 7, Proposition 7.2]). In general,  $G_\infty$  has finitely many connected components, and we denote by  $G_\infty^0$  the connected component of the identity in  $G_\infty$ .

For  $v \in V_K$ , we denote by  $m_v$  the Haar measure on  $G_v$ , and when  $v \in V_K^f$  we normalize it so that  $m_v(G(O_v)) = 1$ . Then for a subset  $S \subset V_K^f$ , the product measure  $m_S := \prod_{v \in S} m_v$  defines a Haar measure on  $G_S$  such that  $m_S(\mathcal{I}_S) = 1$ . We also denote by  $m_\infty$  a Haar measure on  $G_\infty$ . Under the diagonal embedding  $\Gamma_S \hookrightarrow G_\infty \times G_S$ , the group  $\Gamma_S$  is a discrete subgroup with finite covolume in  $G_\infty \times G_S$  (cf. [PR94, Ch. 5]). We normalize the measure  $m_\infty$ , so that  $\Gamma_S$  has covolume one with respect  $m_\infty \times m_S$ .

We recall the strong approximation property [PR94, § 7.4].

---

<sup>1</sup> Note that  $O_{\{v\}} \neq O_v$ , the ring of integers just defined!

**THEOREM 2.1** (Strong approximation). *Let  $G$  be a simply connected  $K$ -simple algebraic group defined over  $K$ . Then if  $G$  is isotropic over  $S \subset V_K^f$ , the image of  $\Gamma_S$  with respect to the embedding  $\Gamma_S \hookrightarrow G_\infty \times \mathcal{I}^S$  is dense. More generally, the embedding  $G(K) \hookrightarrow G_\infty \times G_{V_K^f \setminus S}$  has dense image.*

This result fails if the group  $G$  is not simply connected even for the embedding  $\Gamma_S \hookrightarrow G_\infty$ . Nonetheless, one can deduce the following result about the closure.

**COROLLARY 2.2.** *Let  $G$  be a connected  $K$ -simple algebraic group defined over  $K$ . Then if  $G$  is isotropic over  $S \subset V_K^f$ , the closure of  $\Gamma_S$  in  $G_\infty$  is a finite index open subgroup. In particular,  $\overline{\Gamma_S} \supset G_\infty^0$ .*

*Proof.* Let us consider the simply connected cover  $\mathfrak{p} : \tilde{G} \rightarrow G$ . Then the groups

$$U_v := \mathfrak{p}^{-1}(G(O_v)) \cap \tilde{G}(O_v) \quad \text{with } v \in V_K^f,$$

are compact open subgroups of  $\tilde{G}_v$ . It follows from Theorem 2.1 that the image of the group

$$\Gamma := \tilde{G}(O_S) \cap \left( \tilde{G}_\infty \times \prod_{v \in V_K^f \setminus S} U_v \right)$$

is dense in  $\tilde{G}_\infty$ . Since  $\mathfrak{p}(\Gamma) \subset \Gamma_S$ , it follows that the closure  $\overline{\Gamma_S}$  in  $G_\infty$  contains  $\mathfrak{p}(\tilde{G}_\infty)$  which is an open and closed subgroup of finite index in  $G_\infty$  (cf. [PR94, § 3.2]). □

### 2.2 Automorphic representations

Let  $G$  be a connected  $K$ -simple algebraic group defined over a number field  $K$ . We consider the Hilbert space

$$\mathcal{H}_G := L^2(G(\mathbb{A}_K)/G(K))$$

consisting of square-integrable functions on the space  $G(\mathbb{A}_K)/G(K)$  equipped with the invariant probability measure  $\mu$ . Let

$$\mathcal{H}_G^0 := L_0^2(G(\mathbb{A}_K)/G(K)) = \left\{ \phi \in \mathcal{H}_G : \int_{G(\mathbb{A}_K)/G(K)} \phi \, d\mu = 0 \right\}.$$

A continuous unitary character  $\chi$  of  $G(\mathbb{A}_K)$  is called *automorphic* if  $\chi(G(K)) = 1$ . Then  $\chi$  can be considered as an element of  $\mathcal{H}_G$ . We denote by  $\mathcal{H}_G^{00}$  the subspace of  $\mathcal{H}_G$  orthogonal to all automorphic characters. We note that when  $G$  is simply connected there no non-trivial automorphic characters and  $\mathcal{H}_G^{00} = \mathcal{H}_G^0$ .

Now we describe our choice of maximal compact subgroups  $U_v$  of  $G_v$ . For all but finitely many  $v \in V_K^f$ :

- (i)  $G(O_v)$  is a hyperspecial, good maximal compact subgroup of  $G_v$ ;
- (ii) the group  $G$  is unramified over  $K_v$  (that is,  $G$  is quasi-split over  $K_v$  and split over an unramified extension of  $K_v$ ).

We say that  $G$  is *unramified* over such  $v$ . For those  $v$ , we set  $U_v := G(O_v)$ . For the remaining (finite) set of finite places  $v$ , we fix a good special maximal compact subgroup  $U_v$  of  $G_v$ . For any subset  $S \subset V_K^f$ , we set

$$U_S := \prod_{v \in S} U_v \quad \text{and} \quad U^S := \prod_{v \in V_K^f \setminus S} U_v.$$

For places  $v \in V_K^f$ , we denote by  $\pi_v^{\text{aut}} = \pi_v$  the unitary representation of the group  $G_v$  on the space  $\mathcal{H}_G$ . The *spherical integrability exponent* of the representations  $\pi_v$ , with respect to the

subgroup  $U_v$ , is defined by

$$\mathfrak{q}_v(\mathbf{G}) := \inf \left\{ q \geq 2 : \forall U_v\text{-inv. } \phi \in \mathcal{H}_{\mathbf{G}}^{00} \right. \\ \left. \langle \pi_v(g)\phi, \phi \rangle \in L^q(G_v) \right\}. \tag{2.1}$$

It is a fundamental result in the theory of automorphic representations that the integrability exponents  $\mathfrak{q}_v(\mathbf{G})$  are finite and, moreover,  $\mathfrak{q}_v(\mathbf{G})$  is uniformly bounded over  $v$ , see [Clo03]. These exponents can be estimated in terms of the Satake parameters of the corresponding spherical automorphic representations. We refer to [Clo07, Sar05] for surveys of some of these results.

More generally, for  $S \subset V_K^f$ , we denote by  $\pi_S^{\text{aut}} = \pi_S$  the unitary representation of the group  $G_S$  on the space  $\mathcal{H}_{\mathbf{G}}$ . We define

$$\mathfrak{q}_S(\mathbf{G}) := \inf \left\{ q \geq 2 : \forall U_S\text{-inv. } \phi \in \mathcal{H}_{\mathbf{G}}^{00} \right. \\ \left. \langle \pi_S(g)\phi, \phi \rangle \in L^q(G_S) \right\}. \tag{2.2}$$

This integrability exponent is also finite and it can be estimated in terms of the exponents  $\mathfrak{q}_v(\mathbf{G})$ ,  $v \in S$ , see [GGN13, Cor. 3.5].

Given a strongly continuous unitary representation  $\pi : G_S \rightarrow \mathcal{U}(\mathcal{H})$  on a Hilbert space  $\mathcal{H}$  and a finite Borel measure  $\nu$  on  $G_S$ , we define the averaging operator

$$\pi_S(\nu) : \mathcal{H} \rightarrow \mathcal{H} : \phi \mapsto \int_{G_S} \pi(g)\phi \, d\nu(g).$$

We recall the following estimate on the norm of the averaging operators.

**THEOREM 2.3** [GGN13, Proposition 3.8]. *Let  $\beta$  be a Haar-uniform probability measure supported on a  $U_S$ -bi-invariant bounded subset  $B$  of  $G_S$ . Then*

$$\|\pi_S^{\text{aut}}(\beta)|_{\mathcal{H}_{\mathbf{G}}^{00}}\| \ll_{S,\eta} m_S(B)^{-1/\mathfrak{q}_S(\mathbf{G})+\eta} \quad \text{for all } \eta > 0.$$

We note that although the measure  $m_S$  in [GGN13] was normalized differently (namely, so that  $m_S(U_S) = 1$ ), this gives the same bound up to a multiplicative constant.

An important fact, underlying our considerations below regarding ramified places, is that this result also holds for more general averaging operators. The crucial ingredient here is finiteness of a more general integrability exponent, which we define as follows:

$$\mathfrak{p}_S(\mathbf{G}) := \inf \left\{ p \geq 2 : \forall \phi_1, \phi_2 \text{ in a dense subspace of } \mathcal{H}_{\mathbf{G}}^{00} \right. \\ \left. \langle \pi_S^{\text{aut}}(g)\phi_1, \phi_2 \rangle \in L^p(G_S) \right\}. \tag{2.3}$$

One says that a unitary representation  $\pi : G_S \rightarrow \mathcal{U}(\mathcal{H})$  is  $L^p$ -integrable if for  $\phi_1, \phi_2$  in a dense subset of  $\mathcal{H}$ , the functions  $g \mapsto \langle \pi(g)\phi_1, \phi_2 \rangle$  is in  $L^p(G_S)$ . Thus,

$$\mathfrak{p}_S(\mathbf{G}) = \inf \{ p \geq 2 : \pi_S^{\text{aut}}|_{\mathcal{H}_{\mathbf{G}}^{00}} \text{ is } L^p\text{-integrable} \}.$$

It was proved in [GMO08] that this exponent is finite provided that  $\mathbf{G}$  is either simply connected or adjoint (see [GMO08, Theorem 3.20 and Theorem 3.7]). We shall show in Theorem 5.1 below that the exponent is finite for general  $K$ -simple groups.

We define

$$\mathfrak{n}_S(\mathbf{G}) := \begin{cases} \text{the least even integer } \geq \mathfrak{p}_S(\mathbf{G})/2, & \text{if } \mathfrak{p}_S(\mathbf{G}) > 2, \\ 1, & \text{if } \mathfrak{p}_S(\mathbf{G}) = 2. \end{cases}$$

With this notation, we have the following result.

**THEOREM 2.4.** *Let  $W_S$  be a compact open subgroup of  $G_S$ , and  $\beta$  a Haar-uniform probability measure supported on a  $W_S$ -bi-invariant bounded subset  $B$  of  $G_S$ . Then*

$$\|\pi_S^{\text{aut}}(\beta)|_{\mathcal{H}_G^{00}}\| \ll_{W_S, \eta} m_S(B)^{-1/4\text{ns}(G)+\eta} \quad \text{for all } \eta > 0.$$

*Proof.* The proof is a generalization of the proof of [GN12, Cor. 6.7], and so we only provide an outline and refer to [GN12, §6] for further details. The main idea, which originated in [CHH88] and [Nev98], is to observe that a suitable tensor power of  $\pi_S$  restricted to  $\mathcal{H}_G^{00}$  is weakly contained in (a multiple of) the regular representation  $\lambda_S$  of  $G_S$  on  $L^2(G_S)$ , and then use a generalization of the Kunze–Stein convolution inequality valid in  $L^2(G_S)$ . More precisely, the representation  $(\pi_S|_{\mathcal{H}_G^{00}})^{\otimes \text{ns}(G)}$  is  $L^p$ -integrable for all  $p > 2$ , so that it is weakly contained in the regular representation  $\lambda_G$  (by [CHH88]), which allows us to deduce (by [Nev98]) that for any probability density  $\beta'$  on  $G_S$

$$\|\pi_S^{\text{aut}}(\beta')|_{\mathcal{H}_G^{00}}\| \leq \|\lambda_S(\beta')\|^{1/\text{ns}(G)}. \tag{2.4}$$

Now let

$$B' := U_S B U_S$$

and denote by  $\beta'$  the Haar-uniform probability measure supported on  $B'$ . Since  $U_S \cap W_S$  has finite index in both  $U_S$  and  $W_S$ , it is clear that there exist  $c_1, c_2 > 0$ , depending only on  $W_S$ , such that

$$c_1 m_S(B') \leq m_S(B) \leq c_2 m_S(B'),$$

so that

$$\|\lambda_S(\beta)\| \leq c \|\lambda_S(\beta')\| \tag{2.5}$$

for some  $c > 0$ . Finally, using the decomposition of  $m_S$  with respect to the Iwasawa decomposition on  $G_S$ , we deduce (cf. [GN12, Theorem 6.6]) that  $\|\lambda_S(\beta')\|$  can be estimated in terms of the Harish-Chandra function on  $G_S$ . The Harish-Chandra function in this case is  $L^{4+\eta}$ -integrable for all  $\eta > 0$  (by [GN12, Proposition 6.3]), and we deduce that

$$\|\lambda_S(\beta')\| \ll_{S, \eta} m(B')^{-1/4+\eta} \quad \text{for all } \eta > 0. \tag{2.6}$$

We note that this argument applies to groups that possess Cartan and Iwasawa decompositions, and it does not require that the group be simply connected. Combining (2.4), (2.5) and (2.6), we deduce the theorem. □

### 2.3 Mean ergodic theorem for simply connected groups

In our discussion below we aim to consider approximation by elements of  $\Gamma_S$  subject to arbitrary additional congruence conditions. Such a condition is determined by a compact open subset  $W \subset \mathcal{I}^S$ , and to facilitate this discussion we will now reformulate Theorem 2.4 in a more general and explicit form.

**LEMMA 2.5.** *For a compact open subset  $W \subset \mathcal{I}^S$  there exists a compact open subgroup  $U(W)$  of  $\mathcal{I}^S$  such that  $W$  is bi-invariant under  $U(W)$ , and  $U(W)$  is a maximal subgroup of  $\mathcal{U} = \mathcal{I}^S$  with this property.*

*Proof.* Since  $W$  is open, for every  $w \in W$ , there exists a compact open subgroup  $\mathcal{U}_w$  of  $\mathcal{I}^S$  such that  $\mathcal{U}_w w \subset W$ . Then by compactness,  $W = \bigcup_{i=1}^{\ell} \mathcal{U}_{w_i} w_i$ . This implies that  $W$  is left-invariant under the compact open subgroup  $\mathcal{U}' = \bigcap_{i=1}^{\ell} \mathcal{U}_{w_i}$ . A similar argument shows that  $W$  is also right-invariant under a compact open subgroup  $\mathcal{U}''$ . Therefore,  $U'(W) = \mathcal{U}' \cap \mathcal{U}''$  is compact and

open and leaves  $W$  bi-invariant. It is clear that there exists a maximal open (and, hence, closed and compact) subgroup with this property, and we denote this subgroup by  $U(W)$ .  $\square$

Note that  $U(W)$  depends on  $S$  also, but we suppress this dependence in the notation. Let

$$\Gamma_S(W) := \Gamma_S \cap (G_\infty \times G_S \times U(W)).$$

Since  $U(W)$  is a finite-index subgroup of  $\mathcal{I}^S$ , it follows that  $\Gamma_S(W)$  has finite index in  $\Gamma_S$ . In particular,  $\Gamma_S(W)$  is a lattice subgroup of  $G_\infty \times G_S$ . We consider the homogeneous space

$$Y_{S,W} := (G_\infty \times G_S) / \Gamma_S(W) \tag{2.7}$$

equipped with the Haar probability measure  $\mu_{S,W}$ .

The group  $G_S$  naturally acts on the space  $Y_{S,W}$  by left translations, and we introduce averaging operators

$$\pi_{S,W}(\beta) : L^2(Y_{S,W}) \rightarrow L^2(Y_{S,W}) : \phi \mapsto \frac{1}{m_S(B)} \int_B \phi(g^{-1}y) dm_S(g) \tag{2.8}$$

defined for measurable subsets  $B$  of  $G_S$  with finite positive measures.

For simplicity, we skip the index  $W$  in the above notation if  $W = \mathcal{I}^S$ .

**THEOREM 2.6.** *Assume that  $G$  is simply connected, and isotropic over  $S \subset V_K^f$ . Let  $W \subset \mathcal{I}^S$  be a compact open subset, and let  $W_S$  be a compact open subgroup of  $G_S$ . Let  $\beta$  be the Haar-uniform probability measure supported on a  $W_S$ -bi-invariant bounded subset  $B$  of  $G_S$  with positive measure. Then there exists  $\mathfrak{k}_S(G) > 0$  such that for every  $\phi \in L^2(Y_{S,W})$ , and any  $\eta > 0$*

$$\left\| \pi_{S,W}(\beta)\phi - \int_{Y_{S,W}} \phi d\mu_{S,W} \right\|_{L^2(Y_{S,W})} \ll_{\eta, W_S} m_S(B)^{-\mathfrak{k}_S(G)+\eta} \|\phi\|_{L^2(Y_{S,W})}.$$

*Remark 2.7.* The exponent  $\mathfrak{k}_S(G)$  can be taken to be  $\mathfrak{k}_S(G) = 1/4n_S(G)$  (cf. Theorem 2.4). Furthermore, when the set  $B$  is  $U_S$ -bi-invariant, we may take the better exponent  $\mathfrak{k}_S(G) = 1/q_S(G)$  (cf. Theorem 2.3).

We note that the existence of the exponent  $\mathfrak{k}_S(G)$  and its uniformity over  $W \subset \mathcal{I}^S$ , is a deep property of the automorphic representation. It is a consequence of the spectral gap property of automorphic representations  $\pi_v$  (explained above) holding uniformly over congruence subgroups. In the case of  $SL_2$  for example, this property is known as the Ramanujan–Petersson–Selberg eigenvalue bounds. We refer to [BS91], [BLS92], [Clo03], [Sar05] [COU01], [CU04] and [Clo07] for further discussion.

*Proof of Theorem 2.6.* To simplify notation, we set

$$G := G(\mathbb{A}_K) \quad \text{and} \quad \Gamma := G(K).$$

We consider the action of the group  $G_\infty \times G_S$  on the double-coset space  $U(W) \backslash G / \Gamma$ . The orbits of this action are open (and, consequently, also closed). Since  $G$  is assumed to be isotropic over  $S$ , it follows from the strong approximation property (Theorem 2.1) that the projection of  $\Gamma$  to  $G_{V_K^f \setminus S}$  is dense. Therefore, the above orbits are also dense. Hence, we conclude that this action is transitive, and

$$Y_{S,W} \simeq U(W) \backslash G / \Gamma$$

as  $(G_\infty \times G_S)$ -spaces. In particular, we also deduce equivalence of unitary representations

$$L^2(Y_{S,W}) \simeq L^2(U(W) \backslash G / \Gamma) \tag{2.9}$$

of  $G_S$ . Furthermore, the space  $L^2(U(W)\backslash G/\Gamma)$  can be identified with the subspace  $L^2(G/\Gamma)^{U(W)}$  consisting of  $U(W)$ -invariant functions in  $L^2(G/\Gamma)$ . Since  $G$  is assumed to be simply connected, there are no non-trivial automorphic characters (see, for instance, [GGN13, Lemma 4.1]). Hence, it follows from Theorem 2.4 (or from Theorem 2.3 if  $B$  is  $U_S$ -bi-invariant) that there exists  $\mathfrak{k}_S(G) > 0$  such that for every  $\psi \in L^2(G/\Gamma)$ , and any  $\eta > 0$

$$\|\pi_S^{\text{aut}}(\beta)\psi - \mathcal{P}(\psi)\|_{L^2(G/\Gamma)} \ll_{W(S),\eta} m_S(B)^{-\mathfrak{k}_S(G)+\eta} \|\psi\|_{L^2(G/\Gamma)},$$

where  $\mathcal{P}$  denotes the orthogonal projection on the space of constant functions. Hence, the statement of the theorem follows from (2.9). □

We shall also prove and utilize a version of Theorem 2.6 for general  $K$ -simple groups, but we will postpone this until § 5.2 below.

**2.4 Riemannian local volume and distance estimates**

We now note two local properties of distance and volume in any almost connected semisimple Lie group with finite center, denoted by  $G_\infty$ , which will be used in our arguments below. We fix a right-invariant Riemannian metric  $\rho$  on  $G_\infty$  and consider the corresponding balls

$$B(g, r) := \{x \in G_\infty : \rho(x, g) \leq r\}.$$

We first establish the following local estimates for the volume  $m_\infty(B(g, r))$ . By right invariance, it clearly suffices to consider only the case where the center is the identity element  $e$ . Recall that we use the notation  $d = \dim_{\mathbb{R}} G_\infty$ .

LEMMA 2.8. (a) *There exist  $c_1, c_2 > 0$  and  $r_0 > 0$  such that*

$$c_1 r^d \leq m_\infty(B(e, r)) \leq c_2 r^d \quad \text{for all } r \in (0, r_0).$$

(b) *For  $0 < c_0 < 1$ , there exist  $c, r'_0 > 0$  such that for all  $\epsilon \in (0, c_0 r)$  and  $r \in (0, r'_0)$ ,*

$$m_\infty(B(e, r + \epsilon)) - m_\infty(B(e, r)) \leq c \frac{\epsilon}{r} m_\infty(B(e, r)).$$

*Proof.* According to the volume formula for Riemannian balls [Sak96, p. 66], for sufficiently small  $r$ ,

$$m_\infty(B(e, r)) = \int_0^r \omega(s) ds,$$

where  $\omega$  is a continuous function satisfying

$$c'_1 s^{d-1} \leq \omega(s) \leq c'_2 s^{d-1}$$

for some  $c'_1, c'_2 > 0$ . This implies the first estimate. In addition,

$$m_\infty(B(e, r + \epsilon)) - m_\infty(B(e, r)) = \int_r^{r+\epsilon} \omega(s) ds \ll (r + \epsilon)^{d-1} \epsilon,$$

which gives the second bound. □

Now let us define two metrics  $\rho$  and  $\rho'$  on  $G_\infty$  to be locally equivalent if for every compact neighborhood  $Q \subset G_\infty$  of  $I$ , there exists a constant  $C_Q$  such that for any two points  $x \neq y \in Q$  satisfying  $xy^{-1} \in Q$ , we have  $C_Q^{-1} \leq \rho(x, y)/\rho'(x, y) \leq C_Q$ .

Assuming that  $G_\infty \subset GL_d(\mathbb{R})$  (namely fixing a faithful linear representation of  $G_\infty$ , but suppressing it from the notation), fix a (vector space) norm on  $\text{Mat}_d(\mathbb{R})$  and consider the distance on  $G_\infty$  given by  $\rho'(x, y) = \|x - y\|$ ,  $x, y \in G_\infty$ .

LEMMA 2.9. Any right- (or left-)invariant Riemannian distance  $\rho$  on  $G_\infty$  is locally equivalent to the distance  $\rho'$  defined by any (vector space) norm in any faithful linear representation (as defined above).

*Proof.* The right-invariant Riemannian metric on  $G_\infty$  defining the right-invariant distance  $\rho$  is determined by the choice of a positive-definite inner product on the Lie algebra  $\mathfrak{g}_\infty$  of  $G_\infty$ . The inner product determines a Euclidean norm  $|X|$  on the Lie algebra. We have  $\mathfrak{g}_\infty \subset \text{Mat}_d(\mathbb{R})$ , and the exponential map denoted  $X \rightarrow e^X$  takes  $\mathfrak{g}_\infty$  into  $G$  and is a diffeomorphism on a (Euclidean) ball  $D_{\varepsilon_0}(0)$  centered at  $0 \in \mathfrak{g}_\infty$ . For  $X \in D_{\varepsilon_0}(0)$  we denote  $x = e^X \in G$ , and then  $\rho(I, x) = \rho(I, e^X) = |X| + O(|X|^2)$ . In  $\text{Mat}_d(\mathbb{R})$  we have  $\|I - x\| = \|I - e^X\| = \|X\| + O(\|X\|^2)$  when  $\|X\| < 1/2$  (say). Hence, there exists  $C_{Q_0}$  satisfying that  $C_{Q_0}^{-1} \leq \rho(I, x)/\|I - x\| \leq C_{Q_0}$  for  $x \neq I$  in the compact neighborhood of  $I$  given by the set  $Q_0 = \{e^X; X \in D_{\varepsilon_1}(0)\} \subset G_\infty$ , for suitable  $\varepsilon_1 > 0$ .

Since the operator norms of  $y$  and  $y^{-1}$  are bounded above and below when  $y \in Q_0$ , writing  $\|x - y\| = \|(I - xy^{-1})y\| \leq \|y\|_{\text{op}}\|I - xy^{-1}\|$ , and  $\|x - y\| \geq \|I - xy^{-1}\|/\|y^{-1}\|_{\text{op}}$ , we conclude that  $C_{Q_1}^{-1} \leq \rho(y, x)/\|y - x\| \leq C_{Q_1}$  for any  $x \neq y$  in compact neighborhood  $Q_1 \subset Q_0$  of  $I$ . It then follows by compactness and continuity that the same holds (with a different constant  $C_{Q_R}$ ) for  $x \neq y$  in any compact neighborhood  $Q_R$  of  $I$ .

Now fixing any compact set  $Q$ , given any  $x \in G_\infty$  the set  $Qx$  is contained in compact neighborhood  $Q_R$  of  $I$  for some  $R$  (depending on  $x$ ), and so for all  $y \in Qx, y \neq x$  we have  $C_Q^{-1} \leq \rho(y, x)/\|y - x\| \leq C_Q$ , as stated. □

### 3. Asymptotic formula for the counting function of Diophantine approximants

Let  $\mathbf{G}$  be a simply connected  $K$ -simple linear algebraic group defined over a number field  $K$ . We will freely use the notation introduced in §§ 1–2. Our goal in the present section is to establish an analogue of Schmidt’s theorem. We will analyze the number of Diophantine approximants  $r \in \Gamma_S$  to points  $x \in G_\infty$ , collect them into the counting function:

$$N_T(x) := |\{r \in \Gamma_S : \rho(x, r) \leq H_f(r)^{-b}, 1 \leq H_f(r) \leq T\}|. \tag{3.1}$$

and establish its asymptotics. We refer to the exponent  $b$  as the scale of approximation.

#### 3.1 The parameters of effective Diophantine approximation

Fix  $S \subset V_K^f$ . We recall that the group  $\Gamma_S$  embeds diagonally in  $G_\infty \times G_S$  as a lattice subgroup, and the Haar measures  $m_\infty$  and  $m_S$  on the factors are normalized so that  $\Gamma_S$  has covolume one with respect to  $m_\infty \times m_S$ . We expect that  $N_T(x)$  is approximated by the volume sum

$$V_T := \sum_{1 \leq h \leq T} m_\infty(\mathcal{B}_h)m_S(\Sigma_S(h)), \tag{3.2}$$

where

$$\mathcal{B}_h := \{x \in G_\infty : \rho(x, e) \leq h^{-b}\} \quad \text{and} \quad \Sigma_S(h) := \{g \in G_S : H_f(g) = h\}.$$

We recall that  $\mathcal{I}_S = \prod_{v \in S} \mathbf{G}(O_v)$ . Clearly, the set  $\Sigma_S(h)$  is compact and  $\mathcal{I}_S$ -bi-invariant, but it might be empty for some choices of  $h \in \mathbb{N}$ , so that we introduce

$$\mathcal{L}_S := \{h \in \mathbb{N} : \Sigma_S(h) \neq \emptyset\}.$$



The estimates in this section will depend on the following three parameters:

- $\mathfrak{a}$  – the volume growth rate of the sets  $\Sigma_S(h)$  (see (V1));
- $\mathfrak{k}$  – the error term in the ergodic theorem (see (SP));
- $\mathfrak{d}$  – the volume decay rate of balls  $\mathcal{B}_h$  (see (V2)).

We record some basic properties of the sets  $\Sigma_S(h)$  and  $\mathcal{L}_S$ .

LEMMA 3.1. *Let  $S$  be a finite subset of  $V_K^f$ .*

(a) *For every  $\eta > 0$ ,*

$$\sum_{T \in \mathcal{L}_S} T^{-\eta} < \infty \quad \text{and} \quad |\mathcal{L}_S \cap [1, T]| = O_\eta(T^\eta).$$

(b) *The set  $\log(\mathcal{L}_S)$  has bounded gaps.*

(c) *Suppose that  $G$  is isotropic over  $K_v$  for all  $v \in S$ . Then there exists an exponent  $\mathfrak{a} = \mathfrak{a}(S) > 0$  such that*

$$m_S(\Sigma_S(h)) \gg_S h^\mathfrak{a} \quad \text{for all } h \in \mathcal{L}_S. \tag{V1}$$

We note that in order to obtain the lower bound in terms of  $h$  in part (c), it is essential to assume that the group  $G$  is isotropic for all  $v \in S$ . This is the only place where we use this condition. When this condition does not hold, it is still possible to formulate a weaker lower bound.

*Proof of Lemma 3.1.* We observe that for  $g \in G_S$ , all values of the height  $H_f(g)$  are of the form  $\prod_{v \in S} q_v^{n_v}$ , where  $q_v$  denotes the norm of uniformizing parameter of  $K_v$ . Since  $S$  is assumed to be finite, this implies claim (a).

To prove claim (b), it will be convenient to consider  $G$  as a subgroup of  $SL_n$ . We note that when  $S_1 \subset S_2$ , we have  $\mathcal{L}_{S_1} \subset \mathcal{L}_{S_2}$ . Hence, it is sufficient to prove claim (b) when  $S$  consists of a single place  $v$ . Since  $G$  is isotropic over this place, it contains a non-trivial  $K_v$ -split torus  $A$ . There exists  $g \in SL_n(K_v)$  such that  $gAg^{-1}$  is diagonal. Let us fix  $a \in A(K_v)$  such that  $\|gag^{-1}\|_v > 1$ . Since  $\|ga^n g^{-1}\|_v = \|gag^{-1}\|_v^n$ , it is clear that the set  $\{\log \|ga^n g^{-1}\|_v : n \in \mathbb{N}\}$  has bounded gaps. Finally, we note that for some  $c > 1$ ,  $c^{-1}\|z\|_v \leq \|gzg^{-1}\|_v \leq c\|z\|_v$  for all  $z \in \text{Mat}_n(K_v)$ . This implies that the set  $\{\log \|a^n\|_v : n \in \mathbb{N}\}$  has bounded gaps as well.

Claim (c) was established in [GK17, Proposition 4.2] in the case of the field of rationals, and this argument generalizes to general number fields. □

For  $h \in \mathcal{L}_S$ , we denote by  $\sigma_h$  the Haar-uniform probability measure supported on the subset  $\Sigma_S(h)^{-1}$  of  $G_S$  and consider the corresponding averaging operator

$$\pi_S(\sigma_h) : L^2(Y_S) \rightarrow L^2(Y_S)$$

on the space  $Y_S := (G_\infty \times G_S)/\Gamma_S$  defined in (2.8). We note that  $\Sigma_S(h)^{-1} \neq \Sigma_S(h)$  in general, but both sets have the same Haar measure. By Theorem 2.6, there exists  $\mathfrak{k}_S(G) = \mathfrak{k}_S \in (0, 1/2)$  such that for all  $\phi \in L^2(Y_S)$ , and any  $\eta > 0$

$$\|\pi_S(\sigma_h)\phi - \int_{Y_S} \phi d\mu_S\|_{L^2(Y_S)} \ll_{S,\eta} m_S(\Sigma_S(h))^{-\mathfrak{k}_S+\eta} \quad \text{for all } h \in \mathcal{L}_S. \tag{SP}$$

We also recall that by Lemma 2.8(a),

$$h^{-b\mathfrak{d}} \ll m_\infty(\mathcal{B}_h) \ll h^{-b\mathfrak{d}} \quad \text{for all } h \geq h_0, \tag{V2}$$

where  $\mathfrak{d} := \dim_{\mathbb{R}}(G_\infty)$ .

We set

$$b_0 := 2\mathfrak{a}\mathfrak{k}/\mathfrak{d}$$

and for a positive scale  $b < b_0$ , define

$$\theta_0(b) := \frac{(1 - \mathfrak{k})\mathfrak{a} - b\mathfrak{d}/2}{\mathfrak{a} - b\mathfrak{d}} = \frac{1}{2} + \frac{(1/2 - \mathfrak{k})\mathfrak{a}}{\mathfrak{a} - b\mathfrak{d}}.$$

Note that the condition on  $b < b_0$  insures that  $\theta_0(b) \in (0, 1)$ . With this notation, we prove the following result.

**THEOREM 3.2.** *Let  $G$  be a connected simply connected  $K$ -simple algebraic group defined over a number field  $K$  and let  $S$  be a finite set of finite places such that  $G$  is isotropic for all  $v \in S$ . Then for every  $b \in (0, b_0)$  and  $\theta \in (\theta_0(b), 1)$ ,*

$$\|N_T - V_T\|_{L^2(Q)} \ll_{S,Q,\theta} V_T^\theta,$$

where  $Q$  is an arbitrary bounded measurable subset of  $G_\infty$ .

The proof of Theorem 3.2 will be based on the estimates (V1), (V2) and (SP).

Using a Borel–Cantelli argument we will also derive a pointwise bound for  $N_T$ .

**COROLLARY 3.3.** *With the notation of Theorem 3.2, for every  $\theta' > \theta$*

$$N_T(x) = V_T + O_{S,x,\theta'}(V_T^{\theta'}) \quad \text{for a.e. } x \in G_\infty.$$

### 3.2 Towards a best-possible estimate

Let us note that it is often the case that the lower bound  $m_S(\Sigma_S(h)) \ll_S h^\mathfrak{a}$  holds, as well as the upper bound (V1). Then the sum (3.2) is bounded by

$$V_T \ll \sum_{1 \leq h \leq T, h \in \mathcal{L}_S} h^{-b\mathfrak{d}} \cdot h^\mathfrak{a},$$

and it follows from Lemma 3.1(a) that  $V_T$  is uniformly bounded when  $b > \mathfrak{a}/\mathfrak{d}$ . Therefore, the estimate in Theorem 3.2 is only interesting in the range  $b \leq \mathfrak{a}/\mathfrak{d}$ . We highlight that Corollary 3.3 gives the following estimate.

**COROLLARY 3.4.** *Suppose that additionally in Corollary 3.3 the automorphic representation of  $G_S$  is tempered and that  $S$  consists of unramified places. Then for every  $b < \mathfrak{a}/\mathfrak{d}$ ,*

$$N_T(x) = V_T + O_{S,x,\eta}(V_T^{1/2+\eta}) \quad \text{for all } \eta > 0 \text{ and a.e. } x \in G_\infty.$$

Indeed, under the temperedness condition the estimate (SP) holds with  $\mathfrak{k}_S = 1/2$  (see Theorem 2.3), so that in this case Theorem 3.2 covers all the relevant range of parameters  $b < \mathfrak{a}/\mathfrak{d}$  and gives the exponent  $\theta = 1/2 + \eta$  for every  $\eta > 0$ . Hence, Corollary 3.4 is a direct consequence of Corollary 3.3.

Noting the fact that error term in the foregoing estimate is bounded (in essence) by the square root of the main term, it is natural to expect that the exponent it establishes is, in fact, the best possible. A full proof of this fact requires establishing the expected lower bound for  $N_T$ . We will establish lower bounds for discrepancy estimates in a separate paper, but the question of optimality of Corollary 3.4 remains open.

### 3.3 Proof of Theorem 3.2

Consider the function

$$D_T(x) := N_T(x) - V_T, \quad \text{with } x \in G_\infty,$$

and recall that the metric  $\rho$  is right-invariant, and so  $B(x, h^{-b}) = \mathcal{B}_h x = \mathcal{B}_h^{-1} x$ . Therefore,

$$\begin{aligned} D_T(x) &:= \sum_{1 \leq h \leq T} (|\{\gamma \in \Gamma_S \cap B(x, h^{-b}) : H_f(\gamma) = h\}| - m_\infty(\mathcal{B}_h) m_S(\Sigma_S(h))) \\ &= \sum_{1 \leq h \leq T} (|\Gamma_S \cap (\mathcal{B}_h x \times \Sigma_S(h))| - m_\infty(\mathcal{B}_h) m_S(\Sigma_S(h))). \end{aligned}$$

The crucial ingredient of our proof is that  $D_T(x)$  can be represented in terms of the averaging operators  $\pi_S(\sigma_h)$ . Let

$$\chi_h(g_\infty, g_S) := \chi_{\mathcal{B}_h}(g_\infty) \chi_{\mathcal{I}_S}(g_S)$$

denote the characteristic function of the subset  $\mathcal{B}_h \times \mathcal{I}_S$  of  $G_\infty \times G_S$ . The sum

$$\phi_h(g) := \sum_{\gamma \in \Gamma_S} \chi_h(g\gamma) = \sum_{\gamma \in \Gamma_S} \chi_h(g\gamma^{-1})$$

defines a measurable function with compact support on the homogeneous space  $Y_S := (G_\infty \times G_S)/\Gamma_S$ . We observe that for  $x \in G_\infty$  and  $u \in \mathcal{I}_S$ ,

$$\begin{aligned} \int_{a \in \Sigma_S(h)} \phi_h(a(x, u)) dm_S(a) &= \sum_{\gamma \in \Gamma_S} \int_{\Sigma_S(h)} \chi_h(x\gamma^{-1}, au\gamma^{-1}) dm_S(a) \\ &= \sum_{\gamma \in \Gamma_S} \int_{\Sigma_S(h)} \chi_{\mathcal{B}_h}(x\gamma^{-1}) \chi_{\mathcal{I}_S}(au\gamma^{-1}) dm_S(a) \\ &= \sum_{\gamma \in \Gamma_S \cap B(x, h^{-b})} m_S(\mathcal{I}_S \gamma u^{-1} \cap \Sigma_S(h)). \end{aligned}$$

Note that in the above computation, since  $\Gamma_S$  is a discrete subgroup of  $G_\infty \times G_S$ , the non-zero summands in the sum above constitute a finite subset of  $\gamma$ . In order to evaluate the last expression, we use that the set  $\Sigma_S(h)$  is  $\mathcal{I}_S$ -bi-invariant. Therefore, if  $H_f(\gamma) = h$ , then  $\mathcal{I}_S \gamma u^{-1} \subset \Sigma_S(h)$  and  $m_S(\mathcal{I}_S \gamma u^{-1} \cap \Sigma_S(h)) = 1$ . On the other hand, if  $H_f(\gamma) \neq h$ , we have  $\mathcal{I}_S \gamma u^{-1} \cap \Sigma_S(h) = \emptyset$ . Hence, for every  $u \in \mathcal{I}_S$ ,

$$|\{\gamma \in \Gamma_S \cap B(x, h^{-b}) : H_f(\gamma) = h\}| = \int_{a \in \Sigma_S(h)} \phi_h(a(x, u)) dm_S(a).$$

Furthermore,

$$\begin{aligned} \int_{Y_S} \phi_h d\mu_S &= \int_{G_\infty \times G_S} \chi_{\mathcal{B}_h}(g_\infty) \chi_{\mathcal{I}_S}(g_S) dm_\infty(g_\infty) dm_S(g_S) \\ &= m_\infty(\mathcal{B}_h) m_S(\mathcal{I}_S) = m_\infty(\mathcal{B}_h). \end{aligned}$$

Hence, we deduce that for every  $u \in \mathcal{I}_S$ ,

$$D_T(x) = \sum_{1 \leq h \leq T} \left( \int_{a \in \Sigma_S(h)} \phi_h(a(x, u)) dm_S(a) - m_S(\Sigma_S(h)) \int_{Y_S} \phi_h d\mu_S \right).$$

Let  $Q$  be a bounded measurable subset in  $G_\infty$ . Since the first integral in the previous line was just shown to be independent of  $u \in \mathcal{I}_S$ , we obtain

$$\begin{aligned} & \int_Q \left| \int_{a \in \Sigma_S(h)} \phi_h(a(x, e)) dm_S(a) - m_S(\Sigma_S(h)) \int_{Y_S} \phi_h d\mu_S \right|^2 dm_\infty(x) \\ &= \int_{Q \times \mathcal{I}_S} \left| \int_{a \in \Sigma_S(h)} \phi_h(a(x, u)) dm_S(a) - m_S(\Sigma_S(h)) \int_{Y_S} \phi_h d\mu_S \right|^2 dm_\infty(x) dm_S(u). \end{aligned}$$

The set  $Q \times \mathcal{I}_S$  projects onto a measurable subset  $(Q \times \mathcal{I}_S)\Gamma_S$  of  $Y_S$ . Since  $Q \times \mathcal{I}_S$  is bounded, there exists  $N_Q$  such that every point in the image has preimage of cardinality at most  $N_Q$ . This implies that the last integral is bounded by

$$\begin{aligned} & N_Q \int_{(Q \times \mathcal{I}_S)\Gamma_S} \left| \int_{a \in \Sigma_S(h)} \phi_h(ay) dm_S(a) - m_S(\Sigma_S(h)) \int_{Y_S} \phi_h d\mu_S \right|^2 d\mu_S(y) \\ & \leq N_Q \int_{Y_S} \left| \int_{\Sigma_S(h)} \phi_h(ay) dm_S(a) - m_S(\Sigma_S(h)) \int_{Y_S} \phi_h d\mu_S \right|^2 d\mu_S(y) \\ & = N_Q m_S(\Sigma_S(h))^2 \left\| \pi_S(\sigma_h)\phi_h - \int_{Y_S} \phi_h d\mu_S \right\|_{L^2(Y_S)}^2. \end{aligned}$$

Hence, using the bound (SP), we deduce that for any  $\eta > 0$

$$\begin{aligned} \|D_T\|_{L^2(Q)} &\leq \sum_{1 \leq h \leq T} \left\| \int_{a \in \Sigma_S(h)} \phi_h(a(\cdot, e)) dm_S(a) - m_S(\Sigma_S(h)) \int_{Y_S} \phi_h d\mu_S \right\|_{L^2(Q)} \\ &\ll_{S,Q,\eta} \sum_{1 \leq h \leq T} m_S(\Sigma_S(h))^{1-\xi_S+\eta} \|\phi_h\|_{L^2(Y_S)}. \end{aligned}$$

Furthermore, the  $L^2$ -norm of  $\phi_h$  can be estimated as follows:

$$\begin{aligned} \|\phi_h\|_{L^2(Y_S)}^2 &= \int_{(G_\infty \times G_S)/\Gamma_S} \left( \sum_{\gamma \in \Gamma_S} \chi_h(g\gamma) \right)^2 d(m_\infty \times m_S)(g) \\ &= \int_{(G_\infty \times G_S)/\Gamma_S} \sum_{\gamma_1, \gamma_2 \in \Gamma_S} \chi_h(g\gamma_1)\chi_h(g\gamma_2) d(m_\infty \times m_S)(g) \\ &= \int_{(G_\infty \times G_S)/\Gamma_S} \sum_{\gamma, \delta \in \Gamma_S} \chi_h(g\delta)\chi_h(g\delta\gamma) d(m_\infty \times m_S)(g) \\ &= \int_{G_\infty \times G_S} \sum_{\gamma \in \Gamma_S} \chi_h(g)\chi_h(g\gamma) d(m_\infty \times m_S)(g) \\ &= \sum_{\gamma \in \Gamma_S} (m_\infty \times m_S)((\mathcal{B}_h \times \mathcal{I}_S) \cap (\mathcal{B}_h \times \mathcal{I}_S)\gamma^{-1}) \leq N_h m_\infty(\mathcal{B}_h), \end{aligned}$$

where  $N_h$  denotes the number of  $\gamma \in \Gamma_S$  such that  $(\mathcal{B}_h \times \mathcal{I}_S) \cap (\mathcal{B}_h \times \mathcal{I}_S)\gamma^{-1} \neq \emptyset$ . Since the family of sets  $\mathcal{B}_h \times \mathcal{I}_S$  is uniformly bounded, this number is uniformly bounded, and we conclude using (V2) that for any  $\eta > 0$

$$\|D_T\|_{L^2(Q)} \ll_{S,Q,\eta} \sum_{1 \leq h \leq T} m_S(\Sigma_S(h))^{1-\xi_S+\eta} m_\infty(\mathcal{B}_h)^{1/2} \ll \sum_{1 \leq h \leq T} m_S(\Sigma_S(h))^{1-\xi_S+\eta} h^{-b\eta/2}.$$

To complete the proof of Theorem 3.2 we employ the following computation, where we use parameters  $A > 0$  and  $B > 1$  that will be specified later. Using (V1), and writing  $\mathfrak{k}_S - \eta = \mathfrak{k}$  for brevity, we obtain

$$\sum_{1 \leq h \leq T} m_S(\Sigma_S(h))^{1-\mathfrak{k}} h^{-b\mathfrak{d}/2} \ll_S \sum_{1 \leq h \leq T} m_S(\Sigma_S(h))^{1-\mathfrak{k}+A} h^{-(\mathfrak{a}A+b\mathfrak{d}/2)},$$

and applying Hölder’s inequality, we conclude that this sum is bounded by

$$\left( \sum_{1 \leq h \leq T} m_S(\Sigma_S(h))^{(1-\mathfrak{k}+A)B} h^{-(\mathfrak{a}A+b\mathfrak{d}/2)B} \right)^{1/B} |\mathcal{L}_S \cap [1, T]|^{1-1/B}.$$

We choose  $A$  and  $B$  so that

$$(1 - \mathfrak{k} + A)B = 1 \quad \text{and} \quad (\mathfrak{a}A + b\mathfrak{d}/2)B = b\mathfrak{d},$$

namely,

$$A = \frac{(1 - \mathfrak{k})b\mathfrak{d} - b\mathfrak{d}/2}{\mathfrak{a} - b\mathfrak{d}} \quad \text{and} \quad B = \frac{\mathfrak{a} - b\mathfrak{d}}{(1 - \mathfrak{k})\mathfrak{a} - b\mathfrak{d}/2}.$$

Taking into account that  $\mathfrak{k} \in (0, 1/2)$  and  $b < 2\mathfrak{k}\mathfrak{a}/\mathfrak{d}$ , a direct computation verifies that  $A > 0$  and  $B > 1$ . Hence, using Lemma 3.1(a) and estimate (V2), we conclude that for every  $\eta > 0$ ,

$$\|D_T\|_{L^2(Q)} \ll_{S,Q,\eta} \left( \sum_{1 \leq h \leq T} m_S(\Sigma_S(h)) h^{-b\mathfrak{d}} \right)^{1/B} T^{\eta(1-1/B)} \ll_{\eta} V_T^{1/B} T^{\eta(1-1/B)}.$$

Finally, we note that it follows from (V1)–(V2) and Lemma 3.1(b) that

$$V_T \gg_S \sum_{h \in \mathcal{L}_S \cap [1, T]} h^{-b\mathfrak{d}} \cdot h^{\mathfrak{a}} \gg_S T^{\mathfrak{a}-b\mathfrak{d}}. \tag{3.3}$$

Since  $\mathfrak{a} - b\mathfrak{d} > 0$ , we deduce from the previous estimate that for every  $\eta' > 0$ ,

$$\|D_T\|_{L^2(Q)} \ll_{Q,\eta'} V_T^{\theta_0+\eta'},$$

where

$$\theta_0 := 1/B = \frac{(1 - \mathfrak{k})\mathfrak{a} - b\mathfrak{d}/2}{\mathfrak{a} - b\mathfrak{d}}.$$

Since  $B > 1$ , we have  $\theta_0 \in (0, 1)$ . This completes the proof of Theorem 3.2. □

*Proof of Corollary 3.3.* We note that the function  $D_T(x) = N_T(x) - V_T$  is piecewise constant in  $T$  and is determined by its values for  $T \in \mathcal{L}_S$ . Hence, it is sufficient to analyze the values  $D_T(x)$  for  $T \in \mathcal{L}_S$ . Given any bounded measurable subset  $Q$  of  $G_\infty$ , we have established in Theorem 3.2 the bound

$$\int_Q |D_T(x)|^2 dm_\infty(x) \ll_{S,Q,\theta} V_T^{2\theta}.$$

Therefore, for every  $\eta > 0$ , the sets

$$\Omega_T := \{x \in Q : |D_T(x)| \geq V_T^\theta T^\eta\}$$

satisfy

$$m_\infty(\Omega_T) \ll_{S,Q,\theta} T^{-2\eta}.$$

Hence, it follows from Lemma 3.1(a) that

$$\sum_{T \in \mathcal{L}_S} m_\infty(\Omega_T) < \infty,$$

and by the Borel–Cantelli lemma, the lim sup of the sets  $\Omega_T$  with  $T \in \mathcal{L}_S$  has measure zero. This means that for almost every  $x \in Q$  and  $T \in \mathcal{L}_S$ ,

$$|D_T(x)| \leq V_T^\theta T^\eta \quad \text{when } T \geq T_0(x, \eta).$$

Then using the estimate (3.3), we conclude that for almost all  $x \in Q$ ,

$$|D_T(x)| \ll_S V_T^{\theta+\eta/(a-b\delta)} \quad \text{when } T \geq T_0(x, \eta).$$

Since  $G_\infty$  can be exhausted by a countable union of bounded measurable sets  $Q$ , it follows that for every  $\theta' > \theta$  and almost all  $x \in G_\infty$ ,

$$|D_T(x)| \ll_S V_T^{\theta'} \quad \text{when } T \geq T_0(x, \theta').$$

Thus, in particular,

$$|D_T(x)| \ll_{S,x,\theta'} V_T^{\theta'} \quad \text{for all } T,$$

which implies the corollary. □

### 4. Discrepancy bounds for simply connected groups

Let  $G$  be a connected  $K$ -simple algebraic group defined over a number field  $K$ . In the present section, we assume that  $G$  is simply connected, and establish three effective discrepancy estimates for the distribution of rational points, namely mean-square, almost sure and pointwise everywhere estimates, for general sets  $E \subset G_\infty$ .

#### 4.1 Discrepancy of rational points

We fix a subset  $S$  of finite places of  $K$  such that  $G$  is isotropic over  $S$  and consider the  $S$ -arithmetic group  $\Gamma_S := G(O_S)$  which is exhausted by the increasing family of subsets

$$R_S(h) := \{\gamma \in \Gamma_S : H_f(\gamma) \leq h\}.$$

We recall that  $\mathcal{I}_S := \prod_{v \in S} G(O_v)$ , and  $\mathcal{I}^S := \prod_{v \in V_K^f \setminus S} G(O_v)$ , and the Haar measure  $m_S$  on  $G_S$  is normalized so that  $m_S(\mathcal{I}_S) = 1$ . We also denote by  $m^S$  the Haar measure on  $\mathcal{I}^S$  such that  $m^S(\mathcal{I}^S) = 1$ . Under the diagonal embedding

$$\Gamma_S \hookrightarrow G_\infty \times G_S,$$

$\Gamma_S$  is a lattice subgroup, and so the sets  $R_S(h)$ , while infinite, are locally finite namely deposit in every bounded subset of  $G_\infty$  only finitely many elements. The Haar measure  $m_\infty$  in  $G_\infty$  is normalized so that  $\Gamma_S$  has covolume one with respect to  $m_\infty \times m_S$ .

By the strong approximation property (Theorem 2.1), the diagonal embedding

$$\Gamma_S \hookrightarrow G_\infty \times \mathcal{I}^S$$

is dense. Our goal is to analyze the discrepancy of distribution of this dense set. One can show (in fact, it follows from our results here) that the number of points from  $R_S(h)$  contained in a bounded subset  $\Omega$  of  $G_\infty \times \mathcal{I}^S$  grows as

$$v_S(h) := m_S(B_S(h)),$$

where  $B_S(h) := \{g \in G_S : H_f(g) \leq h\}$ . We define the discrepancy function as

$$\mathcal{D}(R_S(h), \Omega) := \left| \frac{|R_S(h) \cap \Omega|}{v_S(h)} - (m_\infty \times m^S)(\Omega) \right|. \tag{4.1}$$

Our goal is to produce an explicit estimate for this quantity for a natural collection of subsets of  $G_\infty \times \mathcal{I}^S$ . Let  $E$  be a subset of  $G_\infty$  and  $W$  a compact open subset of  $\mathcal{I}^S$ . For  $x \in G_\infty$ , we set

$$E(x) := Ex.$$

We will focus on analyzing the discrepancy for  $\Omega = E(x) \times W$ , where we allow an arbitrary congruence condition  $W \subset \mathcal{I}^S$ .

A crucial ingredient of our analysis is the estimate on averaging operators established in § 2.3, which we now recall. We consider the spaces

$$Y_{S,W} := (G_\infty \times G_S)/\Gamma_S(W)$$

equipped with the invariant probability measures  $\mu_{S,W}$ . Let  $\beta_h$  be the uniform probability measure supported on the set  $B_S(h)^{-1}$  and

$$\pi_{S,W}(\beta_h) : L^2(Y_{S,W}) \rightarrow L^2(Y_{S,W})$$

the corresponding averaging operator defined in (2.8). We note that  $B_S(h)^{-1} \neq B_S(h)$  in general. According to Theorem 2.6, there exists an exponent  $\mathfrak{k}_S(G) = \mathfrak{k}_S \in (0, 1/2]$ , which is uniform in  $W$ , such that for every  $\phi \in L^2(Y_{S,W})$ , and every  $\eta > 0$

$$\left\| \pi_{S,W}(\beta_h)\phi - \int_{Y_{S,W}} \phi d\mu_{S,W} \right\|_{L^2(Y_{S,W})} \ll_{W,S,\eta} m_S(B_S(h))^{-\mathfrak{k}_S+\eta} \|\phi\|_{L^2(Y_{S,W})}. \tag{SP}$$

The estimates in this section will depend on the parameters:

- $\mathfrak{k}_S$  – the speed of convergence in the effective mean ergodic theorem in (SP);
- $\mathfrak{d}$  –  $\dim_{\mathbb{R}} G_\infty$ , namely the decay rate of volume of balls of small radius in (V2).

### 4.2 Mean-square discrepancy estimates

We will begin our discussion by establishing an  $L^2$ -bound for the discrepancy function  $\mathcal{D}(R_S(h), E(x) \times W)$ . Remarkably, this bound holds in great generality for measurable subsets  $E$ , and as noted already it is uniform in  $W$ .

For a subset  $E$  of  $G_\infty$ , we define

$$\mathcal{N}(E) := |\{\gamma \in G(O) : m_\infty(E \cap \gamma E) > 0\}| < \infty.$$

Clearly,  $\mathcal{N}(E)$  is uniformly bounded when  $G_\infty$  is compact, and finite if  $E$  is bounded, but this is not the case in general.

**THEOREM 4.1** (Mean square discrepancy bound). *Let  $E$  be any measurable subset of  $G_\infty$  of finite measure satisfying  $\mathcal{N}(E) < \infty$  and  $W$  a compact open subset of  $\mathcal{I}^S$ . Then for any  $\eta > 0$*

$$\|\mathcal{D}(R_S(h), E(\cdot) \times W)\|_{L^2(Q)} \ll_{S,Q,\eta} \mathcal{N}(E)^{1/2} m_\infty(E)^{1/2} m^S(W)^{1/2} v_S(h)^{-\mathfrak{k}_S(G)+\eta}$$

for every bounded measurable subset  $Q$  of  $G_\infty$ .

Note that in foregoing formula, the discrepancy decreases as the size of  $W$  decreases, and this fact reflects our choice of normalization in (4.1). If we choose to normalize the discrepancy of rational points in  $E$  by dividing by the measure of  $W$ , the error estimate on the right-hand side would depend on  $W$  via  $m^S(W)^{-1/2}$ . Then the discrepancy will increase as the congruence conditions imposed become more stringent, as expected.

*Remark 4.2.* We say a subset  $Q$  of  $G_\infty$  is  $\Gamma_S(W)$ -injective if the projection map

$$Q \times \mathcal{I}_S \rightarrow (G_\infty \times G_S)/\Gamma_S(W)$$

is injective. If  $Q$  is  $\Gamma_S(W)$ -injective, then the implicit constant in the above estimate in Theorem 4.1 is independent of  $Q$ .

*Proof of Theorem 4.1.* Let again

$$\chi(g_\infty, g_S) := \chi_{E^{-1}}(g_\infty)\chi_{\mathcal{I}_S}(g_S), \quad \text{for } (g_\infty, g_S) \in G_\infty \times G_S,$$

denote the characteristic function of the subset  $E^{-1} \times \mathcal{I}_S$  of  $G_\infty \times G_S$ . We introduce the subset

$$\Delta(W) := \Gamma_S \cap (G_\infty \times W)$$

of  $\Gamma_S$ . Since  $W$  is bi-invariant under the subgroup  $U(W)$ , it is clear that  $\Delta(W)$  is bi-invariant under the subgroup  $\Gamma_S(W) := \Gamma_S \cap (G_\infty \times G_S \times U(W))$ . Moreover, since the set  $W$  is union of finitely many right cosets of  $U(W)$ , the set  $\Delta(W)$  is a finite union of right cosets of  $\Gamma_S(W)$ . For  $g = (g_\infty, g_S) \in G_\infty \times G_S$ , we define

$$\phi(g) := \sum_{\delta \in \Delta(W)} \chi(g_\infty \delta^{-1}, g_S \delta^{-1}).$$

Since  $\Delta(W)$  is  $\Gamma_S(W)$ -bi-invariant, this defines a measurable function on the space  $Y_{S,W} = (G_\infty \times G_S)/\Gamma_S(W)$ . First, let us compute the integral of  $\phi$ :

$$\begin{aligned} \int_{Y_{S,W}} \phi d\mu_{S,W} &= \int_{(G_\infty \times G_S)/\Gamma_S(W)} \left( \sum_{\delta \in \Delta(W)} \chi(g\delta^{-1}) \right) d\mu_{S,W}(g) \\ &= \int_{(G_\infty \times G_S)/\Gamma_S(W)} \left( \sum_{\delta \in \Delta(W)/\Gamma_S(W)} \sum_{\gamma \in \Gamma_S(W)} \chi(g\gamma^{-1}\delta^{-1}) \right) d\mu_{S,W}(g) \\ &= \sum_{\delta \in \Delta(W)/\Gamma_S(W)} \int_{G_\infty \times G_S} \chi(g\delta^{-1}) \frac{d(m_\infty \times m_S)(g)}{|\Gamma_S/\Gamma_S(W)|} \\ &= \frac{|\Delta(W)/\Gamma_S(W)|}{|\Gamma_S/\Gamma_S(W)|} \int_{G_\infty \times G_S} \chi d(m_\infty \times m_S) \\ &= \frac{|\Delta(W)/\Gamma_S(W)|}{|\Gamma_S/\Gamma_S(W)|} m_\infty(E^{-1})m_S(\mathcal{I}_S) = \frac{|\Delta(W)/\Gamma_S(W)|}{|\Gamma_S/\Gamma_S(W)|} m_\infty(E). \end{aligned}$$

Since the projection of  $\Gamma_S$  to  $\mathcal{I}^S$  is dense, the map  $\gamma \rightarrow \gamma U(W)$ ,  $\gamma \in \Gamma_S$ , defines a bijection between the cosets  $\Gamma_S/\Gamma_S(W)$  and  $\mathcal{I}^S/U(W)$ . In particular,

$$|\Gamma_S/\Gamma_S(W)| = |\mathcal{I}^S/U(W)|. \tag{4.2}$$

In addition, since  $W$  is open, the projection of  $\Delta(W) = \Gamma_S \cap (G_\infty \times G_S \times W)$  is dense in  $W$ , and we obtain that

$$|\Delta(W)/\Gamma_S(W)| = |W/U(W)|. \tag{4.3}$$

We recall that the measure  $m^S$  on  $\mathcal{I}^S$  is normalized so that  $m^S(\mathcal{I}^S) = 1$ . This implies that

$$m^S(W) = \frac{|W/U(W)|}{|\mathcal{I}^S/U(W)|}. \tag{4.4}$$



Hence, we conclude from the above computations that

$$\int_{Y_{S,W}} \phi \, d\mu_{S,W} = m_\infty(E)m^S(W). \tag{4.5}$$

In particular, this shows that  $\phi \in L^1(Y_{S,W})$ . In fact, we will show later in the proof that  $\phi \in L^2(Y_{S,W})$ .

Since  $\phi$  is non-negative and integrable, it follows from the Fubini–Tonelli theorem that for almost every  $y \in Y_{S,W}$ ,

$$\int_{a \in B_S(h)} \phi(ay) \, dm_S(a) < \infty.$$

We shall show that the discrepancy  $\mathcal{D}(R_S(h), E(\cdot) \times W)$  can be approximated by such integrals. By the Fubini–Tonelli theorem again, for  $x \in G_\infty$  and  $u \in \mathcal{I}_S$ ,

$$\begin{aligned} \int_{a \in B_S(h)} \phi(a(x, u)) \, dm_S(a) &= \sum_{\delta \in \Delta(W)} \int_{a \in B_S(h)} \chi(x\delta^{-1}, au\delta^{-1}) \, dm_S(a) \\ &= \sum_{\delta \in \Delta(W)} \int_{B_S(h)} \chi_{E^{-1}}(x\delta^{-1}) \chi_{\mathcal{I}_S}(au\delta^{-1}) \, dm_S(a) \\ &= \sum_{\delta \in \Delta(W) \cap (Ex \times G_S)} m_S(\mathcal{I}_S \delta u^{-1} \cap B_S(h)). \end{aligned}$$

We recall that the sets  $B_S(h) := \{b \in G_S : H_f(b) \leq h\}$  are  $\mathcal{I}_S$ -bi-invariant. Therefore, if  $\delta \in B_S(h)$ , we have  $\mathcal{I}_S \delta u^{-1} \subset B_S(h)$ , and if  $\delta \notin B_S(h)$ , we have  $\mathcal{I}_S \delta u^{-1} \cap B_S(h) = \emptyset$ . Hence, since  $m_S(\mathcal{I}_S) = 1$ , it follows that for every  $u \in \mathcal{I}_S$ ,

$$|R_S(h) \cap (E(x) \times W)| = \int_{a \in B_S(h)} \phi(a(x, u)) \, dm_S(a). \tag{4.6}$$

Combining (4.5) and (4.6), we derive that

$$\begin{aligned} \mathcal{D}(R_S(h), E(x) \times W) &= \left| \frac{1}{m_S(B_S(h))} \int_{a \in B_S(h)} \phi(a(x, e)) \, dm_S(a) - \int_{Y_{S,W}} \phi \, d\mu_{S,W} \right| \\ &= \left| \pi_{S,W}(\beta_h) \phi(x, e) - \int_{Y_{S,W}} \phi \, d\mu_{S,W} \right|. \end{aligned}$$

This representation allows us to apply the estimate (SP). Let  $Q$  be a bounded measurable subset of  $G_\infty$ . Since the integral in (4.6) is independent of  $u \in \mathcal{I}_S$ , we obtain that

$$\begin{aligned} &\int_Q \left| \int_{a \in B_S(h)} \phi(a(x, e)) \, dm_S(a) - m_S(B_S(h)) \int_{Y_{S,W}} \phi \, d\mu_{S,W} \right|^2 dm_\infty(x) \\ &= \int_{Q \times \mathcal{I}_S} \left| \int_{a \in B_S(h)} \phi(a(x, u)) \, dm_S(a) - m_S(B_S(h)) \int_{Y_{S,W}} \phi \, d\mu_{S,W} \right|^2 dm_\infty(x) dm_S(u). \end{aligned}$$

The set  $Q \times \mathcal{I}_S$  projects to a measurable subset  $(Q \times \mathcal{I}_S)\Gamma_S(W)$  in  $Y_{S,W}$ . Since this is a bounded subset of  $G_\infty \times G_S$ , the fibers of this map have cardinalities bounded by a uniform constant  $N_Q$ ,

which is independent of  $W$  because  $\Gamma_S(W) \subset \Gamma_S$ . Hence, the last integral is bounded by

$$\begin{aligned} & N_Q \int_{(Q \times \mathcal{I}_S) / \Gamma_S(W)} \left| \int_{a \in B_S(h)} \phi(ay) dm_S(a) - m_S(B_S(h)) \int_{Y_{S,W}} \phi d\mu_{S,W} \right|^2 d\mu_{S,W}(y) \\ & \leq N_Q \int_{Y_{S,W}} \left| \int_{a \in B_S(h)} \phi(ay) dm_S(a) - m_S(B_S(h)) \int_{Y_{S,W}} \phi d\mu_{S,W} \right|^2 d\mu_{S,W}(y) \\ & = N_Q m_S(B_S(h))^2 \left\| \pi_{S,W}(\beta_h)\phi - \int_{Y_{S,W}} \phi d\mu_{S,W} \right\|_{L^2(Y_{S,W})}^2. \end{aligned}$$

Thus, it follows from (SP) that for every  $\eta > 0$

$$\| \mathcal{D}(R_S(h), E(\cdot) \times W) \|_{L^2(Q)} \ll_{S,Q,\eta} m_S(B_S(h))^{-\epsilon_S + \eta} \| \phi \|_{L^2(Y_{S,W})}. \tag{4.7}$$

The function  $\phi$  is indeed square-integrable, and the  $L^2$ -norm  $\| \phi \|_{L^2(Y_{S,W})}^2$  is computed as follows:

$$\begin{aligned} & \int_{(G_\infty \times G_S) / \Gamma_S(W)} \left( \sum_{\delta_1, \delta_2 \in \Delta(W)} \chi(g\delta_1^{-1})\chi(g\delta_2^{-1}) \right) d\mu_{S,W}(g) \\ & = \int_{(G_\infty \times G_S) / \Gamma_S(W)} \left( \sum_{\delta_1, \delta_2 \in \Delta(W) / \Gamma_S(W)} \sum_{\gamma_1, \gamma_2 \in \Gamma_S(W)} \chi(g\gamma_1^{-1}\delta_1^{-1})\chi(g\gamma_2^{-1}\delta_2^{-1}) \right) d\mu_{S,W}(g) \\ & = \int_{(G_\infty \times G_S) / \Gamma_S(W)} \left( \sum_{\delta_1, \delta_2 \in \Delta(W) / \Gamma_S(W)} \sum_{\sigma, \gamma \in \Gamma_S(W)} \chi(g\sigma\delta_1^{-1})\chi(g\sigma\gamma^{-1}\delta_2^{-1}) \right) d\mu_{S,W}(g) \\ & = \int_{G_\infty \times G_S} \left( \sum_{\delta_1, \delta_2 \in \Delta(W) / \Gamma_S(W)} \sum_{\gamma \in \Gamma_S(W)} \chi(g\delta_1^{-1})\chi(g\gamma^{-1}\delta_2^{-1}) \right) \frac{d(m_\infty \times m_S)(g)}{|\Gamma_S : \Gamma_S(W)|} \\ & = \int_{G_\infty \times G_S} \left( \sum_{\gamma \in \Delta(W) / \Gamma_S(W)} \sum_{\delta \in \Delta(W)} \chi(g\gamma^{-1})\chi(g\delta^{-1}) \right) \frac{d(m_\infty \times m_S)(g)}{|\Gamma_S : \Gamma_S(W)|}. \end{aligned}$$

We observe that for any  $\gamma, \delta \in \Delta(W)$ ,

$$\begin{aligned} \int_{G_\infty \times G_S} \chi(g\gamma^{-1})\chi(g\delta^{-1}) d(m_\infty \times m_S)(g) & = m_\infty(E^{-1}\gamma \cap E^{-1}\delta) m_S(\mathcal{I}_S \gamma \cap \mathcal{I}_S \delta) \\ & \leq m_\infty(E) m_S(\mathcal{I}_S) = m_\infty(E). \end{aligned}$$

Moreover, this integral is zero unless  $m_\infty(E^{-1} \cap E^{-1}\delta\gamma^{-1}) > 0$  and  $\delta\gamma^{-1} \in \mathcal{I}_S$ . But since  $\gamma, \delta \in \Delta(W)$ , we also have  $\delta\gamma^{-1} \in \mathcal{I}^S$ , and so in this case it follows that  $\delta\gamma^{-1} \in \mathbb{G}(O)$ . Recalling the definition of  $\mathcal{N}(E)$ , this implies that for fixed  $\gamma \in \Delta(W)$ ,

$$\left| \left\{ \delta \in \Delta(W) : \int_{G_\infty \times G_S} \chi(g\gamma^{-1})\chi(g\delta^{-1}) d(m_\infty \times m_S)(g) \neq 0 \right\} \right| \leq \mathcal{N}(E).$$

Hence, applying the Fubini–Tonelli theorem once more, we conclude that

$$\| \phi \|_{L^2(Y_{S,W})}^2 \leq \mathcal{N}(E) m_\infty(E) \frac{|\Delta(W) / \Gamma_S(W)|}{|\Gamma_S : \Gamma_S(W)|} = \mathcal{N}(E) m_\infty(E) m^S(W).$$

Here we have used (4.2)–(4.4) in the last step. Combining this estimate with (4.7), this completes the proof of Theorem 4.1. □

**4.3 Almost sure discrepancy estimates**

We now turn to establish an almost sure bound for the discrepancy. We will use the mean-square bound of Theorem 4.1 and the Borel–Cantelli lemma, and the main difficulty here will be to establish an estimate which holds for all  $h$  on a fixed set of full measure. For this argument, we need the following elementary lemma.

LEMMA 4.3. For  $\theta \geq 1$  and  $y_1, \dots, y_n \geq 0$ ,  $y_1^\theta + \dots + y_n^\theta \leq (y_1 + \dots + y_n)^\theta$ .

*Proof.* The proof proceeds by induction on  $n$ . We consider the function

$$f(x) := x^\theta + y_2^\theta + \dots + y_n^\theta - (x + y_2 + \dots + y_n)^\theta.$$

By the inductive assumption,  $f(0) \leq 0$ . Using that  $\theta \geq 1$ , one checks that  $f'(x) \leq 0$  for  $x \geq 0$ . This implies the claim.  $\square$

THEOREM 4.4 (Almost sure discrepancy bound). With notation as in Theorem 4.1, for every  $0 < \epsilon < \epsilon_S(\mathbb{G})$  for almost all  $x \in G_\infty$ , and for every  $\eta > 0$

$$D(R_S(h), E(x) \times W) \ll_{S,E,W,x,\epsilon,\eta} (\log v_S(h))^{3/2+\eta} v(h)^{-\epsilon}.$$

*Proof.* For  $a, b \in \mathbb{Z}_{\geq 0}$ , we consider the intervals

$$I_{a,b} := \{h \geq 1 : 2^b a < v_S(h) \leq 2^b(a + 1)\}.$$

Note that for fixed  $b$ , they define a partition of  $[1, \infty)$ .

For  $s \in \mathbb{Z}_{\geq 0}$ , we set

$$\mathcal{M}_{s,b} := \{I_{a,b} : v_S(I_{a,b}) \subset (0, 2^s]\} \quad \text{and} \quad \mathcal{M}_s := \bigsqcup_{b \leq s} \mathcal{M}_{s,b}.$$

We observe that

$$\bigcup_{I \in \mathcal{M}_s} I = [1, h_s] \quad \text{with} \quad 2^{s-1} < v_S(h_s) \leq 2^s.$$

For  $I \subset [1, \infty)$ , we set

$$R_S(I) := \{\gamma \in \Gamma_S : H_f(\gamma) \in I\} \quad \text{and} \quad B_S(I) := \{g \in G_S : H_f(g) \in I\}.$$

The argument of the proof of Theorem 4.1 gives the following mean bound: for bounded measurable subsets  $Q \subset G_\infty$ ,

$$\int_Q \left| |R_S(I) \cap (E(x) \times W)| - m_\infty(E)m^S(W)m_S(B_S(I)) \right|^2 dm_\infty(x) \ll m_S(B_S(I))^{2-2\epsilon}.$$

The implicit constant here and the following computations depends on  $S, E, W, Q$  and the difference  $\epsilon_S - \epsilon$ . Using Lemma 4.3,

$$\sum_{I \in \mathcal{M}_{s,b}} m_S(B_S(I))^{2-2\epsilon} \leq \left( \sum_{I \in \mathcal{M}_{s,b}} m_S(B_S(I)) \right)^{2-2\epsilon} \leq m_S(B_S(h_s))^{2-2\epsilon} \leq 2^{s(2-2\epsilon)}.$$

Hence,

$$\sum_{I \in \mathcal{M}_s} \int_Q \left| |R_S(I) \cap (E(x) \times W)| - m_\infty(E)m^S(W)m_S(B_S(I)) \right|^2 dm_\infty(x) \ll s 2^{s(2-2\epsilon)}. \tag{4.8}$$

Let  $\eta > 0$ . We consider the sets  $\Upsilon_s$  consisting of  $x \in Q$  such that

$$\sum_{I \in \mathcal{M}_s} \left| |R_S(I) \cap (E(x) \times W)| - m_\infty(E)m^S(W)m_S(B_S(I)) \right|^2 \geq s^{2+\eta} 2^{s(2-2\epsilon)}.$$

We deduce from (4.8) that

$$m_\infty(\Upsilon_s) \ll s^{-1-\eta}.$$

It follows from the Borel–Cantelli lemma that the lim sup of the sets  $\Upsilon_s$  has measure zero. Hence, for almost all  $x \in Q$  and all  $s \geq s_0(x)$ , we have the bound

$$\sum_{I \in \mathcal{M}_s} \left| |R_S(I) \cap (E(x) \times W)| - m_\infty(E)m^S(W)m_S(B_S(I)) \right|^2 \ll s^{2+\eta}2^{s(2-2\epsilon)}. \tag{4.9}$$

We shall use this bound to prove the theorem.

We first consider the case when  $h$  is an end point of one of the intervals  $I_{a,b}$ . We choose the parameter  $s$  such that  $2^{s-1} < v_S(h) \leq 2^s$ . Using the binary representation, the interval  $[1, h]$  can be written as a disjoint interval of at most  $s$  intervals  $I_i$  from  $\mathcal{M}_s$ . Then using (4.9) and the Cauchy–Schwartz inequality, we deduce that for almost all  $x \in Q$  and  $h \geq h_0(x)$ , and for every  $\eta > 0$

$$\begin{aligned} & \left| |R_S(h) \cap (E(x) \times W)| - m_\infty(E)m^S(W)m_S(B_S(h)) \right| \\ & \leq \sum_i \left| |R_S(I_i) \cap (E(x) \times W)| - m_\infty(E)m^S(W)m_S(B_S(I_i)) \right| \\ & \ll_\eta s^{3/2+\eta/2}2^{s(1-\epsilon)} \ll_\eta (\log v_S(h))^{3/2+\eta/2}v_S(h)^{1-\epsilon}. \end{aligned}$$

For a general  $h$ , we observe that there exist  $h_1$  and  $h_2$  as above such that

$$h_1 \leq h \leq h_2 \quad \text{and} \quad v_S(h_2) - v_S(h_1) \leq 1.$$

Then

$$|R_S(h_1) \cap (E(x) \times W)| \leq |R_S(h) \cap (E(x) \times W)| \leq |R_S(h_2) \cap (E(x) \times W)|,$$

and

$$v_S(h_2) \leq v_S(h) + 1 \quad \text{and} \quad v_S(h_1) \geq v_S(h) - 1.$$

This provides upper and lower bounds on  $|R_S(h) \cap (E(x) \times W)|$  that imply that for almost all  $x \in Q$  and all sufficiently large  $h$ ,

$$|R_S(h) \cap (E(x) \times W)| = m_\infty(E)m^S(W)v_S(h) + O((\log v_S(h))^{3/2+\eta/2}v_S(h)^{1-\epsilon}).$$

Clearly, this estimate also holds for all  $h$  with an implicit constant depending on  $x$  and  $\eta$ . Finally, exhausting  $G_\infty$  by a countable union of bounded measurable sets, we deduce that this estimate holds for almost all  $x \in G_\infty$ , which implies Theorem 4.4. □

#### 4.4 Uniform discrepancy estimates

We now turn to establish a uniform pointwise bound on the discrepancy of rational points in the collection of sets  $E(x) \times W$  with  $E(x) := Ex \subset G_\infty$  and  $W \subset \mathcal{I}^S$ . In our discussion the set  $E$  is fixed, the congruence condition  $W$  is arbitrary and we consider *all points*  $x \in G_\infty$ . This requires imposing a regularity condition on the set  $E$ , namely the right-stability property (RS). The bound will now involve the dimension  $\mathfrak{d} := \dim_{\mathbb{R}}(G_\infty)$ , whereas the almost sure pointwise bound in Theorem 4.4 did not.

We recall that by Lemma 2.8(a),

$$\epsilon^{\mathfrak{d}} \ll m_\infty(B(e, \epsilon)) \ll \epsilon^{\mathfrak{d}} \quad \text{for all } \epsilon \in (0, \epsilon_0). \tag{V}$$

**THEOREM 4.5** (Uniform discrepancy bound). *With notation as in Theorem 4.1, let  $E$  be a right-stable finite-measure subset of  $G_\infty$  satisfying  $\mathcal{N}(E_{\epsilon_0}^+) < \infty$  and  $W$  compact open subset*

of  $\mathcal{I}^S$ . For every  $0 < \mathfrak{k} < \mathfrak{k}_S(\mathbf{G})$  and for  $x \in G_\infty$  the following pointwise bound for the discrepancy holds:

$$\mathcal{D}(R_S(h), E(x) \times W) \ll_{S,E,x,\mathfrak{k}} m^S(W)^{(\mathfrak{d}+1)/(\mathfrak{d}+2)} v_S(h)^{-2\mathfrak{k}/(\mathfrak{d}+2)},$$

provided that  $m^S(W) \gg_{S,\mathfrak{k}} v_S(h)^{-2\mathfrak{k}}$ .

When  $m_\infty(E) > 0$ , this condition is also equivalent to the condition that the error term in the asymptotic expansion of the solution counting function:

$$\begin{aligned} |R_S(h) \cap (E(x) \times W)| &= m_\infty(E)m^S(W)v_S(h) \\ &\quad + O_{S,E,x,\mathfrak{k}}(m^S(W)^{(\mathfrak{d}+1)/(\mathfrak{d}+2)} v_S(h)^{1-2\mathfrak{k}/(\mathfrak{d}+2)}), \end{aligned}$$

is bounded by the main term.

Explicitly, if the volume growth satisfies  $v_S(h) \gg_{S,\mathfrak{a}} h^\mathfrak{a}$ , with some  $\mathfrak{a} > 0$ , then the estimate holds provided the height  $h$  satisfies  $h \gg_{S,\mathfrak{a},\mathfrak{k}} m^S(W)^{-1/2\mathfrak{a}\mathfrak{k}}$ . Moreover, the above estimate is uniform for  $x$  ranging in compact subsets of  $G_\infty$ .

In the proof, we use the estimates (SP), (RS) and (V).

*Proof of Theorem 4.5.* The starting point of our argument is the  $L^2$ -bound established in Theorem 4.1. Let  $Q$  be a bounded measurable subset of  $G_\infty$ , and define the open set  $Q' := B(e, \epsilon_0)Q$ . According to Theorem 4.1, the following bound holds: for any measurable subset  $F$  of  $G_\infty$  with finite measure, and any given  $0 < \mathfrak{k} < \mathfrak{k}_S(\mathbf{G})$ ,

$$\int_{Q'} \mathcal{D}(R_S(h), F(y) \times W)^2 dm_\infty(y) \leq C \mathcal{N}(F)m_\infty(F)m^S(W)v_S(h)^{-2\mathfrak{k}}$$

for some  $C = C_{Q,S,\epsilon_0,\mathfrak{k}} > 0$ . This implies that for every  $\delta > 0$ ,

$$\begin{aligned} m_\infty(\{y \in Q' : \mathcal{D}(R_S(h), F(y) \times W) > \delta\}) \\ \leq C\delta^{-2}\mathcal{N}(F)m_\infty(F)m^S(W)v_S(h)^{-2\mathfrak{k}}. \end{aligned} \tag{4.10}$$

We introduce a parameter  $\epsilon \in (0, \epsilon_0)$  which we assume to satisfy

$$m_\infty(B(e, \epsilon)) > C\delta^{-2}\mathcal{N}(E_\epsilon^+)m_\infty(E_\epsilon^+)m^S(W)v_S(h)^{-2\mathfrak{k}}. \tag{4.11}$$

Here  $\epsilon_0$  is determined by conditions (RS) and (V). We note that for  $x \in Q$ , we have  $B(e, \epsilon)x \subset Q'$ . Hence, it follows from (4.10) applied to  $F = E_\epsilon^+$  that for every  $x \in Q$ , there exists  $g \in B(e, \epsilon)$  such that

$$\mathcal{D}(R_S(h), E_\epsilon^+(gx) \times W) \leq \delta. \tag{4.12}$$

We observe when  $g \in B(e, \epsilon) = B(e, \epsilon)^{-1}$  that it follows from the definition of  $E_\epsilon^-$  and  $E_\epsilon^+$  (cf. (RS)) that

$$E_\epsilon^- \cdot g \subset E \subset E_\epsilon^+ \cdot g, \text{ since } E_\epsilon^- \cdot B(e, \epsilon) \subset E \subset E \cdot B(e, \epsilon)^{-1} = E_\epsilon^+$$

and upon right-multiplication by  $x$ :

$$E_\epsilon^- \cdot gx = E_\epsilon^-(gx) \subset E \cdot x = E(x) \subset E_\epsilon^+ \cdot gx = E_\epsilon^+(gx). \tag{4.13}$$

Hence, taking  $g \in B(e, \epsilon)$  as in (4.12), we obtain that

$$\begin{aligned} |R_S(h) \cap (E(x) \times W)| &\leq |R_S(h) \cap (E_\epsilon^+(gx) \times W)| \\ &\leq (m_\infty(E_\epsilon^+(gx))m^S(W) + \mathcal{D}(R_S(h), E_\epsilon^+(gx) \times W))v_S(h) \\ &\leq (m_\infty(E_\epsilon^+)m^S(W) + \delta)v_S(h). \end{aligned}$$

Since the set  $E$  is assumed to be right stable, we deduce that

$$|R_S(h) \cap (E(x) \times W)| \leq (m_\infty(E)m^S(W) + O(\epsilon m^S(W)) + \delta)v_S(h)$$

for all  $\epsilon \in (0, \epsilon_0)$  satisfying (4.11).

We also apply a similar argument to deduce a lower bound. Since it follows from (4.11) that also

$$m_\infty(B(e, \epsilon)) > C \delta^{-2} \mathcal{N}(E_\epsilon^-) m_\infty(E_\epsilon^-) m^S(W) v(h)^{-2\mathfrak{k}},$$

we deduce from (4.10) with  $F = E_\epsilon^-$  that for every  $x \in Q$ , there exists  $g \in B(e, \epsilon)$  such that

$$\mathcal{D}(R_S(h), E_\epsilon^-(gx) \times W) \leq \delta.$$

Hence, we deduce as above that

$$\begin{aligned} |R_S(h) \cap (E(x) \times W)| &\geq |R_S(h) \cap (E_\epsilon^-(gx) \times W)| \\ &\geq (m_\infty(E_\epsilon^-(gx))m^S(W) - \mathcal{D}(R_S(h), E_\epsilon^-(gx) \times W))v_S(h) \\ &\geq (m_\infty(E)m^S(W) - O(\epsilon m^S(W)) + \delta)v_S(h). \end{aligned}$$

Combining the above estimates on  $|R_S(h) \cap (E(x) \times W)|$ , we deduce that for every  $x \in Q$ ,

$$\mathcal{D}(R_S(h), E(x) \times W) \leq \delta + O(\epsilon m^S(W))$$

provided that (4.11) is satisfied. To arrange (4.11), it is sufficient to pick  $\delta$  of the form

$$\delta = c m_\infty(B(e, \epsilon))^{-1/2} \mathcal{N}(E_\epsilon^+)^{1/2} m_\infty(E_\epsilon^+)^{1/2} m^S(W)^{1/2} v_S(h)^{-\mathfrak{k}}$$

with sufficiently large  $c = c_{Q,S,\epsilon_0,\mathfrak{k}} > 0$ . Then we deduce using (V) that for all  $x \in Q$ ,

$$\mathcal{D}(R_S(h), E(x) \times W) \ll_{E,Q} \epsilon^{-\mathfrak{d}/2} m^S(W)^{1/2} v_S(h)^{-\mathfrak{k}} + \epsilon m^S(W). \tag{4.14}$$

We balance the two summands in the estimate and choose the parameter  $\epsilon$  as

$$\epsilon = (m^S(W)^{-1/2} v_S(h)^{-\mathfrak{k}})^{1/(\mathfrak{d}/2+1)}.$$

Therefore, when  $h$  is sufficiently large, we have  $\epsilon \in (0, \epsilon_0)$  provided that

$$v_S(h)^{2\mathfrak{k}} m^S(W) > \epsilon_0^{-(\mathfrak{d}+2)}, \quad \text{namely } v_S(h)^{2\mathfrak{k}} \gg m^S(W)^{-1},$$

the latter condition is equivalent to

$$m^S(W) \gg m^S(W)^{(\mathfrak{d}+1)/(\mathfrak{d}+2)} v_S(h)^{-2\mathfrak{k}/(\mathfrak{d}+2)},$$

and substituting this expression in the bound just established for the discrepancy we deduce that for every  $x \in Q$ ,

$$\mathcal{D}(R_S(h), E(x) \times W) \ll_{S,E,Q,\mathfrak{k}} m^S(W)^{(\mathfrak{d}+1)/(\mathfrak{d}+2)} v_S(h)^{-2\mathfrak{k}/(\mathfrak{d}+2)}.$$

This completes the proof of Theorem 4.5. □

### 4.5 Uniform discrepancy at arbitrarily small scales

We now turn to establish a bound on the discrepancy which is uniform over a family of balls  $B(x, \ell)$  in  $G_\infty$  of arbitrarily small radius, for every  $x \in G_\infty$ . Let  $\ell > 0$ , and  $W$  be a compact open subset of  $\mathcal{I}^S$ . We will consider the family of subsets  $\Omega$  given by  $B(x, \ell) \times W \subset G_\infty \times \mathcal{I}^S$ , and our goal is to prove an explicit pointwise bound for the number of  $\Gamma_S$ -points contained in them.

**THEOREM 4.6** (Pointwise discrepancy bound at small scales). *With notation as in Theorem 4.1, let  $W$  be a compact open subset of  $\mathcal{I}^S$ . Fix  $0 < \mathfrak{k} < \mathfrak{k}_S(\mathbf{G})$ , let  $x \in G_\infty$  and  $\ell \in (0, \ell_0)$  (for suitable  $\ell_0 > 0$  independent of  $x$ ) and set*

$$\mathcal{E}_{\ell,W}(h) := m_\infty(B(e, \ell))^{\mathfrak{d}/(\mathfrak{d}+2)} m^S(W)^{(\mathfrak{d}+1)/(\mathfrak{d}+2)} v_S(h)^{1-2\mathfrak{k}/(\mathfrak{d}+2)}.$$

Then

$$|R_S(h) \cap B(x, \ell) \times W| = m_\infty(B(e, \ell)) m^S(W) v_S(h) + O_{S,\mathfrak{k},x}(\mathcal{E}_{\ell,W}(h)) \tag{4.15}$$

provided that

$$m_\infty(B(e, \ell))^2 m^S(W) \geq v_S(h)^{-2\mathfrak{k}}. \tag{4.16}$$

This condition is equivalent to the condition that the error term in the foregoing asymptotic formula is bounded by the main term.

Explicitly, if  $v_S(h) \gg_{S,\mathfrak{a}} h^\mathfrak{a}$ , with some  $\mathfrak{a} > 0$ , then the result holds at any scale  $0 < \ell < \ell_0$ , provided  $h$  satisfies

$$h \gg_{S,\mathfrak{a},\mathfrak{k}} \ell^{-\mathfrak{d}/\mathfrak{k}\mathfrak{a}} m^S(W)^{-(\mathfrak{d}+2)/2\mathfrak{k}\mathfrak{a}}.$$

Moreover, this estimate is uniform for  $x$  ranging in compact subsets of  $G_\infty$ .

*Proof.* We adopt the argument from the proof of Theorem 4.5 with some modifications. We note that there exists  $r_0 > 0$  such that when  $r \in (0, r_0)$ ,

$$B(e, r) \cap \gamma B(e, r) = \emptyset \quad \text{for } \gamma \in \mathbf{G}(O) \setminus \{e\},$$

so that  $\mathcal{N}(B(e, r)) = 1$ .

We fix a bounded measurable subset  $Q$  of  $G_\infty$  and define the open set  $Q' := B(e, r_0)Q$ . Applying Theorem 4.1 to the balls  $B(y, \ell) = B(e, \ell)y$ , we deduce that for every  $r \in (0, r_0)$  and  $\delta > 0$ , and some  $C = C_{S,Q,r_0}$

$$\begin{aligned} & m_\infty(\{y \in Q' : \mathcal{D}(R_S(h), B(x, \ell) \times W) > \delta\}) \\ & \leq C \delta^{-2} m_\infty(B(e, r)) m^S(W) v_S(h)^{-2\mathfrak{k}}. \end{aligned} \tag{4.17}$$

We introduce a positive parameter  $\epsilon$  (to be chosen later) satisfying

$$\epsilon < \ell, \tag{4.18}$$

as well as

$$m_\infty(B(e, \epsilon)) > C \delta^{-2} m_\infty(B(e, \ell + \epsilon)) m^S(W) v_S(h)^{-2\mathfrak{k}}. \tag{4.19}$$

We choose  $\ell_0$  so that  $0 < 2\ell_0 < r_0$  and also so that Lemma 2.8 is applicable for  $r < r_0$ . Then it follows from (4.17) that for every  $x \in Q$ , there exists  $g \in B(e, \epsilon)$  such that

$$\mathcal{D}(R_S(h), B(gx, \ell + \epsilon) \times W) \leq \delta.$$

It follows from the triangle inequality that

$$B(gx, \ell - \epsilon) \times W \subset B(x, \ell) \times W \subset B(gx, \ell + \epsilon) \times W,$$

and so,

$$\begin{aligned} |R_S(h) \cap (B(x, \ell) \times W)| & \leq |R_S(h) \cap (B(gx, \ell + \epsilon) \times W)| \\ & \leq (m_\infty(B(e, \ell + \epsilon)) m^S(W) + \mathcal{D}(R_S(h), B(gx, \ell + \epsilon) \times W)) v_S(h) \\ & \leq (m_\infty(B(e, \ell + \epsilon)) m^S(W) + \delta) v_S(h). \end{aligned}$$

Since it follows from Lemma 2.8(b) that

$$m_\infty(B(e, \ell + \epsilon)) \leq \left(1 + c \frac{\epsilon}{\ell}\right) m_\infty(B(e, \ell)), \tag{4.20}$$

we deduce that

$$|R_S(h) \cap (B(x, \ell) \times W)| \leq \left(m_\infty(B(e, \ell))m^S(W) + \delta + c \frac{\epsilon}{\ell} m_\infty(B(e, \ell))m^S(W)\right) v_S(h).$$

A similar argument also gives the lower bound

$$|R_S(h) \cap (B(x, \ell) \times W)| \geq \left(m_\infty(B(e, \ell))m^S(W) - \delta - c \frac{\epsilon}{\ell} m_\infty(B(e, \ell))m^S(W)\right) v_S(h).$$

It follows that the discrepancy

$$\mathcal{D}(R_S(h), B(x, \ell) \times W) = \left| \frac{|R_S(h) \cap (B(x, \ell) \times W)|}{v_S(h)} - m_\infty(B(e, \ell))m^S(W) \right|$$

satisfies

$$\mathcal{D}(R_S(h), B(x, \ell) \times W) \leq \delta + c \frac{\epsilon}{\ell} m_\infty(B(e, \ell))m^S(W).$$

The parameter  $\delta$  have to satisfy (4.19), and since  $m_\infty(B(e, \ell + \epsilon)) \leq (1 + c)m_\infty(B(e, \ell))$  (using (4.20) and  $\epsilon < \ell$ ) it is sufficient to pick  $\delta$  of the form

$$\delta = b_0 m_\infty(B(e, \epsilon))^{-1/2} m_\infty(B(e, \ell))^{1/2} m^S(W)^{1/2} v_S(h)^{-\mathfrak{t}}$$

with sufficiently large  $b_0 > 0$ . Then using Lemma 2.8(a), we deduce the bound

$$\mathcal{D}(R_S(h), B(x, \ell) \times W) \ll \epsilon^{-\mathfrak{d}/2} m_\infty(B(e, \ell))^{1/2} m^S(W)^{1/2} v_S(h)^{-\mathfrak{t}} + \frac{\epsilon}{\ell} m_\infty(B(e, \ell))m^S(W). \tag{4.21}$$

The two summands in the foregoing bound are balanced precisely when  $\epsilon$  is a constant multiple of

$$\ell^{2/(\mathfrak{d}+2)} m_\infty(B(e, \ell))^{-1/(\mathfrak{d}+2)} m^S(W)^{-1/(\mathfrak{d}+2)} v_S(h)^{-2\mathfrak{t}/(\mathfrak{d}+2)}, \tag{4.22}$$

and then the estimate that arises in (4.21) is given by a constant multiple of

$$\ell^{-\mathfrak{d}/(\mathfrak{d}+2)} m_\infty(B(e, \ell))^{(\mathfrak{d}+1)/(\mathfrak{d}+2)} m^S(W)^{(\mathfrak{d}+1)/(\mathfrak{d}+2)} v_S(h)^{-2\mathfrak{t}/(\mathfrak{d}+2)}.$$

By Lemma 2.8(a)  $\ell \ll m_\infty(B(e, \ell))^{1/\mathfrak{d}} \ll \ell$ , and so let us define the error term in (4.15) by

$$\mathcal{E}_{\ell, W}(h) := m_\infty(B(e, \ell))^{\mathfrak{d}/(\mathfrak{d}+2)} m^S(W)^{(\mathfrak{d}+1)/(\mathfrak{d}+2)} v_S(h)^{1-2\mathfrak{t}/(\mathfrak{d}+2)}. \tag{4.23}$$

We now claim that under the condition (4.16) stated in Theorem 4.6 namely  $m_\infty(B(e, \ell))^2 m^S(W) \geq v_S(h)^{-2\mathfrak{t}}$ , the conclusion of Theorem 4.6 holds:

$$|R_S(h) \cap (B(e, \ell) \times W)| = m_\infty(B(e, \ell))m^S(W)v_S(h) + O_Q(\mathcal{E}_{\ell, W}(h)). \tag{4.24}$$

Indeed, (4.16) is equivalent to the condition that the error term  $\mathcal{E}_{\ell, W}(h)$  defined in (4.23) is bounded by the main term in (4.24), namely,

$$m_\infty(B(e, \ell))m^S(W)v_S(h) \geq \mathcal{E}_{\ell, W}(h). \tag{4.25}$$

Furthermore, these two conditions are equivalent to the condition

$$m_\infty(B(e, \ell))^{-2/(\mathfrak{d}+2)} m^S(W)^{-1/(\mathfrak{d}+2)} v_S(h)^{-2\mathfrak{t}/(\mathfrak{d}+2)} \leq 1. \tag{4.26}$$

To establish (4.24) we apply (4.21) with the choice of the parameter  $\epsilon$  being given by (4.22):

$$\epsilon = c'_0 \ell^{2/(\mathfrak{d}+2)} m_\infty(B(e, \ell))^{-1/(\mathfrak{d}+2)} m^S(W)^{-1/(\mathfrak{d}+2)} v_S(h)^{-2\mathfrak{t}/(\mathfrak{d}+2)},$$



where  $c'_0 > 0$  is a sufficiently small positive constant so that Lemma 2.8(b) holds. It follows from Lemma 2.8(a) and (4.26) that

$$\epsilon \ll c'_0 \ell^{-(\mathfrak{d}-2)/(\mathfrak{d}+2)} m^S(W)^{-1/(\mathfrak{d}+2)} v_S(h)^{-2\ell/(\mathfrak{d}+2)} \ll c'_0 \ell.$$

Hence, with sufficiently small  $c'_0$ , condition (4.18) holds, and so the above estimates apply. Now the bound (4.24) follows from (4.21) and this completes the proof of Theorem 4.6.  $\square$

### 5. Discrepancy bounds for general groups

In the present section, our goal is to extend the results of the previous sections to the general case, where  $G$  is any connected  $K$ -simple algebraic group defined over a number field  $K$ . As above, we fix a subset  $S$  of finite places of  $K$  such that  $G$  is isotropic over  $S$ , consider the  $S$ -arithmetic group  $\Gamma_S := G(O_S)$ , and aim to analyze the distribution of  $\Gamma_S$  embedded in  $G_\infty$ .

More generally, the method that we develop can be used to analyze the distribution of  $\Gamma_S$  embedded in  $G_T$  for a finite set of places  $T$  disjoint from  $S$ , but we will not consider this extension here to avoid cumbersome notation.

#### 5.1 Finiteness of the integrability exponent

Our first task is to show that the integrability exponent defined in (2.3) is finite.

THEOREM 5.1. *The integrability exponent  $\mathfrak{p}_S(G)$  is finite.*

*Proof.* When  $G$  is simply connected, this was already established in [GMO08], and we will reduce the proof to this case. We first consider the case when  $S$  is finite, and then deal with the general case.

We recall that when  $G$  is simply connected, there is no non-trivial automorphic characters (see, for instance, [GGN13, Lemma 4.1]), and it follows from [GMO08, Theorem 3.20 and Theorem 3.7] that for all functions  $\phi, \psi \in L^2_0(G(\mathbb{A}_K)/G(K))$  that are  $U_\infty$ -finite for a maximal compact subgroup  $U_\infty$  of  $G_\infty$  and  $\mathcal{W}$ -invariant for a compact open subgroup  $\mathcal{W}$  of  $G_f$ , the functions

$$g \mapsto \langle \pi_S^{\text{aut}}(g)\phi, \psi \rangle = \int_{G(\mathbb{A}_K)/G(K)} \phi(g^{-1}x)\overline{\psi(x)} d\mu(x) \tag{5.1}$$

are in  $L^p(G_S)$  for a uniform  $p > 1$ .

Now suppose that  $G$  is not necessarily simply connected, and that  $S$  is finite. We fix a maximal compact subgroup  $U_\infty$  of  $G_\infty$  and a compact open subgroup  $\mathcal{W}$  of  $G_f$ . We shall show that for any compactly supported  $\phi, \psi \in \mathcal{H}_G^{00}$  that are  $U_\infty$ -finite and  $\mathcal{W}$ -invariant, the matrix coefficients (5.1) are in  $L^p(G_S)$ . Since the span of such functions is dense in  $\mathcal{H}_G^{00}$  as we vary  $\mathcal{W}$ , this will imply that the integrability exponent is finite.

We consider the simply connected cover  $\mathfrak{p} : \tilde{G} \rightarrow G$  that induces the map

$$\mathfrak{p} : \tilde{Y} := \tilde{G}(\mathbb{A}_K)/\tilde{G}(K) \longrightarrow Y := G(\mathbb{A}_K)/G(K).$$

We denote by  $\tilde{\mu}$  and  $\mu$  the invariant probability measures on the spaces  $\tilde{Y}$  and  $Y$ , respectively. According to [PR94, Ch. 8, Proposition 8.8], there is an exact sequence

$$\tilde{G}(\mathbb{A}_K) \xrightarrow{\mathfrak{p}} G(\mathbb{A}_K) \longrightarrow \prod_{v \in V_K} H^1(K_v, Z(G)),$$

so that  $\mathfrak{p}(\tilde{G}(\mathbb{A}_K))$  is a normal co-Abelian subgroup of  $G(\mathbb{A}_K)$ . Let us consider the group  $L := \mathcal{W}\mathfrak{p}(\tilde{G}(\mathbb{A}_K))G(K)$ . Clearly, it is also a normal co-Abelian subgroup of  $G(\mathbb{A}_K)$ . Furthermore,  $L$  is

open in  $G(\mathbb{A}_K)$ . We also consider a subset  $Y_0 := L/G(K)$  of  $Y$ . Then

$$Y = \bigsqcup_{\gamma \in \Delta} \gamma Y_0,$$

where  $\Delta$  is a set of coset representatives for  $L$  in  $G(\mathbb{A}_K)$ . Since  $Y_0$  is open,  $\mu(Y_0) > 0$ . Thus, using that  $\mu(Y) < \infty$ , we conclude that  $\Delta$  is finite. (In fact, this can be also deduced from the finiteness of class number of  $G$ .)

Let  $m_{\mathcal{W}}$  be the Haar probability measure on  $\mathcal{W}$ . The unique  $L$ -invariant probability measure  $\mu_0$  on  $Y_0$  can be given as

$$\int_{Y_0} f d\mu_0 = \int_{w \in \mathcal{W}} \int_{\tilde{y} \in \tilde{Y}} f(w\mathfrak{p}(\tilde{y})) d\tilde{\mu}(\tilde{y}) dm_{\mathcal{W}}(w) \quad \text{for } f \in C_c(Y_0). \tag{5.2}$$

Indeed, the invariance of this measure is easy to check using that  $\mathfrak{p}(\tilde{G}(\mathbb{A}_K))$  is co-Abelian,  $\mathfrak{p}$  is equivariant, and  $\tilde{\mu}$  is invariant. On the other hand, it also follows from invariance that

$$\mu_0 = |G(\mathbb{A}_K) : L| \cdot \mu|_{Y_0}. \tag{5.3}$$

Similarly, considering the exact sequence

$$\tilde{G}_S \xrightarrow{\mathfrak{p}} G_S \longrightarrow \prod_{v \in S} H^1(K_v, Z(G)),$$

we deduce that  $H := \mathfrak{p}(\tilde{G}_S)$  is an open, normal, co-Abelian subgroup of  $G_S$ . Moreover, since  $S$  is finite,  $H$  has finite index in  $G_S$ . We have the decomposition

$$G_S = \bigsqcup_{\delta \in \Omega} \delta H,$$

where  $\Omega$  is a finite subset of coset representatives for  $H$  in  $G_S$ . Let  $\tilde{m}_S$  be a Haar measure on  $\tilde{G}_S$ . Since  $H$  is open in  $G_S$ , a Haar measure on  $H$  is given by the restriction of the Haar measure  $m_S$  on  $G_S$ . It follows from the uniqueness of Haar measure that for some  $c > 0$ ,

$$\int_H f dm_S = c \int_{\tilde{g} \in \tilde{G}_S} f(\mathfrak{p}(\tilde{g})) d\tilde{m}_S(\tilde{g}) \quad \text{for } f \in C_c(H). \tag{5.4}$$

Indeed, since  $S$  is finite, the kernel of the map  $\mathfrak{p} : \tilde{G}_S \rightarrow G_S$  is finite, so that this map is proper. Thus, the right-hand side of (5.4) defines a Haar measure on  $H$ .

For  $\phi, \psi \in \mathcal{H}_G^{00}$  satisfying the invariance properties prescribed above, and  $g \in G_S$ ,

$$\langle \pi_S^{\text{aut}}(g)\phi, \psi \rangle = \int_Y \phi(g^{-1}y)\overline{\psi(y)} d\mu(y) = \sum_{\gamma \in \Delta} \int_{Y_0} \phi(g^{-1}\gamma y)\overline{\psi(\gamma y)} d\mu(y),$$

and by the triangle inequality for  $L^p$ -norms,

$$\begin{aligned} & \left( \int_{G_S} |\langle \pi_S^{\text{aut}}(g)\phi, \psi \rangle|^p dm_S(g) \right)^{1/p} \\ & \leq \sum_{\gamma \in \Delta} \left( \int_{G_S} \left| \int_{Y_0} \phi(g^{-1}\gamma y) \overline{\psi(\gamma y)} d\mu(y) \right|^p dm_S(g) \right)^{1/p} \\ & \leq \sum_{\gamma \in \Delta} \left( \sum_{\delta \in \Omega} \int_H \left| \int_{Y_0} \phi(h^{-1}\delta^{-1}\gamma y) \overline{\psi(\gamma y)} d\mu(y) \right|^p dm_S(h) \right)^{1/p} \\ & = \sum_{\gamma \in \Delta} \left( \sum_{\delta \in \Omega} \int_H \left| \int_{Y_0} \phi(\delta^{-1}\gamma h^{-1}y) \overline{\psi(\gamma y)} d\mu(y) \right|^p dm_S(h) \right)^{1/p}. \end{aligned} \tag{5.5}$$

Here in the last step we have used the fact that  $L$  is co-Abelian. Indeed, denoting  $\delta^{-1}\gamma = z$ , this property implies  $h^{-1}z = zh^{-1}l'$  with  $l' \in L$ . Now  $Y_0 = L/G(K)$  and writing a coset  $y \in Y_0$  as  $y = lG(K)$  with  $l \in L$ , we have  $h^{-1}z lG(K) = zh^{-1}l' lG(K)$ . Therefore, the integrals over  $Y_0 = L/G(K)$  in the last step above gives the same function of  $h \in H$  in both cases.

We claim that for every  $r_1, r_2 \in G(\mathbb{A}_K)$ , the functions

$$c_{r_1, r_2}(h) := \int_{Y_0} \phi(r_1 h y) \overline{\psi(r_2 y)} d\mu(y)$$

are in  $L^p(H)$ . In view of (5.2) and (5.3),

$$c_{r_1, r_2}(h) = |G(\mathbb{A}_K) : L|^{-1} \int_{\tilde{y} \in \tilde{Y}} \phi(r_1 h \mathbf{p}(\tilde{y})) \overline{\psi(r_2 \mathbf{p}(\tilde{y}))} d\tilde{\mu}(\tilde{y}),$$

For a function  $f$  on  $Y$  and  $g \in G(\mathbb{A}_K)$ , we define a function on  $\tilde{Y}$  by

$$\tilde{f}_g(\tilde{y}) := f(g\mathbf{p}(\tilde{y})) \quad \text{for } \tilde{y} \in \tilde{Y}.$$

Then when  $f$  is  $\mathcal{W}$ -invariant, using (5.2), (5.3) and that  $L$  is co-Abelian, we deduce that

$$\begin{aligned} \int_{\tilde{Y}} \tilde{f}_g d\tilde{\mu} &= \int_{\mathcal{W}} \int_{\tilde{Y}} f(gw\mathbf{p}(\tilde{y})) d\tilde{\mu}(\tilde{y}) dm_{\mathcal{W}}(w) = \int_{\mathcal{W}} \int_{\tilde{Y}} f(wg\mathbf{p}(\tilde{y})) d\tilde{\mu}(\tilde{y}) dm_{\mathcal{W}}(w) \\ &= \int_{Y_0} f(gz) d\mu_0(z) = |G(\mathbb{A}_K) : L| \int_{gY_0} f d\mu. \end{aligned}$$

Let  $\mathcal{X}(G, L)$  denote the set of continuous unitary characters  $\chi$  of  $G(\mathbb{A}_K)$  such that  $\chi(L) = 1$ . It follows from the properties of characters of finite Abelian groups that

$$\chi_{Y_0} = |G(\mathbb{A}_K) : L|^{-1} \sum_{\chi \in \mathcal{X}(G, L)} \chi,$$

so that if  $f \in \mathcal{H}_G^{00}$ , using the previous two identities we deduce

$$\begin{aligned} \int_{\tilde{Y}} \tilde{f}_g d\tilde{\mu} &= \int_Y (f \circ g) \left( \sum_{\chi \in \mathcal{X}(G, L)} \chi \right) d\mu = \int_Y f \left( \sum_{\chi \in \mathcal{X}(G, L)} \chi \circ g^{-1} \right) d\mu \\ &= \sum_{\chi \in \mathcal{X}(G, L)} \chi(g)^{-1} \int_Y f \chi d\mu = 0. \end{aligned}$$

This shows that if  $f \in \mathcal{H}_G^{00}$ , then  $\tilde{f}_g \in L^2_0(\tilde{Y})$ . In particular,  $\tilde{\phi}_{r_1}, \tilde{\psi}_{r_2} \in L^2_0(\tilde{Y})$ . Using Hölder’s inequality, the fact that  $L$  is co-Abelian, and (5.4), we conclude that

$$\begin{aligned} \int_H |c_{r_1, r_2}|^p dm_S &= \int_H \left| \int_{\mathcal{W}} \int_{\tilde{Y}} \phi(r_1 h w \mathbf{p}(\tilde{y})) \overline{\psi(r_2 w \mathbf{p}(\tilde{y}))} d\tilde{\mu}(\tilde{y}) dm_{\mathcal{W}}(w) \right|^p dm_S(h) \\ &\leq \int_H \int_{\mathcal{W}} \left| \int_{\tilde{Y}} \phi(r_1 h w \mathbf{p}(\tilde{y})) \overline{\psi(r_2 w \mathbf{p}(\tilde{y}))} d\tilde{\mu}(\tilde{y}) \right|^p dm_{\mathcal{W}}(w) dm_S(h) \\ &\ll \int_{\mathcal{W}} \int_H \left| \int_{\tilde{Y}} \phi(r_1 w h \mathbf{p}(\tilde{y})) \overline{\psi(r_2 w \mathbf{p}(\tilde{y}))} d\tilde{\mu}(\tilde{y}) \right|^p dm_S(h) dm_{\mathcal{W}}(w) \\ &\ll \int_{\mathcal{W}} \int_{\tilde{G}_S} \left| \int_{\tilde{Y}} \phi(r_1 w \mathbf{p}(\tilde{g}) \mathbf{p}(\tilde{y})) \overline{\psi(r_2 w \mathbf{p}(\tilde{y}))} d\tilde{\mu}(\tilde{y}) \right|^p d\tilde{m}_S(\tilde{g}) dm_{\mathcal{W}}(w) \\ &= \int_{\mathcal{W}} \int_{\tilde{G}_S} \left| \langle \tilde{\pi}_S^{\text{aut}}(\tilde{g}^{-1}) \tilde{\phi}_{r_1 w}, \tilde{\phi}_{r_2 w} \rangle \right|^p d\tilde{m}_S(\tilde{g}) dm_{\mathcal{W}}(w). \end{aligned}$$

It is easy to see that  $\tilde{\phi}_{r_1 w}$  and  $\tilde{\phi}_{r_2 w}$  are  $\tilde{U}_\infty$ -finite for suitable maximal compact subgroup  $\tilde{U}_\infty$  of  $\tilde{G}_\infty$  and  $\tilde{\mathcal{W}}$ -finite for suitable compact open subgroup  $\tilde{\mathcal{W}}$  of  $\tilde{G}$ . We observe that the integrand in the above formula is locally constant in  $w \in \mathcal{W}$ , and it follows from the simply connected case that the functions  $g \mapsto \langle \tilde{\pi}_S^{\text{aut}}(\tilde{g}^{-1}) \tilde{\phi}_{r_1 w}, \tilde{\phi}_{r_2 w} \rangle$  are in  $L^p(\tilde{G}_S)$ . Hence, we conclude that

$$\int_{\mathcal{W}} \int_H |c_{r_1 w, r_2 w}|^p dm_S dm_{\mathcal{W}}(w) < \infty,$$

and by (5.5) also

$$\int_{G_S} |\langle \pi_S^{\text{aut}}(g) \phi, \psi \rangle|^p dm_S(g) < \infty.$$

This proves that the representation  $\pi_S^{\text{aut}}|_{\mathcal{H}_G^{00}}$  is  $L^p$ -integrable when  $S$  is finite, and  $p$  is uniform and independent of  $S$ .

To deal with the general case with  $S$  arbitrary and possibly infinite, we use that the representation  $\pi_S^{\text{aut}}$  has the direct integral decomposition

$$\pi_S^{\text{aut}}|_{\mathcal{H}_G^{00}} = \int_{\hat{G}_S} \rho^{\oplus n(\rho)} d\Pi(\rho) \tag{5.6}$$

with respect to a measure  $\Pi$  on the unitary dual  $\hat{G}_S$ , with  $n(\rho) \in \mathbb{N} \cup \{\infty\}$  denoting the multiplicity. The irreducible representations  $\rho$  in this decomposition are restricted tensor products of the form  $\rho = \otimes_{v \in S} \rho_v$ , where  $\rho_v$  are irreducible representations of  $G_v$ , which are  $U_v$ -spherical for almost all  $v \in S$  (see [Fla79]). For each  $v \in S$ , we also have the decomposition

$$\pi_v^{\text{aut}}|_{\mathcal{H}_G^{00}} = \int_{\hat{G}_v} \rho_v^{\oplus n_v(\rho_v)} d\Pi_v(\rho_v). \tag{5.7}$$

We can conclude from the discussion in the previous paragraph that the representation  $\pi_v^{\text{aut}}|_{\mathcal{H}_G^{00}}$  is  $L^p$ -integrable. This condition implies that  $\Pi_v$ -almost all  $\rho_v$  is  $L^{p'}$ -integrable for some finite  $p' \geq p$ . Indeed,  $L^p$ -integrability implies that a suitable tensor power  $(\pi_v^{\text{aut}}|_{\mathcal{H}_G^{00}})^{\otimes N}$  is weakly contained in the regular representation of  $G_v$ . Since  $\rho_v$  is weakly contained in  $\pi_v^{\text{aut}}|_{\mathcal{H}_G^{00}}$ , it follows that for  $\Pi_v$ -almost all  $\rho_v$ , the tensor power  $\rho_v^{\otimes N}$  is weakly contained in the regular representation of  $G_v$ . It follows that the  $N$ th power of the matrix coefficients of  $U_v$ -finite vectors  $\rho_v$  satisfy the pointwise bound given by [CHH88, Theorem 2]. Therefore,  $\rho_v$  is  $p'$ -integrable for  $\Pi_v$ -almost all  $\rho_v$ , and the

matrix coefficients of  $U_v$ -finite vectors  $\phi_v, \psi_v$  of  $\rho_v$  satisfy, for suitable  $k \in \mathbb{N}$ ,

$$|\langle \rho_v(g_v)\phi_v, \psi_v \rangle| \leq d_v(\phi_v)^{1/2} d_v(\psi_v)^{1/2} \|\phi_v\| \|\psi_v\| \Xi_v(g_v)^{1/k} \quad \text{for } g_v \in G_v, \tag{5.8}$$

where  $d_v(\phi_v) := \dim \langle \rho_v(U_v)\phi_v \rangle$ , and  $\Xi_v$  denotes the Harish-Chandra function on  $G_v$ . Since the measure  $\Pi_v$  in (5.7) is the image of the measure  $\Pi$  from (5.7) under the restriction map,  $\Pi$ -almost every representation  $\rho$  appearing in the decomposition (5.6) is of the form  $\rho = \otimes_{v \in S} \rho_v$  where the representations  $\rho_v$  satisfy the bound (5.8). It follows from the description of the space of  $U_S$ -finite vectors for tensor products (see [Fla79]), that for all  $U_S$ -finite vectors  $\phi$  and  $\psi$ , there exists  $c(\phi, \psi) > 0$  such that

$$|\langle \rho(g)\phi, \psi \rangle| \leq c(\phi, \psi) \Xi_S(g)^{1/k} \quad \text{for } g \in G_S, \tag{5.9}$$

where  $\Xi_S(g) := \prod_{v \in S} \Xi_v(g_v)$  is the Harish-Chandra function on  $G_S$ . We recall that the Harish-Chandra function  $\Xi_S$  is  $L^{4+\eta}$ -integrable for all  $\eta > 0$  (see [GN12, Proposition 6.3]). We note that the argument in [GN12] utilizes only the Cartan and Iwasawa decompositions of the group, and does not require that the group be simply connected. Hence, the estimate (5.9) implies that  $\Pi$ -almost every representation  $\rho$  appearing in the decomposition (5.6) is  $L^q$ -integrable with a uniform  $q$ , and with the  $L^q$ -norm uniformly bounded. We conclude that the representation  $\pi_S^{\text{aut}}|_{\mathcal{H}_G^0}$  is  $L^q$ -integrable.  $\square$

### 5.2 Mean ergodic theorem for general groups

We note that Theorem 2.6 fails if  $G$  is not simply connected and the corresponding action of  $G_S$  is not even ergodic. Nonetheless, it turns out that an analogue of this estimate holds if we consider actions on a smaller space.

Let  $\mathcal{X}(G, \mathcal{I}_f)$  denote the set of continuous unitary characters  $\chi$  of  $G(\mathbb{A}_K)$  such that  $\chi(\mathcal{I}_f) = \chi(\Gamma) = 1$ . This set is known to be a finite Abelian group, and its kernel

$$G^{\text{ker}} := \{g \in G(\mathbb{A}_K) : \chi(g) = 1 \text{ for all } \chi \in \mathcal{X}(G, \mathcal{I}_f)\}.$$

is a finite index subgroup of  $G(\mathbb{A}_K)$  (see [GGN13, Lemma 4.4]). Let  $G_\infty^0$  denote the connected component of identity in  $G_\infty$ . Since  $G_\infty^0$  is a connected semisimple Lie group, it is clear that  $G_\infty^0 \subset G_\infty \cap G^{\text{ker}}$ . We also set

$$G_S^{\text{ker}} := G_S \cap G^{\text{ker}}.$$

We note that  $G_S^{\text{ker}}$  is a finite index closed (and open) subgroup of  $G_S$ . Let

$$\Gamma_S^{\text{ker}} := \Gamma_S \cap (G_\infty^0 \times G_S).$$

LEMMA 5.2. We have  $\Gamma_S^{\text{ker}} \subset G_\infty^0 \times G_S^{\text{ker}}$ .

*Proof.* We consider  $\Gamma_S^{\text{ker}}$  as a subgroup of  $G_\infty^0 \times G_S \times \mathcal{I}^S \subset G(\mathbb{A}_K)$  embedded diagonally. Then for every  $\chi \in \mathcal{X}(G, \mathcal{I}_f)$  and  $\gamma \in \Gamma_S^{\text{ker}}$ , we have  $\chi(\gamma, \gamma, \gamma) = 1$  and  $\chi(G_\infty^0) = \chi(\mathcal{I}^S) = 1$ , so that it also follows that  $\chi(e, \gamma, e) = 1$ . Hence,  $\Gamma_S^{\text{ker}}$  is, in fact, contained in  $G_\infty^0 \times G_S^{\text{ker}} \times \mathcal{I}^S$ .  $\square$

We consider the space

$$Y_S^{\text{ker}} := (G_\infty^0 \times G_S^{\text{ker}}) / \Gamma_S^{\text{ker}}$$

equipped with the unique invariant probability measure  $\mu_S^{\text{ker}}$ , and the corresponding unitary representations  $\pi_S$  of  $G_S^{\text{ker}}$  on  $L^2(Y_S^{\text{ker}})$ .

THEOREM 5.3. Let  $\beta$  be a Haar-uniform probability measure supported on  $\mathcal{I}_S$ -bi-invariant bounded subset  $B$  of  $G_S^{\text{ker}}$ . Then there exists  $\mathfrak{k}_S(G) > 0$  such that for every given  $\eta > 0$ ,

for  $\phi \in L^2(Y_S^{\text{ker}})$ ,

$$\left\| \pi_S(\beta)\phi - \int_{Y_S^{\text{ker}}} \phi d\mu_S^{\text{ker}} \right\|_{L^2(Y_S^{\text{ker}})} \ll_{S,\eta} m_S(B)^{-\mathfrak{k}_S(\mathbb{G})+\eta} \|\phi\|_{L^2(Y_S^{\text{ker}})}.$$

*Proof.* Part I. We first consider the case of  $S = V_K^f$ , the full set of finite places.

To simplify notation, we set

$$G := \mathbb{G}(\mathbb{A}_K), \quad \Gamma = \mathbb{G}(K), \quad X := G/\Gamma, \quad X^{\text{ker}} := G^{\text{ker}}/\Gamma.$$

We equip  $X$  and  $X^{\text{ker}}$  with the Haar probability measures  $\mu$  and  $\mu^{\text{ker}}$ , respectively. Then by invariance,

$$\mu^{\text{ker}} = \mu(X^{\text{ker}})^{-1} \cdot \mu|_{X^{\text{ker}}} = |G : G^{\text{ker}}| \cdot \mu|_{X^{\text{ker}}}. \tag{5.10}$$

Let  $B^{\text{ker}}$  denote a bounded  $\mathcal{I}_f$ -bi-invariant subset of  $G_f \cap G^{\text{ker}}$ , and  $\beta^{\text{ker}}$  is the Haar-uniform probability measure. We will first prove an effective mean ergodic theorem for the operators  $\pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})$  acting on  $L^2(X)$  for test functions  $\psi$  with  $\text{supp}(\psi) \subset X^{\text{ker}}$ .

Our argument proceeds similarly to [GGN13, Thm. 4.5]. We observe that the space  $L^2(X)$  has the decomposition

$$L^2(X) = \mathcal{H}_G^{\text{char}} \oplus \mathcal{H}_G^{00},$$

where  $\mathcal{H}_G^{\text{char}}$  is the closure of the span of automorphic characters, and  $\mathcal{H}_G^{00}$  is its orthogonal complement. We also write

$$\mathcal{H}_G^{\text{char}} = \mathcal{H} \oplus \mathcal{H}',$$

where  $\mathcal{H}$  is the finite-dimensional space spanned by  $\mathcal{X}(G, \mathcal{I}_f)$ , the automorphic characters and  $\mathcal{H}'$  its orthogonal complement. Since the measure  $\beta^{\text{ker}}$  is  $\mathcal{I}_f$ -bi-invariant, for all automorphic characters  $\chi$  and  $u \in \mathcal{I}_f$ ,

$$\pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})\chi = \pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}} * \delta_u)\chi = \chi(u^{-1})\pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})\chi.$$

In particular, it follows that  $\pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})\chi = 0$  when  $\chi$  is not  $\mathcal{I}_f$ -invariant, and

$$\pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})|_{\mathcal{H}'} = 0.$$

It follows from Theorem 2.4 with  $S = V_K^f$  that

$$\left\| \pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})|_{\mathcal{H}_G^{00}} \right\| \ll_{S,\eta} m_S(B)^{-\mathfrak{k}_S(\mathbb{G})+\eta}$$

for some explicit  $\mathfrak{k}_S(\mathbb{G}) > 0$ . Hence, we conclude that for every  $\psi \in L^2(X)$ ,

$$\left\| \pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})\psi - \pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})P_{\mathcal{H}}\psi \right\|_{L^2(X)} \ll_S m_S(B)^{-\mathfrak{k}_S(\mathbb{G})} \|\psi\|_{L^2(X)},$$

where  $P_{\mathcal{H}}$  denotes the orthogonal projection on the space  $\mathcal{H}$ .

Using the fact that  $\mathcal{X}(G, \mathcal{I}_f)$  forms an orthonormal basis of  $\mathcal{H}$ , under our additional assumption that  $\text{supp}(\psi)$  is contained in the intersection of the kernels of  $\chi \in \mathcal{X}(G, \mathcal{I}_f)$ , we obtain that

$$P_{\mathcal{H}}\psi = \sum_{\chi \in \mathcal{X}(G, \mathcal{I}_f)} \langle \psi, \chi \rangle_{L^2(X)} \chi = \left( \int_X \psi d\mu \right) \sum_{\chi \in \mathcal{X}(G, \mathcal{I}_f)} \chi = \left( \int_X \psi d\mu \right) \xi,$$

where

$$\xi(g) := \sum_{\chi \in \mathcal{X}(G, \mathcal{I}_f)} \chi(g) \quad \text{for } g \in G.$$

Since  $\text{supp}(\beta^{\text{ker}}) \subset G^{\text{ker}}$ , it follows that

$$\pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})\chi = \chi \quad \text{for } \chi \in \mathcal{X}(G, \mathcal{I}_f).$$

Hence, we conclude that for all  $\psi \in L^2(X^{\text{ker}})$ ,

$$\left\| \pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})\psi - \left( \int_X \psi \, d\mu \right) \xi \right\|_{L^2(X)} \ll_S m_S(B)^{-\mathfrak{k}_S(G)} \|\psi\|_{L^2(X)}.$$

Since

$$\xi|_{X^{\text{ker}}} = |\mathcal{X}(G, \mathcal{I}_f)| = |G : G^{\text{ker}}|,$$

we also deduce, using (5.10), that for all  $\psi \in L^2(X^{\text{ker}})$ ,

$$\left\| \pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})\psi - \int_{X^{\text{ker}}} \psi \, d\mu^{\text{ker}} \right\|_{L^2(X^{\text{ker}})} \ll_S m_S(B)^{-\mathfrak{k}_S(G)} \|\psi\|_{L^2(X^{\text{ker}})}. \tag{5.11}$$

Part II. We now consider a general set of places  $S \subset V_K^f$  with  $G$  isotropic over  $S$ . Let  $Z_S$  denote the orbit of  $G_\infty^0 \times G_S^{\text{ker}} \times \mathcal{I}^S \subset G^{\text{ker}}$  acting on the identity coset in the space  $X^{\text{ker}}$ . It is open and closed subset of  $X^{\text{ker}}$ . We equip  $Z_S$  with the probability measure  $\nu_S := \mu^{\text{ker}}(Z_S)^{-1} \mu^{\text{ker}}$ . We have an isomorphism

$$Y_S^{\text{ker}} \simeq \mathcal{I}^S \setminus Z_S$$

of  $(G_\infty^0 \times G_S^{\text{ker}})$ -spaces. Therefore, the unitary representation  $\pi_S|_{G_S^{\text{ker}}}$  on  $L^2(Y_S^{\text{ker}})$  is equivalent to the unitary representation  $G_S^{\text{ker}}$  on  $L^2(\mathcal{I}^S \setminus Z_S)$ , which is also equivalent to the unitary representation of  $G_S^{\text{ker}}$  on the space  $L^2(Z_S)^{\mathcal{I}^S}$  consisting of  $\mathcal{I}^S$ -invariant functions in  $L^2(Z_S)$ . More explicitly, given a function  $\phi$  on the space  $Y_S^{\text{ker}}$ , we get a  $\mathcal{I}^S$ -invariant function

$$\phi^{\text{ker}}((g_\infty, g_S, u)\Gamma) := \phi((g_\infty, g_S)\Gamma_S^{\text{ker}}), \quad \text{for } (g_\infty, g_S, u) \in G_\infty^{\text{ker}} \times G_S^{\text{ker}} \times \mathcal{I}^S,$$

on the space  $Z_S$ . This defines a  $(G_\infty^0 \times G_S^{\text{ker}})$ -equivariant isometry between  $L^2(Y_S^{\text{ker}})$  and  $L^2(Z_S)^{\mathcal{I}^S}$ .

Recall that we denote by  $\beta$  the Haar-uniform probability measure supported on  $\mathcal{I}_S$ -bi-invariant bounded subset  $B$  of  $G_S^{\text{ker}}$ . Denote by  $\beta^{\text{ker}}$  the Haar-uniform probability measure supported on  $B \times \mathcal{I}^S \subset G_f$ . Then since  $\phi^{\text{ker}}$  is  $\mathcal{I}^S$ -invariant, we obtain that

$$\pi_{V_K^f}^{\text{aut}}(\beta^{\text{ker}})\phi^{\text{ker}} = \pi_S(\beta)\phi.$$

Finally, we apply the estimate (5.11) to the case when  $\psi = \phi^{\text{ker}}$  for  $\phi \in L^2(Y_S^{\text{ker}})$ . Since

$$\int_{X^{\text{ker}}} \phi^{\text{ker}} \, d\mu^{\text{ker}} = \mu^{\text{ker}}(Z_S) \int_{Z_S} \phi^{\text{ker}} \, d\nu_S = \mu^{\text{ker}}(Z_S) \int_{Y_S^{\text{ker}}} \phi \, d\mu_S^{\text{ker}},$$

using the above identifications, we conclude that (for any  $\eta > 0$ )

$$\left\| \pi_S(\beta)\phi - \mu^{\text{ker}}(Z_S) \int_{Y_S^{\text{ker}}} \phi \, d\mu_S^{\text{ker}} \right\|_{L^2(Y_S^{\text{ker}})} \ll_{S,\eta} m_S(B)^{-\mathfrak{k}_S(G)+\eta} \|\phi\|_{L^2(Y_S^{\text{ker}})}.$$

We note that this estimate holds for any bounded  $\mathcal{I}_S$ -bi-invariant subsets of  $G_S^{\text{ker}}$ , so that taking  $\phi = 1$  and  $m_S(B) \rightarrow \infty$ , we deduce from the above estimate that  $\mu^{\text{ker}}(Z_S) = 1$ . This completes the proof of the Theorem 5.3. □

### 5.3 Discrepancy estimates for general groups

When  $G$  is not simply connected, the set  $\Gamma_S$  may be not dense in  $G_\infty$ . Nonetheless, according to Corollary 2.2, its closure is a subgroup of finite index in  $G_\infty$ . In particular,  $\Gamma_S \cap G_\infty^0$  is dense

in  $G_\infty^0$ , the connected component of  $G_\infty$ , and we estimate the discrepancy for  $\Gamma_S$ -points with respect to subsets of  $G_\infty^0$ . As in the previous results, we parametrize  $\Gamma_S$  by the subsets

$$R_S(h) := \{\gamma \in \Gamma_S : H_f(\gamma) \leq h\}.$$

Let  $m_\infty^0$  be the Haar measure on  $G_\infty^0$ , which we choose to normalize so that  $\Gamma_S$  has covolume one in  $G_\infty^0 \times G_S^{\ker}$  with respect to  $m_\infty^0 \times m_S$ . We define

$$v_S^{\ker}(h) := m_S(B_h^{\ker}(S)),$$

where  $B_h^{\ker}(S) := \{g \in G_S^{\ker} : H_f(g) \leq h\}$ . For  $\Omega \subset G_\infty^0$ , we introduce the discrepancy function:

$$\mathcal{D}(R_S(h), \Omega) := \left| \frac{|R_S(h) \cap \Omega|}{v_S^{\ker}(h)} - m_\infty^0(\Omega) \right|.$$

We emphasize here that the correct normalization is, in fact, by  $v_S^{\ker}(h)$  and not by  $v_S(h)$  as in the simply connected case. While  $v_S^{\ker}(h)$  is comparable with  $v_S(h) = m_S(B_h(S))$  up to multiplicative constants, to get the correct main term it is essential to take into account the contribution of automorphic characters. This contribution manifests itself through the volume function  $v_S^{\ker}(h)$  of  $B_h^{\ker} \subset G_S^{\ker}$ . Note that looking only at the group of rational points  $\Gamma_S$  which is a lattice in  $G_\infty \times G_S$  as before, the fact  $B_S^{\ker}(h)$  is the correct choice here is not obvious in advance. It reflects subtle algebraic information regarding the behavior of the automorphic characters when restricted to  $G_S$ , since, in fact,  $\Gamma_S \subset G_\infty^0 \times G_S^{\ker}$ .

Using the methods presented in § 4, it is possible to establish discrepancy results in the present case given congruence constraints determined by suitable compact-open subgroups  $W \subset \mathcal{I}^S$ . But since in the non-simply connected case certain congruence obstructions are bound to arise, the full analysis here is longer and requires additional notation. For brevity, we state the results only for the case of  $W = \mathcal{I}^S$ , namely in the absence of congruence constraints.

As above for  $x \in G_\infty^0$  and  $E \subset G_\infty^0$ , we set  $E(x) := Ex$ .

Using Theorem 5.3, we establish a mean-square discrepancy bound.

**THEOREM 5.4** (Mean-square discrepancy bound). *Let  $E$  be any measurable subset of  $G_\infty^0$  of positive finite measure satisfying  $\mathcal{N}(E) < \infty$ . Then, for any  $\eta > 0$ ,*

$$\|\mathcal{D}(R_S(h), E(\cdot))\|_{L^2(Q)} \ll_{S,Q,\eta} \mathcal{N}(E)^{1/2} m_\infty^0(E)^{1/2} v_S^{\ker}(h)^{-\epsilon_S(\mathbb{G})+\eta}$$

for every bounded measurable subset  $Q$  of  $G_\infty^0$ .

*Proof.* The argument proceeds along the lines of the proof of Theorem 4.1. We note that since  $E(x)$  is a subset of  $G_\infty^0$ , by Lemma 5.2,

$$\Gamma_S \cap (E(x) \times B_S(h)) = \Gamma_S \cap (E(x) \times B_S^{\ker}(h)) = \Gamma_S^{\ker} \cap (E(x) \times B_S^{\ker}(h)).$$

We consider the function

$$\phi(g) := \sum_{\delta \in \Gamma_S^{\ker}} \chi_{E^{-1}}(g_\infty \delta^{-1}) \chi_{\mathcal{I}^S}(g_S \delta^{-1})$$

on  $Y_S^{\ker} = (G_\infty^0 \times G_S^{\ker})/\Gamma_S^{\ker}$ . As in the proof of Theorem 4.1, one verifies that

$$|\Gamma_S^{\ker} \cap (E(x) \times B_S^{\ker}(h))| = \int_{a \in B_S^{\ker}(h)} \phi(a(x, e)) dm_S(b),$$

and

$$\mathcal{D}(R_S(h), E(x)) = \left| \frac{1}{m_S(B_S^{\ker}(h))} \int_{a \in B_S^{\ker}(h)} \phi(a(x, e)) dm_S(b) - \int_{Y_S^{\ker}} \phi d\mu_S \right|.$$



We observe that since the subgroup  $G^{\ker}$  is defined as a kernel of a set of  $\mathcal{I}_S$ -invariant characters, it is  $\mathcal{I}_S$ -bi-invariant. Hence, it follows that the sets  $B_S^{\ker}(h) = G^{\ker} \cap B_S(h)$  are also  $\mathcal{I}_S$ -bi-invariant, and we can apply Theorem 5.3. The remaining proof proceeds exactly as the proof of Theorem 4.1.  $\square$

Once the mean-square discrepancy bound has been established, one can also deduce exactly as § 4 generalizations of the almost-sure discrepancy bound (Theorem 4.4), the uniform discrepancy bound for right-stable sets (Theorem 4.5), and the uniform discrepancy bound for balls of arbitrarily small radius (Theorem 4.6). Here the pointwise estimates depend as before on the dimension  $\mathfrak{d} := \dim_{\mathbb{R}}(G_{\infty})$ . We state these results as follows.

**THEOREM 5.5** (*Almost-sure discrepancy bound*). *With notation as in Theorem 5.4, for every  $\eta > 0$  and almost all  $x \in G_{\infty}^0$ ,*

$$\mathcal{D}(R_S(h), E(x)) \ll_{S,E,x,\mathfrak{k},\eta} (\log v_S^{\ker}(h))^{3/2+\eta} v_S^{\ker}(h)^{-\mathfrak{k}}.$$

**THEOREM 5.6** (*Uniform discrepancy bound*). *For every right-stable subsets  $E$  of  $G_{\infty}$  of finite measure such that  $\mathcal{N}(E_{e_0}^+) < \infty$ , for every  $0 < \mathfrak{k} < \mathfrak{k}_S(\mathbf{G})$  and for  $x \in G_{\infty}$ , the following pointwise bound for the discrepancy holds:*

$$\mathcal{D}(R_S(h), E(x)) \ll_{S,E,x,\mathfrak{k}} v_S^{\ker}(h)^{-2\mathfrak{k}/(\mathfrak{d}+2)}$$

*provided that  $h \geq h_0(S)$ . Moreover, this estimate is uniform for  $x$  ranging in compact subsets of  $G_{\infty}$ .*

**THEOREM 5.7** (*Discrepancy bound for balls*). *Let  $x \in G_{\infty}$  and  $\ell \in (0, \ell_0)$ , for suitable  $\ell_0$  (independent of  $x$ ). For every  $0 < \mathfrak{k} < \mathfrak{k}_S(\mathbf{G})$ :*

$$|R_S(h) \cap B(x, \ell)| = m_{\infty}(B(e, \ell))v_S^{\ker}(h) + O_{S,x,\mathfrak{k}}(m_{\infty}(B(e, \ell))^{\mathfrak{d}/(\mathfrak{d}+2)}v_S(h)^{1-2\mathfrak{k}/(\mathfrak{d}+2)})$$

*provided that  $m_{\infty}(B(e, \ell))^2 \gg_{S,\mathfrak{k}} v_S(h)^{-2\mathfrak{k}}$ .*

*Explicitly, if the volume growth satisfies  $v_S(h) \gg_{S,\mathfrak{a}} h^{\mathfrak{a}}$ , with some  $\mathfrak{a} > 0$ , then the estimate holds provided that the height  $h$  satisfies  $h \gg_{S,\mathfrak{a},\mathfrak{k}} \ell^{-\mathfrak{d}/\mathfrak{k}\mathfrak{a}}$ . Moreover, this estimate is uniform for  $x$  ranging in compact subsets of  $G_{\infty}$ .*

Since the proofs of these results proceed as in § 4, we omit the details.

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CONFLICTS OF INTEREST

None.

REFERENCES

BO12 Y. Benoist and H. Oh, *Effective equidistribution of  $S$ -integral points on symmetric varieties*, Ann. Inst. Fourier (Grenoble) **62** (2012), 1889–1942.  
 BLS92 M. Burger, J.-S. Li and P. Sarnak, *Ramanujan duals and automorphic spectrum*, Bull. Amer. Math. Soc. (N.S.) **26** (1992), 253–257.  
 BS91 M. Burger and P. Sarnak, *Ramanujan duals. II*, Invent. Math. **106** (1991), 1–11.

- Clo02 L. Clozel, *Automorphic forms and the distribution of points on odd-dimensional spheres*, Israel J. Math. **132** (2002), 175–187.
- Clo03 L. Clozel, *Démonstration de la conjecture  $\tau$* , Invent. Math. **151** (2003), 297–328.
- Clo07 L. Clozel, *Spectral theory of automorphic forms*, in *Automorphic forms and applications*, IAS/Park City Mathematics Series, vol. 12 (American Mathematical Society, Providence, RI, 2007), 43–93.
- COU01 L. Clozel, H. Oh and E. Ullmo, *Hecke operators and equidistribution of Hecke points*, Invent. Math. **144** (2001), 327–351.
- CU04 L. Clozel and E. Ullmo, *Équidistribution des points de Hecke*, in *Contributions to automorphic forms, geometry, and number theory* (Johns Hopkins University Press, Baltimore, MD, 2004), 193–254.
- Con11 B. Conrad, *Modular forms and the Ramanujan conjecture* (Cambridge University Press, 2011).
- CHH88 M. Cowling, U. Haagerup and R. Howe, *Almost  $L^2$  matrix coefficients*, J. Reine Angew. Math. **387** (1988), 97–110.
- Duk03 W. Duke, *Rational points on the sphere. Rankin memorial issues*, Ramanujan J. **7** (2003), 235–239.
- Erd59 P. Erdős, *Some results on Diophantine approximation*, Acta Arith. **5** (1959), 359–369.
- Fla79 D. Flath, *Decomposition of representations into tensor products*, in *Automorphic forms, representations and L-functions, Part 1*, Proceedings of Symposia in Pure Mathematics, vol. XXXIII (American Mathematical Society, Providence, RI, 1979), 179–183.
- GGN13 A. Ghosh, A. Gorodnik and A. Nevo, *Diophantine approximation and automorphic spectrum*, Int. Math. Res. Not. IMRN **21** (2013), 5002–5058.
- GGN14 A. Ghosh, A. Gorodnik and A. Nevo, *Metric Diophantine approximation on homogeneous varieties*, Comput. Math. **150** (2014), 1435–1456.
- GGN22 A. Ghosh, A. Gorodnik and A. Nevo, *Counting solutions to intrinsic Diophantine inequalities in simple algebraic groups*, Israel J. Math. **251** (2022), 443–466.
- GK17 A. Gorodnik and S. Kadyrov, *Diophantine approximation with restricted numerators and denominators on semisimple groups*, J. Théor. Nombres Bordeaux **29** (2017), 1–28.
- GMO08 A. Gorodnik, F. Maucourant and H. Oh, *Manin’s and Peyre’s conjectures on rational points and adelic mixing*, Ann. Sci. Éc. Norm. Supér. **41** (2008), 383–435.
- GN12 A. Gorodnik and A. Nevo, *Counting lattice points*, J. Reine Angew. Math. **663** (2012), 127–176.
- Lan65 S. Lang, *Report on Diophantine approximation*, Bull. Soc. Math. France **93** (1965), 177–192.
- LeV59 W. J. LeVeque, *On the frequency of small fractional parts in certain real sequences. III*, J. Reine Angew. Math. **202** (1959), 215–220.
- Nev98 A. Nevo, *Spectral transfer and pointwise ergodic theorems for semi-simple Kazhdan groups*, Math. Res. Lett. **5** (1998), 305–325.
- PR94 V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139 (Academic Press, Boston, MA, 1994).
- Sak96 T. Sakai, *Riemannian geometry*, Translations of Mathematical Monographs, vol. 149 (American Mathematical Society, Providence, RI, 1996).
- Sar05 P. Sarnak, *Notes on the generalized Ramanujan conjectures*, in *Harmonic analysis, the trace formula, and Shimura varieties*, Clay Mathematics Proceedings, vol. 4 (American Mathematical Society, Providence, RI, 2005), 659–685.
- Sch60 W. Schmidt, *A metrical theorem in Diophantine approximation*, Canad. J. Math. **12** (1960), 619–631.

EXPLICIT DISCREPANCY ESTIMATES

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