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THREE-DIMENSIONAL WAVE-FREE POTENTIALS IN THE THEORY OF WATER WAVES

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Abstract

Problems of wave interaction with a body with arbitrary shape floating or submerged in water are of immense importance in the literature on the linearized theory of water waves. Wave-free potentials are used to construct solutions to these problems involving bodies with circular geometry, such as a submerged or half-immersed long horizontal circular cylinder (in two dimensions) or sphere (in three dimensions). These are singular solutions of Laplace's equation satisfying the free surface condition and decaying rapidly away from the point of singularity. Wave-free potentials in two and three dimensions for infinitely deep water as well as water of uniform finite depth with a free surface are known in the literature. The method of constructing wave-free potentials in three dimensions is presented here in a systematic manner, neglecting or taking into account the effect of surface tension at the free surface or for water with an ice cover modelled as a thin elastic plate floating on the water. The forms of the wave motion at the upper surface (free surface or ice-covered surface) related to these wave-free potentials are depicted graphically in a number of figures for all the cases considered.

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1. Introduction

Various water wave problems involving an infinitely long horizontal cylinder submerged or floating on the surface of water have been investigated in the literature using linear theory by employing a general expansion for the wave potential. This expansion involves a general combination of a regular wave, a wave source, a wave dipole and a regular wave-free part. The wave-free part can be further expanded in terms of wave-free multipoles which are termed wave-free potentials. These are singular solutions of Laplace's equation satisfying the free surface condition and decaying rapidly away

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from the point of singularity. Ursell [37] considered the problem of heaving motion of a circular cylinder on the surface of a fluid using the method of multipole expansion of the time-harmonic stream function. The corresponding velocity potential also has a similar expansion. Ursell [38, 39] considered the problem of surface waves in deep water in the presence of a submerged circular cylinder and constructed a different set of multipoles. These multipole potentials are represented in integral forms where the wave terms and local disturbance terms are not separated. Ursell [43] showed that for an infinitely long horizontal cylinder of arbitrary cross section floating on the surface of water, the potential function in general can be written as the sum of a wave source, a wave dipole, regular standing waves and wave-free potentials regular at infinity as also mentioned earlier. Ursell [40, 41] considered problems where the potential function is expanded in terms of wave source and wave-free potentials. Bolton and Ursell [2] considered the effect of a heaving cylinder in oblique sea employing the multipole method. Mandal and Goswami [27] considered the problem of scattering of surface waves obliquely incident on a fixed half-immersed circular cylinder in deep water and solved it by reducing it to the solution of an integral equation involving the unknown scattered velocity potential on the cylinder by the use of Green's integral theorem in the fluid medium. The same problem was solved again by forming the scattered velocity potential by the method of multipoles using the general expansion theorem of Ursell for the scattered potential [2]. Both methods produced almost the same numerical results.

For three-dimensional problems, such as waves from a submerged sphere, the potential function can be expanded in terms of wave source, wave dipole and wave-free potentials. Expansions in terms of wave source and an infinite set of wave-free potentials were introduced for the three-dimensional problem involving a floating sphere half-immersed and making periodic heaving oscillations by Havelock [23]. This work was later improved and extended to the case of sway by Hulme [24]. Velocity potentials due to the presence of different types of singularity were given by Thorne [36] for two- and three-dimensional motion in infinitely deep water as well as in water of uniform finite depth. Modifications of these results by including the effect of surface tension at the free surface were given by Rhodes-Robinson [33]. Wehausen and Laitone [45] also gave expressions for wave source potentials in three and two dimensions for infinitely deep water which behave as outgoing waves far away from the source and mentioned for three dimensions a particular linear combination of the multipole potentials which decay rapidly at infinity. These are actually wave-free potentials, although this nomenclature was not used there. Barakat [1] studied the vertical motion (heave) of a freely floating sphere half-immersed in infinitely deep water under the action of an incident sine wave. Both radiation and diffraction problems involving half-immersed spheres were studied by Barakat [1] who employed three-dimensional wave-free potentials in the expansion of the velocity potential describing the motion in the fluid. The analytical form of the velocity potential of a half-immersed heaving sphere was studied by Ursell [42] where the potential was expanded in terms of wave source and of wave-free potentials. Wang [44] used the method of Havelock [23] to examine the radiation and diffraction problems for a submerged sphere in deep water. Taylor and Hu [14] described multipole expansion of the velocity potential for two- and three-dimensional wave diffraction and radiation problems. Chatjigeorgiou [4, 5] investigated water wave scattering by an oblate and a prolate spheroid submerged in deep water by employing the method of multipole expansions. The analytical process employed Thorne's [36] multipole expansion terms which were transformed into oblate as well as prolate spheroidal coordinates. Other works on the submerged sphere include those of Gray [22], Srokosz [35], Wu and Eatock Taylor [47], Linton [28], Wu [46], Rahman [32] and Liu et al. [31] who, however, did not separate the wave-free part in the expansion of the velocity potential.

All the above works are related to an ocean with a free surface. For more than two decades there has been considerable interest in the investigation of ice–wave interaction problems due to an increase in scientific activities in polar regions. The ice cover is modelled as a thin ice sheet of which a still smaller part is immersed in water, and is composed of materials having elastic properties. Already quite a number of researchers have considered various types of water wave problems in an ocean with an ice cover modelled as a thin elastic plate [3, 6, 13, 15–20, 29, 34]. Das and Mandal [7] investigated wave radiation by a sphere in deep water as well as water of uniform finite depth with an ice cover by using the method of multipoles. They [10] also studied wave radiation by a sphere submerged in either layer of a two-layer fluid by using the same method. Recently, Das and Thakur [12] investigated the problem of water wave scattering by a submerged sphere in water of uniform finite depth with an ice cover, the ice cover being modelled as a thin elastic plate. Das and Mandal [8] earlier investigated wave scattering by a circular cylinder half-immersed in water with an ice cover. They employed the method of multipoles by using the general expansion theorem for the wave potential involving wave-free potentials for which only expressions were given. Thus, many researchers use wave-free potentials in the mathematical analysis of various classes of water wave problems. In most of these works only the expressions of wave-free potentials are given, without their method of derivation. However, Linton and McIver [30] indicated briefly how these could be constructed in the case of water with a free surface. Also, Mandal and Das [9, 26] presented the construction of wave-free potentials in two dimensions for infinitely deep water and in water of uniform finite depth with a free surface, considering the effect of surface tension at the free surface, and also in water with an ice cover modelled as a thin elastic plate. In this paper, we extend the problems considered by Mandal and Das [9, 26] to three dimensions. Construction of wave-free potentials for cases of deep water and water of uniform finite depth with a free surface are presented in a systematic manner. The effect of surface tension at the free surface and a floating ice cover modelled as a thin elastic plate are also considered.

2. Statement of the problem

With the origin at the mean free surface, the x and z axes horizontal and the y axis vertical, y increasing with depth, we define the angles θ, θ' and α by the relations

$$\tan\theta = \frac{R}{y-f}, \quad \tan\theta' = -\frac{R}{y+f}, \quad \tan\alpha = \frac{z}{x},$$

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where $R = \sqrt{x^2 + z^2}$. Let r and r' denote the radial distances of the point (x, y, z) from the points $(0, f, 0)$ and $(0, -f, 0)$, respectively. If $\mathrm{Re}(\phi(x, y, z)e^{-i\omega t})$ denotes the velocity potential having singularity at $(0, f, 0)$, $f > 0$, describing the motion in the fluid, then $\phi(x, y, z)$ satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \text{in the fluid region except at } (0, f, 0). \tag{2.1}$$

Solutions to Laplace's equation in three dimensions which are singular at $r = 0$ are

$$\frac{P_n^m(\cos\theta)}{r^{n+1}} \begin{cases} \cos(m\alpha) \\ \sin(m\alpha) \end{cases} \quad \text{where } n \geq m \geq 0.$$

We note the following integral representations [21]:

$$\frac{P_n^m(\cos\theta)}{r^{n+1}} = \begin{cases} \dfrac{(-1)^m}{(n-m)!} \displaystyle\int_0^\infty e^{-k(y-f)} k^n J_m(kR)\, dk & \text{if } y > f \\ \dfrac{(-1)^n}{(n-m)!} \displaystyle\int_0^\infty e^{-k(f-y)} k^n J_m(kR)\, dk & \text{if } y < f, \end{cases} \tag{2.2}$$

$$\frac{P_n^m(\cos\theta')}{r'^{n+1}} = \frac{(-1)^n}{(n-m)!} \int_0^\infty e^{-k(y+f)} k^n J_m(kR)\, dk \quad \text{where } y + f > 0. \tag{2.3}$$

Singular solutions for ϕ (three-dimensional multipoles) are given by

$$\phi(x, y, z) = \phi_n^m \begin{cases} \cos(m\alpha) \\ \sin(m\alpha), \end{cases}$$

where ϕ_n^m, $n \geq m \geq 0$, are constructed below for different cases. The wave-free potentials ψ_n^m are then constructed by appropriate linear combinations of ϕ_n^m such that they tend to zero as $R \to \infty$.

3. Wave-free potentials for water of infinite depth

The bottom condition for water of infinite depth is given by

$$\nabla\phi(x, y, z) \to 0 \quad \text{as } y \to \infty. \tag{3.1}$$

Also, $\phi(x, y, z)$ behaves as outgoing waves as $R \to \infty$.

3.1. Water with a free surface The potential function $\phi(x, y, z)$ satisfies (2.1), (3.1) and also the linearized boundary condition at the free surface which is

$$K\phi + \phi_y = 0 \quad \text{on } y = 0, \tag{3.2}$$

where $K = \omega^2/g$ with g the acceleration due to gravity. In this case, to take care of the condition at $y = 0$, we choose

$$\phi_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \int_0^\infty A(k)e^{-ky} J_m(kR)\, dk, \tag{3.3}$$

and $A(k)$ is a function of k to be obtained such that the integral exists in some sense and the boundary condition (3.2) is satisfied. The free surface condition (3.2) is satisfied if we choose $A(k)$ as

$$A(k) = \frac{(-1)^n}{(n-m)!}\frac{k+K}{k-K}k^n e^{-kf}$$

after using the representations (2.2). Thus multipoles singular at $(0, f, 0)$, $f > 0$, are of the form $\phi_n^m \cos(m\alpha)$ and $\phi_n^m \sin(m\alpha)$, where

$$\phi_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{(-1)^n}{(n-m)!}\int_0^\infty \frac{k+K}{k-K}k^n e^{-k(y+f)} J_m(kR)\, dk,$$

the contour in the integral being indented below the pole $k = K$ on the real k axis to take care of their outgoing behaviour as $R \to \infty$. The far-field form of the multipole is given by

$$\phi_n^m \sim 2\pi i\, a_n^m e^{-k(y+f)} \sqrt{\frac{2}{\pi KR}}\, e^{i(KR-\pi/4)} \quad \text{as } R \to \infty,$$

where

$$a_n^m = \frac{(-1)^n}{(n-m)!}K^{n+1}e^{-im\pi/2}. \tag{3.4}$$

Now from (3.4), since

$$a_{n+1}^m + \frac{K}{n-m+1}a_n^m = 0$$

the combination $\phi_{n+1}^m + (K/n-m+1)\phi_n^m$ does not contribute anything as $R \to \infty$ so that it is wave free. Now using the representation (2.3) it can be shown that

$$\begin{aligned}\psi_n^m &\equiv \phi_{n+1}^m + \frac{K}{n-m+1}\phi_n^m \\ &= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{P_{n+1}^m(\cos\theta')}{r'^{n+2}} + \frac{K}{n-m+1}\left(\frac{P_n^m(\cos\theta)}{r^{n+1}} - \frac{P_n^m(\cos\theta')}{r'^{n+1}}\right).\end{aligned} \tag{3.5}$$

For different n and m, $n \geq m$, these are wave-free potentials with singularity at $(0, f, 0)$. When the singularity is on the free surface ($f \to 0, \theta' \to \pi - \theta$) these potentials reduce to zero unless $n + m$ is odd. In that case we obtain two distinct sets of wave-free potentials as given by Linton and McIver [30]:

$$\chi_{2n-1}^{2m} = \frac{P_{2n}^{2m}(\cos\theta)}{r^{2n+1}} + \frac{K}{2n-2m}\frac{P_{2n-1}^{2m}(\cos\theta)}{r^{2n}},$$

$$\chi_{2n}^{2m+1} = \frac{P_{2n+1}^{2m+1}(\cos\theta)}{r^{2n+2}} + \frac{K}{2n-2m}\frac{P_{2n}^{2m+1}(\cos\theta)}{r^{2n+1}}.$$

3.2. Effect of surface tension at the free surface Here the potential function $\phi(x, y, z)$ satisfies (2.1), (3.1), and the linearized free surface condition

$$K\phi + M\phi_{yyy} + \phi_y = 0 \quad \text{on } y = 0, \tag{3.6}$$

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where $M = T/\rho g$ with T the coefficient of surface tension at the free surface and ρ the density of water. Setting $\phi_n^m(x, y, z)$ as in (3.3) and following a similar procedure, it can be shown that the combination $\phi_{n+1}^m + (\kappa/n - m + 1)\phi_n^m$ does not contribute anything as $R \to \infty$. Here κ is the only real positive root of the dispersion equation

$$k(1 + Mk^2) - K = 0$$

relevant to this case. Thus wave-free potentials having singularity at $(0, f, 0)$ are

$$\begin{aligned}\psi_n^m &\equiv \phi_{n+1}^m + \frac{\kappa}{n-m+1}\phi_n^m \\ &= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\kappa}{n-m+1}\frac{P_n^m(\cos\theta)}{r^{n+1}} \\ &\quad + \frac{(-1)^{n+1}}{(n-m+1)!}\int_0^\infty \frac{k(1+Mk^2)+K}{[M(k^2+k\kappa+\kappa^2)+1]}k^n e^{-k(y+f)}J_m(kR)\,dk. \qquad (3.7)\end{aligned}$$

Taking $f \to 0$ in (3.7), we find wave-free potentials having singularity on the free surface given by

$$\begin{aligned}\chi_n^m = &\frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\kappa}{n-m+1}\frac{P_n^m(\cos\theta)}{r^{n+1}} \\ &+ \frac{(-1)^{n+1}}{(n-m+1)!}\int_0^\infty \frac{k(1+Mk^2)+K}{[M(k^2+k\kappa+\kappa^2)+1]}k^n e^{-ky}J_m(kR)\,dk.\end{aligned}$$

These can be identified with the results of Rhodes-Robinson [33] for $m = 0$.

3.3. Water with an ice cover In this case the potential function $\phi(x, y, z)$ satisfies (2.1), (3.1) and the linearized condition at the ice cover modelled as a thin elastic plate given by

$$K\phi + [D\nabla_{x,z}^4 + 1 - \epsilon K]\phi_y = 0 \quad \text{on } y = 0. \qquad (3.8)$$

Here

$$D = \frac{Eh_0^3}{12(1-\nu)\rho_1 g}$$

is the flexural rigidity of the ice cover, where E and ν are respectively Young's modulus and Poisson's ratio of the elastic material of the ice cover, h_0 is the very small thickness of the ice cover, $\epsilon = (\rho_0/\rho_1)h_0$ with ρ_0 and ρ_1 the densities of the ice and water, respectively, and

$$\nabla_{x,z}^4 = \left[\nabla_R^2 + \frac{1}{R^2}\frac{\partial^2}{\partial\alpha^2}\right]^2 \quad \text{with } \nabla_R^2 = \left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right)\right].$$

Again following a similar method, it can be shown that the combination $\phi_{n+1}^m + (k_1/n - m + 1)\phi_n^m$ does not contribute anything as $R \to \infty$ so that it is wave free. Here k_1 is the only real root for the relevant dispersion equation

$$(Dk^4 + 1 - \epsilon K)k - K = 0.$$

Thus the wave-free potentials having singularity at $(0, f, 0)$ are

$$
\begin{aligned}
\psi_n^m &\equiv \phi_{n+1}^m + \frac{k_1}{n-m+1}\phi_n^m \\
&= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{k_1}{n-m+1}\frac{P_n^m(\cos\theta)}{r^{n+1}} \\
&\quad + \frac{(-1)^{n+1}}{(n-m+1)!}\int_0^\infty g_1(k)k^n e^{-k(y+f)} J_m(kR)\,dk,
\end{aligned} \tag{3.9}
$$

where

$$
g_1(k) = \frac{[(Dk^4 + 1 - \epsilon K)k + K]}{[D(k^4 + kk_1^3 + k^2k_1^2 + k^3k_1 + k_1^4) + 1 - \epsilon K]}.
$$

Taking $f \to 0$ in (3.9) we obtain the wave-free potentials having singularity on the ice cover given by

$$
\chi_n^m = \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{k_1}{n-m+1}\frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{(-1)^{n+1}}{(n-m+1)!}\int_0^\infty g_1(k)k^n e^{-ky} J_m(kR)\,dk.
$$

4. Wave-free potentials for water of uniform finite depth

The bottom condition for water of uniform finite depth is given by

$$
\frac{\partial}{\partial y}\phi(x, y, z) = 0 \quad \text{on } y = h, \tag{4.1}
$$

where h is the uniform finite depth of water.

4.1. Water with a free surface The potential function $\phi(x, y, z)$ satisfies (2.1), (4.1) and also the linearized boundary condition at the free surface which is

$$
K\phi + \phi_y = 0 \quad \text{on } y = 0. \tag{4.2}
$$

In this case, we choose

$$
\phi_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \int_0^\infty [A(k)\sinh(ky) + B(k)\cosh(h-y)]J_m(kR)\,dk,
$$

where $A(k)$ and $B(k)$ are functions of k to be obtained such that the integral exists in some sense and the boundary conditions (4.1) and (4.2) are satisfied. The boundary conditions (4.1) and (4.2) are satisfied if we choose $A(k)$ and $B(k)$ as

$$
\begin{aligned}
A(k) &= \frac{(-1)^m}{(n-m)!}\frac{k^n}{\cosh(kh)}e^{-k(h-f)}, \\
B(k) &= \frac{k^n}{(n-m)!}\left[\frac{(-1)^n(K+k)\cosh(kh)e^{-kf} + (-1)^m ke^{-k(h-f)}}{\cosh(kh)(k\sinh(kh) - K\cosh(kh))}\right]
\end{aligned}
$$

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after using the representations (2.3). Thus, we find

$$\phi_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{1}{(n-m)!}\int_0^\infty \frac{k^n f_n^m(k,y) J_m(kR)}{k\sinh(kh) - K\cosh(kh)}\,dk,$$

where the contour is indented below the pole $k = k_0$ on the real k axis to take care of their outgoing behaviour as $R \to \infty$, k_0 being the only real positive root of the dispersion equation

$$k\sinh(kh) - K\cosh(kh) = 0$$

and

$$f_n^m(k,y) = (-1)^m e^{-k(h-f)}(k\cosh(ky) - K\sinh(ky)) + (-1)^n (K+k)e^{-kf}\cosh(k(h-y)).$$

The far-field form of the multipole is given by

$$\phi_n^m \sim \pi i a_n^m \frac{\cosh(k_0(h-y))}{2hN_0^2}\sqrt{\frac{2}{\pi k_0 R}}\,e^{i(k_0R-\pi/4)} \quad \text{as } R \to \infty,$$

where

$$a_n^m = \frac{k_0^n[(-1)^m e^{-k_0(h-f)} + (-1)^n e^{k_0(h-f)}]e^{-im\pi/2}}{(n-m)!}, \qquad N_0^2 = \frac{2k_0h + \sinh(2k_0h)}{4k_0h}.$$

Since

$$a_{n+1}^m + \frac{k_0}{n-m+1}\tanh(k_0(h-f))a_n^m = 0, \quad n+m \text{ even},$$
$$a_{n+1}^m + \frac{k_0}{n-m+1}\coth(k_0(h-f))a_n^m = 0, \quad n+m \text{ odd},$$

the combinations

$$\phi_{n+1}^m + \frac{k_0}{n-m+1}\tanh(k_0(h-f))\phi_n^m, \quad n+m \text{ even},$$
$$\phi_{n+1}^m + \frac{k_0}{n-m+1}\coth(k_0(h-f))\phi_n^m, \quad n+m \text{ odd},$$

do not contribute anything as $R \to \infty$ so that they are wave free. These are given by Linton and McIver [30]. Therefore,

$$\begin{aligned}
\psi_n^m &\equiv \phi_{n+1}^m + \frac{k_0}{n-m+1}\tanh(k_0(h-f))\phi_n^m \\
&= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{k_0}{n-m+1}\tanh(k_0(h-f))\frac{P_n^m(\cos\theta)}{r^{n+1}} \\
&\quad + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{k\sinh(kh) - K\cosh(kh)} \\
&\quad \times \{(-1)^m e^{-k(h-f)}(k\cosh(ky) - K\sinh(ky))(k + k_0\tanh(k_0(h-f))) \\
&\quad + (-1)^{n+1}(K+k)e^{-kf}\cosh(k(h-y))(k - k_0\tanh(k_0(h-f)))\}\,dk
\end{aligned} \tag{4.3}$$

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for $n+m$ even, and

$$\begin{aligned}
\psi_n^m &\equiv \phi_{n+1}^m + \frac{k_0}{n-m+1}\coth(k_0(h-f))\phi_n^m \\
&= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{k_0}{n-m+1}\coth(k_0(h-f))\frac{P_n^m(\cos\theta)}{r^{n+1}} \\
&\quad + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{k\sinh(kh)-K\cosh(kh)} \\
&\quad \times \{(-1)^m e^{-k(h-f)}(k\cosh(ky)-K\sinh(ky))(k+k_0\coth(k_0(h-f))) \\
&\quad + (-1)^{n+1}(K+k)e^{-kf}\cosh(k(h-y))(k-k_0\coth(k_0(h-f)))\}\,dk
\end{aligned} \tag{4.4}$$

for $n+m$ odd. These are wave-free potentials with singularity at $(0, f, 0)$. Taking $f \to 0$ in (4.3) and (4.4) we obtain wave-free potentials having singularity on the free surface given by

$$\begin{aligned}
\chi_n^m &= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{k_0}{n-m+1}\tanh(k_0h)\frac{P_n^m(\cos\theta)}{r^{n+1}} \\
&\quad + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{k\sinh(kh)-K\cosh(kh)} \\
&\quad \times \{(-1)^m e^{-kh}(k\cosh(ky)-K\sinh(ky))(k+k_0\tanh(k_0h)) \\
&\quad + (-1)^{n+1}(K+k)\cosh(k(h-y))(k-k_0\tanh(k_0h))\}\,dk
\end{aligned}$$

for $n+m$ even, and

$$\begin{aligned}
\chi_n^m &= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{k_0}{n-m+1}\coth(k_0h)\frac{P_n^m(\cos\theta)}{r^{n+1}} \\
&\quad + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{k\sinh(kh)-K\cosh(kh)} \\
&\quad \times \{(-1)^m e^{-kh}(k\cosh(ky)-K\sinh(ky))(k+k_0\coth(k_0h)) \\
&\quad + (-1)^{n+1}(K+k)\cosh(k(h-y))(k-k_0\coth(k_0h))\}\,dk
\end{aligned}$$

for $n+m$ odd.

4.2. Effect of surface tension at the free surface In this case the potential function $\phi(x, y, z)$ satisfies (2.1), (3.6) and (4.1). Following a similar procedure to that above, it can be shown that the combinations

$$\begin{aligned}
&\phi_{n+1}^m + \frac{\kappa_0}{n-m+1}\tanh(\kappa_0(h-f))\phi_n^m, \quad n+m \text{ even}, \\
&\phi_{n+1}^m + \frac{\kappa_0}{n-m+1}\coth(\kappa_0(h-f))\phi_n^m, \quad n+m \text{ odd},
\end{aligned}$$

are wave free. Here κ_0 is the unique positive real root of the dispersion equation

$$(1+Mk^2)k\sinh(kh) - K\cosh(kh) = 0.$$

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Therefore, the wave-free potentials having singularity at $(0, f, 0)$ are

$$\begin{aligned}
\psi_n^m &\equiv \phi_{n+1}^m + \frac{\kappa_0}{n-m+1}\tanh(\kappa_0(h-f))\phi_n^m \\
&= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\kappa_0}{n-m+1}\tanh(\kappa_0(h-f))\frac{P_n^m(\cos\theta)}{r^{n+1}} \\
&\quad + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{(1+Mk^2)k\sinh(kh)-K\cosh(kh)} \\
&\quad \times \{(-1)^m e^{-k(h-f)}((1+Mk^2)k\cosh(ky)-K\sinh(ky))(k+\kappa_0\tanh(\kappa_0(h-f))) \\
&\quad + (-1)^{n+1}(K+k(1+Mk^2))e^{-kf}\cosh(k(h-y))(k-\kappa_0\tanh(\kappa_0(h-f)))\}\,dk
\end{aligned} \tag{4.5}$$

for $n+m$ even, and

$$\begin{aligned}
\psi_n^m &\equiv \phi_{n+1}^m + \frac{\kappa_0}{n-m+1}\coth(\kappa_0(h-f))\phi_n^m \\
&= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\kappa_0}{n-m+1}\coth(\kappa_0(h-f))\frac{P_n^m(\cos\theta)}{r^{n+1}} \\
&\quad + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{(1+Mk^2)k\sinh(kh)-K\cosh(kh)} \\
&\quad \times \{(-1)^m e^{-k(h-f)}((1+Mk^2)k\cosh(ky)-K\sinh(ky))(k+\kappa_0\coth(\kappa_0(h-f))) \\
&\quad + (-1)^{n+1}(K+k(1+Mk^2))e^{-kf}\cosh(k(h-y))(k-\kappa_0\coth(\kappa_0(h-f)))\}\,dk
\end{aligned} \tag{4.6}$$

for $n+m$ odd. Taking $f \to 0$ in (4.5) and (4.6), we obtain wave-free potentials having singularity on the free surface given by

$$\begin{aligned}
\chi_n^m = {}& \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\kappa_0}{n-m+1}\tanh(\kappa_0 h)\frac{P_n^m(\cos\theta)}{r^{n+1}} \\
& + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{(1+Mk^2)k\sinh(kh)-K\cosh(kh)} \\
& \times \{(-1)^m e^{-kh}((1+Mk^2)k\cosh(ky)-K\sinh(ky))(k+\kappa_0\tanh(\kappa_0 h)) \\
& + (-1)^{n+1}(K+k(1+Mk^2))\cosh(k(h-y))(k-\kappa_0\tanh(\kappa_0 h))\}\,dk
\end{aligned}$$

for $n+m$ even, and

$$\begin{aligned}
\chi_n^m = {}& \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\kappa_0}{n-m+1}\coth(\kappa_0 h)\frac{P_n^m(\cos\theta)}{r^{n+1}} \\
& + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{(1+Mk^2)k\sinh(kh)-K\cosh(kh)} \\
& \times \{(-1)^m e^{-kh}((1+Mk^2)k\cosh(ky)-K\sinh(ky))(k+\kappa_0\coth(\kappa_0 h)) \\
& + (-1)^{n+1}(K+k(1+Mk^2))\cosh(k(h-y))(k-\kappa_0\coth(\kappa_0 h))\}\,dk
\end{aligned}$$

for $n+m$ odd.

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4.3. Water with an ice cover Here the potential function $\phi(x, y, z)$ satisfies (2.1), (3.8) and (4.1). Proceeding as before, it can be shown that the combinations

$$\phi_{n+1}^m + \frac{\kappa_1}{n-m+1}\tanh(\kappa_1(h-f))\phi_n^m, \quad n+m \text{ even},$$
$$\phi_{n+1}^m + \frac{\kappa_1}{n-m+1}\coth(\kappa_1(h-f))\phi_n^m, \quad n+m \text{ odd}$$

are wave free, where κ_1 is the unique positive root of the relevant dispersion equation

$$k(Dk^4 + 1 - \epsilon K)\sinh(kh) - K\cosh(kh) = 0.$$

Thus, in this case, the wave-free potentials having singularity at $(0, f, 0)$ are given by

$$\begin{aligned}\psi_n^m &\equiv \phi_{n+1}^m + \frac{\kappa_1}{n-m+1}\tanh(\kappa_1(h-f))\phi_n^m \\ &= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\kappa_1}{n-m+1}\tanh(\kappa_1(h-f))\frac{P_n^m(\cos\theta)}{r^{n+1}} \\ &\quad + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{k(Dk^4+1-\epsilon K)\sinh(kh) - K\cosh(kh)} \\ &\quad \times \{(-1)^m e^{-k(h-f)}(k(Dk^4+1-\epsilon K)\cosh(ky) \\ &\quad - K\sinh(ky))(k+\kappa_1\tanh(\kappa_1(h-f))) + (-1)^{n+1}(K + k(Dk^4+1-\epsilon K))e^{-kf} \\ &\quad \times \cosh(k(h-y))(k-\kappa_1\tanh(\kappa_1(h-f)))\}\,dk \end{aligned} \tag{4.7}$$

for $n+m$ even, and

$$\begin{aligned}\psi_n^m &\equiv \phi_{n+1}^m + \frac{\kappa_1}{n-m+1}\coth(\kappa_1(h-f))\phi_n^m \\ &= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\kappa_1}{n-m+1}\coth(\kappa_1(h-f))\frac{P_n^m(\cos\theta)}{r^{n+1}} \\ &\quad + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{k(Dk^4+1-\epsilon K)\sinh(kh) - K\cosh(kh)} \\ &\quad \times \{(-1)^m e^{-k(h-f)}(k(Dk^4+1-\epsilon K)\cosh(ky) \\ &\quad - K\sinh(ky))(k+\kappa_1\coth(\kappa_1(h-f))) + (-1)^{n+1}(K + k(Dk^4+1-\epsilon K))e^{-kf} \\ &\quad \times \cosh(k(h-y))(k-\kappa_1\coth(\kappa_1(h-f)))\}\,dk \end{aligned} \tag{4.8}$$

for $n+m$ odd. Taking $f \to 0$ in (4.7) and (4.8), we obtain wave-free potentials having singularity on the ice cover given by

$$\begin{aligned}\chi_n^m &= \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\kappa_1}{n-m+1}\tanh(\kappa_1 h)\frac{P_n^m(\cos\theta)}{r^{n+1}} \\ &\quad + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{k(Dk^4+1-\epsilon K)\sinh(kh) - K\cosh(kh)} \\ &\quad \times \{(-1)^m e^{-kh}(k(Dk^4+1-\epsilon K)\cosh(ky) - K\sinh(ky))(k+\kappa_1\tanh(\kappa_1 h)) \\ &\quad + (-1)^{n+1}(K + k(Dk^4+1-\epsilon K))\cosh(k(h-y))(k-\kappa_1\tanh(\kappa_1 h))\}\,dk \end{aligned}$$

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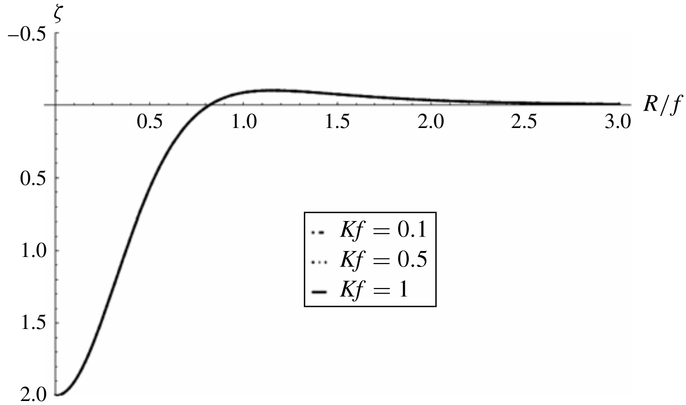

FIGURE 1. Form of the free surface for infinitely deep water with $n = 2$.

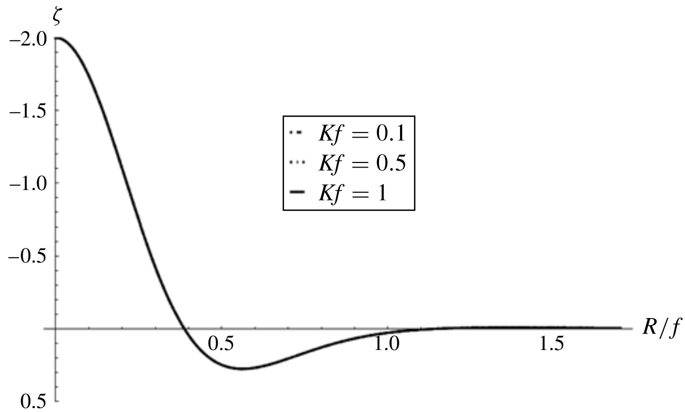

FIGURE 2. Form of the free surface for infinitely deep water with $n = 5$.

for $n + m$ even, and

$$\begin{aligned}\chi_n^m = {} & \frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\kappa_1}{n-m+1}\coth(\kappa_1 h)\frac{P_n^m(\cos\theta)}{r^{n+1}} \\ & + \frac{1}{(n-m+1)!}\int_0^\infty \frac{k^n J_m(kR)}{k(Dk^4+1-\epsilon K)\sinh(kh) - K\cosh(kh)} \\ & \times \{(-1)^m e^{-kh}(k(Dk^4+1-\epsilon K)\cosh(ky) - K\sinh(ky))(k+\kappa_1\coth(\kappa_1 h)) \\ & + (-1)^{n+1}(K + k(Dk^4+1-\epsilon K))\cosh(k(h-y))(k-\kappa_1\coth(\kappa_1 h))\}\,dk\end{aligned}$$

for $n + m$ odd.

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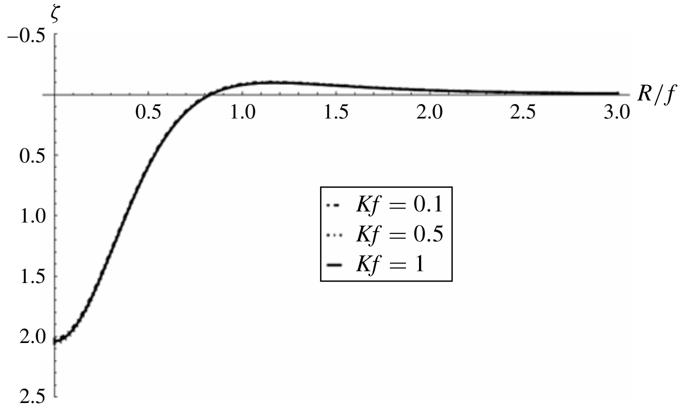

FIGURE 3. Form of the free surface for uniform finite depth water with $h/f = 3$ and $n = 2$.

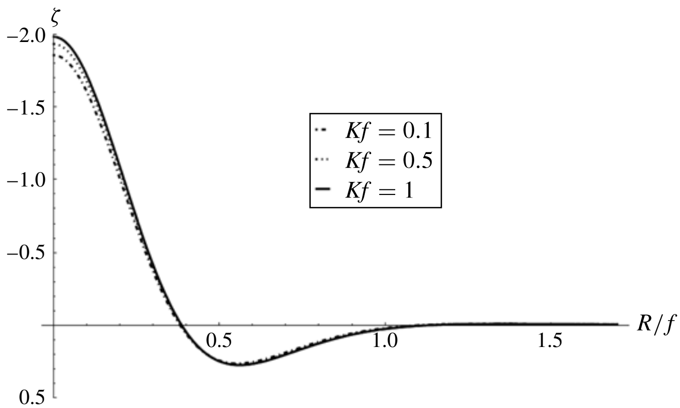

FIGURE 4. Form of the free surface for uniform finite depth water with $h/f = 3$ and $n = 5$.

5. Numerical results

The velocity potential $\Phi(x, y, z, t)$ defining the motion in the fluid corresponding to a wave-free potential $\Psi_n^m(x, y, z, t)$ can be written as

$$\Phi(x, y, z, t) = \frac{g f^{n+3}}{\omega} \Psi_n^m(x, y, z, t),$$

where g is the acceleration due to gravity and ω is the circular frequency assuming time-harmonic dependence of the wave-free potential. We can write

$$\Psi_n^m(x, y, z, t) = \mathrm{Re}(\psi_n^m(x, y, z)e^{-i\omega t}),$$

so that

$$\Phi(x, y, z, t) = \mathrm{Re}(\phi(x, y, z)e^{-i\omega t}).$$

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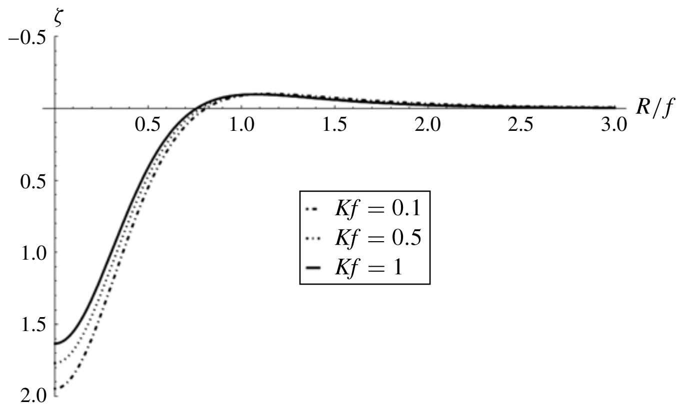

FIGURE 5. Form of the free surface due to the presence of surface tension at the free surface for infinitely deep water with $n = 2$ and $M/f^2 = 1$.

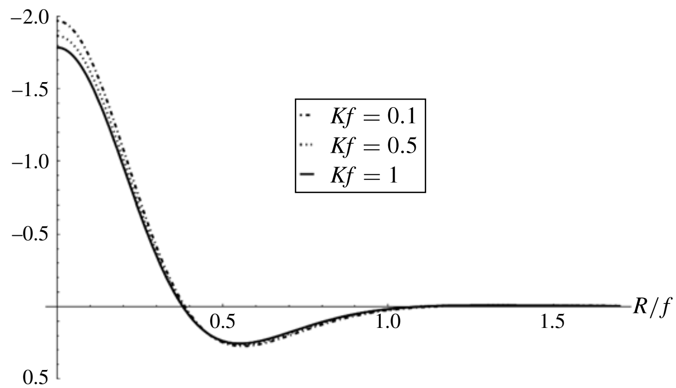

FIGURE 6. Form of the free surface due to the presence of surface tension at the free surface for infinitely deep water with $n = 5$ and $M/f^2 = 1$.

Then the depression $\eta(x, z, t)$ of the upper surface is given by

$$\begin{aligned}\eta(x, z, t) &= \frac{1}{g}\frac{\partial\Phi}{\partial t}(x, 0, z, t)\\ &= \frac{1}{g}\operatorname{Re}(-i\omega\phi(x, 0, z)(\cos(\omega t) - i\sin(\omega t)))\\ &= -f^{n+3}\psi_n^m(x, 0, z)\sin(\omega t).\end{aligned}$$

We write $\psi_n^m(x, 0, z)$ as $\psi_n^m(R, 0)$ where $R = (x^2 + z^2)^{1/2}$, since a wave-free potential is a function of R and y only, so that

$$\zeta(R, t) \equiv \frac{\eta(R, t)}{f} = -f^{n+2}\psi_n^m(R, 0)\sin(\omega t)$$

is the nondimensionalized depression of the upper surface of the fluid.

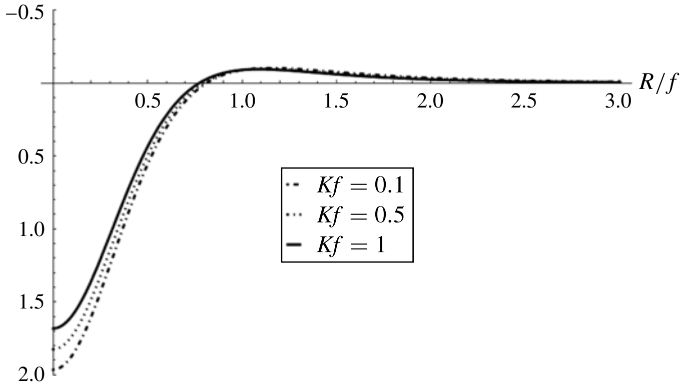

FIGURE 7. Form of the free surface due to the presence of surface tension at the free surface for uniform finite depth water with $h/f = 3$, $n = 2$ and $M/f^2 = 1$.

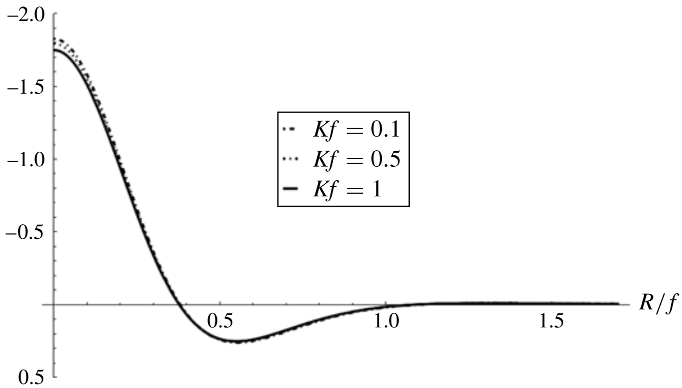

FIGURE 8. Form of the free surface due to the presence of surface tension at the free surface for uniform finite depth water with $h/f = 3$, $n = 5$ and $M/f^2 = 1$.

For numerical computation we choose $\omega t = \pi/2$ and $m = 0$, and the nondimensionalized upper surface depression ζ is plotted against R/f in a number of figures. Other values of ωt and m can also be taken. In Figures 1, 2, 5, 6, 9, 10, 11 and 14, ζ is depicted against R/f for infinitely deep water and $n = 2, 5$ and $Kf = 0.1, 0.5, 1$. In Figures 3, 4, 7, 8, 12 and 13, ζ is depicted against R/f for uniform finite depth water with $h/f = 3$, $n = 2, 5$ and $Kf = 0.1, 0.5, 1$. Figures 1, 3, 5, 7, 10, 12 correspond to the case $n = 2$, and Figures 2, 4, 6, 8, 11, 13 to the case $n = 5$.

Figures 1–4 correspond to the case of water with a free surface, observed for $y = 0$, $r = r'$ and $\theta = \theta'$. Expression (3.5) shows that $\psi_n^m(R, 0)$ is independent of the wave number K, so that ζ is likewise. That is why there is only one curve for ζ in Figures 1 and 2 corresponding to any value of Kf. From Figure 3, it is seen that the wave profiles for $Kf = 0.1, 0.5, 1$ almost coincide and no significant change in amplitude is observed.

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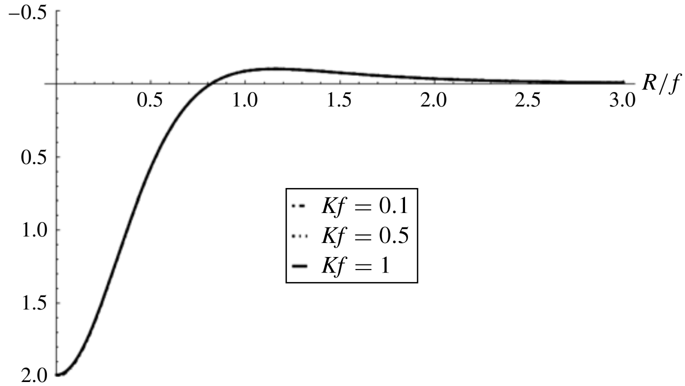

FIGURE 9. Form of the free surface due to the presence of surface tension at the free surface for infinitely deep water with $n = 2$ and $M/f^2 = 0.001$.

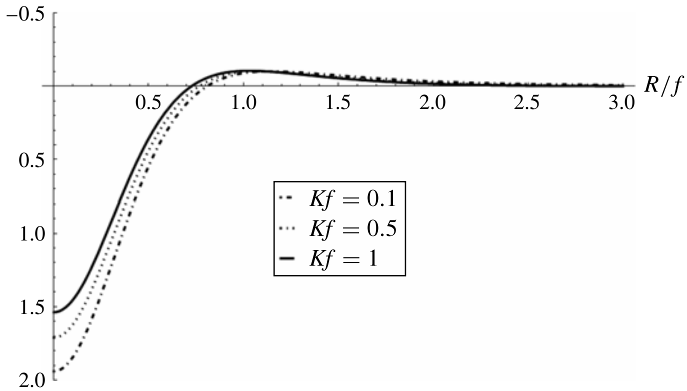

FIGURE 10. Form of the upper surface due to the presence of an ice cover for infinitely deep water with $n = 2$, $D/f^4 = 1$ and $\epsilon/f = 0.01$.

From Figure 4, it is observed that the amplitude of the wave profiles increases as Kf increases from 0.1 to 1. In all of these figures it is observed that as R/f increases, the wave profile decays to zero rapidly and ultimately dies out.

In Figures 5–9, ζ is depicted against R/f, taking into account the effect of surface tension at the free surface. The surface tension parameter M/f^2 is chosen to be 1 in Figures 5–8, and $Kf = 0.1, 0.5, 1$. From Figure 5, it is observed that as Kf increases from 0.1 to 1, the amplitude of the wave profile decreases. In Figure 6, the wave profiles are just the opposite to those in Figure 5. The same nature of the wave profiles is observed in Figures 5 and 7 and in Figures 6 and 8. In Figure 9, M/f^2 is taken to be very small (0.001) so that there is almost no effect of the surface tension at the free surface. The curves in Figure 9 (for $Kf = 0.1, 0.5, 1$) become almost a single curve and coincide with the curve in Figure 1, as should have been expected. From Figures 1–4

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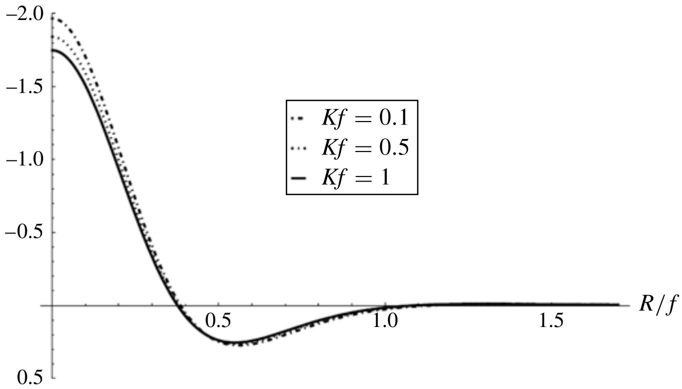

FIGURE 11. Form of the upper surface due to the presence of an ice cover for infinitely deep water with $n = 5$, $D/f^4 = 1$ and $\epsilon/f = 0.01$.

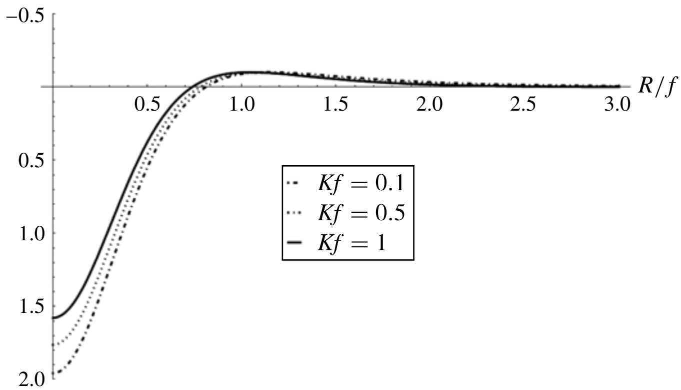

FIGURE 12. Form of the upper surface due to the presence of an ice cover for uniform finite depth water with $h/f = 3$, $n = 2$, $D/f^4 = 1$ and $\epsilon/f = 0.01$.

and 5–8 it is observed that the presence of surface tension at the free surface decreases the amplitude of the free surface profile.

In Figures 10–14, the case of water with an ice cover modelled as a thin elastic plate floating on the water is considered. In Figures 10–13, the stiffness and mass parameters, D/f^4 and ϵ/f respectively, are taken as $D/f^4 = 1$ and $\epsilon/f = 0.01$. Here also it is observed from Figure 10 that as Kf increases from 0.1 to 1, the amplitude of the wave profile decreases. The wave profiles in Figure 11 are opposite to those in Figure 10. The same nature of the wave profiles is observed in Figures 10 and 12 and in Figures 11 and 13. In Figure 14, the stiffness parameter D/f^4 is chosen to be small ($D/f^4 = 0.001$ and $\epsilon/f = 0.01$) so that the ice cover becomes almost a free surface and n is taken to be 2. The curves in Figure 14 shrink to a single curve which is almost

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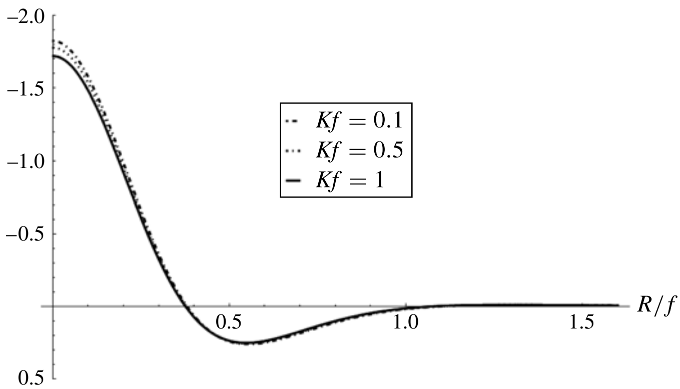

FIGURE 13. Form of the upper surface due to the presence of an ice cover for uniform finite depth water with $h/f = 3$, $n = 5$, $D/f^4 = 1$ and $\epsilon/f = 0.01$.

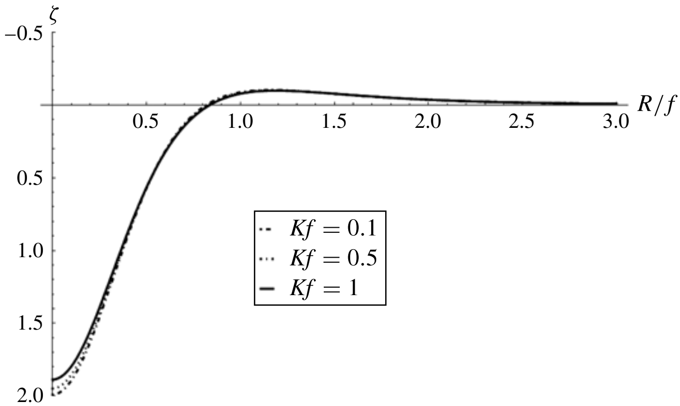

FIGURE 14. Form of the upper surface due to the presence of an ice cover for infinitely deep water with $n = 2$, $D/f^4 = 0.001$ and $\epsilon/f = 0.01$.

the same as in Figure 1. It is observed from Figures 10–13 that the amplitude of the wave motion decreases due to the presence of the ice cover.

6. Conclusion

Wave-free potentials in deep water and water of uniform finite depth with a free surface for three dimensions are constructed in a systematic manner. These are also obtained taking into account the effect of the presence of surface tension at the free surface and also in the presence of an ice cover modelled as a thin elastic pate. The form of the free surface or ice cover due to a multipole at $(0, f, 0)$ is depicted against the distance for different values of Kf and n, $\omega t = \pi/2$ and $m = 0$ being always taken. It is observed from all of these figures that the free surface depression/elevation is

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maximum above the singularity and tends to zero as the distance from the origin becomes large. This is quite plausible. The results can also be extended to free surface boundary conditions with higher-order derivatives (compare with the work of Manam et al. [25] and Das et al. [11]).

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