Proceedings of the Edinburgh Mathematical Society (2018) **61**, 201–214
doi:10.1017/S0013091517000190

AN ELLIPTIC PDE WITH CONVEX SOLUTIONS

JON WARREN
Department of Statistics, University of Warwick, Coventry CV4 7AL, UK
(j.warren@warwick.ac.uk)

(Received 5 January 2017)

Abstract Using a mixture of classical and probabilistic techniques, we investigate the convexity of solutions to the elliptic partial differential equation associated with a certain generalized Ornstein–Uhlenbeck process.

Keywords: convexity; elliptic partial differential equation; Brownian motion

2010 *Mathematics subject classification:* Primary 35J25

1. Introduction and results

We study solutions to the elliptic partial differential equation

$$\frac{1}{2}\sum_{i,j=1}^{d}(\delta_{ij}+x_ix_j)\frac{\partial^2 u}{\partial x_i \partial x_j} = c, \qquad x \in \mathbf{R}^d, \tag{1.1}$$

c being an arbitrary constant, and δ_{ij} denoting Kronecker's delta. This equation arose in a probabilistic context, that of an advection-diffusion model, motivated by the work of Gawędzki and Horvai [**4**], in which particles are carried by a stochastic flow but with each particle experiencing an independent Brownian perturbation. The generator of the diffusion process describing the motion of such a system of particles (in a certain limiting regime) is the operator, which we will denote by $\mathcal{A}$, appearing on the left-hand side of (1.1). The purpose of this paper is to prove the convexity of certain solutions to (1.1). This convexity property plays an essential part in [**14**], where it is used to prove the convergence in law of the particle motions in the advection-diffusion model to a family of sticky Brownian motions.

The operator $\mathcal{A}$ is associated with a linear stochastic differential equation, and consequently is related to a random evolution on (a subgroup of) the affine group of $\mathbf{R}^d$. Random walks on the affine group, and particularly their invariant measures, have been studied in considerable detail, and are important in a variety of applications; see the recent book [**1**] and the references therein. However, it seems that, with the exception of the classical Ornstein–Uhlenbeck (OU) processes, the corresponding diffusions have not received much attention.

We will consider solutions of (1.1) that grow linearly as $|x| \to \infty$ and admit 'boundary values'

$$\frac{u(x)}{|x|} \to g(x/|x|) \quad \text{as } |x| \to \infty, \tag{1.2}$$

where function g defined on the sphere $S^{d-1} = \{x \in \mathbf{R}^d : |x| = 1\}$ satisfies $\int g(\theta)\,\mathrm{d}\theta = c/\gamma_d$, where c is the constant appearing on the right-hand side of (1.1), and

$$\gamma_d = \frac{1}{\sqrt{\pi}} \frac{\Gamma((d+1)/2)}{\Gamma(d/2)}. \tag{1.3}$$

We will assume that the dimension $d \geq 2$. Here, the integral over the sphere is taken with respect to Lebesgue measure normalized so $\int 1\,\mathrm{d}\theta = 1$.

Our first result is that the 'Dirichlet problem' is solvable for continuous boundary data, with convergence to the boundary values occurring uniformly.

Theorem 1.1. *Suppose that $g \in C(S^{d-1})$ and let $c = \gamma_d \int g(\theta)\,\mathrm{d}\theta$, then there exists a unique solution to the partial differential equation (1.1), with $u(0) = 0$ and such that*

$$\lim_{r\to\infty} \sup_{\theta \in S^{d-1}} |u(r\theta)/r - g(\theta)| = 0.$$

Taking the constant c to be zero, this result looks at first sight as if it might be related to a Martin boundary result for the operator $\mathcal{A}$. In fact, the corresponding diffusion process is recurrent, and the only positive solutions to $\mathcal{A}u = 0$ on $\mathbf{R}^d$ are the constant solutions. Thus, the theory of Martin boundaries as usually developed for transient processes, see, for example [**12**], is not directly applicable.

It seems plausible that one could transform (1.1) into an elliptic equation on the ball $\Omega = \{x \in \mathbf{R}^d : |x| \leq 1\}$ with g becoming the boundary data on $\partial\Omega$, and then deduce Theorem 1 from standard results on the Dirichlet boundary problem for such equations, as described in [**5**]. However if this approach were to work, then there would have to be some solution corresponding to g being identically a non-zero constant, and no such solution to (1.1) and (1.2) with $c = 0$ exists. Instead, our strategy for proving Theorem 1 is to take advantage of the spherical symmetry of the operator $\mathcal{A}$ to write a series expansion for solutions involving spherical harmonic functions. This evidently associates to any function g defined on the sphere the appropriate solution of (1.1). Then, the more delicate part of the argument proves the uniform convergence of the solution to the boundary data, making use of an appropriate analogue of the maximum principle in the context of linear growth at infinity.

Convexity of the solutions to elliptic partial differential equations has been studied a great deal in the literature, see, for example [**7**, **8**]. Here we will follow one of the established approaches to proving convexity: making use of the fact the corresponding parabolic equation is convexity preserving. General conditions are known [**6**, **9**] that ensure this. However, in our problem we can see directly that the semigroup generated by $\mathcal{A}$ preserves convexity because the associated diffusion process can be extended to a stochastic flow of affine maps. Then to complete the argument for proving the following result, we must show convergence of the solution to the parabolic equation to that of the elliptic boundary value problem.

Theorem 1.2. *Suppose that $g \in C(S^{d-1})$ and $u \in C^2(\mathbf{R}^d)$ is the solution to elliptic boundary problem (1.1) and (1.2) with $u(0) = 0$. Then u is convex if and only if $v \in C(\mathbf{R}^d)$, given by*

$$v(x) = |x|g(x/|x|) \qquad x \in \mathbf{R}^d,$$

is convex also.

2. Separation of variables and properties of the radial equation

We may rewrite the operator $\mathcal{A}$ in spherical coordinates as

$$\mathcal{A} = \frac{r^2}{2}\frac{\partial^2}{\partial r^2} + \frac{1}{2}\nabla^2 = \frac{1}{2}(1+r^2)\frac{\partial^2}{\partial r^2} + \frac{(d-1)}{2r}\frac{\partial}{\partial r} + \frac{1}{2r^2}\Delta_{S^{d-1}} = \mathcal{A}_R + \frac{1}{2r^2}\Delta_{S^{d-1}}, \tag{2.1}$$

where $\Delta_{S^{d-1}}$ is the Laplace–Beltrami operator on the sphere S^{d-1}. The evident spherical symmetry suggests a solution by the separation of variables, taking the form

$$u(x) = u(r\theta) = \sum_{l \geq 0} f_l(r) g_l(\theta). \tag{2.2}$$

Suppose that $g \in L^2(S^{d-1})$ and take g_l to be the projection in $L^2(S^{d-1})$ of g onto the space of spherical harmonic functions of degree l, see [**11**]. Then g_l satisfies

$$\Delta_{S^{d-1}} g_l = -l(l+d-2)g_l, \tag{2.3}$$

and consequently for $l \geq 1$, we would like f_l to solve

$$\mathcal{A}_R f_l - \frac{l(l+d-2)}{2r^2} f_l = 0 \tag{2.4}$$

with $f_l(r)/r \to 1$ as $r \to \infty$ and $f_l(0+) = 0$. In fact, such f_l may be expressed in terms of hypergeometric functions, see Lemma 2.1.

For $l = 0$ we define f_l differently, one reason for this being that non-constant solutions to (2.4) with $l = 0$ all have a singularity at the origin. Instead, we take f_0 to solve

$$\mathcal{A}_R f_0 = \gamma_d \tag{2.5}$$

with $f_0(r)/r \to 1$ as $r \to \infty$ and $f_0(0+) = 0$. This has a solution

$$f_0(r) = 2\gamma_d \int_0^r \left(\frac{u^2}{1+u^2}\right)^{-(d-1)/2} \int_0^u \frac{v^{d-1}}{(1+v^2)^{(d+1)/2}}\,\mathrm{d}v\,\mathrm{d}u \tag{2.6}$$

which may be verified by simple calculus, noting that

$$\int_0^\infty \frac{v^{d-1}}{(1+v^2)^{(d+1)/2}}\,\mathrm{d}v = \frac{1}{2\gamma_d}.$$

Using Euler's integral representation of the hypergeometric function it is straightforward to check (see Lemma 2.1) that $f_l(r)$ decays to 0 geometrically fast for r in compact

https://doi.org/10.1017/S0013091517000190 Published online by Cambridge University Press

sets as l tends to infinity. On the other hand, $g_l(\theta)$ grows at most polynomially as l tends to infinity, as can be seen from the integral representation for g_l [11, p. 42]. In conjunction, these facts guarantee that the series (2.2) converges uniformly on compact sets of $\mathbf{R}^d$ and does indeed define a smooth solution to $\mathcal{A}u = c$, except possibly at the origin. However, since $\{0\}$ is a polar set for the diffusion process associated with $\mathcal{A}$, any bounded solution to $\mathcal{A}u = c$ in a punctured ball $\{x \in \mathbf{R}^d : 0 < |x| < r\}$ extends to a solution on the entire ball, and so (2.2) defines a solution on all of $\mathbf{R}^d$. To see that this is so, first note that by classical partial differential equation results the Dirichlet problem $\mathcal{A}u = c$ in a ball $\{x \in \mathbf{R}^d : |x| < r\}$ with continuous boundary data possesses a solution. So by linearity it is enough to know that any bounded solution to $\mathcal{A}w = 0$ in the punctured ball which extends continuously to the outer boundary with $w(x) = 0$ on that boundary, is in fact identically zero in the whole ball. This is a consequence of $\{0\}$ being polar, which implies that the Poisson kernel for the punctured ball, that is, the exit distribution of the associated diffusion process, does not charge $\{0\}$.

Lemma 2.1. *The solution to*

$$\mathcal{A}_R f - \frac{l(l+d-2)}{2r^2} f = 0,$$

satisfying boundary conditions $f(0) = 0$ and $f(r)/r \to 1$ as $r \to \infty$ is

$$f(r) = f_l(r) = r^l \frac{\Gamma((l+d+1)/2)\Gamma(l/2)}{\Gamma(l+d/2)\Gamma(1/2)} \, {}_2F_1(l/2, (l-1)/2; l+d/2; -r^2).$$

Moreover, for each $R > 0$, there exists $\delta_R \in (0,1)$ so that

$$\sup_{r \le R} f_l(r) \le \delta_R^l \text{ for all sufficiently large } l.$$

Proof. Substituting $f(r) = r^l y(-r^2)$ and $x = -r^2$ into

$$\frac{1}{2}(1+r^2) f'' + \frac{d-1}{2} f' - \frac{l(l+d-2)}{2r^2} f = 0$$

gives

$$x(1-x)y'' + \left\{ l + \frac{d}{2} - x\left(l + \frac{1}{2}\right) \right\} y' - \frac{l(l-1)}{4} y = 0,$$

which is the standard form of the hypergeometric equation with parameters $a = l/2$, $b = (l-1)/2$ and $c = l + d/2$. The boundary condition $f(0) = 0$ is satisfied by taking $y(x)$ proportional to ${}_2F_1(a, b; c; x)$. Now, to choose the constant of proportionality to get the behaviour as $r \to \infty$ correct we combine Pfaff's transformation with Gauss's formula

https://doi.org/10.1017/S0013091517000190 Published online by Cambridge University Press

for ${}_2F_1(a, b; c; 1)$ to deduce that

$$\lim_{x\to-\infty}(1-x)^b\ {}_2F_1(a,b;c;x) = {}_2F_1(c-a,b;c;1) = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}.$$

Next, using Euler's integral representation for the hypergeometric function

$$f_l(r) = r^l \frac{\Gamma(l/2)}{\Gamma((l-1)/2)\Gamma(1/2)} \int_0^1 t^{(l-3)/2}(1-t)^{(d+l-1)/2}(1+r^2t)^{-l/2}\,\mathrm{d}t.$$

Now the ratio of gamma functions appearing here grows sublinearly with l, whereas we can estimate the integral as being less than

$$\sup_{0\le t\le 1}\left(\frac{1-t}{1+r^2t}\right)^{l/2} \le \left(\frac{1}{1+r^2}\right)^{l/2}.$$

Consequently, the statement of the lemma holds choosing $\delta_R > R/\sqrt{1+R^2}$. □

3. The associated diffusion process

Associated with the operator $\mathcal{A}$ is a diffusion process. We will make use of this to study solutions of (1.1). In fact, the stochastic differential equation corresponding to $\mathcal{A}$ is linear, and consequently the diffusion process can be constructed explicitly as in the following lemma. Of particular importance is that this representation of the diffusion process actually defines a stochastic flow of affine maps of $\mathbf{R}^d$.

Lemma 3.1. *Let B be a standard one-dimensional Brownian motion, and W a standard Brownian motion in $\mathbf{R}^d$ starting from 0 and independent of B. For $x \in \mathbf{R}^d$, let*

$$X^x(t) = x\exp\{B(t)-t/2\} + \int_0^t \exp\{(B(t)-B(s))-(t-s)/2\}\,\mathrm{d}W(s) \tag{3.1}$$

then $\big(X^x(t); t \ge 0\big)$ is a diffusion process with generator $\mathcal{A}$ starting from x.

Proof. This follows by applying Itô's formula to X^x, which shows that

$$X^x(t) = x + W(t) + \int_0^t X^x(s)\,\mathrm{d}B(s).$$

This is an example of a linear stochastic differential equation; see Proposition 2.3 of Chapter IX of [**13**]. Since W and B are independent, the quadratic covariation of the ith and jth components of $X^x(t)$ is given by

$$\langle X_i^x(t), X_j^x(t)\rangle = \int_0^t \big(\delta_{ij} + X_i^x(s)X_j^x(s)\big)\,\mathrm{d}s,$$

and accordingly X^x is a diffusion process with generator $\mathcal{A}$. □

https://doi.org/10.1017/S0013091517000190 Published online by Cambridge University Press

It is easy to see from this lemma that the diffusion process is recurrent rather than transient. Indeed, we have for every $x \in \mathbf{R}^d$, as $t \to \infty$,

$$\begin{aligned} X^x(t) &= x \exp\{B(t) - t/2\} + \int_0^t \exp\{(B(t) - B(s)) - (t-s)/2\} \,\mathrm{d}W(s) \\ &\stackrel{\text{law}}{=} x \exp\{B(t) - t/2\} + \int_0^t \exp\{B(s) - s/2\} \,\mathrm{d}W(s) \\ &\stackrel{\text{a.s.}}{\to} \int_0^\infty \exp\{B(s) - s/2\} \,\mathrm{d}W(s), \end{aligned} \tag{3.2}$$

where the last stochastic integral is almost surely convergent because its quadratic variation is almost surely finite. The above convergence in distribution is plainly inconsistent with transience, and by the usual dichotomy between transience and recurrence [**12**] we deduce that our diffusion process is recurrent. Indeed, the fact that the right-hand side of (3.2) gives an invariant distribution follows directly from writing the decomposition

$$\begin{aligned} &\int_0^\infty \exp\{B(s) - s/2\} \,\mathrm{d}W(s) \\ &\quad = \exp\{B(t) - t/2\} \int_t^\infty \exp\{(B(u) - B(t)) - (u-t)/2\} \,\mathrm{d}W(s) \\ &\qquad + \int_0^t \exp\{B(s) - s/2\} \,\mathrm{d}W(s) \end{aligned} \tag{3.3}$$

and comparing with (3.1) with the integral $\int_t^\infty \exp\{(B(u) - B(t)) - (u-t)/2\} \,\mathrm{d}W(s)$ playing the part of a random starting point x.

The process X^x defined by (3.1) is an example of a generalized OU process. See [**2**] for a general discussion of these processes and in particular their invariant measures. The particular case of the generalized OU process constructed from two one-dimensional Brownian motions, which corresponds to (3.1) with $d = 1$, was studied in [**15**]. There is a close relationship between the generalized OU processes and exponential functionals of Lévy processes, in our case, exponential functionals of Brownian motion. These have been extensively studied, see the survey article [**10**]. In particular, we will have need of the following observations. The invariant measure given at (3.2) can be re-written in the form

$$\int_0^\infty \exp\{B(s) - s/2\} \,\mathrm{d}W(s) \stackrel{\text{law}}{=} W(A_\infty) \stackrel{\text{law}}{=} \sqrt{A_\infty} W(1), \tag{3.4}$$

where A_∞ denotes the exponential functional $\int_0^\infty \exp\{2B(s) - s\} \,\mathrm{d}s$. The first of these equalities in law is (a special case of) Knight's theorem on orthogonal martingales, see Theorem 1.9 of Chapter 5 of [**13**], and the second is the Brownian scaling property, noting that W and B are independent. The distribution of A_∞ is known to be a stable distribution of index 1/2, see [**3**], also Theorem 6.2 of [**10**]. Consequently we recognize (3.4) as the exit distribution for Brownian motion from a half space, given by the Poisson kernel. Thus the invariant measure has an explicit density

$$\rho(x)=\frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}\frac{1}{(1+|x|^2)^{(d+1)/2}}. \tag{3.5}$$

In fact, one may verify easily that $\mathcal{A}^*\rho=0$ where $\mathcal{A}^*$ is the formal adjoint with respect to the Lebesgue measure of $\mathcal{A}$. Moreover, with respect to the measure with density ρ, $\mathcal{A}$ is formally self-adjoint.

It follows from (3.5) or (3.4) that if $X(\infty)$ is a $\mathbf{R}^d$ valued random variable whose distribution is the invariant measure at (3.2), then,

$$\mathbf{E}[|X(\infty)|^p]<\infty \text{ for } p<1 \quad \text{and } \mathbf{E}[|X(\infty)|]=\infty. \tag{3.6}$$

Moreover, the convergence at (3.2) occurs in L^p for every $p<1$. On the other hand, the random variable $\int_0^t \exp\{B(s)-s/2\}\,\mathrm{d}s$ has finite first moment, and so for every finite time $t<\infty$ we have

$$\mathbf{E}[|X^x(t)|^2]<\infty. \tag{3.7}$$

4. Proof of Theorem 1

In order to prove Theorem 1.1, we must show that the solution u, given by the series (2.2), has the correct boundary behaviour. If g is a finite linear combination of spherical harmonic functions, then this follows immediately from the asymptotic behaviour of f_l. However, in general it is more difficult to verify the limit behaviour of u. The key tool we use is the following result, which plays the part of a maximum principle in our setting.

Lemma 4.1. *There exists a constant K such that for every $g\in C(S^{d-1})$ satisfying $\int_{S^{d-1}} g\,\mathrm{d}\theta=0$ the function u given by (2.2) and corresponding to g satisfies*

$$|u(x)|\le K(1+|x|)\sup_{\theta\in S^{d-1}}|g(\theta)| \quad \textit{for all } x\in\mathbf{R}^d.$$

Admitting this result, we can prove the convergence statement of Theorem 1.1 as follows. Fix an arbitrary $g\in C(S^{d-1})$. Finite linear combinations of spherical harmonics are dense in $C(S^{d-1})$ by the Stone–Weierstrass theorem, and hence given any $\epsilon>0$ we can find g_ϵ, a finite linear combination of spherical harmonics, satisfying $\int_{S^{d-1}} g_\epsilon\,\mathrm{d}\theta=\int_{S^{d-1}} g\,\mathrm{d}\theta$ and with

$$||g_\epsilon-g||_\infty\le\epsilon.$$

But then if u_ϵ is the solution to (1.1), which corresponds to g_ϵ given by a finite series of the form (2.2), as we have remarked already,

$$\lim_{r\to\infty}\sup_{\theta\in S^{d-1}}|u_\epsilon(r\theta)/r-g_\epsilon(\theta)|=0.$$

Now $u-u_\epsilon$ corresponds to $g-g_\epsilon$, which has mean 0, and applying the previous lemma to this we obtain

$$|u(x)-u_\epsilon(x)|\le K\epsilon(1+|x|),$$

https://doi.org/10.1017/S0013091517000190 Published online by Cambridge University Press

and hence

$$\limsup_{r\to\infty} \sup_{\theta\in S^{d-1}} |u(r\theta)/r - g(\theta)| \leq (K+1)\epsilon.$$

Since ϵ is arbitrary, this proves the desired uniform convergence.

Proof of Lemma 4.1. We begin by solving the equation $\mathcal{A}_R h = 0$. By elementary means we find that the general solution is a linear combination of a constant and the function

$$h(r) = \int_1^r \left(\frac{1+u^2}{u^2}\right)^{(d-1)/2} \mathrm{d}u. \tag{4.1}$$

Notice that $h(r)/r \to 1$ as $r \to \infty$. Now, for $R > |x| > r$, let

$$\tau_{r,R} = \inf\{t > 0 : X_t^x \notin (r, R)\}.$$

Taking expectations of the martingale $h(|X^x_{t\wedge\tau_{r,R}}|)$, we obtain,

$$\mathbf{P}(|X^x_{\tau_{r,R}}| = R) = \frac{h(|x|) - h(r)}{h(R) - h(r)}. \tag{4.2}$$

Now note that for each x, $u(x)$ varies continuously with $g \in C(S^{d-1})$. In fact, there exist constants K_R so that

$$\sup_{|x|\leq R} |u(x)| \leq K_R \sup_{\theta\in S^{d-1}} |g(\theta)| \tag{4.3}$$

as can be seen by estimating the terms in the series (2.2) using Lemma 2.1. Consequently, it is enough to prove the inequality for g belonging to the dense subset consisting of $g \in C(S^{d-1})$ formed of finite linear combinations of spherical harmonics with $\int_{S^{d-1}} g\,\mathrm{d}\theta = 0$. Fix such a g and let u be the corresponding solution of $\mathcal{A}u = 0$. Considering the martingale $u(X^x_{t\wedge\tau_{1,R}})$, where $1 < |x| < R$, we obtain

$$u(x) = \mathbf{E}[u(X^x_{\tau_{1,R}})],$$

whence, using (4.2),

$$|u(x)| \leq \sup_{|y|=1} |u(y)| + \frac{h(|x|)}{h(R)} \sup_{|y|=R} |u(y)|. \tag{4.4}$$

Recall that, as we have observed previously, since u is formed from a finite linear combination of spherical harmonics,

$$\lim_{r\to\infty} \sup_{\theta\in S^{d-1}} |u(r\theta)/r - g(\theta)| = 0.$$

Consequently, letting $R \to \infty$ in (4.6) we obtain,

$$|u(x)| \leq \sup_{|y|=1} |u(y)| + h(|x|) \sup_{\theta\in S^{d-1}} |g(\theta)|.$$

Now we apply the estimate (4.3) to the first of these terms, and we deduce that the statement of the lemma holds if K is chosen greater than both $\sup_{r\geq 1} h(r)/r$ and K_1. $\square$

https://doi.org/10.1017/S0013091517000190 Published online by Cambridge University Press

It remains to prove the uniqueness assertion of the theorem. This we can do adapting the argument just used in the proof of the lemma. Suppose that u_1 and u_2 are two solutions to $\mathcal{A}u = 0$ satisfying

$$\lim_{r\to\infty} \sup_{\theta\in S^{d-1}} |u_i(r\theta)/r - g(\theta)| = 0$$

for the same choice of g. Then $u = u_1 - u_2$ solves $\mathcal{A}u = 0$ with

$$\lim_{r\to\infty} \sup_{\theta\in S^{d-1}} |u(r\theta)/r| = 0. \tag{4.5}$$

Considering the martingale $u(X^x_{t\wedge\tau_{r,R}})$ we obtain

$$u(x) = \mathbf{E}[u(X^x_{\tau_{r,R}})],$$

whence, using (4.2),

$$|u(x)| \leq \sup_{|y|=r} |u(y)| + \frac{h(|x|) - h(r)}{h(R) - h(r)} \sup_{|y|=R} |u(y)|. \tag{4.6}$$

Now letting $R \to \infty$, holding r fixed, and using (4.5) gives

$$|u(x)| \leq \sup_{|y|=r} |u(y)|.$$

But then letting $r \downarrow 0$ and noting $u(0) = 0$, we deduce that u is identically zero.

5. Proof of Theorem 2

We now define the semigroup $(P_t; t \geq 0)$ via $P_t u(x) = \mathbf{E}[u(X^x(t))]$ whenever u is such that the random variable $u(X^x(t))$ is integrable for all $x \in \mathbf{R}^d$. Recall, in particular, (3.7) stating that $\mathbf{E}[|X^x(t)|^2] < \infty$.

Each random map $x \mapsto X^x(t)$ is affine and, consequently, if u is a convex function then the random function $x \mapsto u(X^x(t))$ is also convex with probability one. Taking expectations we have, for any $x, y \in \mathbf{R}^d$ and $\alpha \in [0, 1]$,

$$\begin{aligned} P_t u(\alpha x + (1-\alpha)y) &= \mathbf{E}[u(\alpha X^x(t) + (1-\alpha)X^y(t))] \\ &\leq \mathbf{E}[\alpha u(X^x(t)) + (1-\alpha)u(X^y(t))] = \alpha P_t u(x) + (1-\alpha)P_t u(y), \end{aligned}$$

and thus P_t preserves convexity. This will be a key ingredient in the proof of our second theorem. We note in passing that the semigroup of any generalized OU process is convexity preserving.

Our strategy for the proof of Theorem 2 is to study the behaviour of $P_t v$ as $t \to \infty$ where $v(x) = |x|g(x/|x|)$. To begin, first note that the probabilistic analogue of (2.1) is the skew-product decomposition for the diffusion process $(X^x(t); t \geq 0)$

$$X^x(t) = R^{(r)}(t)\Theta\left(\int_0^t \frac{\mathrm{d}s}{R^{(r)}(s)^2}\right) \tag{5.1}$$

where $R^{(r)}(t) = |X^x(t)|$ is a diffusion process on $(0, \infty)$ with generator $\mathcal{A}_R$ starting from $r = |x| \neq 0$, and $(\Theta(t); t \geq 0)$ an independent Brownian motion on the sphere S^{d-1} starting from $x/|x|$. An elegant argument for establishing this skew-product is to write $X^x(t)$

https://doi.org/10.1017/S0013091517000190 Published online by Cambridge University Press

as a time change

$$X^x(t) = \mathrm{e}^{B(t)-t/2}\hat{W}\left(\int_0^t \mathrm{e}^{-2B(s)+s}\,\mathrm{d}s\right) \tag{5.2}$$

of a d-dimensional Brownian motion $\hat{W}$ satisfying $\hat{W}(0) = x$, and then apply the usual skew-product decomposition of d-dimensional Brownian motion to $\hat{W}$,

$$\hat{W}(u) = |\hat{W}(u)|\Theta\left(\int_0^u \frac{\mathrm{d}v}{|\hat{W}(v)|^2}\right).$$

On making the time change $u = \int_0^t \mathrm{e}^{-2B(s)+s}\,\mathrm{d}s$, this yields a representation for the radial part of $X^x(t)$,

$$R^{(r)}(t) = \mathrm{e}^{B(t)-t/2}|\hat{W}|\left(\int_0^t \mathrm{e}^{-2B(s)+s}\,\mathrm{d}s\right),$$

and then noting that

$$\int_0^t \frac{\mathrm{d}s}{R^{(r)}(s)^2} = \int_0^u \frac{\mathrm{d}v}{|\hat{W}(v)|^2},$$

we obtain (5.1).

Equations (2.4) and (2.5) imply that the processes

$$f_l\big(R^{(r)}(t)\big)\exp\left(-\frac{l(l+d-2)}{2}\int_0^t \frac{\mathrm{d}s}{R^{(r)}(s)^2}\right) \tag{5.3}$$

for $l \geq 1$, and

$$f_0\big(R^{(r)}(t)\big) - \gamma_d t \tag{5.4}$$

are local martingales. In fact, they are true martingales because f_l' being bounded together with (3.7) implies that their quadratic variations are square integrable.

Now define $f_l(t, r)$ by,

$$f_l(t,r) = \mathbf{E}\left[R^{(r)}(t)\exp\left(-\frac{l(l+d-2)}{2}\int_0^t \frac{\mathrm{d}s}{R^{(r)}(s)^2}\right)\right]. \tag{5.5}$$

Lemma 5.1. *For $l \geq 1$ we have for all $r \geq 0$,*

$$\lim_{t\to\infty} f_l(t,r) = f_l(r).$$

Moreover, we have $f_l(r) \leq f_l(t,r) \leq r$ for all $t \geq 0$ and $l \geq 1$. The case $l = 0$ satisfies

$$\lim_{t\to\infty}\big(f_0(t,r) - \gamma_d t\big) = f_0(r) + \lambda_d$$

for all $r \geq 0$, where λ_d is a constant not depending on r.

Proof. Fix $l \geq 1$. Since $f_l(r)/r \to 1$ as $r \to \infty$, for any $\epsilon > 0$ there exists a K so that for all $r \geq 0$,

$$(1-\epsilon)f_l(r) - K \leq r \leq (1+\epsilon)f_l(r) + K.$$

Replacing r by $R^{(r)}(t)$, multiplying by

$$\exp\left(-\frac{l(l+d-2)}{2}\int_0^t \frac{\mathrm{d}s}{R^{(r)}(s)^2}\right)$$

and taking expectations, we deduce that

$$(1-\epsilon)f_l(r) - K\delta_l(t,r) \leq f_l(t,r) \leq (1+\epsilon)f_l(r) + K\delta_l(t,r), \tag{5.6}$$

where

$$\delta_l(t,r) = \mathbf{E}\left[\exp\left(-\frac{l(l+d-2)}{2}\int_0^t \frac{\mathrm{d}s}{R^{(r)}(s)^2}\right)\right].$$

Now the diffusion process $X^x(t)$ being recurrent implies that $\int_0^\infty (\mathrm{d}s/R^{(r)}(s)^2) = \infty$ with probability one, and hence $\delta_l(t,r) \to 0$ as $t \to \infty$. Thus, in (5.6), if we let $t \to \infty$ and then $\epsilon \downarrow 0$, we deduce that $\lim_{t\to\infty} f_l(t,r) = f_l(r)$ as desired.

For $l \geq 1$ applying Itô's formula to

$$R^{(r)}(t)\exp\left(-\frac{l(l+d-2)}{2}\int_0^t \frac{\mathrm{d}s}{R^{(r)}(s)^2}\right)$$

shows this process to be a supermartingale, and hence $f_l(t,r)$ is a decreasing function of t. This shows that $f_l(r) \leq f_l(t,r) \leq f_l(0,r) = r$.

Set $\hat{f}_0(r) = r - f_0(r)$. Using (2.6), it is easy to check that there exists constants A and B so that

$$|\hat{f}_0(r)| \leq A + B\log(1+r). \tag{5.7}$$

Now

$$\begin{aligned}
&\mathbf{E}\big[\hat{f}_0(R^{(r)}(t))\big] \\
&\quad = \mathbf{E}\big[\hat{f}_0(|X^x(t)|)\big] \\
&\quad = \mathbf{E}\left[\hat{f}_0\left(\left|x\exp\{B(t)-t/2\} + \int_0^t \exp\{(B(t)-B(s))-(t-s)/2\}\,\mathrm{d}W(s)\right|\right)\right] \qquad (5.8)\\
&\quad = \mathbf{E}\left[\hat{f}_0\left(\left|x\exp\{B(t)-t/2\} + \int_0^t \exp\{B(s)-s/2\}\,\mathrm{d}W(s)\right|\right)\right] \\
&\quad \to \mathbf{E}\left[\hat{f}_0\left(\left|\int_0^\infty \exp\{B(s)-s/2\}\,\mathrm{d}W(s)\right|\right)\right].
\end{aligned}$$

This convergence of expectations is justified by the uniform integrability of the random variables which follows from the bound (5.7) and the fact that the convergence at (3.2)

https://doi.org/10.1017/S0013091517000190 Published online by Cambridge University Press

occurs in L^p for any $0 < p < 1$. Now define the constant λ_d to be the value of the limit at (5.8), which does not depend on r. Then we have

$$\begin{aligned} f_0(t,r) &= \mathbf{E}\big[R^{(r)}(t)\big] = \mathbf{E}\big[f_0(R^{(r)}(t)) + \hat{f}_0(R^{(r)}(t))\big] \\ &= f_0(r) + \gamma_d t + \mathbf{E}\big[\hat{f}_0(R^{(r)}(t))\big] \to f_0(r) + \gamma_d t + \lambda_d. \end{aligned} \tag{5.9}$$

□

In the following lemma we establish the convergence of (a shift of) $P_t v$ to the solution u of the elliptic equation. We expect this convergence to be locally uniform, but it is enough for our purposes to prove it in a weaker L^2 sense.

Lemma 5.2. *Suppose that $g \in C(S^{d-1})$, and let $c = \gamma_d \int g(\theta)\,\mathrm{d}\theta$ and $b = \lambda_d \int g(\theta)\,\mathrm{d}\theta$. Let $v(x) = |x| g(x/|x|)$ for $x \in \mathbf{R}^d$ and let u be the solution of (1.1) corresponding to g. Then, as $t \to \infty$,*

$$\int_{S^{d-1}} (P_t v(r\theta) - u(r\theta) - ct - b)^2 \mathrm{d}\theta \to 0,$$

for every $r > 0$.

Proof. Letting g_l be the projection of g into the subspace of spherical harmonics of degree l as usual, we claim we can expand $P_t v$ as a series,

$$P_t v(r\theta) = \sum_{l=0}^{\infty} f_l(t,r) g_l(\theta), \tag{5.10}$$

with the series converging in $L^2(S^{d-1}(r))$ for each $r > 0$. This convergence is guaranteed by the inequality $0 \le f_l(t,r) \le r$.

To verify the claim that (5.10) is valid, first note that it holds for g that are a finite linear combination of spherical harmonics, by virtue of the skew product (5.1), the fact that g_l is an eigenfunction of the Laplacian on the sphere, and the definition (5.5) of $f_l(t,r)$. In more detail, suppose that $g = g_l$ for some l, then

$$\begin{aligned} \mathbf{E}\big[v(X^x(t))\big] &= \mathbf{E}\left[|X^x(t)| g_l\big(X^x(t)/|X^x(t)|\big)\right] = \mathbf{E}\left[R^{(r)}(t) g_l\left(\Theta\left(\int_0^t \frac{\mathrm{d}s}{R^{(r)}(s)^2}\right)\right)\right] \\ &= \mathbf{E}\left[R^{(r)}(t) \exp\left(-\frac{l(l+d-2)}{2}\int_0^t \frac{\mathrm{d}s}{R^{(r)}(s)^2}\right) g_l(\theta)\right] \\ &= f_l(t,r) g_l(\theta), \end{aligned}$$

where we use the independence of Θ and $R^{(r)}$ to compute the expectation in two steps.

Now consider, for a fixed $r > 0$ and $t > 0$, the applications

$$g \in C(S^{d-1}) \mapsto P_t v(r\cdot) \in L^2(S^{d-1}),$$

https://doi.org/10.1017/S0013091517000190 Published online by Cambridge University Press

and

$$g \in C(S^{d-1}) \mapsto \sum_{l=0}^{\infty} f_l(t,r) g_l(\cdot) \in L^2(S^{d-1}).$$

Both are continuous (equipping $C(S^{d-1})$ with the uniform norm) and they agree on the dense subspace of finite linear combinations of spherical harmonics. Thus (5.10) holds for any $g \in C(S^{d-1})$.

With the help of (5.10), we can now compute, noting that $g_0 = \int_{S^{d-1}} g(\theta)\,\mathrm{d}\theta$,

$$\int_{S^{d-1}} (P_t v(r\theta) - u(r\theta) - ct - b)^2 \,\mathrm{d}\theta$$
$$= (f_0(t,r)g_0 - f_0(r)g_0 - ct - b)^2 + \sum_{l=1}^{\infty} (f_l(t,r) - f_l(r))^2 ||g_l||^2_{S^{d-1}},$$

which tends to 0 as $t \to \infty$ using Lemma 5.1 and the dominated convergence theorem. □

Proof of Theorem 2. Recall that v being convex implies that $P_t v$ is also convex for every $t \geq 0$. Because L^2 convergence implies almost everywhere convergence along some subsequence, it follows from Lemma 5.2 that, for all but a null set of $x, y \in \mathbf{R}^d$ and $\alpha \in [0,1]$,

$$u(\alpha x + (1-\alpha)y) \leq \alpha u(x) + (1-\alpha)u(y).$$

But u is continuous, so this inequality extends to all $x, y \in \mathbf{R}^d$ and $\alpha \in [0,1]$.

To prove the converse implication, consider arbitrary $x, y \in \mathbf{R}^d \setminus \{0\}$ and $\alpha \in [0,1]$ with $\alpha x + (1-\alpha)y \neq 0$. Then u being convex implies that, for every $r > 0$,

$$\alpha u(rx) + (1-\alpha)u(ry) \geq u(\alpha rx + (1-\alpha)ry).$$

Dividing through by r, and then letting $r \to \infty$, we obtain from (1.2) that

$$\alpha|x|g(x/|x|) + (1-\alpha)|y|g(y/|y|) \geq |\alpha x + (1-\alpha)y|\, g\left(\frac{\alpha x + (1-\alpha)y}{|\alpha x + (1-\alpha)y|}\right)$$

which in view of the definition of v implies that it is convex. □

References

1. D. Buraczewski, E. Damek and T. Mikosch, *Stochastic models with power-law tails* (Springer, 2016).
2. P. Carmona, F. Petit and M. Yor, Exponential functionals of Lévy Processes, in *Lévy processes* (ed. O. E. Barndorff-Nielson, T. Mikosch and S. I. Resnick), pp. 41–55 (Birkhäuser, Boston, 2001).
3. D. Dufresne, The distribution of a perpetuity, with applications to risk theory and pension funding, *Scand. Actuar. J.* **1990**(1) (1990), 39–79.
4. K. Gawędzki and P. Horvai, Sticky behavior of fluid particles in the compressible Kraichnan model, *J. Stat. Phys.* **116**(5–6) (2004), 1247–1300.
5. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order* (Springer, 2001).

6. S. Janson and J. Tysk, Preservation of convexity of solutions to parabolic equations, *J. Differential Equations* **206** (2004), 182–226.
7. B. Kawohl, *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Mathematics, Volume 1150 (Springer, 1985).
8. N. J. Korevaar, Convexity properties of solutions to elliptic PDEs, in *Variational methods for free surface interfaces* (ed. P. Concus and R. Finn), pp. 115–121 (Springer, New York, 1987).
9. P. L. Lions and M. Musliela, Convexity of solutions to parabolic equations, *C. R. Acad. Sci. Paris, Ser.* I **342** (2006), 215–921.
10. H. Matsumoto and M. Yor, Exponential functionals of Brownian motion, I: Probability laws at fixed time, *Probab. Surv.* **2** (2005), 312–347.
11. C. Müller, *Spherical harmonics*, Lecture Notes in Mathematics, Volume 17 (Springer, 1966).
12. R. G. Pinsky, *Positive harmonic functions and diffusion* (Cambridge University Press, 1995)
13. D. Revuz and M. Yor, *Continuous martingales and Brownian motion* (Springer, 1999).
14. J. Warren, Sticky particles and stochastic flows, in *Memoriam Marc Yor – Séminaire de probabilités XLVII*, (ed. C. Donati-Martin, A. Lejay and A. Rouault) Lecture Notes in Mathematics, Volume 2137, pp. 17–35 (Springer, 2015).
15. M. Yor, Interpretations in terms of Brownian and Bessel meanders of the distribution of a subordinated perpetuity (ed. O. E. Barndorff-Nielson, T. Mikosch and S. I. Resnick), in *Lévy processes*, pp. 361–375 (Birkhäuser, Boston, 2001).