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EQUIVALENCE OF ELLIPTICITY AND THE FREDHOLM PROPERTY IN THE WEYL-HÖRMANDER CALCULUS

STEVAN PILIPOVIĆ AND BOJAN PRANGOSKI

Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
(stevan.pilipovic@dmi.uns.ac.rs)

Department of Mathematics, Faculty of Mechanical Engineering-Skopje, University "Ss. Cyril and Methodius", Karposh 2 b.b., 1000 Skopje, Macedonia
(bprangoski@yahoo.com)

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Abstract The main result is that the ellipticity and the Fredholm property of a ΨDO acting on Sobolev spaces in the Weyl-Hörmander calculus are equivalent when the Hörmander metric is geodesically temperate and its associated Planck function vanishes at infinity. The proof is essentially related to the following result that we prove for geodesically temperate Hörmander metrics: If $\lambda \mapsto a_\lambda \in S(1,g)$ is a $\mathcal{C}^N$, $0 \leq N \leq \infty$, map such that each a_λ^w is invertible on L^2, then the mapping $\lambda \mapsto b_\lambda \in S(1,g)$, where b_λ^w is the inverse of a_λ^w, is again of class $\mathcal{C}^N$. Additionally, assuming also the strong uncertainty principle for the metric, we obtain a Fedosov-Hörmander formula for the index of an elliptic operator. At the very end, we give an example to illustrate our main result.

Keywords and phrases: Hörmander metric; geodesic temperance; Sobolev spaces *H(M,g)*

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1. Introduction

The question of spectral invariance is of a significant importance in the theory of pseudodifferential operators. Recall that a pseudodifferential calculus is said to be spectrally invariant if for every ΨDO with 0 order symbol (consequently, continuous on L^2) that is invertible on L^2 its inverse is again a ΨDO with a 0 order symbol. This property has been proved by several authors for various global (and local) calculi including the Shubin calculus, the SG (scattering) calculus, the Beals-Fefferman calculus, etc. (see [5, 14, 16, 26, 25, 34]). In their seminal paper [9], Bony and Chemin (see also [10]) generalised these results by proving the spectral invariance for the Weyl-Hörmander calculus [22, 23] when the Hörmander metric satisfies the so-called geodesic temperance (see [9, 27]). In the first part of this article (Section 3) we slightly improve the proof of [27, Lemma 2.6.25, p. 155], also given in [9], related to the norm estimate of a composition of operators changing the right-hand side of the estimate. Subsequently, we avail ourselves of these results to prove the following fact, which sheds more light on the spectral invariance of the Weyl-Hörmander calculus: the process of taking inverses in $S(1,g)$ preserves the regularity. To be more precise, if $\lambda \mapsto a_\lambda$ is a $\mathcal{C}^N$-mapping with values in $S(1,g)$ such that a_λ^w is invertible on L^2, then the mapping $\lambda \mapsto b_\lambda$, where b_λ^w is the inverse of a_λ^w, is also of class $\mathcal{C}^N$, $0 \leq N \leq \infty$. In fact, we prove this result for matrix-valued symbols. In the second part of the article (Section 4), we investigate the Fredholm properties of ΨDOs with symbols in the Weyl-Hörmander classes when acting between the Sobolev spaces naturally associated to them. The main result is that the Fredholm property of a ΨDO can be characterised by the ellipticity of the symbol; that is, a ΨDO is a Fredholm operator between appropriate Sobolev spaces if and only if its symbol is elliptic (see [7, 30, 25, 26] for similar types of results concerning special instances of the Weyl-Hörmander calculus). This result heavily relies on the vanishing at infinity of the Planck function associated to the Hörmander metric as well as on the main result of Section 3, which, in turn, depends on the spectral invariance and the geodesic temperance of the metric. As a consequence of the proof of the main result, we derive that elliptic operators always have parametrices when the Hörmander metric is geodesically temperate and its associated Planck function vanishes at infinity. For geodesically temperate metrics, to the best of our knowledge, this is an improvement over already existing results because it does not require the strong uncertainty principle (which is the condition usually imposed for the construction of parametrices).

If the metric satisfies the strong uncertainty principle, then the Fedosov-Hörmander integral formula (see [18, 17, 19, 22]) expresses the index of an elliptic operator with symbol $a \in S(1,g)$ as the integral of the form $\operatorname{tr}(a^{-1}da)$. As a consequence of this result, in the final part of Section 4, we prove the same holds true for elliptic operators with symbols in $S(M,g)$ for any admissible weight M. Of course, this agrees with the Atiyah-Singer index theorem [1, Theorem 2.12] (cf. [22, p. 422] and [18, p. 320]).

Finally, in Section 5, as an illustrative example, we consider the operator

$$-\Delta + \langle x \rangle^{-2s}, \quad 0 < s < 1.$$

This is not an elliptic operator in any of the 'classical' calculi (like the Shubin calculus, the SG calculus and the Beals-Fefferman calculus; see Lemma 5.2 and the comments after it), but it is elliptic in the Weyl-Hörmander calculus when one chooses an appropriate metric. Consequently, we can apply the results of the article to describe its Fredholm property.

2. Preliminaries

Let V be an n-dimensional real vector space with V' being its dual. The $2n$-dimensional vector space $W = V \times V'$ is symplectic with the symplectic form $[(x,\xi),(y,\eta)] = \langle \xi, y\rangle - \langle \eta, x\rangle$. We will always denote the points in W with capital letters $X, Y, Z, \dots$. Let $X \mapsto g_X$ be a Borel measurable symmetric covariant 2-tensor field on W that is positive definite at every point. We will always denote the corresponding positive-definite quadratic form at $X \in W$ by the same symbol g_X; that is, $g_X(T) = g_X(T,T)$, $T \in T_X W$. Denoting by

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Q_X the corresponding linear map $W \to W'$ and by $\sigma : W \to W'$ the linear map induced by the symplectic form, one defines the symplectic dual of Q_X by $Q_X^\sigma = \sigma^* Q_X^{-1} \sigma$. The corresponding symmetric covariant 2-tensor field $X \mapsto g_X^\sigma$ is again Borel measurable and positive definite at every point and can be given by $g_X^\sigma(T) = \sup_{S \in W \setminus \{0\}} [T,S]^2 / g_X(S)$. We say that $X \mapsto g_X$ is a Hörmander metric (i.e., an admissible metric in the terminology of [8, 27]) if the following three conditions are satisfied:

(i) (slow variation) there exist $C \geq 1$ and $r > 0$ such that

$$g_X(X-Y) \leq r^2 \Rightarrow C^{-1} g_Y(T) \leq g_X(T) \leq C g_Y(T),\ \forall X, Y, T \in W;$$

(ii) (temperance) there exist $C \geq 1$ and $N \in \mathbb{N}$ such that

$$(g_X(T)/g_Y(T))^{\pm 1} \leq C(1 + g_X^\sigma(X-Y))^N,\ \forall X, Y, T \in W;$$

(iii) (the uncertainty principle) $g_X(T) \leq g_X^\sigma(T)$, $\forall X, T \in W$.

We point out that in [12, 33] $X \mapsto g_X$ is called feasible if it satisfies only (i) and (iii) and strongly feasible if it satisfies (i), (ii) and (iii). We call C, r and N the structure constants of g. We say that g is symplectic if $g = g^\sigma$. Denote $\lambda_g(X) = \inf_{T \in W \setminus \{0\}} (g_X^\sigma(T)/g_X(T))^{1/2}$; it is Borel measurable and $\lambda_g(X) \geq 1$, $\forall X \in W$. Given $Y \in W$ and $r > 0$, denote $U_{Y,r} = \{X \in W \,|\, g_Y(X-Y) \leq r^2\}$ and define

$$\delta_r(X,Y) = 1 + g_X^\sigma \wedge g_Y^\sigma(U_{X,r} - U_{Y,r}),\ X, Y \in W,$$

where $g_X^\sigma \wedge g_Y^\sigma$ denotes the harmonic mean of the positive-definite quadratic forms g_X^σ and g_Y^σ. The function $(X,Y) \mapsto \delta_r(X,Y)$ is Borel measurable on $W \times W$ and when $r \leq r'$, where r' depends only on the structure constants of g, the function δ_r enjoys very useful properties (see [27, Section 2.2.6] for the complete account).

A positive Borel measurable function M on W is said to be g-admissible if there are $C \geq 1$, $r > 0$ and $N \in \mathbb{N}$ such that

$$g_X(X-Y) \leq r^2 \Rightarrow C^{-1} M(Y) \leq M(X) \leq C M(Y);$$
$$(M(X)/M(Y))^{\pm 1} \leq C(1 + g_X^\sigma(X-Y))^N,\ \forall X, Y \in W.$$

We denote by $g_X^\#$ the geometric mean of g_X and g_X^σ; that is, $g_X^\# = \sqrt{g_X \cdot g_X^\sigma} = \sqrt{g_X^\sigma \cdot g_X}$ (cf. [27, Definition 4.4.26, p. 341]). Then, $X \mapsto g_X^\#$ is a symplectic Hörmander metric called the symplectic intermediate of g; every g-admissible weight is also $g^\#$-admissible (see [33, Theorem 5.6]; see also [27, Proposition 2.2.20, p. 78]). Furthermore, $g_X \leq g_X^\# \leq g_X^\sigma$.

For any Banach space E with norm $\|\cdot\|_E$, we denote by $\mathcal{L}_b(E) = \mathcal{L}_b(E,E)$ the Banach space of all continuous operators on E equipped with the operator norm $\|\cdot\|_{\mathcal{L}_b(E)}$ induced by $\|\cdot\|_E$. When E is finite-dimensional, we drop the index b and employ the notations $\mathcal{L}(E)$ and $\|\cdot\|_{\mathcal{L}(E)}$ instead of $\mathcal{L}_b(E)$ and $\|\cdot\|_{\mathcal{L}_b(E)}$, respectively.

Given a g-admissible weight M, the space of symbols $S(M,g)$ is defined as the space of all $a \in \mathcal{C}^\infty(W)$ for which

$$\|a\|_{S(M,g)}^{(k)} = \sup_{l \leq k} \sup_{\substack{X \in W \\ T_1, \dots, T_l \in W \setminus \{0\}}} \frac{|a^{(l)}(X; T_1, \dots, T_l)|}{M(X) \prod_{j=1}^{l} g_X(T_j)^{1/2}} < \infty,\ \forall k \in \mathbb{N}. \quad (2.1)$$

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With this system of seminorms, $S(M,g)$ becomes a Fréchet space. One can always regularise the metric making it smooth (hence Riemannian) without changing the notion of g-admissibility of a weight and the space $S(M,g)$. The same can be done for any g-admissible weight (see [22], [27, Remark 2.2.8, p. 71]). In fact, given any g-admissible weight M, there exists a smooth g-admissible weight $\tilde{M} \in S(M,g)$ and $C > 0$ such that $M(X) \leq C\tilde{M}(X)$, $\forall X \in W$. The definition of $S(M,g)$ can be naturally extended to matrix-valued symbols. Namely, let $\tilde{V}$ be a finite-dimensional complex Banach space with norm $\|\cdot\|_{\tilde{V}}$ and denote by $\|\cdot\|_{\mathcal{L}(\tilde{V})}$ the induced norm on $\mathcal{L}(\tilde{V})$. One defines the space of $\mathcal{L}(\tilde{V})$-valued symbols $S(M,g;\mathcal{L}(\tilde{V}))$ as the space of all $a \in \mathcal{C}^\infty(W;\mathcal{L}(\tilde{V}))$ for which $\|a\|^{(k)}_{S(M,g;\mathcal{L}(\tilde{V}))} < \infty$ where the latter norms are defined as in (2.1) with $\|a^{(l)}(X;T_1,\ldots,T_l)\|_{\mathcal{L}(\tilde{V})}$ in place of $|a^{(l)}(X;T_1,\ldots,T_l)|$. Then, $S(M,g;\mathcal{L}(\tilde{V})) = S(M,g) \otimes \mathcal{L}(\tilde{V})$ is a Fréchet space (the topology on the tensor product is $\pi = \epsilon$ because $\mathcal{L}(\tilde{V})$ is finite-dimensional).

For any $a \in \mathcal{S}(W)$ (or $a \in \mathcal{S}(W;\mathcal{L}(\tilde{V}))$), the Weyl quantisation a^w is the operator

$$a^w\varphi(x) = \frac{1}{(2\pi)^n}\int_{V'}\int_V e^{i\langle x-y,\xi\rangle} a((x+y)/2,\xi)\varphi(y)\,dyd\xi,\ \varphi \in \mathcal{S}(V)\ (\text{respectively}\ \varphi \in \mathcal{S}(V;\tilde{V})),$$

where dy is a left–right Haar measure on V with $d\xi$ being its dual measure defined on V' so that the Fourier inversion formula holds with the standard constants. Consequently, a^w as well as the product measure $dyd\xi$ on W are unambiguously defined; a^w extends to a continuous operator from $\mathcal{S}'(V)$ into $\mathcal{S}(V)$ (respectively from $\mathcal{S}'(V;\tilde{V}) = \mathcal{S}'(V) \otimes \tilde{V}$ into $\mathcal{S}(V;\tilde{V}) = \mathcal{S}(V) \otimes \tilde{V}$; again, $\pi = \epsilon$). The definition of the Weyl quantisation extends to symbols in $\mathcal{S}'(W)$ (respectively $\mathcal{S}'(W;\mathcal{L}(\tilde{V}))$) and in this case $a^w : \mathcal{S}(V) \to \mathcal{S}'(V)$ (respectively $a^w : \mathcal{S}(V;\tilde{V}) \to \mathcal{S}'(V;\tilde{V})$) is continuous. When $a \in S(M,g)$ (respectively $a \in S(M,g;\mathcal{L}(\tilde{V}))$) for a g-admissible weight M, then a^w is continuous as an operator on $\mathcal{S}(V)$ (respectively on $\mathcal{S}(V;\tilde{V})$) and uniquely extends to an operator on $\mathcal{S}'(V)$ (respectively on $\mathcal{S}'(V;\tilde{V})$; cf. [22]). Furthermore, if $a, b \in \mathcal{S}(W)$ (respectively $a, b \in \mathcal{S}(W;\mathcal{L}(\tilde{V}))$), then $a^w b^w = (a\#b)^w$, where $a\#b \in \mathcal{S}(W)$ (respectively $a\#b \in \mathcal{S}(W;\mathcal{L}(\tilde{V}))$) is given by

$$a\#b(X) = \frac{1}{\pi^{2n}}\int_{W\times W} e^{-2i[X-Y_1,X-Y_2]} a(Y_1)b(Y_2)\,dY_1dY_2.$$

The bilinear map $\#$ uniquely extends to a weakly continuous bilinear map $S(M_1,g) \times S(M_2,g) \to S(M_1M_2,g)$ (in the sense of [22, Theorem 4.2]) and it is also continuous when these spaces are equipped with the Fréchet topologies described above. This holds equally well in the $\mathcal{L}(\tilde{V})$-valued case (see [22]).

Given $Y \in W$ and $r > 0$, denote $U_{Y,r} = \{X \in W \mid g_Y(X-Y) \leq r^2\}$. We say that $a \in \mathcal{C}^\infty(W)$ is g_Y-confined in $U_{Y,r}$ (see [9, 27]) if

$$\|a\|^{(k)}_{g_Y,U_{Y,r}} = \sup_{l\leq k}\ \sup_{\substack{X\in W\\ T_1,\ldots,T_l\in W\backslash\{0\}}} \frac{|a^{(l)}(X;T_1,\ldots,T_l)|(1+g_Y^\sigma(X-U_{Y,r}))^{k/2}}{\prod_{j=1}^l g_X(T_j)^{1/2}} < \infty,\ \forall k \in \mathbb{N}.$$

We will use the same notations even when a is $\mathcal{L}(\tilde{V})$-valued (of course, instead of the absolute value one employs $\|\cdot\|_{\mathcal{L}(\tilde{V})}$ in the above definition); from the context, it will always be clear whether we are considering scalar or $\mathcal{L}(\tilde{V})$-valued symbols. For fixed Y and r, the set of g_Y-confined symbols in $U_{Y,r}$ coincides with $\mathcal{S}(W)$ (respectively with $\mathcal{S}(W;\mathcal{L}(\tilde{V}))$). A family $\mathcal{S}(W)\ni\varphi_Y,\ Y\in W$, (respectively $\mathcal{S}(W;\mathcal{L}(\tilde{V}))\ni\varphi_Y,\ Y\in W$) is said to be uniformly g_Y-confined in $U_{Y,r}$ if $\sup_{Y\in W}\|\varphi_Y\|^{(k)}_{g_Y,U_{Y,r}}<\infty$, $\forall k\in\mathbb{N}$. There is $r_0>0$ that depends only on the structure constants of g such that for each $r\leq r_0$ there is a smooth uniformly g_Y-confined family in $U_{Y,r}$, $Y\mapsto\varphi_Y$, $W\to\mathcal{S}(W)$, such that $\operatorname{supp}\varphi_Y\subseteq U_{Y,r}$, $\varphi_Y\geq 0$ and

$$\int_W \varphi_Y(X)|g_Y|^{1/2}dY=1,\ \forall X\in W, \tag{2.2}$$

where $|g_Y|=\det g_Y$ computed in (any) basis of W comprised of a basis of V and the corresponding dual basis of V' (see [27, Theorem 2.2.7, p. 70]); notice that $|g_Y|^{1/2}dY$ is the volume density induced by g. Let $a_j\in S(M_j,g)$ (respectively $a_j\in S(M_j,g;\mathcal{L}(\tilde{V}))$), $j=1,2$, and $a_{j,Y}=a_j\varphi_Y,\ Y\in W$. Then

$$a_1\#a_2(X)=\int_{W\times W}a_{1,Y_1}\#a_{2,Y_2}(X)|g_{Y_1}|^{1/2}|g_{Y_2}|^{1/2}dY_1dY_2,\ \forall X\in W$$

(cf. the proof of [27, Theorem 2.3.7, p. 91]). Furthermore, given $a\in S(M,g)$ (respectively $a\in S(M,g;\mathcal{L}(\tilde{V}))$), and denoting as before $a_Y=a\varphi_Y$, we have

$$a^w u=\int_W a^w_Y u|g_Y|^{1/2}dY,\ u\in\mathcal{S}(V)\ (\text{respectively}\ u\in\mathcal{S}(V;\tilde{V})),$$

where the equality holds if we interpret the integral in Bochner sense as well as pointwise. Furthermore, for $\varphi_Y,\ Y\in W$ (as above) and any $r'>r$, there exist two strongly Borel measurable uniformly g_Y-confined families in $U_{Y,r'}$, $Y\mapsto\psi_Y$, $Y\mapsto\theta_Y$, $W\to\mathcal{S}(W)$, such that $\varphi_Y=\psi_Y\#\theta_Y,\ Y\in W$ (see [27, Theorem 2.3.15, p. 98]). The Sobolev space $H(M,g)$, with a g-admissible weight M, is the space of all $u\in\mathcal{S}'(V)$ such that

$$\int_W M(Y)^2\|\theta^w_Y u\|^2_{L^2(V)}|g_Y|^{1/2}dY<\infty. \tag{2.3}$$

It is a Hilbert space with inner product

$$(u,v)_{H(M,g)}=\int_W M(Y)^2(\theta^w_Y u,\theta^w_Y v)_{L^2(V)}|g_Y|^{1/2}dY \tag{2.4}$$

and its definition and topology do not depend on the choice of the partition of unity φ_Y, $Y\in W$, and the families $\psi_Y,\theta_Y,\ Y\in W$. The space $\mathcal{S}(V)$ is continuously and densely included in $H(M,g)$ and the latter is continuously and densely included in $\mathcal{S}'(V)$. If $a\in S(M',g)$, a^w restricts to a continuous operator from $H(M,g)$ into $H(M/M',g)$. In particular, if M_1/M_2 is bounded from below, then $H(M_1,g)$ is continuously (and densely) included in $H(M_2,g)$. Furthermore, $H(1,g)$ is just $L^2(V)$. (We refer to [27, Section 2.6] and [9] for the proofs of these properties of the Sobolev spaces $H(M,g)$.) The definition of $H(M,g;\tilde{V})$ is similar: $u\in\mathcal{S}'(V;\tilde{V})$ is in $H(M,g;\tilde{V})$ if (2.3) is finite with $\|\theta^w_Y u\|_{L^2(V)}$ replaced by $\|(\theta_Y I)^w u\|_{L^2(V;\tilde{V})}$, where $I:\tilde{V}\to\tilde{V}$ is the identity operator. It is a Banach space because it is topologically isomorphic to $H(M,g)\otimes\tilde{V}$. Fixing an inner product on $\tilde{V}$ naturally induces an inner product on $L^2(V;\tilde{V})$, which, in turn, induces an inner product

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on $H(M, g; \tilde{V})$ (similarly as in (2.4)) and the latter becomes a Hilbert space. Moreover, the above isomorphism verifies that all facts we mentioned for the scalar-valued case remain true in the vector-valued case as well.

For any $A \in \mathcal{L}(\mathcal{S}(V), \mathcal{S}'(V))$ and any linear form L on W, we denote by $\operatorname{ad} L^w \cdot A$ the commutator of L^w and A; that is, $\operatorname{ad} L^w \cdot A = L^w A - AL^w \in \mathcal{L}(\mathcal{S}(V), \mathcal{S}'(V))$. When $LX = [T, X]$, for some $T \in W$, it will be convenient to identify the linear form L with T. If $a \in S(M, g)$, the following seminorms are always finite for all uniformly g_Y-confined in $U_{Y,r}$ families ϕ_Y, $Y \in W$,

$$\|a^w\|^{(k)}_{op(M,g)} = \sup_{Y \in W} \sup_{\substack{l \leq k \\ g_Y(L_1) \leq 1, \ldots, g_Y(L_l) \leq 1}} M(Y)^{-1} \|\operatorname{ad} L^w_1 \ldots \operatorname{ad} L^w_l \cdot \phi^w_Y a^w\|_{\mathcal{L}(L^2)} < \infty, \ \forall k \in \mathbb{N},$$

where $L_j X = [T_j, X]$ and, as mentioned above, we identified L_j with T_j. In fact, a result of Bony and Chemin [9, Theorem 5.5] (see also [27, Theorem 2.6.12, p. 145]) proves that the converse is also true. Namely, if $A \in \mathcal{L}(\mathcal{S}(V), \mathcal{S}'(V))$ is such that for all families ϕ_Y, $Y \in W$, that are uniformly g_Y-confined in $U_{Y,r}$, $\operatorname{ad} L^w_1 \ldots \operatorname{ad} L^w_k \cdot \phi^w_Y A \in \mathcal{L}(L^2(V))$, $\forall Y \in W$, $\forall k \in \mathbb{N}$, and the seminorms $\|A\|^{(k)}_{op(M,g)}$ are finite for all $k \in \mathbb{N}$, then $A = a^w$, for some $a \in S(M, g)$. In fact, with φ_Y, ψ_Y, θ_Y, $Y \in W$, as before, one needs to check this only for the uniformly confined family θ_Y, $Y \in W$, and

$$\forall k \in \mathbb{N}, \exists C > 0, \exists l \in \mathbb{N}, \ \|a\|^{(k)}_{S(M,g)} \leq C \|a^w\|^{(l)}_{op(M,g)},$$

with $\|a^w\|^{(l)}_{op(M,g)}$ defined via θ_Y, $Y \in W$. The facts presented above for the scalar case hold equally well in the vector-valued case with ϕ^w_Y and $\operatorname{ad} L^w$ replaced by $(\phi_Y I)^w$ and $\operatorname{ad}(LI)^w$, respectively. In fact, the validity of these results is a direct consequence of the topological isomorphism $S(M, g; \mathcal{L}(\tilde{V})) \cong S(M, g) \otimes \mathcal{L}(\tilde{V})$.

On a couple of occasions we will impose the following additional assumption on g; we will always emphasise when we assume it. We say the Hörmander metric g is geodesically temperate if there exist $C \geq 1$ and $N \in \mathbb{N}$ such that

$$g_X(T) \leq C g_Y(T)(1 + d(X, Y))^N, \ \forall X, Y, T \in W, \tag{2.5}$$

where $d(\cdot, \cdot)$ stands for the geodesic distance on W induced by $g^{\#}$. A number of metrics that correspond to different calculi are geodesically temperate: the $S^m_{\rho,\delta}$-calculus, the semi-classical, the Shubin calculus (see [9, Example 7.3], [27, Lemmas 2.6.22 and 2.6.23, p. 154]). In fact, [8, Theorem 5 (i)] proves that if the positive Borel measurable functions φ and Φ on $\mathbb{R}^{2n}$ are such that

$$g_{x,\xi} = \varphi(x, \xi)^{-2} |dx|^2 + \Phi(x, \xi)^{-2} |d\xi|^2$$

is a Hörmander metric, then g satisfies (2.5) with $d(\cdot, \cdot)$ standing for the geodesic distance induced by g^σ. Applying this result to $g^{\#}_{x,\xi} = \Phi \varphi^{-1} |dx|^2 + \varphi \Phi^{-1} |d\xi|^2$, we conclude that the latter is geodesically temperate. As $g = \varphi^{-1} \Phi^{-1} g^{\#}$, [27, Lemma 2.6.22, p. 154] verifies that g is also geodesically temperate ($\varphi \Phi \geq 1$ because g is a Hörmander metric). In particular, the geodesic temperance is valid for the Beals-Fefferman calculus [2, 3, 4] (cf. [22, Example 3]) as well as the Nicola-Rodino calculus [29].

3. Inverse smoothness in $S(1, g; \mathcal{L}(\tilde{V}))$

The result of Bony and Chemin [9, Theorem 7.6] (see also [27, Theorem 2.6.27, p. 158]) verifies that the Weyl-Hörmander calculus is spectrally invariant provided the Hörmander metric g is geodesically temperate. That is, given $a \in S(1, g)$ such that a^w is invertible on $L^2(V)$, its inverse is a pseudodifferential operator with symbol in $S(1, g)$ (i.e., the operators with symbols in $S(1, g)$ form a Ψ^*-algebra in the C^*-algebra $\mathcal{L}_b(L^2(V))$; cf. [20, 30] and the notation there). In this section, we prove that this process of taking inverses preserves the regularity in the following sense. If $\lambda \mapsto a_\lambda$ is of class $\mathcal{C}^N$, $0 \leq N \leq \infty$, with values in $S(1, g; \mathcal{L}(\tilde{V}))$ such that a_λ^w is invertible in $\mathcal{L}_b(L^2(V; \tilde{V}))$, then the mapping $\lambda \mapsto b_\lambda$, where b_λ^w is the inverse of a_λ^w, is also of class $\mathcal{C}^N$.

Before we state and prove this result, we need the following technical results. They have the same form as [27, Lemma 2.6.25, p. 155] and [27, Lemma 2.6.26, p. 156] (see also [9, Lemmas 7.4 and 7.5]) but, as we noted in the introduction, for the first one, the right-hand side of the estimate in our article has a precise form. Moreover, the second lemma is a slightly more general variant of the corresponding second cited lemma.

Lemma 3.1. *Let g be a Hörmander metric. Then $\forall N_0 \geq 0$, $\exists C_0 > 0$, $\exists k_0 \in \mathbb{N}$, $\forall N_1 \geq 0$, $\exists C_1 > 0$, $\exists k_1 \in \mathbb{N}$, $\forall \nu \in \mathbb{Z}_+$, $\forall J \subseteq \{0, \ldots, \nu - 1\}$, $J \neq \emptyset$, $\forall c_0, \ldots, c_\nu \in \mathcal{S}(W; \mathcal{L}(\tilde{V}))$, $\forall Y_0, \ldots, Y_\nu \in W$ it holds that*

$$\begin{aligned}
&\| c_0^w \ldots c_\nu^w \|_{\mathcal{L}_b(L^2(V; \tilde{V}))} \\
&\quad \leq C_0^{\nu - |J|} C_1^{1+|J|} \| c_0 \|^{(k_1)}_{g_{Y_0}, U_{Y_0, r}} \| c_\nu \|^{(k_1)}_{g_{Y_\nu}, U_{Y_\nu, r}} \left(\max_{j \in K'} \| c_j \|^{(k_0)}_{g_{Y_j}, U_{Y_j}, r} \right)^{\nu - |J \cup \{0, \nu - 1\}|} \\
&\quad \cdot \left(\max_{j \in K} \| c_j \|^{(k_1)}_{g_{Y_j}, U_{Y_j}, r} \right)^{|(J \cup \{0\}) \setminus \{\nu - 1\}|} \prod_{j=0}^{\nu - 1} \delta_r(Y_j, Y_{j+1})^{-N_0} \prod_{j \in J} \delta_r(Y_j, Y_{j+1})^{-N_1},
\end{aligned}$$

with $K = (J \cup (J+1)) \setminus \{0, \nu\}$ and $K' = (\mathbb{N} \cap [1, \nu - 1]) \setminus (J \cap (J+1))$.

Remark 3.2. In the previous lemma, $K = \emptyset$ implies $|(J \cup \{0\}) \setminus \{\nu - 1\}| = 0$ and in this case we set

$$\left(\max_{j \in K} \| c_j \|^{(k_1)}_{g_{Y_j}, U_{Y_j}, r} \right)^{|(J \cup \{0\}) \setminus \{\nu - 1\}|} = 1.$$

Similarly, $K' = \emptyset$ implies $\nu - |J \cup \{0, \nu - 1\}| = 0$ and in this case we set

$$\left(\max_{j \in K'} \| c_j \|^{(k_0)}_{g_{Y_j}, U_{Y_j}, r} \right)^{\nu - |J \cup \{0, \nu - 1\}|} = 1.$$

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Proof. Applying the same technique as in the proof of [9, Lemma 7.4] (see also the proof of [27, Lemma 2.6.25, p. 155]) we infer[1]

$$
\begin{aligned}
&\|c_0^w \dots c_\nu^w\|^2_{\mathcal{L}_b(L^2(V;\tilde{V}))} \leq C_0^{2\nu-2|J|} C_1^{2+2|J|} \left(\|c_0\|^{(k_1)}_{g_{Y_0},U_{Y_0},r}\right)^2 \left(\|c_\nu\|^{(k_1)}_{g_{Y_\nu},U_{Y_\nu},r}\right)^2\\
&\cdot \left(\prod_{j\in J\setminus\{0\}} \|c_j\|^{(k_1)}_{g_{Y_j},U_{Y_j},r}\right)\left(\prod_{j\in (J+1)\setminus\{\nu\}} \|c_j\|^{(k_1)}_{g_{Y_j},U_{Y_j},r}\right)\left(\prod_{\substack{j=1\\ j\notin J}}^{\nu-1} \|c_j\|^{(k_0)}_{g_{Y_j},U_{Y_j},r}\right)\left(\prod_{\substack{j=1\\ j\notin J+1}}^{\nu-1} \|c_j\|^{(k_0)}_{g_{Y_j},U_{Y_j},r}\right)\\
&\cdot \left(\prod_{j=0}^{\nu-1} \delta_r(Y_j,Y_{j+1})^{-2N_0}\right)\left(\prod_{j\in J} \delta_r(Y_j,Y_{j+1})^{-2N_1}\right), \qquad (3.1)
\end{aligned}
$$

with C_0 and $k_0 \geq 2n+1$ depending only on N_0 and $C_1 \geq C_0$ and $k_1 \geq k_0$ depending on N_0+N_1. Now, one can deduce the claim of the lemma by considering the four cases, whether 0 or $\nu-1$ belongs to J or not. We illustrate the main ideas for the case when $0 \notin J$ and $\nu-1 \notin J$. Because $J \neq \emptyset$, it follows that $\nu \geq 3$. Denote $s = \min\{j|j \in J\}$, $t = \max\{j|j \in J\}$. Clearly $1 \leq s \leq t \leq \nu-2$; $K, K' \neq \emptyset$; $t+1 \notin J$; $s \notin J+1$. Denote $\tilde{c} = \max_{j\in K} \|c_j\|^{(k_1)}_{g_{Y_j},U_{Y_j},r}$, $\tilde{\tilde{c}} = \max_{j\in K'} \|c_j\|^{(k_0)}_{g_{Y_j},U_{Y_j},r}$. The products in (3.1) are equal to

$$
\begin{aligned}
&\left(\prod_{j\in J\setminus\{0\}} \|c_j\|^{(k_1)}_{g_{Y_j},U_{Y_j},r}\right)\left(\prod_{j\in (J+1)\setminus\{\nu\}} \|c_j\|^{(k_1)}_{g_{Y_j},U_{Y_j},r}\right)\left(\prod_{\substack{j=1\\ j\neq t+1,\, j\notin J}}^{\nu-1} \|c_j\|^{(k_0)}_{g_{Y_j},U_{Y_j},r}\right)\\
&\cdot\left(\prod_{\substack{j=1\\ j\neq s,\, j\notin J+1}}^{\nu-1} \|c_j\|^{(k_0)}_{g_{Y_j},U_{Y_j},r}\right)\|c_{t+1}\|^{(k_0)}_{g_{Y_{t+1}},U_{Y_{t+1}},r}\|c_s\|^{(k_0)}_{g_{Y_s},U_{Y_s},r}\\
&\leq \tilde{c}^{2|J|}\tilde{\tilde{c}}^{2(\nu-|J|-2)}\|c_{t+1}\|^{(k_0)}_{g_{Y_{t+1}},U_{Y_{t+1}},r}\|c_s\|^{(k_0)}_{g_{Y_s},U_{Y_s},r}.
\end{aligned}
$$

Because $t+1, s \in K$ and $k_0 \leq k_1$, we infer $\|c_s\|^{(k_0)}_{g_{Y_s},U_{Y_s},r} \leq \tilde{c}$, $\|c_{t+1}\|^{(k_0)}_{g_{Y_{t+1}},U_{Y_{t+1}},r} \leq \tilde{c}$ and the above is bounded by $\tilde{c}^{2(|J|+1)}\tilde{\tilde{c}}^{2(\nu-|J|-2)}$. Because $|J|+1 = |(J\cup\{0\})\setminus\{\nu-1\}|$ and $\nu-|J|-2 = \nu-|J\cup\{0,\nu-1\}|$, we deduce the claim of the lemma. □

The following is a slight generalisation of [9, Lemma 7.5] (cf. [27, Lemma 2.6.26, p. 156]).

Lemma 3.3. *Let g be a geodesically temperate symplectic Hörmander metric. There exist $C_0 > 0$ and $k_0 \in \mathbb{Z}_+$ that depend only on the structure constants of g, and for all $k \in \mathbb{N}$ there exist $C_1, N_1 > 0$, $k_1 \in \mathbb{Z}_+$ such that for $\nu \in \mathbb{Z}_+$ and $a_1, \dots, a_\nu \in S(1,g;\mathcal{L}(\tilde{V}))$ it holds*

[1]Here and throughout the article we use the principle of vacuous (empty) product; that is, $\prod_{j=1}^0 r_j = 1$.

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that

$$\|a_1^w \dots a_\nu^w\|^{(k)}_{op(1,g)} \le C_1(\nu+1)^{N_1}\left(C_0 \max_{j=1,\dots,\nu}\|a_j\|^{(k_0)}_{S(1,g;\mathcal{L}(\tilde V))}\right)^{\nu-4k}\left(C_1 \max_{j=1,\dots,\nu}\|a_j\|^{(k_1)}_{S(1,g;\mathcal{L}(\tilde V))}\right)^{4k}.$$

Proof. Let $\varphi_Y \in \mathcal{S}(W)$, $Y \in W$, be the decomposition of unity given in [27, Theorem 2.2.7, p. 70]; that is, $Y \mapsto \varphi_Y$, $W \to \mathcal{S}(W)$, is a smooth family of nonnegative functions such that $\operatorname{supp}\varphi_Y \subseteq U_{Y,r}$ and (2.2) holds true. Denote $a_{j,Y} = \varphi_Y a_j$, $j = 1,\dots,\nu$. Set $a_{0,Y} = \theta_Y I$ and $a_{\nu+1,Y} = \varphi_Y I$, with $I : \tilde V \to \tilde V$ being the identity operator. Let $k \in \mathbb{Z}_+$. Fix $Y_0 \in W$ and let $L_j(X) = [T_j, X]$, with $g_{Y_0}(T_j) = 1$, $j = 1,\dots,k$. Employing the same technique as in the proof of [27, Lemma 2.6.26, p. 156] (see also the proof of [9, Lemma 7.5]), we deduce that $\|\operatorname{ad}(L_1 I)^w \dots \operatorname{ad}(L_k I)^w \cdot (\theta_{Y_0} I)^w a_1^w \dots a_\nu^w\|_{\mathcal{L}_b(L^2(V;\tilde V))}$ is bounded by a sum of $(\nu+2)^k$ terms $\tilde\omega_k$ of the form

$$\tilde\omega_k = \int_{W^{\nu+1}} \|b_0^w \dots b_{\nu+1}^w\|_{\mathcal{L}_b(L^2(V;\tilde V))} |g_{Y_1}|^{1/2} \dots |g_{Y_{\nu+1}}|^{1/2} dY_1 \dots dY_{\nu+1}, \tag{3.2}$$

where $b_j = (\prod_{\alpha\in E_j} \partial_{T_\alpha}) a_{j,Y_j}$ and E_j, $j = 0,\dots,\nu+1$, are disjoint possibly empty subsets of $\{1,\dots,k\}$.[2] Let $J = \{j \in \mathbb{N} | E_j \ne \emptyset\}$. Clearly, $|J| \le \sum_j |E_j| \le k$. Similarly as in the proof of [27, Lemma 2.6.26, p. 156], we define $c_j = b_j = a_{j,Y_j}$ for $j \notin J$ and $c_j = b_j(\prod_{\alpha\in E_j} g_{Y_j}(T_\alpha)^{-1/2})$ for $j \in J$. Employing the geodesic temperance of $g(= g^\#)$, in an analogous fashion, as in the proof of the quoted result, we infer

$$g_{Y_j}(T_\alpha) \le C(\nu+1)^{N-1} \sum_{l=0}^{j-1} \delta_r(Y_l, Y_{l+1})^{N^2},\ j = 1,\dots,\nu+1, \alpha = 1,\dots,k,$$

where C and N depend only on the structure constants of g and the constants in (2.5). If $J\backslash\{0\} \ne \emptyset$, we infer

$$\begin{aligned}\|b_0^w \dots b_{\nu+1}^w\|_{\mathcal{L}_b(L^2(V;\tilde V))} &\le C^k(\nu+1)^{(N-1)k} \|c_0^w \dots c_{\nu+1}^w\|_{\mathcal{L}(L_b^2(V;\tilde V))} \\ &\quad \cdot \prod_{j\in J\backslash\{0\}} \left(\sum_{l=0}^{j-1} \delta_r(Y_l, Y_{l+1})^{N^2}\right)^{|E_j|} \\ &= C^k(\nu+1)^{(N-1)k} \|c_0^w \dots c_{\nu+1}^w\|_{\mathcal{L}(L_b^2(V;\tilde V))} \sum_\mu F_\mu, \end{aligned} \tag{3.3}$$

where the very last sum has at most $(\nu+1)^k$ terms and each F_μ $(0 \le \mu \le (\nu+1)^k)$ is of the form

$$F_\mu = \prod_{j=0}^{\nu} \delta_r(Y_j, Y_{j+1})^{m_{j,\mu} N^2}$$

with $m_{j,\mu} \in \mathbb{N}$ satisfying $\sum_{j=0}^{\nu} m_{j,\mu} \le \sum_{j\in J} |E_j| \le k$. If $J = \{0\}$ (i.e., only E_0 is nonempty), then (3.3) remains true if the sum over μ has only one term $F_\mu = 1$; that is, $m_{j,\mu} = 0$,

[2] We employ the convention $\prod_{s\in\emptyset} B_s = \mathrm{Id}$.

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$j=0,\dots,\nu$. For each μ, let $J_\mu=\{j\in\mathbb{N}|\, m_{j,\mu}\neq 0\}$; clearly, $|J_\mu|\leq\sum_j m_{j,\mu}\leq k$. Define

$$\tilde{F}_\mu=F_\mu\prod_{\substack{j\in((J\cup\{\nu\})\backslash\{\nu+1\})\cup((J-1)\cap\mathbb{N})\\ j\notin J_\mu}}\delta_r(Y_j,Y_{j+1})^{N^2}=\prod_{j=0}^{\nu}\delta_r(Y_j,Y_{j+1})^{\tilde{m}_{j,\mu}N^2}.$$

Then, $\sum_{j=0}^{\nu}\tilde{m}_{j,\mu}\leq 3k+1$. Let $\tilde{J}_\mu=\{j\in\mathbb{N}|\,\tilde{m}_{j,\mu}\neq 0\}$. Then, $|\tilde{J}_\mu|\leq 3k+1$, $\nu\in\tilde{J}_\mu$ and

$$\|b_0^w\dots b_{\nu+1}^w\|_{\mathcal{L}(L^2(V;\tilde{V}))}\leq C^k(\nu+1)^{(N-1)k}\sum_\mu\|c_0^w\dots c_{\nu+1}^w\|_{\mathcal{L}(L^2(V;\tilde{V}))}\tilde{F}_\mu; \tag{3.4}$$

$$J\backslash\{0,\nu+1\}\subseteq\tilde{J}_\mu\cap(\tilde{J}_\mu+1). \tag{3.5}$$

For each μ we apply Lemma 3.1 with $N_0\geq 0$ such that $\sup_{Y\in W}\int_W\delta_r(Y,Z)^{-N_0}|g_Z|^{1/2}dZ<\infty$, $N_1=kN^2$ and $\tilde{J}_\mu\subseteq\{0,\dots,\nu\}$ (with $\nu+1$ in place of ν) to obtain

$$\begin{aligned}\|c_0^w\dots c_{\nu+1}^w\|_{\mathcal{L}(L^2(V;\tilde{V}))}&\leq C_0^{\nu+1-|\tilde{J}_\mu|}C_1^{1+|\tilde{J}_\mu|}\|c_0\|^{(k_1)}_{g_{Y_0},U_{Y_0,r}}\|c_{\nu+1}\|^{(k_1)}_{g_{Y_{\nu+1}},U_{Y_{\nu+1},r}}\\ &\cdot\left(\max_{j\in\tilde{K}'_\mu}\|c_j\|^{(k_0)}_{g_{Y_j},U_{Y_j,r}}\right)^{\nu+1-|\tilde{J}_\mu\cup\{0,\nu\}|}\left(\max_{j\in\tilde{K}_\mu}\|c_j\|^{(k_1)}_{g_{Y_j},U_{Y_j,r}}\right)^{|(\tilde{J}_\mu\cup\{0\})\backslash\{\nu\}|}\\ &\cdot\prod_{j=0}^{\nu}\delta_r(Y_j,Y_{j+1})^{-N_0}\prod_{j\in\tilde{J}_\mu}\delta_r(Y_j,Y_{j+1})^{-kN^2},\end{aligned}$$

where $\tilde{K}_\mu=(\tilde{J}_\mu\cup(\tilde{J}_\mu+1))\backslash\{0,\nu+1\}\neq\emptyset$ (because $\nu\in\tilde{J}_\mu$) and $\tilde{K}'_\mu=(\mathbb{N}\cap[1,\nu])\backslash(\tilde{J}_\mu\cap(\tilde{J}_\mu+1))$; of course, we may assume $k_0\leq k_1$. Notice that $\tilde{K}'_\mu=\emptyset$ if and only if $\tilde{J}_\mu=\{0,\dots,\nu\}$. By construction, there exists $C'\geq 1$, which depends only on the structure constants of g, such that $\|c_j\|^{(l)}_{g_{Y_j},U_{Y_j,r}}\leq C'^k\|a_{j,Y_j}\|^{(l+k)}_{g_{Y_j},U_{Y_j,r}}$, for all $l\in\mathbb{N}$, $j=1,\dots,\nu$. Furthermore, if $j\in\tilde{K}'_\mu$, then (3.5) implies $c_j=a_{j,Y_j}$. Finally, notice that the seminorms of c_0 and $c_{\nu+1}$ depend only on the structure constants of g (recall $a_{0,Y_0}=\theta_{Y_0}I$, $a_{\nu+1,Y}=\varphi_Y I$). Plugging these estimates in (3.4), we infer (because $\nu\in\tilde{J}_\mu$)

$$\begin{aligned}\|b_0^w\dots b_{\nu+1}^w\|_{\mathcal{L}(L^2(V;\tilde{V}))}&\leq C'_1(\nu+1)^{(N-1)k}\prod_{j=0}^{\nu}\delta_r(Y_j,Y_{j+1})^{-N_0}\\ &\cdot\sum_\mu\left(C'_0\max_{j=1\dots,\nu}\|a_j\|^{(k_0)}_{S(1,g;\mathcal{L}(\tilde{V}))}\right)^{\nu+1-|\tilde{J}_\mu\cup\{0\}|}\left(C'_1\max_{j=1,\dots,\nu}\|a_j\|^{(k_1+k)}_{S(1,g;\mathcal{L}(\tilde{V}))}\right)^{|\tilde{J}_\mu\cup\{0\}|-1},\end{aligned} \tag{3.6}$$

with C'_0 depending only on the structure constants of g and C'_1 independent of ν and $a_1,\dots,a_\nu$. As $|\tilde{J}_\mu\cup\{0\}|\leq 3k+2\leq 4k+1$, we deduce

$$\begin{aligned}&\left(C'_0\max_{j=1\dots,\nu}\|a_j\|^{(k_0)}_{S(1,g;\mathcal{L}(\tilde{V}))}\right)^{\nu+1-|\tilde{J}_\mu\cup\{0\}|}\left(C'_1\max_{j=1,\dots,\nu}\|a_j\|^{(k_1+k)}_{S(1,g;\mathcal{L}(\tilde{V}))}\right)^{|\tilde{J}_\mu\cup\{0\}|-1}\\ &\leq\left(C'_0\max_{j=1\dots,\nu}\|a_j\|^{(k_0)}_{S(1,g;\mathcal{L}(\tilde{V}))}\right)^{\nu-4k}\left(C'_1\max_{j=1,\dots,\nu}\|a_j\|^{(k_1+k)}_{S(1,g;\mathcal{L}(\tilde{V}))}\right)^{4k}.\end{aligned}$$

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Having in mind the latter and the fact that the sum over μ has at most $(\nu+1)^k$ terms, we can employ the estimate (3.6) in (3.2) to conclude the claim of the lemma; the estimates for $\|(\theta_{Y_0}I)^w a_1^w \dots a_\nu^w\|_{\mathcal{L}_b(L^2(V;\tilde{V}))}$ (when $k=0$) can be obtained in an analogous fashion as for the case when $k \in \mathbb{Z}_+$. □

The following remarks will prove useful throughout the rest of the article; we will frequently tacitly apply them.

Remark 3.4. If E_j, $j=1,2$, are two locally compact Hausdorff topological spaces and $\mathbf{f}_j : E_j \to \mathcal{S}(W;\mathcal{L}(\tilde{V}))$, $j=1,2$, are continuous mappings, then the mapping

$$(\lambda,\mu) \mapsto \mathbf{f}_1(\lambda)\#\mathbf{f}_2(\mu),\ E_1 \times E_2 \to \mathcal{S}(W;\mathcal{L}(\tilde{V})), \tag{3.7}$$

is continuous. This is a direct consequence of [27, Corollary 2.3.3, p. 85]. Consequently, if $E_1 = E_2 = E$, the mapping $\lambda \mapsto \mathbf{f}_1(\lambda)\#\mathbf{f}_2(\lambda)$, $E \to \mathcal{S}(W;\mathcal{L}(\tilde{V}))$, is continuous.

If E_j, $j=1,2$, are as above and $\mathbf{f}_j : E_j \to S(M_j, g;\mathcal{L}(\tilde{V}))$, $j=1,2$, are continuous mappings where M_1 and M_2 are admissible weights for g, then [27, Theorem 2.3.7, p. 91] verifies that the mapping

$$(\lambda,\mu) \mapsto \mathbf{f}_1(\lambda)\#\mathbf{f}_2(\mu),\ E_1 \times E_2 \to S(M_1M_2, g;\mathcal{L}(\tilde{V})) \tag{3.8}$$

is continuous. Again, if $E_1 = E_2 = E$, this implies that the mapping $\lambda \mapsto \mathbf{f}_1(\lambda)\#\mathbf{f}_2(\lambda)$, $E \to S(M_1M_2, g;\mathcal{L}(\tilde{V}))$, is also continuous.

Remark 3.5. If E_j, $j=1,2$, are two smooth manifolds without boundary (we always assume the smooth manifolds are second-countable and thus paracompact) and $\mathbf{f}_j : E_j \to \mathcal{S}(W;\mathcal{L}(\tilde{V}))$, $j=1,2$, are of class $\mathcal{C}^N$, $0 \le N \le \infty$, then so is the map (3.7). This can be easily derived from [27, Corollary 2.3.3, p. 85]; in fact, because the problem is local in nature, it is enough to prove it when E_1 and E_2 are Euclidean spaces. If X_j is a smooth vector field on E_j and $\mathbf{f}_j$ is smooth, $j=1,2$, then

$$X_1 \times X_2(\mathbf{f}_1(\lambda)\#\mathbf{f}_2(\mu)) = X_1\mathbf{f}_1(\lambda)\#\mathbf{f}_2(\mu) + \mathbf{f}_1(\lambda)\#X_2\mathbf{f}_2(\mu) \tag{3.9}$$

(of course, $X_1\mathbf{f}_1(\lambda)$ and $X_2\mathbf{f}_2(\mu)$ are smooth maps into $\mathcal{S}(W;\mathcal{L}(\tilde{V}))$). Consequently, if $E_1 = E_2 = E$, the mapping $\lambda \mapsto \mathbf{f}_1(\lambda)\#\mathbf{f}_2(\lambda)$, $E \to \mathcal{S}(W;\mathcal{L}(\tilde{V}))$, is smooth and the smooth vector fields on E are derivations of the algebra $\mathcal{C}^\infty(E;\mathcal{S}(W;\mathcal{L}(\tilde{V})))$ (with the associative product $\#$). If $\mathbf{f}_j$, $j=1,2$, are of class $\mathcal{C}^N$, $N \ge 1$, and X a smooth vector field on E, we still have

$$X(\mathbf{f}_1(\lambda)\#\mathbf{f}_2(\lambda)) = X\mathbf{f}_1(\lambda)\#\mathbf{f}_2(\lambda) + \mathbf{f}_1(\lambda)\#X\mathbf{f}_2(\lambda). \tag{3.10}$$

If E_1 and E_2 are as above and $\mathbf{f}_j : E_j \to S(M_j, g;\mathcal{L}(\tilde{V}))$, $j=1,2$, are of class $\mathcal{C}^N$, $0 \le N \le \infty$, where M_j, $j=1,2$, are admissible weights for g, then the mapping (3.8) is also of class $\mathcal{C}^N$ (by [27, Theorem 2.3.7, p. 91]). When $\mathbf{f}_j$, $j=1,2$, are smooth, (3.9) holds true. In particular, if $E_1 = E_2 = E$ and $M_1 = M_2 = 1$, the smooth vector fields on E are derivations of the unital algebra $\mathcal{C}^\infty(E;S(1,g;\mathcal{L}(\tilde{V})))$ (with the associative product $\#$). Furthermore, if $\mathbf{f}_1$ and $\mathbf{f}_2$ are of class $\mathcal{C}^N$, $N \ge 1$, and X is a smooth vector field on E, then (3.10) remains valid.

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The main result of the section is the following.

Theorem 3.6. *Assume that g is a geodesically temperate Hörmander metric. Let E be a Hausdorff topological space and $\mathbf{f}: E \to S(1,g;\mathcal{L}(\tilde{V}))$ be a continuous mapping. If for each $\lambda \in E$, $\mathbf{f}(\lambda)^w$ is an invertible operator on $L^2(V;\tilde{V})$, then there exists a unique continuous mapping $\tilde{\mathbf{f}}: E \to S(1,g;\mathcal{L}(\tilde{V}))$ such that*

$$\tilde{\mathbf{f}}(\lambda)\#\mathbf{f}(\lambda) = \mathbf{f}(\lambda)\#\tilde{\mathbf{f}}(\lambda) = I, \ \forall \lambda \in E. \tag{3.11}$$

If E is a smooth manifold without boundary and $\mathbf{f}: E \to S(1,g;\mathcal{L}(\tilde{V}))$ is of class $\mathcal{C}^N$, $0 \leq N \leq \infty$, then $\tilde{\mathbf{f}}: E \to S(1,g;\mathcal{L}(\tilde{V}))$ is also of class $\mathcal{C}^N$.

Proof. The existence of $\tilde{\mathbf{f}}: E \to S(1,g;\mathcal{L}(\tilde{V}))$ that satisfies (3.11) is a direct consequence of [27, Theorem 2.6.27, p. 158][3] and the uniqueness easily follows from the fact that $S(1,g;\mathcal{L}(\tilde{V}))$ is a unital associative algebra. We need to prove the continuity and the fact that $\tilde{\mathbf{f}}$ is of class $\mathcal{C}^N$, respectively. Throughout the proof, we fix an inner product on $\tilde{V}$ and denote by $\|\cdot\|_{\tilde{V}}$ and $\|\cdot\|_{\mathcal{L}(\tilde{V})}$ the induced norms.

Once its existence has been established, the continuity of $\tilde{\mathbf{f}}$ follows from general facts concerning Fréchet algebras. The set of invertible elements of the Banach algebra $\mathcal{L}_b(L^2(V;\tilde{V}))$ is open and thus its inverse image under the continuous mapping $a \mapsto a^w$, $S(1,g;\mathcal{L}(\tilde{V})) \to \mathcal{L}_b(L^2(V;\tilde{V}))$, is open in $S(1,g;\mathcal{L}(\tilde{V}))$ and it coincides with the set of invertible elements of $S(1,g;\mathcal{L}(\tilde{V}))$ because of spectral invariance [27, Theorem 2.6.27, p. 158]. Hence, [35, Chapter 7, Proposition 2, p. 113] implies that the inversion on this set (equipped with the topology induced by $S(1,g;\mathcal{L}(\tilde{V}))$) is continuous, which implies that $\tilde{\mathbf{f}}$ is continuous. However, we will give a direct proof of the continuity of $\tilde{\mathbf{f}}$ in the case E is a locally compact Hausdorff topological space without invoking [35, Chapter 7, Proposition 2, p. 113] by extending the arguments employed in the proof of [27, Theorem 2.6.27, p. 158].

Assume first that g is symplectic; thus, $g = g^\# = g^\sigma$. Let E be a locally compact Hausdorff topological space and let $\mathbf{r}: E \to S(1,g;\mathcal{L}(\tilde{V}))$ be a continuous mapping such that $\|\mathbf{r}(\lambda)^w\|_{\mathcal{L}_b(L^2(V;\tilde{V}))} < 1$, $\forall \lambda \in E$. Then $(\mathrm{Id} - \mathbf{r}(\lambda)^w)^{-1} = \mathrm{Id} + \sum_{m=1}^{\infty}(\mathbf{r}(\lambda)^w)^m$ as operators on $L^2(V;\tilde{V})$. Fix $\lambda_0 \in E$ and a compact neighbourhood K of λ_0. There exists $0 < \varepsilon < 1$ such that $\sup_{\lambda\in K}\|\mathbf{r}(\lambda)^w\|_{\mathcal{L}_b(L^2(V;\tilde{V}))} \leq \varepsilon$, and for every $k \in \mathbb{N}$ there exists $\tilde{C}_k \geq 1$ such that $\sup_{\lambda\in K}\|\mathbf{r}(\lambda)\|^{(k)}_{S(1,g;\mathcal{L}(\tilde{V}))} \leq \tilde{C}_k$. Now, by employing Lemma 3.3 in an analogous way as in the first part of the proof of [27, Theorem 2.6.27, p. 158] one deduces that for each $k \in \mathbb{N}$ there exists $0 < \varepsilon_k < 1$ and $\tilde{C}'_k \geq 1$ such that $\sup_{\lambda\in K}\|\mathbf{r}(\lambda)^{\#m}\|^{(k)}_{S(1,g;\mathcal{L}(\tilde{V}))} \leq \tilde{C}'_k\varepsilon_k^m$. Thus, $I + \sum_{m=1}^{\infty}\mathbf{r}(\lambda)^{\#m}$ converges to a continuous function $\mathbf{R}: E \to S(1,g;\mathcal{L}(\tilde{V}))$ such that $\mathbf{R}(\lambda)^w$ is the inverse of $\mathrm{Id} - \mathbf{r}(\lambda)^w$ in $L^2(V;\tilde{V})$. A direct inspection also yields $(I - \mathbf{r}(\lambda))\#\mathbf{R}(\lambda) = \mathbf{R}(\lambda)\#(I - \mathbf{r}(\lambda)) = I$, for all $\lambda \in E$.

Let $\mathbf{f}$ be as in the statement of the theorem. We continue to assume that g is symplectic. Fix $\lambda_0 \in E$. Let K be a compact neighbourhood of λ_0. Because $(\mathbf{f}(\lambda)^w)^*\mathbf{f}(\lambda)^w$ is positive invertible on $L^2(V;\tilde{V})$, $\mathbf{f}$ is continuous and K compact, it follows that there exists $C > 0$

[3]This result is given only for the scalar-valued case, but one can easily verify that the same proof works in the vector-valued case as well.

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and for each $\lambda \in K$ there exists $0 < c_\lambda \leq C$ such that

$$c_\lambda \|u\|^2_{L^2(V;\tilde{V})} \leq ((\mathbf{f}(\lambda)^w)^* \mathbf{f}(\lambda)^w u, u) \leq C\|u\|^2_{L^2(V;\tilde{V})}, \ \forall u \in L^2(V;\tilde{V}), \forall \lambda \in K.$$

Define $\mathbf{r}(\lambda) = I - C^{-1}\mathbf{f}(\lambda)^* \# \mathbf{f}(\lambda)$ and, thus, $\mathbf{r}(\lambda)^w = \mathrm{Id} - C^{-1}(\mathbf{f}(\lambda)^w)^* \mathbf{f}(\lambda)^w$. The mapping $\mathbf{r} : K \to S(1, g; \mathcal{L}(\tilde{V}))$ is continuous and $\|\mathbf{r}(\lambda)^w\|_{\mathcal{L}_b(L^2(V;\tilde{V}))} < 1$, $\forall \lambda \in K$. As K is compact, we infer $\sup_{\lambda \in K} \|\mathbf{r}(\lambda)^w\|_{\mathcal{L}_b(L^2(V;\tilde{V}))} < 1$. The first part now implies that there exists a continuous mapping $\mathbf{R}_K : K \to S(1, g; \mathcal{L}(\tilde{V}))$ such that

$$\mathbf{R}_K(\lambda) \# \mathbf{f}(\lambda)^* \# \mathbf{f}(\lambda) = I = \mathbf{f}(\lambda)^* \# \mathbf{f}(\lambda) \# \mathbf{R}_K(\lambda), \ \forall \lambda \in K. \tag{3.12}$$

Similarly, there exists a continuous mapping $\tilde{\mathbf{R}}_K : K \to S(1, g; \mathcal{L}(\tilde{V}))$ such that

$$\tilde{\mathbf{R}}_K(\lambda) \# \mathbf{f}(\lambda) \# \mathbf{f}(\lambda)^* = I = \mathbf{f}(\lambda) \# \mathbf{f}(\lambda)^* \# \tilde{\mathbf{R}}_K(\lambda), \ \forall \lambda \in K. \tag{3.13}$$

Now, (3.12) and (3.13) imply that $\mathbf{R}_K(\lambda) \# \mathbf{f}(\lambda)^* = \mathbf{f}(\lambda)^* \# \tilde{\mathbf{R}}_K(\lambda)$, $\forall \lambda \in K$. Thus, by defining $\tilde{\mathbf{f}}_K(\lambda) = \mathbf{R}_K(\lambda) \# \mathbf{f}(\lambda)^*$, we deduce that $\tilde{\mathbf{f}}_K : K \to S(1, g; \mathcal{L}(\tilde{V}))$ is continuous and satisfies the conclusion of the theorem on K. Covering E by compact neighbourhoods and noticing that, when $K \cap K' \neq \emptyset$,

$$\tilde{\mathbf{f}}_K(\lambda) = \tilde{\mathbf{f}}_K(\lambda) \# \mathbf{f}(\lambda) \# \tilde{\mathbf{f}}_{K'}(\lambda) = \tilde{\mathbf{f}}_{K'}(\lambda), \ \forall \lambda \in K \cap K',$$

we conclude the proof of the first part of the theorem when g is symplectic.

Assume now that g is a general geodesically temperate Hörmander metric. Then $g^\#$ is also a Hörmander metric by [27, Proposition 2.2.20, p. 78], $g^\#$ is geodesically temperate (cf. [27, Remark 2.6.21, p. 153]) and every admissible weight for g is also admissible for $g^\#$. Let E be a locally compact Hausdorff topological space. Because $S(1, g; \mathcal{L}(\tilde{V}))$ is continuously included in $S(1, g^\#; \mathcal{L}(\tilde{V}))$, the first part of the proof verifies the existence of a continuous mapping $\tilde{\mathbf{f}} : E \to S(1, g^\#; \mathcal{L}(\tilde{V}))$ such that (3.11) holds. We need to prove that $\tilde{\mathbf{f}}$ is well-defined and continuous as a mapping into $S(1, g; \mathcal{L}(\tilde{V}))$. Let $k \in \mathbb{Z}_+$. Given $S \in W$ and $T_j \in W$, $j = 1, \ldots, k$, satisfying $g_S(T_j) = 1$, define $M_S^{T_{l_1},\ldots,T_{l_m}}(X) = \prod_{j=1}^m g_X(T_{l_j})^{1/2}$, $X \in W$. One easily verifies that $M_S^{T_{l_1},\ldots,T_{l_m}}$, $\{l_1, \ldots, l_m\} \subseteq \{1, \ldots, k\}$, are admissible weights for g and $g^\#$ with uniform structure constants for g and $g^\#$ (cf. the proof of [27, Theorem 2.6.27, p. 158]); of course, the structure constants depend on k. Notice that $\partial_{T_{l_1}} \ldots \partial_{T_{l_m}} \mathbf{f}(\lambda) \in S(M_S^{T_{l_1},\ldots,T_{l_m}}, g^\#; \mathcal{L}(\tilde{V}))$, $\forall \lambda \in E$. Moreover,

$$\|\partial_{T_{l_1}} \ldots \partial_{T_{l_m}} \mathbf{f}(\lambda)\|^{(q)}_{S(M_S^{T_{l_1},\ldots,T_{l_m}}, g^\#; \mathcal{L}(\tilde{V}))} \leq \|\mathbf{f}(\lambda)\|^{(q+m)}_{S(1,g;\mathcal{L}(\tilde{V}))}, \ \forall q \in \mathbb{N}, \lambda \in E. \tag{3.14}$$

Applying ∂_{T_1} to the identity (3.11), we infer $\partial_{T_1}\tilde{\mathbf{f}}(\lambda) \# \mathbf{f}(\lambda) + \tilde{\mathbf{f}}(\lambda) \# \partial_{T_1}\mathbf{f}(\lambda) = 0$ and thus $\partial_{T_1}\tilde{\mathbf{f}}(\lambda) = -\tilde{\mathbf{f}}(\lambda) \# \partial_{T_1}\mathbf{f}(\lambda) \# \tilde{\mathbf{f}}(\lambda)$. By induction, one can verify that $\partial_{T_1} \ldots \partial_{T_k}\tilde{\mathbf{f}}(\lambda)$ is a finite sum of terms of the form

$$\pm f_\lambda^{(1)} \# \ldots \# f_\lambda^{(s)}, \tag{3.15}$$

where each $f_\lambda^{(j)}$ is either $\tilde{\mathbf{f}}(\lambda)$ or $\partial_{T_{l_1}} \ldots \partial_{T_{l_m}}\mathbf{f}(\lambda)$ and $s \leq 2k+1$; furthermore, each ∂_{T_j}, $j = 1 \ldots, k$, appears exactly once in (3.15). Fix $\lambda_0 \in E$ and a compact neighbourhood K

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of λ_0. Then $\partial_{T_1}\dots\partial_{T_k}\tilde{\mathbf{f}}(\lambda)-\partial_{T_1}\dots\partial_{T_k}\tilde{\mathbf{f}}(\lambda_0)$ is a finite sum of terms of the form

$$\pm\left(f_\lambda^{(1)}\#\dots\#f_\lambda^{(s)}-f_{\lambda_0}^{(1)}\#\dots\#f_{\lambda_0}^{(s)}\right) \tag{3.16}$$

with $f_\lambda^{(j)}$ and $f_{\lambda_0}^{(j)}$ as above and $s\leq 2k+1$. The quantity (3.16) is equal to

$$\pm\sum_{j=1}^{s}f_{\lambda_0}^{(1)}\#\dots\#f_{\lambda_0}^{(j-1)}\#(f_\lambda^{(j)}-f_{\lambda_0}^{(j)})\#f_\lambda^{(j+1)}\#\dots\#f_\lambda^{(s)}.$$

We take the norm $\|\cdot\|^{(0)}_{S(M_S^{T_1,\dots,T_k},g^\#;\mathcal{L}(\tilde{V}))}$ of the above sum. Because of [27, Theorem 2.3.7, p. 91], there exist $p\in\mathbb{Z}_+$ and $C>0$ independent of S and T_j (because $M_S^{T_{l_1},\dots,T_{l_m}}$ have uniform structure constants with respect to $g^\#$ and $s\leq 2k+1$) such that this norm is dominated by

$$C\sum_{j=1}^{s}\left(\prod_{l=1}^{j-1}\|f_{\lambda_0}^{(l)}\|^{(p)}_{S(\tilde{M}_l,g^\#;\mathcal{L}(\tilde{V}))}\right)\|f_\lambda^{(j)}-f_{\lambda_0}^{(j)}\|^{(p)}_{S(\tilde{M}_j,g^\#;\mathcal{L}(\tilde{V}))}\left(\prod_{l=j+1}^{s}\|f_\lambda^{(l)}\|^{(p)}_{S(\tilde{M}_l,g^\#;\mathcal{L}(\tilde{V}))}\right), \tag{3.17}$$

where $\tilde{M}_j$, $j=1,\dots,s$, are given as follows: when $f_\lambda^{(j)}=\tilde{\mathbf{f}}(\lambda)$ then $\tilde{M}_j(X)=1$, $\forall X\in W$, and when $f_\lambda^{(j)}=\partial_{T_{l_1}}\dots\partial_{T_{l_m}}\mathbf{f}(\lambda)$ then $\tilde{M}_j(X)=M_S^{T_{l_1},\dots,T_{l_m}}(X)$, $\forall X\in W$. Because $\tilde{\mathbf{f}}$ and $\mathbf{f}$ are continuous with values in $S(1,g^\#;\mathcal{L}(\tilde{V}))$ and $S(1,g;\mathcal{L}(\tilde{V}))$, respectively, and K is compact, it follows that (3.17) tends to 0 as $\lambda\to\lambda_0$ uniformly in $S\in W$, $T_j\in W$, $j=1,\dots,k$, satisfying $g_S(T_j)=1$ (cf. (3.14)). Thus,

$$\sup_{\substack{S\in W\\ T_1,\dots,T_k\in W,\, g_S(T_j)=1}}\|\partial_{T_1}\dots\partial_{T_k}\tilde{\mathbf{f}}(\lambda)(S)-\partial_{T_1}\dots\partial_{T_k}\tilde{\mathbf{f}}(\lambda_0)(S)\|_{\mathcal{L}(\tilde{V})}\to 0,\ \text{as}\ \lambda\to\lambda_0. \tag{3.18}$$

In an analogous fashion, one also deduces

$$\sup_{\substack{S\in W\\ T_1,\dots,T_k\in W,\, g_S(T_j)=1}}\|\partial_{T_1}\dots\partial_{T_k}\tilde{\mathbf{f}}(\lambda)(S)\|_{\mathcal{L}(\tilde{V})}<\infty,\ \forall\lambda\in K.$$

Consequently $\tilde{\mathbf{f}}(K)\subseteq S(1,g,\mathcal{L}(\tilde{V}))$ and (3.18) implies $\tilde{\mathbf{f}}$ is continuous at λ_0 as an $S(1,g;\mathcal{L}(\tilde{V}))$-valued mapping.

Assume now that E is a smooth p-dimensional manifold with $\mathbf{f}$ being of class $\mathcal{C}^N$, $1\leq N\leq\infty$. The above verifies that $\tilde{\mathbf{f}}:E\to S(1,g;\mathcal{L}(\tilde{V}))$ is continuous. In order to prove $\tilde{\mathbf{f}}$ is of class $\mathcal{C}^N$ as an $S(1,g;\mathcal{L}(\tilde{V}))$-valued mapping, let $\lambda_0\in E$ and let K be a regular compact set (i.e., $K=\overline{\operatorname{int}K}$) containing λ_0 in its interior and K is contained in a coordinate neighbourhood U of λ_0 with local coordinates $(\lambda^1,\dots,\lambda^p)$. The maps

$$\tilde{\mathbf{f}}_j:U\to S(1,g;\mathcal{L}(\tilde{V})),\ \tilde{\mathbf{f}}_j(\lambda)=-\tilde{\mathbf{f}}(\lambda)\#\partial_{\lambda^j}\mathbf{f}(\lambda)\#\tilde{\mathbf{f}}(\lambda),\quad j=1,\dots,p,$$

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are well defined and continuous. We will prove that

$$|\lambda-\lambda_0|^{-1}\left(\tilde{\mathbf{f}}(\lambda)-\tilde{\mathbf{f}}(\lambda_0)-\sum_{j=1}^{p}(\lambda^j-\lambda_0^j)\tilde{\mathbf{f}}_j(\lambda_0)\right)\to 0,\ \text{as}\ \lambda\to\lambda_0,\ \text{in}\ S(1,g;\mathcal{L}(\tilde{V})). \quad (3.19)$$

Notice that

$$\begin{aligned}
&\tilde{\mathbf{f}}(\lambda)-\tilde{\mathbf{f}}(\lambda_0)-\sum_{j=1}^{p}(\lambda^j-\lambda_0^j)\tilde{\mathbf{f}}_j(\lambda_0)\\
&\quad=-\tilde{\mathbf{f}}(\lambda)\#\left(\mathbf{f}(\lambda)-\mathbf{f}(\lambda_0)-\sum_{j=1}^{p}(\lambda^j-\lambda_0^j)\partial_{\lambda^j}\mathbf{f}(\lambda_0)\right)\#\tilde{\mathbf{f}}(\lambda_0)\\
&\qquad-\sum_{j=1}^{p}(\lambda^j-\lambda_0^j)(\tilde{\mathbf{f}}(\lambda)-\tilde{\mathbf{f}}(\lambda_0))\#\partial_{\lambda^j}\mathbf{f}(\lambda_0)\#\tilde{\mathbf{f}}(\lambda_0).
\end{aligned}$$

Thus, by [27, Theorem 2.3.7, p. 91], for each $k\in\mathbb{Z}_+$ there exists $k'\in\mathbb{Z}_+$ and $C'>0$ such that

$$\begin{aligned}
&|\lambda-\lambda_0|^{-1}\left\|\tilde{\mathbf{f}}(\lambda)-\tilde{\mathbf{f}}(\lambda_0)-\sum_{j=1}^{p}(\lambda^j-\lambda_0^j)\tilde{\mathbf{f}}_j(\lambda_0)\right\|^{(k)}_{S(1,g;\mathcal{L}(\tilde{V}))}\\
&\leq\frac{C'}{|\lambda-\lambda_0|}\left\|\mathbf{f}(\lambda)-\mathbf{f}(\lambda_0)-\sum_{j=1}^{p}(\lambda^j-\lambda_0^j)\partial_{\lambda^j}\mathbf{f}(\lambda_0)\right\|^{(k')}_{S(1,g;\mathcal{L}(\tilde{V}))}\\
&\quad\cdot\|\tilde{\mathbf{f}}(\lambda)\|^{(k')}_{S(1,g;\mathcal{L}(\tilde{V}))}\|\tilde{\mathbf{f}}(\lambda_0)\|^{(k')}_{S(1,g;\mathcal{L}(\tilde{V}))}\\
&\quad+C'\|\tilde{\mathbf{f}}(\lambda)-\tilde{\mathbf{f}}(\lambda_0)\|^{(k')}_{S(1,g;\mathcal{L}(\tilde{V}))}\|\tilde{\mathbf{f}}(\lambda_0)\|^{(k')}_{S(1,g;\mathcal{L}(\tilde{V}))}\sum_{j=1}^{p}\|\partial_{\lambda^j}\mathbf{f}(\lambda_0)\|^{(k')}_{S(1,g;\mathcal{L}(\tilde{V}))}.
\end{aligned}$$

As $\tilde{\mathbf{f}}:E\to S(1,g;\mathcal{L}(\tilde{V}))$ is continuous and $\mathbf{f}:E\to S(1,g;\mathcal{L}(\tilde{V}))$ is of class $\mathcal{C}^N$, we deduce the validity of (3.19). Because $\lambda_0\in\operatorname{int}K$ is arbitrary, we conclude that $\tilde{\mathbf{f}}$ is $\mathcal{C}^1$ as an $S(1,g;\mathcal{L}(\tilde{V}))$-valued mapping whose partial derivatives are $\tilde{\mathbf{f}}_j$ (recall that these are continuous $S(1,g;\mathcal{L}(\tilde{V}))$-valued mappings). In the same way one proves that $\tilde{\mathbf{f}}$ is $\mathcal{C}^k$, for every $k\in\mathbb{Z}_+$, $k\leq N$; that is, it is of class $\mathcal{C}^N$ on $\operatorname{int}K$ and, as K is arbitrary, it is of class $\mathcal{C}^N$ on E as an $S(1,g;\mathcal{L}(\tilde{V}))$-valued mapping. □

As a consequence of this theorem, we have the following result.

Corollary 3.7. *Assume that g is a geodesically temperate Hörmander metric and M and M_1 are g-admissible weights. Let E be a Hausdorff topological space and $\mathbf{f}:E\to S(M,g;\mathcal{L}(\tilde{V}))$ be a continuous mapping. If for each $\lambda\in E$, $\mathbf{f}(\lambda)^w$ restricts to an invertible operator from $H(M_1,g;\tilde{V})$ onto $H(M_1/M,g;\tilde{V})$, then there exists a unique continuous mapping $\tilde{\mathbf{f}}:E\to S(1/M,g;\mathcal{L}(\tilde{V}))$ such that*

$$\tilde{\mathbf{f}}(\lambda)\#\mathbf{f}(\lambda)=\mathbf{f}(\lambda)\#\tilde{\mathbf{f}}(\lambda)=I,\ \forall\lambda\in E.$$

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If E is a smooth manifold without boundary and $\mathbf{f}: E \to S(M,g;\mathcal{L}(\tilde{V}))$ is of class $\mathcal{C}^N$, $0 \le N \le \infty$, then $\tilde{\mathbf{f}}: E \to S(1/M,g;\mathcal{L}(\tilde{V}))$ is also of class $\mathcal{C}^N$.

Proof. By [27, Corollary 2.6.16, p. 150], we can find $a_1 \in S(1/M_1,g)$, $b_1 \in S(M_1,g)$, $a \in S(M_1/M,g)$, $b \in S(M/M_1,g)$ such that $b_1 \# a_1 = 1 = a_1 \# b_1$ and $b \# a = 1 = a \# b$. If $\mathbf{f}$ is of class $\mathcal{C}^N$, $0 \le N \le \infty$, the mapping

$$\mathbf{f}_1(\lambda) = (aI)\#\mathbf{f}(\lambda)\#(a_1 I),\ E \to S(1,g,\mathcal{L}(\tilde{V})),$$

is also of class $\mathcal{C}^N$, and it satisfies the assumptions of Theorem 3.6. Hence, there exists $\tilde{\mathbf{f}}_1 : E \to S(1,g,\mathcal{L}(\tilde{V}))$ of class $\mathcal{C}^N$ such that $\tilde{\mathbf{f}}_1(\lambda)\#\mathbf{f}_1(\lambda) = I = \mathbf{f}_1(\lambda)\#\tilde{\mathbf{f}}_1(\lambda)$. Then $\tilde{\mathbf{f}}(\lambda) = (a_1 I)\#\tilde{\mathbf{f}}_1(\lambda)\#(aI)$, $E \mapsto S(1/M,g;\mathcal{L}(\tilde{V}))$, is the sought-after mapping. The proof of the uniqueness is easy and we omit it. □

4. Ellipticity and the Fredholm property

The relationship between the class of pseudodifferential operators that are Fredholm between appropriate Sobolev spaces and those that are elliptic is the subject of this section. We will prove the equivalence of these two properties provided the metric is geodesically temperate and its associate function λ_g tends to infinity at infinity.

We start with the following simple but useful result.

Lemma 4.1. *Let g be a Hörmander metric and M_1, M_2 and M g-admissible weights. If MM_2/M_1 vanishes at infinity, then for any $a \in S(M,g;\mathcal{L}(\tilde{V}))$, a^w restricts to a compact operator from $H(M_1,g;\tilde{V})$ into $H(M_2,g;\tilde{V})$.*

Proof. By [27, Corollary 2.6.16, p. 150], we can choose $a_j \in S(M_j,g)$, $\tilde{a}_j \in S(1/M_j,g)$ satisfying $a_j \# \tilde{a}_j = 1 = \tilde{a}_j \# a_j$, $j = 1,2$. Then, $a^w = (\tilde{a}_2 I)^w((a_2 I)\#a\#(\tilde{a}_1 I))^w(a_1 I)^w$. Because $(a_2 I)\#a\#(\tilde{a}_1 I) \in S(MM_2/M_1,g;\mathcal{L}(\tilde{V}))$ and MM_2/M_1 vanishes at infinity, [22, Theorem 5.5] yields that $((a_2 I)\#a\#(\tilde{a}_1 I))^w$ is compact on $L^2(V;\tilde{V})$ and the result of the lemma follows. □

The ellipticity is defined in a usual way.

Definition 4.2. Let g be a Hörmander metric and M a g-admissible weight. We say that $a \in S(M,g;\mathcal{L}(\tilde{V}))$ is $S(M,g;\mathcal{L}(\tilde{V}))$-elliptic if there exist a compact neighbourhood of the origin $K \subseteq W$ and $C > 0$ such that $|\det a(X)| \ge CM(X)^{\dim \tilde{V}}$, for all $X \in W \backslash K$.

Remark 4.3. Of course, in the scalar-valued case, this definition reduces to the familiar one when working in the frequently used calculi (the Shubin calculus, the SG calculus, etc.; cf. [29, 31]); see also [11] for the notion of hypoellipticity in the scalar-valued setting of the Weyl-Hörmander calculus.

Remark 4.4. When $a \in S(M,g;\mathcal{L}(\tilde{V}))$, we always have $\det a \in S(M^{\dim \tilde{V}},g)$. Thus, for a given $a \in S(M,g;\mathcal{L}(\tilde{V}))$, the $S(M,g;\mathcal{L}(\tilde{V}))$-ellipticity of a is equivalent to the $S(M^{\dim \tilde{V}},g)$-ellipticity of $\det a$.

Remark 4.5. There exists $c_0' \geq 1$ that depends only on $\dim \tilde{V}$ and $\|\cdot\|_{\tilde{V}}$ such that for any invertible $A : \tilde{V} \to \tilde{V}$ we have $1/\|A\|_{\mathcal{L}(\tilde{V})} \leq \|A^{-1}\|_{\mathcal{L}(\tilde{V})} \leq c_0' \|A\|^{\dim \tilde{V}-1}_{\mathcal{L}(\tilde{V})}/|\det A|$. Consequently, for $a \in S(M, g; \mathcal{L}(\tilde{V}))$ the $S(M, g; \mathcal{L}(\tilde{V}))$-ellipticity of a is equivalent to the following: There exists a compact neighbourhood of the origin $K \subseteq W$ and $C > 0$ such that $a(X)$ is invertible on $W\backslash K$ and $\|a(X)^{-1}\|_{\mathcal{L}(\tilde{V})} \leq C/M(X)$, $\forall X \in W\backslash K$.

Theorem 4.6. *Let g be a Hörmander metric satisfying $\lambda_g \to \infty$ and M be a g-admissible weight. If $a \in S(M, g; \mathcal{L}(\tilde{V}))$ is elliptic, then for any g-admissible weight M_1, a^w restricts to a Fredholm operator from $H(M_1, g; \tilde{V})$ into $H(M_1/M, g; \tilde{V})$ and its index is independent of M_1.*

Proof. Let $\tilde{a} = a^{-1}$ away from the origin and modified on a sufficiently large compact neighbourhood of the origin so as to be a well-defined element of $S(1/M, g; \mathcal{L}(\tilde{V}))$. Then $\tilde{a}\#a - I \in S(1/\lambda_g, g; \mathcal{L}(\tilde{V}))$ and Lemma 4.1 verifies that $\tilde{a}^w a^w - \mathrm{Id}$ is a compact operator on $H(M_1, g; \tilde{V})$. Similarly, $a^w \tilde{a}^w - \mathrm{Id}$ is a compact operator on $H(M_1/M, g; \tilde{V})$. Consequently, $a^w : H(M_1, g; \tilde{V}) \to H(M_1/M, g; \tilde{V})$ is Fredholm. To prove that the index is independent of M_1, let M_2 be another g-admissible weight and denote by A_j the restriction of a^w to $H(M_j, g; \tilde{V}) \to H(M_j/M, g; \tilde{V})$, $j = 1, 2$. Because of [27, Corollary 2.6.16, p. 150], we can choose $b_1 \in S(M_1/M_2, g)$ and $b_2 \in S(M_2/M_1, g)$ such that $b_1\#b_2 = 1 = b_2\#b_1$. Consequently, the restrictions B_1 and B_2 of $(b_1 I)^w$ and $(b_2 I)^w$ to $H(M_1, g; \tilde{V}) \to H(M_2, g; \tilde{V})$ and $H(M_2/M, g; \tilde{V}) \to H(M_1/M, g; \tilde{V})$, respectively, are isomorphisms. Because $(b_2 I)\#a\#(b_1 I) - a \in S(M/\lambda_g, g; \mathcal{L}(\tilde{V}))$ and $\lambda_g \to \infty$ at infinity, Lemma 4.1 implies that $B_2 A_2 B_1 - A_1 : H(M_1, g; \tilde{V}) \to H(M_1/M, g; \tilde{V})$ is compact. Consequently, $\mathrm{ind}\, A_2 = \mathrm{ind}\, B_2 A_2 B_1 = \mathrm{ind}\, A_1$. □

Remark 4.7. When working in the ‘classical’ pseudodifferential calculi, if there exists $C, \delta > 0$ such that $\lambda_g(X) \geq C(1 + g_0(X))^\delta$, $\forall X \in W$ (i.e., if the metric satisfies the strong uncertainty principle), then given an elliptic $a \in S(M, g)$ one can construct a parametrix of a (see [6, 28]; see also [13, 21]) and derive from that the index of $a^w{}_{|H(M_1,g)} : H(M_1, g) \to H(M_1/M, g)$ does not depend on M_1; in fact, one can derive the stronger result that the dimensions of the kernel and cokernel are independent of M_1 (cf. [29, Section 1.6]). The significance of the above result is that the index is independent of M_1 even when only requiring $\lambda_g \to \infty$; however, we cannot say anything about the invariance of the dimensions of the kernel and cokernel. As we will see at the end of this section, if one additionally assumes that g is geodesically temperate, then the invariance of the dimensions of the kernel and cokernel will follow.

Our next goal is to prove a converse result to that of Theorem 4.6. Namely, if a^w restricts to a Fredholm operator between Sobolev spaces, then it is elliptic. The proof relies on Theorem 3.6 and, consequently, on the spectral invariance of the Weyl-Hörmander calculus and the geodesic temperance of g. We first prove this result for symbols in $S(1, g; \mathcal{L}(\tilde{V}))$ and derive the general case from the latter.

Before we proceed, we need the following result whose proof is the same as for [30, Lemma 2.7] and thus we omit it.

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Lemma 4.8. *Let g be a Hörmander metric and $a \in S(1,g;\mathcal{L}(\tilde{V}))$ be such that $A = a^w{}_{|L^2(V;\tilde{V})}$ has finite-dimensional range. Then, there exist $\varphi_j \in \mathcal{S}(V;\tilde{V}')$, $\psi_j \in \mathcal{S}(V;\tilde{V})$, $j=1,\dots,m$, such that $Af = \sum_{j=1}^m \langle f,\varphi_j\rangle \psi_j$, $f \in L^2(V;\tilde{V})$. Consequently, the kernel of A is in $\mathcal{S}(V;\tilde{V}')\otimes\mathcal{S}(V;\tilde{V})$ and, thus, $a \in \mathcal{S}(W;\mathcal{L}(\tilde{V}))$.*

Theorem 4.9. *Let g be a geodesically temperate Hörmander metric satisfying $\lambda_g \to \infty$. If $a \in S(1,g;\mathcal{L}(\tilde{V}))$ is such that a^w restricts to a Fredholm operator on $L^2(V;\tilde{V})$ then a is elliptic.*

Proof. Throughout the proof, we fix an inner product on $\tilde{V}$ and denote by $\|\cdot\|_{\tilde{V}}$ and $\|\cdot\|_{\mathcal{L}(\tilde{V})}$ the induced norms. Denote $A = a^w{}_{|L^2(V;\tilde{V})}$. Because A is Fredholm, 0 is an isolated point of the spectrum of the positive operator A^*A (see [15, Lemma 7.2]). Let Γ be a circle about the origin in $\mathbb{C}$ with radius $r \leq 1$ that contains no other point of the spectrum of A^*A except possibly 0 and define

$$B = \frac{1}{2\pi i}\int_\Gamma (\lambda \mathrm{Id} - A^*A)^{-1}\,d\lambda.$$

Then B is an orthogonal projection and [32, Section 5.10, Theorems 10.2 and 10.1, p. 330] imply that the range of B is $\ker A^*A = \ker A$; that is, B is an orthogonal projection onto $\ker A$ (this trivially holds if $\ker A = \{0\}$). Set $\tilde{a}_\lambda = \lambda I - a^*\#a \in S(1,g;\mathcal{L}(\tilde{V}))$, $\lambda \in \Gamma$. The mapping $\lambda \mapsto \tilde{a}_\lambda$, $\Gamma \to S(1,g;\mathcal{L}(\tilde{V}))$, is continuous (and in fact smooth) and $\tilde{a}^w_\lambda$ is invertible on $L^2(V;\tilde{V})$. Theorem 3.6 yields the existence of a continuous (and in fact smooth) mapping $\lambda \mapsto \tilde{b}_\lambda$, $\Gamma \to S(1,g;\mathcal{L}(\tilde{V}))$, such that $\tilde{b}_\lambda \# \tilde{a}_\lambda = I = \tilde{a}_\lambda \# \tilde{b}_\lambda$, $\lambda \in \Gamma$. Define

$$b(X) = \frac{1}{2\pi i}\int_\Gamma \tilde{b}_\lambda(X)\,d\lambda = \frac{r}{2\pi}\int_0^{2\pi} \tilde{b}_{re^{it}}(X)e^{it}\,dt,\ \ X \in W.$$

Clearly $b \in \mathcal{C}^\infty(W;\mathcal{L}(\tilde{V}))$ and, since $\lambda \mapsto \tilde{b}_\lambda$ is continuous and Γ is compact, one easily derives that $b \in S(1,g;\mathcal{L}(\tilde{V}))$. For each $m \in \mathbb{Z}_+$, define

$$\tilde{c}_{m,t} = \tilde{b}_{re^{2\pi ij/m}}e^{2\pi ij/m},\ \text{when } 2\pi(j-1)/m \leq t < 2\pi j/m,\ j=1,\dots,m.$$

Clearly $c_{m,t} \in S(1,g;\mathcal{L}(\tilde{V}))$. Furthermore,

$$b_m = \frac{r}{2\pi}\int_0^{2\pi} c_{m,t}\,dt = \frac{r}{2\pi}\sum_{j=1}^m \tilde{b}_{re^{2\pi ij/m}}e^{2\pi ij/m}\cdot\frac{2\pi}{m} \in S(1,g;\mathcal{L}(\tilde{V})).$$

Now

$$\begin{aligned}\|b^w - b^w_m\|_{\mathcal{L}_b(L^2(V;\tilde{V}))} &\leq C\|b-b_m\|^{(k)}_{S(1,g;\mathcal{L}(\tilde{V}))}\\ &\leq \frac{C}{2\pi}\sum_{j=1}^m\int_{2\pi(j-1)/m}^{2\pi j/m}\|\tilde{b}_{re^{it}}e^{it} - \tilde{b}_{re^{2\pi ij/m}}e^{2\pi ij/m}\|^{(k)}_{S(1,g;\mathcal{L}(\tilde{V}))}\,dt.\end{aligned}$$

The right-hand side tends to 0 as $m \to \infty$ because $t \mapsto \tilde{b}_{re^{it}}e^{it}$, $[0,2\pi] \to S(1,g;\mathcal{L}(\tilde{V}))$, is uniformly continuous. Consequently, $b^w_m \to b^w$ in $\mathcal{L}_b(L^2(V;\tilde{V}))$. On the other hand, $c^w_{m,t} \to \tilde{b}^w_{re^{it}}e^{it}$, as $m \to \infty$, pointwise in $\mathcal{L}_b(L^2(V;\tilde{V}))$, so the dominated convergence theorem implies $b^w_m \to B$ in $\mathcal{L}_b(L^2(V;\tilde{V}))$. We conclude $b^w{}_{|L^2(V;\tilde{V})} = B$. As the range of B

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is the finite-dimensional space $\ker A$, we can apply Lemma 4.8 to deduce $b \in \mathcal{S}(W;\mathcal{L}(\tilde{V}))$. One easily verifies that $B + A^*A$ is invertible on $L^2(V;\tilde{V})$ and, consequently, there exists $c \in S(1,g;\mathcal{L}(\tilde{V}))$ such that $c^w{}_{|L^2(V;\tilde{V})} = (B+A^*A)^{-1}$. We infer $c\#(b+a^*\#a) = I$, which yields

$$c\#a^*\#a = I - c\#b. \tag{4.1}$$

Because $c\#b \in \mathcal{S}(W;\mathcal{L}(\tilde{V}))$ and $c\#a^*\#a - ca^*a \in S(1/\lambda_g, g;\mathcal{L}(\tilde{V}))$, we deduce $ca^*a - I \in S(1/\lambda_g, g;\mathcal{L}(\tilde{V}))$. As $1/\lambda_g$ vanishes at infinity, the continuity of the determinant on $\mathcal{L}(\tilde{V})$[4] implies

$$|\det c(X)||\det a(X)|^2 = |\det(c(X)a^*(X)a(X) - I + I)| \geq 1/2,$$

for all X outside of a compact neighbourhood of the origin. Because $c \in S(1,g;\mathcal{L}(\tilde{V}))$, the claim in the theorem follows. □

The main result of the section is the following one.

Theorem 4.10. *Let g be a geodesically temperate Hörmander metric satisfying $\lambda_g \to \infty$ and M and M_1 two g-admissible weights. If $a \in S(M,g;\mathcal{L}(\tilde{V}))$ is such that a^w restricts to a Fredholm operator from $H(M_1,g;\tilde{V})$ into $H(M_1/M,g;\tilde{V})$, then a is elliptic.*

Proof. Take an elliptic $b \in S(1/M_1, g)$ and an elliptic $c \in S(M_1/M, g)$. Then $\tilde{a} = (cI)\#a\#(bI) \in S(1,g;\mathcal{L}(\tilde{V}))$ and $\tilde{a}^w = (cI)^w a^w (bI)^w$ is a Fredholm operator on $L^2(V;\tilde{V})$ (cf. Theorem 4.6). By Theorem 4.9, $|\det\tilde{a}(X)| \geq 1/C$ and $\|\tilde{a}(X)^{-1}\|_{\mathcal{L}(\tilde{V})} \leq C$ for all X outside of a compact neighbourhood of the origin $K \subseteq W$ (cf. Remark 4.5). Because $f = \tilde{a} - (cI)a(bI) \in S(1/\lambda_g, g;\mathcal{L}(\tilde{V}))$, we have

$$|\det(cI)a(bI)(X)| = |\det\tilde{a}(X)||\det(I - \tilde{a}(X)^{-1}f(X))|,\ \forall X \in W\backslash K.$$

As $1/\lambda_g$ vanishes at infinity the claim of the theorem follows. □

As a consequence of the proof of Theorem 4.9, we derive that the geodesic temperance of g implies that elliptic operators always have parametrices if one merely requires $\lambda_g \to \infty$. To the best of our knowledge, in the literature the construction of parametrices is always done under the assumption of the strong uncertainty principle (see the references cited in Remark 4.7). In addition to being of an independent interest, this strengthens the conclusion of Theorem 4.6 because it yields the invariance of dimensions of both the kernel and cokernel of an elliptic operator and not just its index.

Theorem 4.11. *Let g be a geodesically temperate Hörmander metric satisfying $\lambda_g \to \infty$ and M a g-admissible weight. If $a \in S(M,g;\mathcal{L}(\tilde{V}))$ is elliptic, then there are $r_1, r_2 \in \mathcal{S}(W,\mathcal{L}(\tilde{V}))$ and elliptic $\tilde{a}_1, \tilde{a}_2 \in S(1/M,g;\mathcal{L}(\tilde{V}))$ such that $\tilde{a}_1\#a = I + r_1$ and $a\#\tilde{a}_2 = I + r_2$ and consequently a^w is globally regular. Furthermore, $r_1^w(\mathcal{S}'(V;\tilde{V}))$ and $r_2^w(\mathcal{S}'(V;\tilde{V}))$ are finite-dimensional subspaces of $\mathcal{S}(V;\tilde{V})$.*

[4]That is, there exists $\varepsilon > 0$ that depends only on $\dim\tilde{V}$ and $\|\cdot\|_{\tilde{V}}$ such that $|\det(I+P)| \geq 1/2$ for all $P \in \mathcal{L}(\tilde{V})$ satisfying $\|P\|_{\mathcal{L}(\tilde{V})} \leq \varepsilon$.

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In particular, $\ker a^w$ *is a finite-dimensional subspace of* $\mathcal{S}(V;\tilde{V})$ *and for any* g*-admissible weight* M_1*,* $\ker(a^w{}_{|H(M_1,g;\tilde{V})}) = \ker a^w$*.*

Proof. To prove the existence of $\tilde{a}_1$ and r_1, assume first $M = 1$; that is, $a \in S(1,g;\mathcal{L}(\tilde{V}))$ is elliptic. Then $A = a^w{}_{|L^2(V;\tilde{V})}$ is a Fredholm operator on $L^2(V;\tilde{V})$ by Theorem 4.6 and we can repeat the proof of Theorem 4.9 verbatim to conclude the existence of $b \in \mathcal{S}(W;\mathcal{L}(\tilde{V}))$ and an elliptic $c \in S(1,g;\mathcal{L}(\tilde{V}))$ such that $B = b^w{}_{|L^2(V;\tilde{V})} : L^2(V;\tilde{V}) \to L^2(V;\tilde{V})$ is an orthogonal projection on the finite-dimensional subspace $\ker A \subseteq L^2(V;\tilde{V})$ and the equality (4.1) holds. Then $\tilde{a}_1 = c\#a^* \in S(1,g;\mathcal{L}(\tilde{V}))$ is elliptic. Set $r_1 = -c\#b \in \mathcal{S}(W;\mathcal{L}(\tilde{V}))$. Lemma 4.8 implies

$$\ker A = b^w(L^2(V;\tilde{V})) = b^w(\mathcal{S}'(V;\tilde{V})) \subseteq \mathcal{S}(V;\tilde{V})$$

and, consequently, $r_1^w(\mathcal{S}'(V;\tilde{V}))$ is a finite-dimensional subspace of $\mathcal{S}(V;\tilde{V})$.

When $a \in S(M,g;\mathcal{L}(\tilde{V}))$ is elliptic for general M, pick $a_1 \in S(1/M,g)$ and $a_2 \in S(M,g)$ such that $a_2\#a_1 = 1 = a_1\#a_2$ and apply the above to the elliptic symbol $(a_1 I)\#a \in S(1,g;\mathcal{L}(\tilde{V}))$.

To prove the existence of $\tilde{a}_2$ and r_2, similarly as above, it suffices to consider the case when $M = 1$. But then one can apply an analogous construction as in the proof of Theorem 4.9 to the elliptic symbol $a^* \in S(1,g;\mathcal{L}(\tilde{V}))$. □

Remark 4.12. With g, M and a as in Theorem 4.11, in view of Theorem 4.6, we can also conclude that the dimensions of the cokernels of the Fredholm operators $a^w{}_{|H(M_1,g;\tilde{V})} : H(M_1,g;\tilde{V}) \to H(M_1/M,g;\tilde{V})$ are the same for any g-admissible weight M_1.

4.1. The Fedosov-Hörmander integral formula for the index

If the Hörmander metric g satisfies the strong uncertainty principle – that is, there are $C,\delta > 0$ such that $\lambda_g(X) \geq C(1+g_0(X))^\delta$, $\forall X \in W$, and $a \in S(1,g;\mathcal{L}(\tilde{V}))$ is elliptic – then $\operatorname{ind} a^w$ can be given by the Fedosov-Hörmander integral formula [22, Theorem 7.3] (see also [17, 19]). As a consequence of this result, we derive that the same is true if a is an elliptic symbol in $S(M,g;\mathcal{L}(\tilde{V}))$. Before we state the result, we fix the notion. Let F be a smooth manifold without boundary. A regular domain in F is a properly embedded codimension 0 submanifold with boundary. If D is a regular domain in F, then the topological boundary and interior of D coincide to its respective manifold boundary and interior (see [24, Proposition 5.46, p. 120]).

Proposition 4.13. *Assume that the Hörmander metric* g *satisfies the strong uncertainty principle and let* a *be an elliptic symbol in* $S(M,g;\mathcal{L}(\tilde{V}))$ *for some* g*-admissible weight* M*. Let* D *be any compact regular domain in* W *that contains in its interior the set where* a *is not invertible. Then*

$$\operatorname{ind} a^w = -\frac{(n-1)!}{(2n-1)!\,(2\pi i)^n}\int_{\partial D} \operatorname{tr}(a^{-1}da)^{2n-1}. \tag{4.2}$$

The orientation of D *is the one induced by* W*, where the latter has the orientation induced by the symplectic form.*

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Remark 4.14. If we fix a basis for V and take the dual basis for V', the orientation on W is given by the nonvanishing $2n$-form $d\xi_1 \wedge dx^1 \wedge \ldots \wedge d\xi_n \wedge dx^n$.

Proof. Because D is a compact regular domain in W, there exists a smooth exhausting function $f : W \to \mathbb{R}$ (i.e., a smooth f such that each sublevel set $f^{-1}(-\infty, c]$, $c \in \mathbb{R}$, is compact) such that $f^{-1}(-\infty, s] = D$ and s is a regular value of f; see [24, Theorem 5.48, p. 121]. Then, $f^{-1}(-\infty, s) = \operatorname{int} D$ and $f^{-1}(s) = \partial D$; of course, without loss of generality, we can assume $s > 1$. Fix an inner product on W and denote by $|\cdot|$ the Euclidean distance induced by it. For each $r > 0$, denote by B_r the open ball with centre at the origin and radius r. There exists $r_1 > 1$ such that $D \subseteq B_{r_1}$. Let $r > r_1$ be arbitrary but fixed. Take $\chi \in \mathcal{D}(W)$ such that $0 \leq \chi \leq 1$, $\operatorname{supp} \chi \Subset B_r$ and $\chi = 1$ on $\overline{B_{r_1}}$. Set

$$f_1(X) = \chi(X) f(X) + \tilde{s}(1 - \chi(X))|X|, \text{ with } \tilde{s} = \max\{f(X) | X \in \overline{B_r}\} > s.$$

Clearly, $f_1 \in \mathcal{C}^\infty(W)$, $f_1^{-1}(s) = \partial D$, $f_1^{-1}(\tilde{s}r) = \partial \overline{B_r}$, $f_1^{-1}([s, \tilde{s}r]) = \overline{B_r} \backslash \operatorname{int} D$ and s and $\tilde{s}r$ are regular values of f_1. Consequently, $\overline{B_r} \backslash \operatorname{int} D$ is a compact regular domain in W (see [24, Proposition 5.47, p. 121]) with $\partial(\overline{B_r} \backslash \operatorname{int} D) = \partial D \cup \partial \overline{B_r}$. Notice that $d((a^{-1} da)^{2n-1})$ is a finite sum of forms of the type $\pm(a^{-1} da)^{2n}$. Thus,

$$d \operatorname{tr}(a^{-1} da)^{2n-1} = \operatorname{tr} d((a^{-1} da)^{2n-1}) = k \operatorname{tr}(a^{-1} da)^{2n},$$

for some $k \in \mathbb{Z}$. But $\operatorname{tr}(a^{-1} da)^{2n} = 0$ because moving a factor $a^{-1} da$ from the end to the beginning produces a minus sign (the trace is invariant under cyclic permutations). Consequently, Stokes' formula applied on the manifold $\overline{B_r} \backslash \operatorname{int} D$ gives

$$\int_{\partial \overline{B_r}} \operatorname{tr}(a^{-1} da)^{2n-1} - \int_{\partial D} \operatorname{tr}(a^{-1} da)^{2n-1} = 0,$$

where the orientations on $\partial \overline{B_r}$ and ∂D are the ones induced by $\overline{B_r}$ and D respectively (and the latter have the symplectic orientations given by W). Notice that the orientation induced by $\overline{B_r} \backslash \operatorname{int} D$ on ∂D is the opposite one. The validity of this follows from the fact that the smooth vector field $\operatorname{grad} f_1 = (df_1)^\sharp$ defined by any Riemannian metric on W (for example, the Euclidean metric) is outward pointing on ∂D when viewed as the boundary of D but inward pointing when viewed as part of the boundary of $\overline{B_r} \backslash \operatorname{int} D$. Consequently, it is enough to prove (4.2) with $\overline{B_r}$ in place of D.

Take a positive elliptic symbol $\tilde{b} \in S(1/M, g)$. Let $\psi \in \mathcal{D}(W)$ be such that $0 \leq \psi \leq 1$, $\operatorname{supp} \psi \subseteq B_{r+2}$ and $\psi = 1$ on $\overline{B_{r+1}}$. Define $b = \psi + (1 - \psi)\tilde{b}$. Then, $b \in S(1/M, g)$ is a positive elliptic symbol; in fact, $b = 1$ on $\overline{B_{r+1}}$ and $b = \tilde{b}$ on $W \backslash B_{r+2}$. Furthermore, $(bI)a$ is an elliptic symbol in $S(1, g; \mathcal{L}(\tilde{V}))$. We claim that $\operatorname{ind}(bI)^w = 0$. To see this, we first note that $(bI)^w$ is formally self-adjoint. Denote by B the operator

$$(bI)^w{}_{|H(1/M, g; \tilde{V})} : H(1/M, g; \tilde{V}) \to L^2(V; \tilde{V}).$$

Then ${}^t B : L^2(V; \tilde{V}')(= L^2(V; \tilde{V})') \to H(1/M, g; \tilde{V})'$ is continuous, $\ker {}^t B$ is isomorphic to $\operatorname{coker} B$ (this holds in general) and ${}^t B u = \overline{(bI')^w \bar{u}}$, $\forall u \in L^2(V; \tilde{V}')$, with I' the identity operator on $\tilde{V}'$. Thus, $\ker {}^t B = \{u \in L^2(V; \tilde{V}') | (bI')^w \bar{u} = 0\}$. The latter space is isomorphic to

$$\ker((bI)^w{}_{|L^2(V; \tilde{V})}) = \{u \in L^2(V; \tilde{V}) | (bI)^w u = 0\}.$$

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Because g satisfies the strong uncertainty principle, there exists $k \in \mathbb{Z}_+$ such that

$$C^{-1}\lambda_g(X)^{-k} \leq M(X) \leq C\lambda_g(X)^k, \ \forall X \in W.$$

As $1/b \in S(M,g)$ (see the proof of [27, Lemma 2.2.22, p. 80]), $(I-(b^{-1}I)\#(bI))^{\#k} \in S(1/\lambda_g^k, g; \mathcal{L}(\tilde{V}))$. If $u \in \ker((bI)^w{}_{|L^2(V;\tilde{V})})$, then

$$u = (\mathrm{Id} - (b^{-1}I)^w (bI)^w)^k u \in H(\lambda_g^k, g; \tilde{V}) \subseteq H(1/M, g; \tilde{V})$$

and, consequently, $u \in \ker B$. Analogously, $u \in \ker B$ implies $u \in \ker((bI)^w{}_{|L^2(V;\tilde{V})})$. We conclude that $\operatorname{ind}(bI)^w = 0$.

Now, $\operatorname{ind} a^w = \operatorname{ind}((bI)\#a)^w = \operatorname{ind}((bI)a)^w$, where the second equality follows from the fact that $((bI)\#a)^w - ((bI)a)^w$ is compact because $(bI)\#a - (bI)a \in S(1/\lambda_g, g; \mathcal{L}(\tilde{V}))$ and $\lambda_g \to \infty$ (see Lemma 4.1). For the operator $((bI)a)^w$ we can apply the Fedosov-Hörmander formula [22, Theorem 7.3] to obtain

$$\operatorname{ind} a^w = \operatorname{ind}((bI)a)^w = -\frac{(n-1)!}{(2n-1)!\,(2\pi i)^n} \int_{\partial \overline{B_r}} \operatorname{tr}(((bI)a)^{-1} d((bI)a))^{2n-1}.$$

Because $b = 1$ on a neighbourhood of $\partial \overline{B_r}$, we infer $((bI)a)^{-1} d((bI)a) = a^{-1} da$ on $\partial \overline{B_r}$. This completes the proof. □

5. Example

In this section we give an example of an operator that is not elliptic in any of the 'classical' calculi, but it is elliptic in the Weyl-Hörmander calculus for an appropriate metric. Consequently, the results from the previous section will imply that it is a Fredholm operator between the appropriate Sobolev spaces associated to the calculus. The operator we consider is

$$-\Delta + \langle x \rangle^{-2s}, \ 0 < s < 1.$$

Remark 5.1. The left symbol of the operator $a(x,\xi) = |\xi|^2 + \langle x \rangle^{-2s}$ is equal to its Weyl symbol (and, in fact, is equal to the τ-symbol for any $\tau \in \mathbb{R}$). This follows from the fact that the Weyl symbol is given by $J^{-1/2}a$, where $J^{-1/2} = e^{-\frac{i}{2}D_x \cdot D_\xi}$; one easily verifies that $J^{-1/2}a = a$.

Lemma 5.2. *The symbol $a(x,\xi) = |\xi|^2 + \langle x \rangle^{-2s}$ of the operator $-\Delta + \langle x \rangle^{-2s}$, $0 < s < 1$, is not elliptic in any symbol class generated by a Hörmander metric g of the form $g_{x,\xi} = \varphi(x,\xi)^{-2}|dx|^2 + \Phi(x,\xi)^{-2}|d\xi|^2$ if Φ is bounded from below; that is, if $\Phi(x,\xi) \geq c$, for all $x,\xi \in \mathbb{R}^n$.*

Proof. Assume that a is elliptic for some Hörmander metric g of this form. Then, there exist $R \geq 1$ and $C \geq 1$ such that

$$2|\xi_1| = |\partial_{\xi_1} a(x,\xi)| \leq C|a(x,\xi)|/\Phi(x,\xi) \leq c^{-1}C(|\xi|^2 + \langle x \rangle^{-2s}), \ \text{if } |x|^2 + |\xi|^2 \geq R^2.$$

Consequently, this inequality is true for all points $(x^{(k)}, \xi^{(k)})$, where $x_1^{(k)} = k$, $\xi_1^{(k)} = 1/k^s$, $x_j^{(k)} = \xi_j^{(k)} = 0$, $j = 2,\dots,n$, $k \geq R$, $k \in \mathbb{Z}_+$. But this implies that $2 \leq c^{-1}C(k^{-s} + k^s(1+k^2)^{-s})$, for all $k \geq R$, which is a contradiction. □

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The lemma implies that $-\Delta + \langle x\rangle^{-2s}$, $0 < s < 1$, is not elliptic in any of the 'classical' calculi like the Shubin, the SG and the Beals-Fefferman calculus, because in each of these calculi the weight Φ is always (as part of the assumptions) bounded from below. However, the operator is elliptic in the Weyl-Hörmander calculus for an appropriate choice of the metric.

Lemma 5.3. *Let $0 < s < 1$. The Riemannian metric $g_{x,\xi} = \langle x\rangle^{-2}|dx|^2 + \langle x\rangle^{2s}\langle\xi\rangle^{-2}|d\xi|^2$ is a geodesically temperate Hörmander metric and $\lambda_g(x,\xi) = \langle x\rangle^{1-s}\langle\xi\rangle$. Furthermore, $M(x,\xi) = \langle x\rangle^{-2s} + |\xi|^2$ is admissible for g.*

Proof. The symplectic dual of g is $g^\sigma_{x,\xi} = \langle\xi\rangle^2\langle x\rangle^{-2s}|dx|^2 + \langle x\rangle^2|d\xi|^2$. To prove that g is slowly varying, it is enough to find $0 < r < 1$ and $C \geq 1$ such that for $x, y, \xi, \eta \in \mathbb{R}^n$ satisfying $|x - y| \leq r\langle x\rangle$ and $|\xi - \eta| \leq r\langle\xi\rangle\langle x\rangle^{-s}$ it holds that

$$C^{-1} \leq \langle y\rangle\langle x\rangle^{-1} \leq C \text{ and } C^{-1} \leq \langle x\rangle^s\langle\xi\rangle^{-1}\langle y\rangle^{-s}\langle\eta\rangle \leq C. \tag{5.1}$$

We claim that one can take any $r \leq \min\{1/4, 1/2^{1/s}\}$. To see this, notice that $\langle y\rangle^2 \leq 1 + 2r^2\langle x\rangle^2 + 2|x|^2 \leq 4\langle x\rangle^2$. Similarly, $\langle x\rangle^2 \leq 2r^2\langle x\rangle^2 + 2\langle y\rangle^2$ and, consequently, $\langle x\rangle^2 \leq (16/7)\langle y\rangle^2$, which proves the first part of (5.1). Analogously, for the second part we infer

$$\langle x\rangle^{2s}\langle\xi\rangle^{-2} \leq (2r^2\langle x\rangle^2 + 2\langle y\rangle^2)^s\langle\xi\rangle^{-2} \leq 2^s r^{2s}\langle x\rangle^{2s}\langle\xi\rangle^{-2} + 2^s\langle y\rangle^{2s}\langle\xi\rangle^{-2},$$

which implies $\langle x\rangle^{2s}\langle\xi\rangle^{-2} \leq 4\langle y\rangle^{2s}\langle\xi\rangle^{-2}$. When $|\eta| \leq 2|\xi|$, the right-hand side of the second part of (5.1) easily follows. The validity of the latter is also trivial when $1 \geq |\eta| \geq 2|\xi|$. Assume $|\eta| \geq 2|\xi|$ and $|\eta| \geq 1$. Then, $|\eta - \xi| \geq |\eta| - |\xi| \geq |\eta|/2$ and thus

$$\langle\eta\rangle \leq \sqrt{2}|\eta| \leq 2\sqrt{2}|\eta - \xi| \leq 2\sqrt{2}r\langle\xi\rangle\langle x\rangle^{-s} \leq \langle\xi\rangle,$$

which together with the above implies the right-hand side of the second part of (5.1). The proof of the left-hand side is similar and we omit it.

Next, we prove that g is temperate. We have to find $C, N \geq 1$ such that

$$\left(\langle y\rangle^2\langle x\rangle^{-2}\right)^{\pm 1} \leq C\left(1 + |x-y|^2\langle\xi\rangle^2\langle x\rangle^{-2s} + |\xi - \eta|^2\langle x\rangle^2\right)^N, \tag{5.2}$$

$$\left(\langle x\rangle^{2s}\langle\xi\rangle^{-2}\langle y\rangle^{-2s}\langle\eta\rangle^2\right)^{\pm 1} \leq C\left(1 + |x-y|^2\langle\xi\rangle^2\langle x\rangle^{-2s} + |\xi - \eta|^2\langle x\rangle^2\right)^N, \tag{5.3}$$

for all $x, y, \xi, \eta \in \mathbb{R}^n$. Because $\langle y\rangle^2\langle x\rangle^{-2} \leq 2 + 2|x-y|^2\langle x\rangle^{-2}$, we immediately deduce the validity of (5.2) for $\langle y\rangle^2\langle x\rangle^{-2}$. To prove it for $\langle x\rangle^2\langle y\rangle^{-2}$, we first infer $\langle x\rangle^2\langle y\rangle^{-2} \leq 2 + 2|x-y|^2\langle y\rangle^{-2}$. This immediately implies the validity of (5.2) when $|x| \leq 2|y|$. Assume $|x| \geq 2|y|$. Then, $|x-y| \geq |x| - |y| \geq |x|/2$, which implies $\langle x\rangle \leq 2\langle x-y\rangle$ and we estimate as follows:

$$\langle x-y\rangle \leq 2^{s/(1-s)}\langle x-y\rangle^{1+s/(1-s)}\langle x\rangle^{-s/(1-s)} \leq 2^{s/(1-s)}\left(1 + |x-y|^2\langle x\rangle^{-2s}\right)^{1/(2-2s)}.$$

Because $\langle x\rangle^2\langle y\rangle^{-2} \leq 2 + 2|x-y|^2$ the validity of (5.2) follows. Next, we prove (5.3). As

$$\langle x\rangle^{2s}\langle\xi\rangle^{-2} \leq 2\langle y\rangle^{2s}\langle\eta\rangle^{-2}\left(\langle x\rangle^2\langle y\rangle^{-2}\right)^s\langle\xi - \eta\rangle^2,$$

(5.2) implies (5.3) when the exponent on the left hand side is 1. The proof when the exponent is -1 is similar and we omit it.

As $\lambda_g(x,\xi) = \langle x\rangle^{1-s}\langle\xi\rangle \geq 1$, we deduce $g \leq g^\sigma$. Consequently, g is a Hörmander metric. The fact that g is geodesically temperate immediately follows from the observations given at the very end of Section 2.

We turn our attention to the g-admissibility of M. Clearly, M is strictly positive. We need to find $0 < r < 1$ and $C, N \geq 1$ such that

$$\left(|x-y| \leq r\langle x\rangle \text{ and } |\xi-\eta| \leq r\langle\xi\rangle\langle x\rangle^{-s}\right) \Rightarrow C^{-1} \leq M(x,\xi)/M(y,\eta) \leq C; \tag{5.4}$$

$$(M(x,\xi)/M(y,\eta))^{\pm 1} \leq C\left(1+|x-y|^2\langle\xi\rangle^2\langle x\rangle^{-2s}+|\xi-\eta|^2\langle x\rangle^2\right)^N, \text{ for all } x,y,\xi,\eta \in \mathbb{R}^n. \tag{5.5}$$

We claim that (5.4) holds true for any fixed $r \leq \min\{1/4, 1/2^{1/s}\}$. To see this, notice that

$$M(x,\xi)/M(y,\eta) \leq \frac{|\xi|^2}{|\eta|^2+\langle y\rangle^{-2s}} + \langle y\rangle^{2s}\langle x\rangle^{-2s}.$$

The term $\langle y\rangle^{2s}\langle x\rangle^{-2s}$ is bounded because of (5.1). When $|\xi| \leq 2|\eta|$, the first term is also bounded. Assume $|\xi| \geq 2|\eta|$. First notice that

$$\frac{|\xi|^2}{|\eta|^2+\langle y\rangle^{-2s}} \leq \frac{2\langle y\rangle^{2s}|\xi-\eta|^2}{\langle y\rangle^{2s}|\eta|^2+1}+2 \leq \frac{C_1\langle x\rangle^{2s}|\xi-\eta|^2}{\langle y\rangle^{2s}|\eta|^2+1}+2 \leq \frac{C_1 r^2\langle\xi\rangle^2}{\langle y\rangle^{2s}|\eta|^2+1}+2,$$

where the second inequality follows from the boundedness of $\langle y\rangle\langle x\rangle^{-1}$. Because $|\xi| \geq 2|\eta|$ implies $|\xi-\eta| \geq |\xi|/2$, we deduce $|\xi| \leq 2r\langle\xi\rangle \leq 2r+2r|\xi|$, which, in turn, implies $|\xi| \leq 1$. Consequently, the upper bound in (5.4) holds true. To prove the lower bound, arguing as before, we deduce that it is enough to check that $|\eta|^2/(|\xi|^2+\langle x\rangle^{-2s})$ is bounded. This follows from

$$\frac{|\eta|^2}{|\xi|^2+\langle x\rangle^{-2s}} \leq \frac{2\langle x\rangle^{2s}|\xi-\eta|^2}{\langle x\rangle^{2s}|\xi|^2+1}+2 \leq \frac{2r^2\langle\xi\rangle^2}{|\xi|^2+1}+2 \leq 3.$$

It remains to prove (5.5). In view of (5.2), it is enough to prove that both

$$|\xi|^2/(|\eta|^2+\langle y\rangle^{-2s}) \text{ and } |\eta|^2/(|\xi|^2+\langle x\rangle^{-2s})$$

are bounded by the right-hand side of (5.5) for some $C, N \geq 1$. We prove this only for the first term; the second term can be treated similarly. We estimate as follows:

$$\begin{aligned}\frac{|\xi|^2}{|\eta|^2+\langle y\rangle^{-2s}} &\leq \frac{2\langle y\rangle^{2s}|\xi-\eta|^2}{\langle y\rangle^{2s}|\eta|^2+1}+2 \leq \frac{2^{s+1}\langle x-y\rangle^{2s}|\xi-\eta|^2}{\langle y\rangle^{2s}|\eta|^2+1}+2^{s+1}|\xi-\eta|^2\langle x\rangle^2+2\\ &\leq 4\langle x-y\rangle^{2s}|\xi-\eta|^2+4|\xi-\eta|^2\langle x\rangle^2+2\\ &\leq 2\langle x-y\rangle^{4s}\langle x\rangle^{-4s}+2|\xi-\eta|^4\langle x\rangle^{4s}+4|\xi-\eta|^2\langle x\rangle^2+2.\end{aligned}$$

The very last term is bounded by the right-hand side of (5.5) for some $C, N \geq 1$. □

Proposition 5.4. *The symbol $a(x,\xi) = |\xi|^2+\langle x\rangle^{-2s}$ of the operator $-\Delta+\langle x\rangle^{-2s}$, $0 < s < 1$, is elliptic in $S(M,g)$ with g and M as in Lemma 5.3.*

Proof. As $a = M$, it is enough to verify $a \in S(M,g)$. The only nontrivial part is to prove that the first derivatives with respect to ξ satisfy the appropriate bounds when $|\xi| \leq 1$.

This follows from

$$|\partial_{\xi_j} a(x,\xi)| \leq 2|\xi|\langle x\rangle^{-s}\langle x\rangle^{s} \leq (|\xi|^2 + \langle x\rangle^{-2s})\langle x\rangle^{s} \leq \sqrt{2}(|\xi|^2 + \langle x\rangle^{-2s})\langle x\rangle^{s}\langle \xi\rangle^{-1}.$$

□

Now, Theorem 4.6 implies (as $\lambda_g \to \infty$) that $A = -\Delta + \langle x\rangle^{-2s}$, $0 < s < 1$, is a Fredholm operator between $H(M_1, g)$ and $H(M_1/M, g)$ for any g-admissible weight M_1 and $M(x,\xi) = |\xi|^2 + \langle x\rangle^{-2s}$. Moreover, its index is independent of M_1. In fact, Proposition 4.13 gives $\operatorname{ind} A = 0$. This is trivial when $n \geq 2$ and when $n = 1$ one can easily calculate the integral on a circle with centre 0. Let $\varphi \in \bigcap_{k=0}^{\infty} H(\lambda_g^k, g)$. For every $\alpha, \beta \in \mathbb{N}^n$, $x^\beta \xi^\alpha$ belongs to $S(\lambda_g^{|\alpha+\beta|/(1-s)}, g)$ and, thus, [27, Theorem 2.3.18, p. 100] verifies that the same holds for $J^{-1/2}(x^\beta \xi^\alpha)$ as well. Because $x^\beta D^\alpha \varphi = (J^{-1/2}(x^\beta \xi^\alpha))^w \varphi \in L^2(\mathbb{R}^n)$, for all $\alpha, \beta \in \mathbb{N}^n$, one has $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The closed graph theorem implies that $\mathcal{S}(\mathbb{R}^n)$ is topologically isomorphic to $\varprojlim_{k\to\infty} H(\lambda_g^k, g)$, where the linking mappings in the projective limit are the compact inclusions $H(\lambda_g^{k+1}, g) \to H(\lambda_g^k, g)$; because the strong dual of $H(\lambda_g^k, g)$ is isomorphic to $H(\lambda_g^{-k}, g)$, we also have $\mathcal{S}'(\mathbb{R}^n) = \varinjlim_{k\to\infty} H(\lambda_g^{-k}, g)$. Employing a similar technique as in the second part of the proof of Proposition 4.13, we deduce that $\ker A \subseteq \mathcal{S}(\mathbb{R}^n)$. But, $(A\varphi, \varphi) > 0$, $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, which implies that A is injective on $\mathcal{S}'(\mathbb{R}^n)$. As $\operatorname{ind} A = 0$ we can immediately deduce that A restricts to an isomorphism from $H(M_1, g)$ onto $H(M_1/M, g)$ for any g-admissible weight M_1. The latter implies that A restricts to a topological isomorphism on $\mathcal{S}(\mathbb{R}^n)$ as well. The above representation of $\mathcal{S}'(\mathbb{R}^n)$ yields that A is also a topological isomorphism on $\mathcal{S}'(\mathbb{R}^n)$. Because g is geodesically temperate (by Lemma 5.3), Corollary 3.7 (i.e., spectral invariance) implies that the inverse of A is a pseudodifferential operator with symbol in $S(1/M, g)$.

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