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TRANSIENT SOLUTION FOR A SIMPLE IMMIGRATION BIRTH–DEATH CATASTROPHE PROCESS

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In this note, we consider a simple immigration birth–death process with total catastrophes and we obtain the transient probabilities. Our approach involves a renewal argument. It is comparatively simpler and leads to more elegant expressions than other approaches that appeared in the literature recently.

1. INTRODUCTION

We consider a simple immigration birth–death process of individuals that is influenced by random total catastrophes which, when they occur, annihilate the entire population. Choosing an appropriate unit of time, it is assumed that when the state of the system is n, the immigration rate is ν, the birth rate is $n\lambda$, the death rate is $n\mu$, and the catastrophe rate is one. The process can then be described as having the following transition rates in the time interval $(t, t + \delta t)$:

Transition	Rate
$n \to n+1$	$\nu + \lambda n \quad (n \geq 0)$
$n \to n-1$	$\mu n \quad (n \geq 1)$
$n \to 0$	$1 \quad (n \geq 1)$

The same process was considered by Kyriakidis [5], Swift [8], and Chao and Zheng [2]. Swift, Chao, and Zheng used the forward Kolmogorov equations to determine the probability $p_n(t)$ that the population size $N(t)$ at time $t \geq 0$ is equal to $n \geq 0$ given

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that $P(N(0) = a) = p_a$, $a \geq 0$. They obtained a partial differential equation for the generating function of $p_n(t)$, $n \geq 0$. Swift used the symbolic software package Mathematica to solve the partial differential equation and then gave expressions for $p_0(t)$ in terms of the hypergeometric function. Chao and Zheng solved the partial differential equation using the method of characteristics and, then, obtained closed-form expressions for $p_n(t)$, $n \geq 0$. Chao and Zheng also studied the stationary probabilities π_n, $n \geq 0$, of the process. They solved an ordinary differential equation for the generating function of π_n, $n \geq 0$, and then obtained expressions for π_n, $n \geq 0$.

In the next section, we obtain expressions for $p_n(t)$, $n \geq 0$, in a comparatively simpler manner by applying a renewal argument, which has been used in [5] for the determination of π_0. The same renewal argument has also been used for the determination of the transient probabilities of the simple immigration catastrophe process (see [6]) and the transient probabilities of a simple immigration–emigration catastrophe process (see [7]). Economou and Fakinos [4] also utilized this argument to study the transient and limiting distribution of a continuous-time Markov chain influenced by a regulating point process.

2. TRANSIENT SOLUTION

Assume that the catastrophes are introduced at rate 1 even if the process is in state 0. This assumption does not change the behavior of the process and implies that the catastrophes occur as a Poisson process with rate 1. Let U_t, $t \geq 0$, be the backward recurrence time (i.e., the length of time measured backward from time t to the last catastrophe at or before t). It is well known (see, e.g., Cox [3, p. 31]) that U_t is exponentially distributed with parameter 1, censored at time t (i.e., the distribution of U_t has a continuous part with density e^{-s} in the interval $(0, t)$ and a probability atom of size e^{-t} at t). Therefore, conditioning on U_t, we obtain

$$p_n(t) = \sum_{a=0}^{\infty} p_a p_{an}(t) = \sum_{a=0}^{\infty} p_a \left[\tilde{p}_{an}(t) e^{-t} + \int_0^t \tilde{p}_{on}(s) e^{-s}\, ds \right], \tag{1}$$

where $p_{an}(t) = P\{N(t) = n | N(0) = a\}$ and $\tilde{p}_{an}(t) = P\{X(t) = n | X(0) = a\}$ for the simple immigration birth–death process $\{X(t); t \geq 0\}$. Let

$$P(x, t) = \sum_{n=0}^{\infty} \tilde{p}_{an}(t) x^n,$$

the probability generating function of $X(t)$. Clearly,

$$\tilde{p}_{an}(t) = \frac{1}{n!} \left[\frac{\partial^n P(x, t)}{\partial x^n} \right]_{x=0}. \tag{2}$$

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It is well known (see relation (8.71) in Bailey [1]) that

$$P(x,t) = \frac{(\lambda-\mu)^{\nu/\mu}\{\mu(e^{(\lambda-\mu)t}-1)-(\mu e^{(\lambda-\mu)t}-\lambda)x\}^a}{\{(\lambda e^{(\lambda-\mu)t}-\mu)-\lambda(e^{(\lambda-\mu)t}-1)x\}^{a+\nu/\lambda}} \quad \text{if } \mu\neq\lambda\neq 0,$$

$$P(x,t) = \{1+(x-1)e^{-\mu t}\}^a \exp\left[\frac{\nu}{\mu}(x-1)(1-e^{-\mu t})\right], \qquad \lambda = 0\neq\mu,$$

$$P(x,t) = \frac{(\lambda t-\lambda t x+x)^a}{(\lambda t+1-\lambda x t)^{\nu/\lambda+a}} \quad \text{if } \lambda=\mu.$$

Using the Leibniz rule, we find expressions for the nth derivative of $P(x,t)$ with respect to x, and from (1) and (2), we obtain the following expressions:

If $\mu\neq\lambda\neq 0$,

$$p_n(t) = e^{-t}\sum_{a=0}^{\infty} p_a\tilde{p}_{an}(t) + |\lambda-\mu|^{\nu/\lambda}\frac{1}{n!}\prod_{i=0}^{n-1}(\nu+\lambda i) \times \int_0^t |\lambda e^{(\lambda-\mu)s}-\mu|^{-\nu/\lambda-n}|e^{(\lambda-\mu)s}-1|^n e^{-s}\,ds, \tag{3}$$

where

$$\tilde{p}_{an}(t) = |\lambda-\mu|^{\nu/\lambda}\sum_{k=0}^{n}\frac{1}{k!(n-k)!} \times \left\{\prod_{i=0}^{n-k-1}(a-i)|\mu(e^{(\lambda-\mu)t}-1)|^{a-n+k} \times \prod_{i=0}^{k-1}[\nu+\lambda(a+i)]|\lambda e^{(\lambda-\mu)t}-\mu|^{-a-k\frac{\nu}{\lambda}}|e^{(\lambda-\mu)t}-1|^k\right\}.$$

If $\lambda = 0\neq\mu$,

$$p_n(t) = e^{-t}\sum_{a=0}^{\infty} p_a\tilde{p}_{an}(t) + \frac{1}{n!}\left(\frac{\nu}{\mu}\right)^n\int_0^t \exp\left[-\frac{\nu}{\mu}(1-e^{-\mu s})\right](1-e^{-\mu s})^n e^{-s}\,ds, \tag{4}$$

where

$$\tilde{p}_{an}(t) = \sum_{k=0}^{n}\frac{1}{k!(n-k)!}\prod_{i=0}^{n-k-1}(a-i)(1-e^{-\mu t})^{a-n+2k} \times \exp\left[-\mu t(n-k)-\frac{\nu}{\mu}(1-e^{-\mu t})\right]\left(\frac{\nu}{\mu}\right)^k.$$

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If $\lambda = \mu \neq 0$,

$$p_n(t) = e^{-t} \sum_{a=0}^{\infty} p_a \tilde{p}_{an}(t) + \prod_{i=0}^{n-1} (\nu + \lambda i) \frac{1}{n!} \int_0^t (1 + \lambda s)^{-\nu/\lambda - n} s^n e^{-s}\, ds, \quad \textbf{(5)}$$

where

$$\tilde{p}_{an}(t) = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \prod_{i=0}^{n-k-1} (a-i) \times \prod_{i=0}^{k-1} [\nu + \lambda(a+i)] \lambda^{a-n+k} t^{a-n+2k} (1-\lambda t)^{n-k} (1+\lambda t)^{-\nu/\lambda - a - k}.$$

It can be shown that (4) and (5) are equal to the corresponding expressions (19) and (20) in [2].

If t tends to infinity in (3)–(5), the first terms in these expressions vanish. Using the transformation $y = e^{-s}$ in the integrals, we obtain the same expressions for the stationary probabilities that Chao and Zheng obtained (see [2, Thm. 3.2]).

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