Nagoya Math. J., **246** (2022), 233–255
DOI 10.1017/nmj.2021.1

POWERS OF BINOMIAL EDGE IDEALS WITH QUADRATIC GRÖBNER BASES

VIVIANA ENE, GIANCARLO RINALDO AND NAOKI TERAI

Abstract. We study powers of binomial edge ideals associated with closed and block graphs.

§1. Introduction

Binomial edge ideals generalize in a natural way the determinantal ideals generated by the two-minors of a generic matrix of type $2 \times n$. They were independently introduced a decade ago in the papers [22] and [34]. Since then, they have been intensively studied and there exists a rich recent literature on this subject. Fundamental results on their Gröbner bases, primary decomposition, and their resolutions are presented in the monograph [23].

Binomial edge ideals with quadratic Gröbner basis are of particular interest since their initial ideals are monomial edge ideals associated with bipartite graphs. Therefore, the theory of monomial edge ideals can be employed in deriving information about binomial edge ideals.

While many questions regarding binomial edge ideals have been already answered, much less is known about their powers. In [28], first steps in studying the regularity of powers of binomial edge ideals have been done. By using quadratic sequences, the authors obtain bounds for the regularity of powers of binomial edge ideals which are almost complete intersection. For the same class of ideals, in the paper [29], the Rees rings are considered. Another direction of research was pursued in [13]. Here, it is shown that binomial edge ideals with quadratic Gröbner basis have the nice property that their ordinary and symbolic powers coincide.

Let G be a simple graph (i.e., an undirected graph with no multiple edges and no loops) on the vertex set $[n] = \{1, 2, \ldots, n\}$ and let $S = K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ be the polynomial ring in $2n$ variables over the field K. For $1 \leq i < j \leq n$, we set $f_{ij} = x_i y_j - x_j y_i$. The binomial edge ideal of the graph G is

$$J_G = (f_{ij} : i < j, \{i, j\} \text{ is an edge of } G).$$

We consider the polynomial ring S endowed with the lexicographic order induced by the natural order of the variables, namely $x_1 > x_2 > \cdots > x_n > y_1 > y_2 \cdots > y_n$. The Gröbner basis of J_G with respect to this order was computed in [22]. The graphs G with the property that J_G has a quadratic Gröbner basis were characterized in the same paper and they were called *closed*. Later on, it turned out that closed graphs coincide with the *proper interval graphs*, see [10], which have a history of about 50 years in combinatorics. In Section 2, we survey various combinatorial characterizations of closed graphs which are very useful in working with their associated binomial edge ideals. Closed graphs with Cohen–Macaulay

Received November 23, 2020. Revised February 13, 2021. Accepted February 18, 2021.
2020 Mathematics subject classification: 13D02, 05E40, 14M05.
Naoki Terai was supported by the JSPS Grant-in Aid for Scientific Research (C) 18K03244.

binomial edge ideals are classified in [14, Th. 3.1]. Roughly speaking, they are “chains” of cliques (i.e., complete graphs) with the property that every two consecutive cliques intersect in one vertex. For Cohen–Macaulay binomial edge ideals of closed graphs, we compute the depth function in Theorem 3.1 and Proposition 3.6. For this class of ideals, we show that

$$\operatorname{depth}\frac{S}{J_G^i}=\operatorname{depth}\frac{S}{\operatorname{in}_<(J_G^i)}$$

and this common value depends on the cardinality of the maximal cliques of G. The proof of Theorem 3.1 follows from several technical lemmas. The basic idea of the proof is the following. Starting with a closed graph G whose binomial edge ideal is Cohen–Macaulay, we consider a disconnected graph G' whose connected components are complete graphs of the same size as the maximal cliques of G. By using the techniques developed in [19] for computing the depth of powers of sums of ideals, we are able to calculate the depth of the powers of $J_{G'}$. Next, by using a regular sequence of linear forms, we can recover the powers of J_G from the powers of $J_{G'}$, and, finally we can compute the depth of the powers of J_G. Similar arguments are used to compute the depth for the powers of $\operatorname{in}_<(J_G)$.

In addition, Proposition 3.6 implies that the depth function of J_G and $\operatorname{in}_<(J_G)$ is nonincreasing. We expect the same behavior for every closed graph, not only for those whose binomial edge ideal is Cohen–Macaulay; see Question 6.1. However, we are able to show that for every closed graph G, the limit depth for J_G and $\operatorname{in}_<(J_G)$ coincide and we compute this value in Theorem 3.10. On the other hand, in Proposition 3.12, we show that the initial ideal $\operatorname{in}_<(J_G)$ has a nonincreasing depth function. An important step in deriving Theorem 3.10 is Proposition 3.9 where we prove that the Rees rings $\mathcal{R}(J_G)$ and $\mathcal{R}(\operatorname{in}_<(J_G))$ are Cohen–Macaulay. This reduces the proof of the equality

$$\lim_{k\to\infty}\operatorname{depth}\frac{S}{J_G^k}=\lim_{k\to\infty}\operatorname{depth}\frac{S}{(\operatorname{in}_<(J_G))^k}$$

by showing that J_G and $\operatorname{in}_<(J_G)$ have the same analytic spread. This is shown by using the Sagbi basis theory.

One of the problems that we have considered at the beginning of this project was to characterize the graphs G such that J_G^k is Cohen–Macaulay for (some) $k\geq 2$. We still do not have a complete solution for this problem which is probably very difficult in the largest generality, but we can solve it if we restrict to closed or connected block graphs; see Proposition 3.7 and Proposition 5.2. In the last part of Section 3, we show that binomial edge ideals of closed graphs have the strong persistence property, as their initial ideals do.

In Section 4, we compute the regularity of the powers of J_G, when G is closed. In Theorem 4.1, we prove that if G is connected, then, for every $k\geq 1$,

$$\operatorname{reg}\frac{S}{J_G^k}=\operatorname{reg}\frac{S}{\operatorname{in}_<(J_G^k)}=\ell+2(k-1),$$

where ℓ is the length of the longest induced path in G. The inequality $\operatorname{reg} S/J_G^k\geq \ell+2(k-1)$ follows from a result in [28]. For the rest of the proof, we combine various known facts about the regularity of the powers of edge ideals of bipartite graphs. The statement is extended to disconnected closed graphs in Proposition 4.2.

In Section 5, we consider block graphs. These are chordal graphs with the property that every two maximal cliques intersect in at most one vertex. For the block graphs whose

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

binomial edge ideal is Cohen–Macaulay, in Theorem 5.1, we show that the symbolic powers coincide with the ordinary ones if and only if the graph is closed. This theorem shows, in particular, that the equality between the symbolic and ordinary powers of binomial edge ideals does not hold for all chordal graphs. Finally, in Proposition 5.2, we show that for every connected block graph G which is not a path, J_G^k is not Cohen–Macaulay for $k \geq 2$.

In the last section of the paper, we discuss a few open questions. The most intriguing is related to a conjecture which appeared in [14] and which is still open. This conjecture states that for every closed graph G, we have $\beta_{ij}(J_G) = \beta_{ij}(\mathrm{in}_<(J_G))$. While doing some calculations with the computer, we observed an interesting phenomenon, namely that the graded Betti numbers are the same also for powers of J_G and $\mathrm{in}_<(J_G)$. Moreover, as we explain in Section 6, the equalities between the graded Betti numbers are true for complete and path graphs. Taking into account also our results on the regularity and depth of the powers of J_G and $\mathrm{in}_<(J_G)$, we were tempted to conjecture that for every closed graph G and every $k \geq 1$, we have $\beta_{ij}(J_G^k) = \beta_{ij}((\mathrm{in}_<(J_G))^k)$.

Another interesting question concerns block graphs. By computer calculation, we observed that the net graph N (Figure 2) which plays an important role in Theorem 5.1 has the property that $J_N^{(2)}$ is Cohen–Macaulay, but J_N^2 is not Cohen–Macaulay. It would be of interest to classify all the block graphs with the property that the second symbolic power of the associated binomial edge ideal is Cohen–Macaulay.

§2. Preliminaries

Let G be a graph[1] on the vertex set $V(G) = [n]$ and edge set $E(G)$. Let $S = K[x_1,\dots,x_n,y_1,\dots,y_n]$ be the polynomial ring in $2n$ variables over the field K. The binomial edge ideal J_G associated with G is generated by the binomials $f_{ij} = x_iy_j - x_jy_i \in S$ where $\{i,j\} \in E(G)$. In other words, J_G is generated by the maximal minors of the generic $2\times n$-matrix $X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$ whose column indices correspond to the edges of G. Two simple examples of binomial edge ideals are the ideal $I_2(X)$ generated by all the maximal minors of X which in our notation is denoted by J_{K_n} where K_n is the complete graph on the vertex set $[n]$, and the ideal generated by the adjacent minors of X which coincides with J_{P_n}, where P_n is the path graph with edges $\{i,i+1\}, 1 \leq i \leq n-1$.

We consider the polynomial ring S endowed with the lexicographic order induced by the natural order of the variables. Let $\mathrm{in}_<(J_G)$ be the initial ideal of J_G with respect to this monomial order. By [22, Cor. 2.2], J_G is a radical ideal. In the same paper, it was shown that the minimal prime ideals can be characterized in terms of the combinatorics of the graph G. In order to recall this characterization, we introduce the following notation. Let $W \subset [n]$ be a (possibly empty) subset of $[n]$, and let $G_1,\dots,G_{c(W)}$ be the connected components of $G_{[n]\setminus W}$ where $G_{[n]\setminus W}$ is the induced subgraph of G on the vertex set $[n]\setminus W$, and $c(W)$ denotes the number of connected components of $G_{[n]\setminus W}$. For $1 \leq i \leq c(W)$, let $\widetilde{G}_i$ be the complete graph on the vertex set $V(G_i)$. Let

$$P_W(G) = (\{x_i,y_i\}_{i\in W}) + J_{\widetilde{G}_1} + \cdots + J_{\widetilde{G}_{c(W)}}.$$

[1] In this paper, by a graph we always mean a simple graph, that is, an undirected graph with no multiple edges and no loops.

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

Then $P_W(G)$ is a prime ideal of height equal to $n - c(W) + |W|$ for every $W \subset [n]$, [22, Lem. 3.1].

By [22, Th. 3.2], $J_G = \bigcap_{W \subset [n]} P_W(G)$. In particular, the minimal primes of J_G are among the prime ideals $P_W(G)$ with $W \subset [n]$.

Proposition 2.1. [22, Cor. 3.9] *$P_W(G)$ is a minimal prime of J_G if and only if either $W = \emptyset$ or W is nonempty and for each $i \in W, c(W \setminus \{i\}) < c(W)$.*

In graph theoretical terms, $P_W(G)$ is a minimal prime ideal of J_G if and only if W is empty or W is nonempty and is a *cut-point set* of G, that is, i is a cut point of the induced subgraph $G_{([n]\setminus W)\cup\{i\}}$ for every $i \in W$. Let $\mathcal{C}(G)$ be the set of all sets $W \subset [n]$ such that $P_W(G) \in \mathrm{Min}(J_G)$, where $\mathrm{Min}(J_G)$ is the set of minimal prime ideals of J_G.

In particular, it follows that

$$\dim S/J_G = \max\{n + c(W) - |W| : W \in \mathcal{C}(G)\}. \tag{1}$$

For $W = \emptyset, c = c(\emptyset)$ is the number of connected components $G_1, \ldots, G_c$ of G. In addition, one can easily see that $P_\emptyset(G) = J_{\widetilde{G}_1} + \cdots + J_{\widetilde{G}_c}$ is a minimal prime of J_G. Therefore, if J_G is unmixed (which is the case, for instance, if J_G is Cohen–Macaulay), then all the minimal primes of J_G have dimension equal to $n + c$. In particular, if G is connected, then J_G is unmixed if and only if, for every minimal prime $P_W(G)$ of G, we have $n + c(W) - |W| = n + 1$, that is, $c(W) - |W| = 1$.

By [6, Th. 3.1] and [6, Cor. 2.12], we have

$$\mathrm{in}_<(J_G) = \bigcap_{W \in \mathcal{C}(G)} \mathrm{in}_< P_W(G). \tag{2}$$

In what follows, we are mainly interested in binomial edge ideals with quadratic Gröbner bases. We recall the following result from [22, Th. 1.1].

Theorem 2.2 [22]. *Let G be a graph on the vertex set $[n]$ with the edge set $E(G)$, and let $<$ be the lexicographic order on S induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. Then the following conditions are equivalent:*

(a) *The generators f_{ij} of J_G form a quadratic Gröbner basis.*
(b) *For all edges $\{i,j\}$ and $\{i,k\}$ with $j > i < k$ or $j < i > k$ one has $\{j,k\} \in E(G)$.*

According to [22], a graph G endowed with a labeling which satisfies condition (b) in the above theorem is called *closed with respect to the given labeling*. Therefore, the generators of J_G form a Gröbner basis with respect to the lexicographic order if and only if G is closed with respect to its given labeling. Moreover, a graph G is called *closed* if there exists a labeling of its vertices such that G is closed with respect to it. Later on, Crupi and Rinaldo proved in [10] that closed graphs coincide with the so-called proper interval graphs, a class of graphs with a rich history in combinatorics. However, in this paper, we will call them closed graphs. There are several characterizations of closed graphs. Before discussing them, let us recall some notions of graph theory. A graph is called *chordal* if it has no induced cycle of length greater than or equal to 4. A graph is called *claw-free* if it has no induced subgraph isomorphic to the one displayed in Figure 1. A *clique* of a graph G is a complete subgraph of G. The cliques of G form a simplicial complex $\Delta(G)$ which is called the *clique complex* of G.

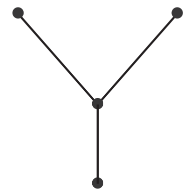
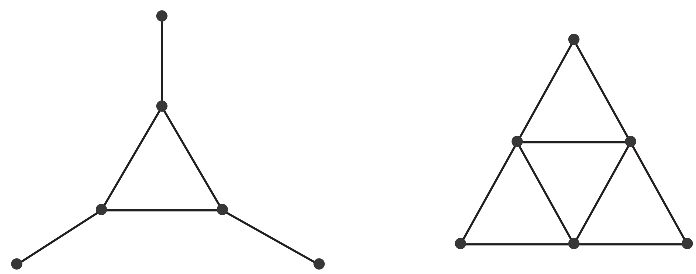


Figure 1.
Claw graph



N : Net graph $\qquad$ T : Tent graph

Figure 2.
Net and tent

The equivalences of the following theorem collects several results proved in [3], [9], [10], [14], [22].

THEOREM 2.3. *Let G be a graph on the vertex set $[n]$. The following statements are equivalent:*

(i) *G is a closed graph with respect to the given labeling, or equivalently, the generators of J_G form a Gröbner basis with respect to the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$;*
(ii) *for all $\{i,j\},\{k,\ell\} \in E(G)$ with $i < j$ and $k < \ell$, one has $\{j,\ell\} \in E(G)$ if $i = k, j \neq \ell$, and $\{i,k\} \in E(G)$ if $j = \ell, i \neq k$;*
(iii) *The facets, say $F_1,\ldots,F_r$, of the clique complex $\Delta(G)$ of G are intervals of the form $F_i = [a_i,b_i]$ which can be ordered such that $1 = a_1 < \cdots < a_r < b_r = n$;*
(iv) *for any $1 \leq i < j < k \leq n$, if $\{i,k\} \in E(G)$, then $\{i,j\},\{j,k\} \in E(G)$; and*
(v) *G is a chordal and claw-free graph which does not contain any subgraph isomorphic to the graphs displayed in Figure 2.*

The connected closed graphs with Cohen–Macaulay binomial edge ideals were characterized in [14].

THEOREM 2.4. [14, Th. 3.1] *Let G be a connected graph on $[n]$ which is closed with respect to the given labeling. Then the following conditions are equivalent:*

(a) *J_G is unmixed;*
(b) *J_G is Cohen–Macaulay;*
(c) $\mathrm{in}_<(J_G)$ *is Cohen–Macaulay;*
(d) *G satisfies the following condition: if $\{i,j+1\},\{j,k+1\} \in E(G)$ with $i < j < k$, then $\{i,k+1\} \in E(G)$; and*

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

(e) *there exist integers* $1 = a_1 < a_2 < \cdots < a_r < a_{r+1} = n$ *and a leaf order of the facets* $F_1, \ldots, F_r$ *of* $\Delta(G)$ *such that* $F_i = [a_i, a_{i+1}]$ *for all* $i = 1, \ldots, r$.

Let us remark that if G is closed and has the connected components $G_1, G_2, \ldots,, G_c$, then

$$\frac{S}{J_G} \cong \frac{S_1}{J_{G_1}} \otimes \frac{S_2}{J_{G_2}} \otimes \cdots \otimes \frac{S_c}{J_{G_c}},$$

where $S_i = K[\{x_j, y_j : j \in V(G_i)\}]$ for $1 \le i \le c$. Thus, J_G is Cohen–Macaulay if and only if each S_i/J_{G_i} is Cohen–Macaulay.

Let G be a closed graph. Then the generators of J_G form a reduced Gröbner basis with respect to the lexicographic order. This implies that $\text{in}_<(J_G) = (x_i y_j : i < j, \{i,j\} \in E(G))$. Thus, $\text{in}_<(J_G)$ is the monomial edge ideal of a bipartite graph, let us call it H, on the vertex set $\{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\}$ whose edges are $\{x_i, y_j\}$ where $\{i,j\} \in E(G)$. Since H is bipartite, it follows that the edge ideal $I(H) = \text{in}_<(J_G)$ has the property that its ordinary powers coincide with the symbolic ones [36, Th. 5.9]. Combining (2) with the proof of [13, Lem. 3.1], it follows that if G is closed, then

$$\text{in}_<(J_G^i) = (\text{in}_<(J_G))^i, \text{ for every } i \ge 1. \tag{3}$$

In other words, if G is closed, then the generators of J_G^i form a Gröbner basis of J_G^i for $i \ge 1$. Moreover, with the same assumption on the graph G, by [13, Cor. 3.4, Prop. 2.5], we have

$$J_G^i = J_G^{(i)} \text{ for every } i \ge 1, \tag{4}$$

where $J_G^{(i)}$ denotes the ith symbolic power of J_G. In other words, for a closed graph G, the symbolic powers of the binomial edge ideal J_G coincide with the ordinary powers. We recall the notion of symbolic power. Let $I \subset R$ be an ideal in a Noetherian ring R, and let $\text{Min}(I)$ be the set of the minimal prime ideals of I. For an integer $k \ge 1$, one defines the *kth symbolic power* of I as follows:

$$I^{(k)} = \bigcap_{\mathfrak{p} \in \text{Min}(I)} (I^k R_{\mathfrak{p}} \cap R).$$

By the definition of the symbolic power, we have $I^k \subseteq I^{(k)}$ for $k \ge 1$. Symbolic powers do not, in general, coincide with the ordinary powers.

§3. Depth of powers

The first main result of this section is the following.

THEOREM 3.1. *Let G be a connected closed graph on the vertex set $[n]$ such that J_G is Cohen–Macaulay. Let $F_1, F_2, \ldots, F_r$ be the maximal cliques of G and $d_i = \dim F_i = \#F_i - 1$ for $1 \le i \le r$. Assume that $d_1 \ge d_2 \ge \cdots \ge d_r \ge 1$.*[2] *Then the following equalities hold:*

(a) $\text{depth}\,\frac{S}{J_G^i} = \text{depth}\,\frac{S}{\text{in}_<(J_G^i)} = n - \sum_{j=1}^{i-1} d_j + i$ *for* $1 \le i \le r$,

(b) $\text{depth}\,\frac{S}{J_G^i} = \text{depth}\,\frac{S}{\text{in}_<(J_G^i)} = r + 2$ *for* $i \ge r + 1$.

For the proof of this theorem, we need a few lemmas. The proof of the first preparatory lemma is a straightforward extension of the proof of [26, Th. 4.4].

[2] Note that this is not necessarily the order of the facets of $\Delta(G)$ from Theorem 2.3.

LEMMA 3.2. *Let G be a complete graph on the vertex set $[n]$ and J_G its binomial edge ideal. Then $y_{n-2}-x_{n-1}, y_{n-1}-x_n, y_n$ is a maximal regular sequence on S/J_G^i for all $i \geq 2$. In particular,* $\operatorname{depth} S/J_G^i = 3$ *for $i \geq 2$.*

Proof. Let $R = S/(y_{n-2}-x_{n-1}, y_{n-1}-x_n, y_n) = K[x_1,\ldots,x_{n-2},y_1,\ldots,y_{n-1}]$. The image of J_G in R is the ideal J' generated by all the 2–minors of the matrix

$$X' = \begin{pmatrix} x_1 & \ldots & x_{n-2} & y_{n-2} & y_{n-1} \\ y_1 & \ldots & y_{n-2} & y_{n-1} & 0 \end{pmatrix}.$$

In order to prove our claim, it is enough to show that $\mathfrak{m}$, that is, the maximal ideal of R is associated to $(J')^i$ for $i \geq 2$. If we show that $((J')^i : y_{n-1}^{2i-1})$ is $\mathfrak{m}$–primary, then $\mathfrak{m} \in \operatorname{Ass}(R/(J')^i)$, which implies that $\operatorname{depth}(R/(J')^i) = 0$ and the claim follows.

Set $y = y_{n-1}$. Then $y^{2i-1} \notin (J')^i$ since $(J')^i$ is generated in degree $2i$. But $y \cdot y^{2i-1} = (y^2)^i \in (J')^i$ since $y^2 \in J'$. Therefore, $y \in (J')^i : y^{2i-1}$. For $1 \leq j \leq n-2$, we have $y_j y \in J'$. Then $y_j^{2i-1} y^{2i-1} \in (J')^{2i-1} \subseteq (J')^i$, thus $y_j^{2i-1} \in (J')^i : y^{2i-1}$ for $1 \leq j \leq n-2$. Finally, since $x_j y - y_j y_{n-2} \in J'$ for $1 \leq j \leq n-2$, we get that for $i \geq 2, y^{2i-2}(x_j y - y_j y_{n-2}) \in (J')^{i-1} \cdot J' = (J')^i$, since $y^{2i-2} = (y^2)^{i-1} \in (J')^{i-1}$. On the other hand, $y_j y_{n-2} y^{2i-2} = (y_j y)(y_{n-2} y)(y^2)^{i-2} \in (J')^i$. It follows that $x_j y^{2i-1} \in (J')^i$, which implies that $x_j \in (J')^i : y^{2i-1}$ for $1 \leq j \leq n-2$. ∎

LEMMA 3.3. *Under the same assumption of Lemma 3.2, we have*

$$\operatorname{depth} \frac{S}{\operatorname{in}_<(J_G^i)} = 3,$$

for all $i \geq 2$.

Proof. By (3), we have $\operatorname{in}_<(J_G^i) = (\operatorname{in}_<(J_G))^i$ for $i \geq 1$. Since y_1 and x_n form a regular sequence on $\operatorname{in}_<(J_G)$ and using Lemma 3.2, we get the following relations:

$$\operatorname{depth} \frac{\overline{S}}{(\operatorname{in}_<(J_G))^i} + 2 = \operatorname{depth} \frac{S}{(\operatorname{in}_<(J_G))^i} = \operatorname{depth} \frac{S}{\operatorname{in}_<(J_G^i)} \leq \operatorname{depth} \frac{S}{J_G^i} = 3,$$

where $\overline{S} = K[\{x_i, y_j : 1 \leq i \leq n-1, 2 \leq j \leq n\}]$. By [37, Th. 4.4], $\operatorname{depth} \overline{S}/(\operatorname{in}_<(J_G))^i \geq 1$ for $i \geq 1$. Therefore, we get the desired equality. ∎

In the following lemma, we use the following notation. If H is a graph on some vertex set $V(H)$, then we denote by $S(H)$ the polynomial ring over K in the variables x_k, y_k, where $k \in V(H)$.

LEMMA 3.4. *Let G be a closed graph on the vertex set $[n]$ with maximal cliques $[a_1,a_2],[a_2,a_3],\ldots,[a_r,a_{r+1}]$ where $1 = a_1 < a_2 < \cdots < a_r < a_{r+1} = n$. Let G' be the graph whose connected components are the mutually disjoint cliques*

$$[a_1,a_2],[a_2+1,a_3+1],\ldots,[a_r+(r-1),a_{r+1}+(r-1)].$$

Then the following hold:

(a) *The sequence of linear forms*

$$\underline{\ell} : \ell_1^y = y_{a_2} - y_{a_2+1}, \ell_1^x = x_{a_2} - x_{a_2+1}, \ell_2^y = y_{a_3+1} - y_{a_3+2}, \ell_2^x = x_{a_3+1} - x_{a_3+2},$$
$$\ldots, \ell_{r-1}^y = y_{a_r+(r-2)} - y_{a_r+(r-1)}, \ell_{r-1}^x = x_{a_r+(r-2)} - x_{a_r+(r-1)}$$

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

is regular on $S(G')/J^j_{G'}$ *and*

$$\frac{\frac{S(G')}{J^j_{G'}}}{(\underline{\ell})\frac{S(G')}{J^j_{G'}}} \cong \frac{S}{J^j_G},$$

for every $j \geq 1$.

(b) *The sequence of variables* $\underline{\mu}: x_{a_2}, y_{a_2+1}, x_{a_3+1}, \ldots, y_{a_r+(r-1)}$ *is regular on* $S(G')/\mathrm{in}_<(J^j_{G'})$ *and*

$$\frac{\frac{S(G')}{\mathrm{in}_<(J^j_{G'})}}{(\underline{\mu})\frac{S(G')}{\mathrm{in}_<(J^j_{G'})}} \cong \frac{S}{\mathrm{in}_<(J^j_G)},$$

for every $j \geq 1$.

Proof. (a) Let $j \geq 1$ be an integer. We prove by induction on $2 \leq i \leq r$ that the sequence $\underline{\ell}_{i-1}:\ \ell^y_1, \ell^x_1, \ldots, \ell^y_{i-1}, \ell^x_{i-1}$ is regular on $S(G')/J^j_{G'}$ and

$$\frac{\frac{S(G')}{J^j_{G'}}}{(\underline{\ell}_{i-1})\frac{S(G')}{J^j_{G'}}} \cong \frac{S(\widetilde{G}_{i-1})}{J^j_{\widetilde{G}_{i-1}}}.$$

where, after relabeling the vertices, $\widetilde{G}_{i-1}$ is a closed graph with maximal cliques

$$[a_1, a_2], [a_2, a_3], \ldots, [a_i, a_{i+1}], [a_{i+1}+1, a_{i+2}+1], \ldots, [a_r+(r-i), a_{r+1}+(r-i)].$$

Let us first check the claim for $i = 2$. We have to show that ℓ^y_1, ℓ^x_1 is regular on $S(G')/J^j_{G'}$. Note that $J_{G'}$ is a prime ideal since it is the sum of r prime ideals in pairwise disjoint sets of variables corresponding to the r connected components of G'; see [23, Lem. 7.14].

Let $h \in S(G')$ such that $\ell^y_1 h \in J^j_{G'}$. Since $\ell^y_1 \notin J_{G'}$, because J'_G is generated in degree 2, it follows that $h \in J^{(j)}_{G'} = J^j_{G'}$, thus ℓ^y_1 is regular on $S(G')/J^j_{G'}$. Now we show that ℓ^x_1 is regular on $S(G')/(J^j_{G'} + (\ell^y_1))$. We have

$$\frac{S(G')}{J^j_{G'} + (\ell^y_1)} \cong \frac{S(G')}{\overline{J} + (\ell^y_1)},$$

where $\overline{J}$ is the ideal in $S(G')$ generated by the polynomials $\overline{g}_1, \ldots, \overline{g}_m$ obtained from the generators $g_1, \ldots, g_m$ of $J^j_{G'}$ as follows. If g_k is a generator which contains the variable y_{a_2+1}, we replace it by y_{a_2} and denote the new binomial by $\overline{g}_k$. If g_k contains the variable y_{a_2}, we replace it by y_{a_2+1}, and denote the new binomial by $\overline{g}_k$. If a generator g_k of $J^j_{G'}$ contains both variables y_{a_2} and y_{a_2+1}, then we exchange these variables and denote the new binomial by $\overline{g}_k$. Finally, if g_k is a generator of $J^j_{G'}$ which does not contain any of the variables y_{a_2}, y_{a_2+1}, we simply set $\overline{g}_k = g_k$. Then $\overline{g}_1, \ldots, \overline{g}_m$ are the generators of the jth power of the binomial edge ideal associated with the graph G' and the matrix

$$X' = \begin{pmatrix} x_1 & \cdots & x_{a_2-1} & x_{a_2} & x_{a_2+1} & x_{a_2+2} & \cdots & x_{a_{r+1}+r-1} \\ y_1 & \cdots & y_{a_2-1} & y_{a_2+1} & y_{a_2} & y_{a_2+2} & \cdots & y_{a_{r+1}+r-1} \end{pmatrix}.$$

Since G' consists of r complete graphs, by (3) it follows that $\mathrm{in}_<(\overline{J})$ is generated by the monomials $\mathrm{in}_<(\overline{g}_1), \ldots, \mathrm{in}_<(\overline{g}_m)$ where $<$ is the lexicographic order on $S(G')$. Note that $\mathrm{in}_<(\overline{g}_k)$ differs from $\mathrm{in}_<(g_k)$ if and only if $y_{a_2} | \mathrm{in}_<(g_k)$ and, in this case, $\mathrm{in}_<(\overline{g}_k)$ is obtained

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

from $\mathrm{in}_<(g_k)$ by replacing the variable y_{a_2} with y_{a_2+1}. Then it follows that none of the generators of the initial ideal of $\overline{J}+(\ell_1^y)$ is divisible by x_{a_2} since $\{a_2,a_2+1\}$ is not an edge in G'. Therefore, x_{a_2} is regular on $\mathrm{in}_<(\overline{J}+(\ell_1^y))$ and further, $x_{a_2}-x_{a_2+1}$ is regular on $S(G')/(J^j_{G'}+(\ell_1^y))$. Moreover, we get

$$\frac{S(G')}{J^j_{G'}+(\ell_1^y,\ell_1^x)}\cong\frac{S(\widetilde{G}_1)}{J^j_{\widetilde{G}_1}},$$

where $\widetilde{G}_1$ is obtained from G' by identifying the vertices a_2 and a_2+1 and by relabeling the vertices k with $k-1$ for $k\geq a_2+2$. Thus, $\widetilde{G}_1$ has the maximal cliques

$$[a_1,a_2],[a_2,a_3],[a_3+1,a_4+1],\ldots,[a_r+(r-2),a_{r+1}+(r-2)].$$

In particular, $\widetilde{G}_1$ is a closed graph which has $r-1$ connected components.

Assume that the sequence $\underline{\ell}_{i-1}:\ \ell_1^y,\ell_1^x,\ldots,\ell_{i-1}^y,\ell_{i-1}^x$ is a regular sequence on $S(G')/J^j_{G'}$ and

$$\frac{\frac{S(G')}{J^j_{G'}}}{(\underline{\ell}_{i-1})\frac{S(G')}{J^j_{G'}}}\cong\frac{S(\widetilde{G}_{i-1})}{J^j_{\widetilde{G}_{i-1}}},$$

where the graph $\widetilde{G}_{i-1}$ has the first connected component consisting of the maximal cliques $[a_1,a_2],[a_2,a_3],\ldots,[a_i,a_{i+1}]$ and the other connected components are pairwise disjoint cliques. We have to show that ℓ_i^y,ℓ_i^x is a regular sequence on $S(\widetilde{G}_{i-1})/J^j_{\widetilde{G}_{i-1}}$. In the closed graph $\widetilde{G}_{i-1}$, the vertices a_{i+1} and $a_{i+1}+1$ are free, thus ℓ_i^y does not belong to any minimal prime ideal of $\widetilde{G}_{i-1}$ by [35, Prop. 2.1] or [23, Prop. 7.22]. This implies that ℓ_i^y is regular on $S(\widetilde{G}_{i-1})/J^j_{\widetilde{G}_{i-1}}$ since $J^j_{\widetilde{G}_{i-1}}$ has no embedded component by (4). It remains to show that ℓ_i^x is regular on $S(\widetilde{G}_{i-1})/(J^j_{\widetilde{G}_{i-1}}+(\ell_i^y))$. The argument is very similar to the induction basis. We observe that $J^j_{\widetilde{G}_{i-1}}+(\ell_i^y)=\overline{J}+(\ell_i^y)$ where $\overline{J}$ is obtained as follows. Let $g_1,\ldots,g_m$ be the generators of $J^j_{\widetilde{G}_{i-1}}$ and denote by $\overline{g}_1,\ldots,\overline{g}_m$ the polynomials obtained in the following way. If g_k contains the variable $y_{a_{i+1}}$, (respectively $y_{a_{i+1}+1}$) we replace it by $y_{a_{i+1}+1}$ (respectively by $y_{a_{i+1}}$), and define $\overline{g}_k$ to be this new binomial. If g_k contains both variables $y_{a_{i+1}}$ and $y_{a_{i+1}+1}$, then we exchange these variables and denote the new binomial by $\overline{g}_k$. Finally, if g_k does not contain any of the variables $y_{a_{i+1}},y_{a_{i+1}+1}$, we simply define $\overline{g}_k=g_k$. Then $\overline{J}=(\overline{g}_1,\ldots,\overline{g}_m)$ is the jth power of the binomial edge ideal corresponding to the closed graph $\widetilde{G}_{i-1}$ and the matrix

$$X'=\begin{pmatrix} x_1 & \cdots & x_{a_{i+1}-1} & x_{a_{i+1}} & x_{a_{i+1}+1} & x_{a_{i+1}+2} & \cdots & x_{a_{r+1}+r-i}\\ y_1 & \cdots & y_{a_{i+1}-1} & y_{a_{i+1}+1} & y_{a_{i+1}} & y_{a_{i+1}+2} & \cdots & y_{a_{r+1}+r-i}\end{pmatrix}.$$

It follows that the initial ideal of $\overline{J}$ is minimally generated by the monomial generators of $\mathrm{in}_<(J^j_{\widetilde{G}_{i-1}})$ in which we replaced the variable $y_{a_{i+1}}$ with $y_{a_{i+1}+1}$. Hence $\overline{g}_1,\ldots,\overline{g}_m,\ell_i^y$ is a Gröbner basis of $\overline{J}+(\ell_i^y)$. This implies that all the monomial minimal generators of $\mathrm{in}_<(\overline{J}+(\ell_i^y))$ are not divisible by $x_{a_{i+1}}$. Therefore, $x_{a_{i+1}}$ is regular on $\mathrm{in}_<(\overline{J}+(\ell_1^y))$ and,

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

consequently, ℓ_i^x is regular on $S/(\overline{J}+(\ell_i^y))$. Moreover, we get the following isomorphism:

$$\frac{S(\widetilde{G}_{i-1})}{J^j_{\widetilde{G}_{i-1}}+(\ell_i^x,\ell_i^y)} \cong \frac{S(\widetilde{G}_i)}{J^j_{\widetilde{G}_i}},$$

where $\widetilde{G}_i$ is a closed graph which is obtained from $\widetilde{G}_{i-1}$ by identifying the vertex $a_{i+1}+1$ with a_{i+1} and by relabeling the vertex k with $k-1$ for $k \geq a_{i+1}+2$. Thus, the new graph $\widetilde{G}_i$ has the maximal cliques

$$[a_1,a_2],\ldots,[a_i,a_{i+1}],[a_{i+1},a_{i+2}],[a_{i+2}+1,a_{i+3}+1],\ldots,[a_r+(r-i-1),a_{r+1}+(r-i-1)].$$

(b) Since the variables from $\underline{\mu}$ do not appear in the support of the minimal generators of $\mathrm{in}_<(J_{G'})$, it obviously follows that $\underline{\mu}$ is a regular sequence on $S(G')/(\mathrm{in}_<(J_{G'}))^j = S(G')/\mathrm{in}_<(J^j_{G'})$ and the desired conclusion follows. ∎

LEMMA 3.5. *Let G' be the graph with the connected components $H_1,H_2,\ldots,H_r$, where each H_i is a complete graph with $d_i+1 \geq 2$ vertices. Assume that $d_1 \geq d_2 \geq \cdots \geq d_r \geq 1$. Let $J_{G'}$ be the binomial edge ideal of G' in the polynomial ring $S' = K[\{x_i,y_i : i \in V(G')\}]$. Then:*

(a) $\operatorname{depth} \frac{S'}{J^i_{G'}} = \operatorname{depth} \frac{S'}{\mathrm{in}_<(J^i_{G'})} = d_i+d_{i+1}+\cdots+d_r+2r+i-1,$ *for* $1 \leq i \leq r$ *and*
(b) $\operatorname{depth} \frac{S'}{J^i_{G'}} = \operatorname{depth} \frac{S'}{\mathrm{in}_<(J^i_{G'})} = 3r,$ *for* $i \geq r+1$.

Proof. We proceed by induction on i. To simplify the notation, we set $J_k = J_{H_k}$ for $1 \leq k \leq r$. For $i=1$, we have

$$\operatorname{depth} \frac{S'}{J_{G'}} = \operatorname{depth} \frac{S_1}{J_1} + \cdots + \operatorname{depth} \frac{S_r}{J_r}$$

and

$$\operatorname{depth} \frac{S'}{\mathrm{in}_<(J_{G'})} = \operatorname{depth} \frac{S_1}{\mathrm{in}_<(J_1)} + \cdots + \operatorname{depth} \frac{S_r}{\mathrm{in}_<(J_r)},$$

where $S_k = K[\{x_j,y_j : j \in V(H_k)\}]$ for $1 \leq k \leq r$.

Since J_k and $\mathrm{in}_<(J_k)$ are Cohen–Macaulay for all k and $\operatorname{depth} S_k/\mathrm{in}_<(J_k) = d_k+2$, we get

$$\operatorname{depth} \frac{S'}{J_{G'}} = \operatorname{depth} \frac{S'}{\mathrm{in}_<(J_{G'})} = (d_1+2)+(d_2+2)+\cdots+(d_r+2) = d_1+d_2+\cdots+d_r+2r.$$

The inductive step follows from the same argument for $\operatorname{depth} S'/J^i_{G'}$ and for $\operatorname{depth} S'/\mathrm{in}_<(J^i_{G'})$. We will explain in detail the proof for $\operatorname{depth} S'/J^i_{G'}$ and, in the final part we will point out the difference in the proof for $\operatorname{depth} S'/\mathrm{in}_<(J^i_{G'})$.

Let us assume that

$$\operatorname{depth} \frac{S'}{J^i_{G'}} = d_i+d_{i+1}+\cdots+d_r+2r+i-1$$

and

$$\operatorname{depth} \frac{S'}{\mathrm{in}_<(J^i_{G'})} = \operatorname{depth} \frac{S'}{\mathrm{in}_<(J_{G'})^i} = d_i+d_{i+1}+\cdots+d_r+2r+i-1$$

for $i \leq r-1$.

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

By [19, Th. 3.3], we have

$$(5)\qquad \operatorname{depth}\frac{J_{G'}^i}{J_{G'}^{i+1}}=\min_{j_1+j_2+\cdots+j_r=i}\left\{\operatorname{depth}\frac{J_1^{j_1}}{J_1^{j_1+1}}+\operatorname{depth}\frac{J_2^{j_2}}{J_2^{j_2+1}}+\cdots+\operatorname{depth}\frac{J_r^{j_r}}{J_r^{j_r+1}}\right\}.$$

We know that depth $\frac{S_i}{J_i}=d_i+2\geq 3$ since J_i is Cohen–Macaulay, and by Lemma 3.2 and the Depth Lemma, depth $\frac{J_i}{J_i^2}=3$ and depth $\frac{J_i^j}{J_i^{j+1}}\geq 3$ for $j\geq 2$.

If $i\leq r-1$, in the equality $j_1+j_2+\cdots+j_r=i$, at most i exponents among $j_1,j_2,\dots,j_r$ are not 0. Since $d_1\geq d_2\geq\cdots\geq d_r$, we get

$$\sum_{s=1}^{r}\operatorname{depth}\frac{(J_i)^{j_s}}{(J_i)^{j_s+1}}\geq 3i+(d_{i+1}+2)+\cdots+(d_r+2)=d_{i+1}+\cdots+d_r+2r+i.$$

Moreover, the minimal value $d_{i+1}+\cdots+d_r+2r+i$ is achieved for $j_1=\cdots=j_i=1$ and $j_{i+1}=\cdots=j_r=0$. Hence, equality (5) implies that

$$\operatorname{depth}\frac{J_{G'}^i}{J_{G'}^{i+1}}=d_{i+1}+\cdots+d_r+2r+i.$$

We have the exact sequence of S'-modules:

$$0\to\frac{J_{G'}^i}{J_{G'}^{i+1}}\to\frac{S'}{J_{G'}^{i+1}}\to\frac{S'}{J_{G'}^i}\to 0.$$

By the inductive hypothesis, since $i\leq r-1$, we have depth $\frac{S'}{J_{G'}^i}=d_i+d_{i+1}+\cdots+d_r+2r+(i-1)$. As $d_{i+1}+\cdots+d_r+2r+i\leq d_i+d_{i+1}+\cdots+d_r+2r+(i-1)$, by the Depth Lemma applied to the above exact sequence, it follows that depth $\frac{S'}{J_{G'}^{i+1}}=d_{i+1}+\cdots+d_r+2r+i$. Therefore, we proved part (a) of the statement. In particular, for $i=r$, we have depth $\frac{S'}{J_{G'}^r}=d_r+3r-1$. To prove part (b), we apply again induction on $i\geq r+1$. We have the exact sequence of S'–modules:

$$0\to\frac{J_{G'}^r}{J_{G'}^{r+1}}\to\frac{S'}{J_{G'}^{r+1}}\to\frac{S'}{J_{G'}^r}\to 0.$$

In equality (5), if we consider $j_1+j_2+\cdots+j_r=r$, we derive that

$$\sum_{s=1}^{r}\operatorname{depth}\frac{(J_i)^{j_s}}{(J_i)^{j_s+1}}\geq 3r$$

and the minimal value $3r$ is achieved for $j_1=j_2=\cdots=j_r=1$. Thus, depth $\frac{J_{G'}^r}{J_{G'}^{r+1}}=3r$. Since $3r\leq d_r+3r-1$, the Depth Lemma applied to the above exact sequence yields depth $\frac{S'}{J_{G'}^{r+1}}=3r$. For the inductive step, we consider the exact sequence

$$0\to\frac{J_{G'}^i}{J_{G'}^{i+1}}\to\frac{S'}{J_{G'}^{i+1}}\to\frac{S'}{J_{G'}^i}\to 0$$

for $i\geq r+1$. By hypothesis we have depth $\frac{S'}{J_{G'}^i}=3r$, and we know from equality (5) that depth $\frac{J_{G'}^i}{J_{G'}^{i+1}}=3r$. Then, by the Depth Lemma, we obtain depth $\frac{S'}{J_{G'}^{i+1}}=3r$.

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

As we have already mentioned, the inductive step for depth $S'/\mathrm{in}_<(J^i_{G'})$ works in the same way. The only difference is that we need to apply Lemma 3.3 in order to derive that depth $\mathrm{in}_<(J_i)/(\mathrm{in}_<(J_i))^2 = 3$ and $\mathrm{depth}(\mathrm{in}_<(J_i))^j/(\mathrm{in}_<(J_i))^{j+1} \geq 3$ for $j \geq 2$. ∎

Proof of Theorem 3.1. To begin with, we prove the formulas for the depth of S/J^i_G. Let $[a_1,a_2],[a_2,a_3],\ldots,[a_r,a_{r+1}]$ be the maximal cliques of G, where $1 = a_1 < a_2 < \cdots < a_r < a_{r+1} = n$. Note that this is not necessarily the order with respect to the dimensions of the cliques. Let G' be the graph on $[n+r-1]$ with the connected components $[a_1,a_2],[a_2+1,a_3+1],\ldots,[a_r+(r-1),a_{r+1}+(r-1)]$ and $J_{G'} \subset S' = K[\{x_j,y_j : j \in V(G')\}]$ the associated binomial edge ideal. By Lemma 3.5, we have

$$\mathrm{depth}\,\frac{S'}{J^i_{G'}} = \mathrm{depth}\,\frac{S'}{\mathrm{in}_<(J^i_{G'})} = \begin{cases} d_i + d_{i+1} + \cdots + d_r + 2r + (i-1), & \text{for } 1 \leq i \leq r, \\ 3r, & \text{for } i \geq r+1. \end{cases}$$

By Lemma 3.4, the sequence of $2(r-1)$ linear forms

$$\underline{\ell} : \ell^y_1 = y_{a_2} - y_{a_2+1}, \ell^x_1 = x_{a_2} - x_{a_2+1}, \ell^y_2 = y_{a_3+1} - y_{a_3+2}, \ell^x_2 = x_{a_3+1} - x_{a_3+2},$$

$$\ldots, \ell^y_{r-1} = y_{a_r+(r-2)} - y_{a_r+(r-1)}, \ell^x_{r-1} = x_{a_r+(r-2)} - x_{a_r+(r-1)}$$

is regular on $S'/J^i_{G'}$ and $S'/(J^i_{G'} + (\underline{\ell})) \cong S/J^i_G$ for all $i \geq 1$. In addition, the sequence of $2(r-1)$ elements

$$\underline{\mu} : x_{a_2}, y_{a_2+1}, x_{a_3+1}, \ldots, y_{a_r+(r-1)}$$

is regular on $S(G')/\mathrm{in}_<(J^j_{G'})$ and

$$\frac{\frac{S(G')}{\mathrm{in}_<(J^j_{G'})}}{(\underline{\mu})\frac{S(G')}{\mathrm{in}_<(J^j_{G'})}} \cong \frac{S}{\mathrm{in}_<(J^j_G)}.$$

for every $j \geq 1$. This implies that

$$\mathrm{depth}\,\frac{S}{J^i_G} = \mathrm{depth}\,\frac{S}{\mathrm{in}_<(J^i_G)} = \mathrm{depth}\,\frac{S(G')}{J^i_{G'}} - 2(r-1) =$$

$$= \begin{cases} \sum_{j=i}^r d_j + i + 1 = n - d_1 - d_2 \cdots - d_{i-1} + i, & \text{for } 1 \leq i \leq r, \\ r+2, & \text{for } i \geq r+1, \end{cases}$$

where the second equality holds because $n = \sum_{j=1}^r (d_j+1) - (r-1) = \sum_{j=1}^r d_j + 1$. ∎

With similar arguments as the ones we used for the connected case, we may derive the depth function for the powers of J_G and $\mathrm{in}_<(J_G)$ in the case that G has several connected components, say $G_1,\ldots,G_c$. The only difference is that we do not need to mod out by the entire sequences $\underline{\ell}$ and $\underline{\mu}$ of length $2(r-1)$ but, instead, by sequences of length $2(r-1) - 2(c-1) = 2(r-c)$. Consequently, we get the following.

Proposition 3.6. *Let G be a closed graph on the vertex set $[n]$ with the connected components $G_1,G_2,\ldots,G_c$ such that J_G is Cohen–Macaulay. Let $F_1,F_2,\ldots,F_r$ be the maximal cliques of G and $d_i = \dim F_i = \#F_i - 1$ for $1 \leq i \leq r$. Assume that $d_1 \geq d_2 \geq \cdots \geq d_r \geq 1$. Then:*

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

(a) $\operatorname{depth}\frac{S}{J_G^i} = \operatorname{depth}\frac{S}{\text{in}_<(J_G^i)} = n - \sum_{j=1}^{i-1} d_j + i + c - 1,$ *for* $1 \le i \le r$*and*
(b) $\operatorname{depth}\frac{S}{J_G^i} = \operatorname{depth}\frac{S}{\text{in}_<(J_G^i)} = r + 2c,$ *for* $i \ge r+1$.

PROPOSITION 3.7. *Let G be a closed graph with the property that at least one of its connected components is not a path. Then J_G^i is not Cohen–Macaulay for $i \ge 2$.*

Proof. If J_G is Cohen–Macaulay, then, by Proposition 3.6, it follows that

$$\operatorname{depth}(S/J_G^i) < \operatorname{depth}(S/J_G) = \dim(S/J_G)$$

for $i \ge 2$ since G has cliques with at least three vertices. This implies that J_G^i is not Cohen–Macaulay.

If J_G is not Cohen–Macaulay, then, by Theorem 2.4, J_G is not unmixed. This implies that $J_G^{(i)}$ is not unmixed, thus it is not Cohen–Macaulay. But we know that $J_G^i = J_G^{(i)}$ for all $i \ge 1$, therefore, J_G^i is not Cohen–Macaulay for $i \ge 1$. ∎

Since all the powers of a complete intersection ideal in a polynomial ring are Cohen–Macaulay [1], [8], [38], we get the following consequence of the above proposition.

COROLLARY 3.8. *Let G be a closed graph. Then the following are equivalent:*

(a) *Each connected component of G is a path graph,*
(b) *J_G^i is Cohen–Macaulay for every $i \ge 2$,*
(c) *J_G^i is Cohen–Macaulay for some $i \ge 2$, and*
(d) *J_G^2 is Cohen–Macaulay.*

PROPOSITION 3.9. *Let G be a closed graph and let J_G be the associated binomial edge ideal. Then the Rees algebras $\mathcal{R}(J_G)$ and $\mathcal{R}(\text{in}_<(J_G))$ are Cohen–Macaulay and have the same dimension. In particular, the graded rings of J_G and $\text{in}_<(J_G)$ are Cohen–Macaulay.*

Proof. Since $\text{in}_<(J_G)$ is normally torsion free (by (3) and [13, Lem. 3.1]), it follows that $\mathcal{R}(\text{in}_<(J_G))$ is Cohen–Macaulay by [25] and, by [7, Th. 2.7], we have $\mathcal{R}(\text{in}_<(J_G)) = \text{in}_{<'}(\mathcal{R}(J_G))$. Here, $\text{in}_{<'}(\mathcal{R}(J_G))$ is the initial algebra of $\mathcal{R}(J_G)$ with respect to the monomial order $<'$ on $S[t]$ which extends the lexicographic order $<$ on S as follows: given two monomials $u, v \in S$ and two integers $i, j \ge 0$, we have $ut^i <' vt^j$ if and only if $i < j$ or $i = j$ and $u < v$. Since $\mathcal{R}(\text{in}_<(J_G))$ is Cohen–Macaulay, it follows that $\text{in}_{<'}(\mathcal{R}(J_G))$ is Cohen–Macaulay and this implies that $\mathcal{R}(J_G)$ shares the same property [7, Cor. 2.3]. In addition, as $\text{in}_{<'}(\mathcal{R}(J_G))$ and $\mathcal{R}(J_G)$ have the same Krull dimension [7, Prop. 2.4], it follows that $\mathcal{R}(J_G)$ and $\mathcal{R}(\text{in}_<(J_G))$ have the same dimension.

The last part of the statement follows by [27, Prop. 1.1]. ∎

Theorem 3.1 shows that the depth function of Cohen–Macaulay binomial edge ideals of closed graphs is nonincreasing. Moreover, it coincides with the depth function of their initial ideals. We expect that this behavior holds for every closed graph, but we could not prove it. Instead, in the next theorem we show that, for every closed graph G, the ideals J_G and $\text{in}_<(J_G)$ have the same limit depth and we compute its value. Moreover, in Proposition 3.12, we will show that $\text{in}_<(J_G)$ has a nonincreasing depth function.

Before stating the theorem, let us recall a few notions and results. A classical result of Brodmann [4] states that if I is a homogeneous ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$, then

$$\lim_{k\to\infty} \operatorname{depth}\frac{R}{I^k} \le n - \ell(I), \tag{6}$$

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

where $\ell(I) = \dim \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ is the *analytic spread*of I. Here $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ is the maximal graded ideal of R and $\mathcal{R}(I)$ is the Rees algebra of the ideal I. For an alternative proof of (6) we refer to [20, Th. 1.2]. In [11], it was shown that the equality holds in (6) if the ring $\mathrm{gr}_I(R)$ is Cohen–Macaulay, which is the case if $\mathcal{R}(I)$ is Cohen–Macaulay [27]. We should also recall that if I is generated by some polynomials, say $f_1, \ldots, f_m$, of the same degree, then the fiber ring $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ is isomorphic to $K[f_1, \ldots, f_m]$ since we have the following isomorphism:

$$\frac{\mathcal{R}(I)}{\mathfrak{m}\mathcal{R}(I)} \cong \frac{R}{\mathfrak{m}} \oplus \frac{I}{\mathfrak{m}I} \oplus \frac{I^2}{\mathfrak{m}I^2} \oplus \cdots .$$

On the other hand, we need to recall some graph theoretical terminology. A vertex v of a graph G is called a *free* vertex if it belongs to exactly one maximal clique of G. A connected graph G is called *decomposable* if there exist G_1 and G_2 induced subgraphs of G such that $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v\}$ and v is a free vertex in G_1 and G_2. A connected graph G is *indecomposable* if it is not decomposable. Clearly, every graph G (not necessarily connected) has a unique decomposition up to relabeling of the form $G = G_1 \cup G_2 \cup \cdots \cup G_r$ where $G_1, \ldots, G_r$ are indecomposable graphs and for every $1 \le i < j \le r$, we have either $V(G_i) \cap V(G_j) = \emptyset$ or $V(G_i) \cap V(G_j) = \{v\}$ where v is a free vertex in G_i and G_j. We call $G_1, \ldots, G_r$ the *indecomposable components* of G.

THEOREM 3.10. *Let G be a closed graph and $J_G \subset S$ its binomial edge ideal. Let $g_1, \ldots, g_m$ be the generators of J_G. Then the following hold:*

(a) *The set $\{g_1, \ldots, g_m\}$ is a Sagbi basis of the K-algebra $K[g_1, \ldots, g_m]$ with respect to the lexicographic order on S, that is,*

$$\mathrm{in}_<(K[g_1, \ldots, g_m]) = K[\mathrm{in}_< g_1, \ldots, \mathrm{in}_< g_m].$$

(b) *The ideals J_G and $\mathrm{in}_<(J_G)$ have the same analytic spread.*

(c) $\lim_{k\to\infty} \mathrm{depth}\, \frac{S}{J_G^k} = \lim_{k\to\infty} \mathrm{depth}\, \frac{S}{(\mathrm{in}_<(J_G))^k} = r + 2$, *where r is the number of indecomposable components of G.*

Proof. Let $A = K[g_1, \ldots, g_m]$ and $B = K[\mathrm{in}_< g_1, \ldots, \mathrm{in}_< g_m]$.

(a). In order to show that $\{g_1, \ldots, g_m\}$ is a Sagbi basis of A, we apply a criterion which plays a similar role to the Buchberger criterion in the Gröbner basis theory; see [12, Th. 6.43]. Let $\varphi : K[t_1, \ldots, t_m] \to A$ and $\psi : K[t_1, \ldots, t_m] \to B$ be the K-algebra homomorphisms defined by $\varphi(t_i) = g_i$ and $\psi(t_i) = \mathrm{in}_< g_i$ for $1 \le i \le m$. Let $\mathbf{t}^{\mathbf{a}_1} - \mathbf{t}^{\mathbf{b}_1}, \ldots, \mathbf{t}^{\mathbf{a}_r} - \mathbf{t}^{\mathbf{b}_r}$ be a system of binomial generators for the toric ideal $\ker \psi$. Then $\{g_1, \ldots, g_m\}$ is a Sagbi basis of A if and only if there exist some coefficients $c_{\mathbf{a}}^{(j)} \in K$ such that

$$\mathbf{g}^{\mathbf{a}_j} - \mathbf{g}^{\mathbf{b}_j} = \sum_{\mathbf{a}} c_{\mathbf{a}}^{(j)} \mathbf{g}^{\mathbf{a}}$$

with $\mathrm{in}_<(\mathbf{g}^{\mathbf{a}}) < \mathrm{in}_<(\mathbf{g}^{\mathbf{a}_j})$ for all $\mathbf{a}$, where by $\mathbf{g}^{\mathbf{a}}$ we mean $g_1^{a_1} \cdots g_m^{a_m}$ if $\mathbf{a} = (a_1, \ldots, a_m)$. Thus, we first need to find a set of binomial generators for $\ker \psi$. The K-algebra B is the edge ring of the bipartite graph H on the vertex set $V(H) = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ and edge set $E(H) = \{\{x_i, y_j\} : i < j \text{ and } \{i,j\} \in E(G)\}$. By [16, Lem. 3.3], we know that every induced cycle in H has length 4. By [33], the toric ideal of B is generated by the binomials $\beta_{\gamma_1}, \ldots, \beta_{\gamma_s}$ where $\gamma_1, \ldots, \gamma_s$ are the four-cycles of H. If γ is a four-cycle in H, say $\gamma = (x_i, y_j, x_k, y_\ell)$ with $i < k < j < \ell$, and $x_i y_j = \mathrm{in}_<(g_{i_1}), x_i y_\ell = \mathrm{in}_<(g_{i_2}), x_k y_j = \mathrm{in}_<(g_{i_3}), x_k y_\ell = \mathrm{in}_<(g_{i_4})$, then

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

$\beta_\gamma = t_{i_1}t_{i_4} - t_{i_2}t_{i_3}$. We have to lift the relations determined by the binomials β_γ to A. But this is very easy since

$$g_{i_1}g_{i_4} - g_{i_2}g_{i_3} = g_{i_5}g_{i_6},$$

where $g_{i_5} = x_iy_k - x_ky_i$ and $g_{i_6} = x_jy_\ell - x_\ell y_j$. Note that since $i < k < j < \ell$, and $\{i,\ell\} \in E(G)$, then $\{i,k\}$ and $\{j,\ell\}$ are edges in G as well, by Theorem 2.3 (iv). Moreover,

$$\mathrm{in}_<(g_{i_5}g_{i_6}) = x_ix_jy_ky_\ell < x_ix_ky_jy_\ell = \mathrm{in}_<(g_{i_1}g_{i_4})$$

since $k < j$.

(b) follows from (a) since $\dim A = \dim \mathrm{in}_<(A)$ by [7, Prop. 2.4].

(c) By Proposition 3.9 and [11, Props. 3.1 and 3.3], we have

$$\lim_{k\to\infty} \operatorname{depth} \frac{S}{J_G^k} = \dim S - \ell(J_G) \text{ and } \lim_{k\to\infty} \operatorname{depth} \frac{S}{(\mathrm{in}_<(J_G))^k} = \dim S - \ell(\mathrm{in}_<(J_G)).$$

Therefore, we get the equality of the two limits by (b).

Since y_1 and x_n are isolated vertices in the bipartite graph H whose edge ideal is equal to $\mathrm{in}_<(J_G)$, we have

$$\lim_{k\to\infty} \operatorname{depth} \frac{S}{(\mathrm{in}_<(J_G))^k} = \lim_{k\to\infty} \operatorname{depth} \frac{S'}{I(H)^k} + 2,$$

where S' is the polynomial ring in the variables $x_j, 1 \le j \le n-1$ and $y_j, 2 \le j \le n$. By [37, Th. 4.4] or [21, Cor. 10.3.18],

$$\lim_{k\to\infty} \operatorname{depth} \frac{S'}{I(H)^k} = r,$$

where r is the number of connected components of H. But, taking into account the characterization of closed graphs given in Theorem 2.3 (iii), it is easily seen that this is exactly the number of indecomposable components of G. ∎

Remark 3.11. By using [2, Th. 4.6 and Cor. 4.9], one may derive that the limit depth of the so-called closed determinantal facet ideals and their initial ideals is the same. This class of ideals was introduced in [15].

Proposition 3.12. *Let G be a closed graph and J_G its binomial edge ideal. Then*

$$\operatorname{depth} \frac{S}{(\mathrm{in}_<(J_G))^{k+1}} \le \operatorname{depth} \frac{S}{(\mathrm{in}_<(J_G))^k}$$

for every $k \ge 1$.

Proof. The inequalities follow by [31, Th. 5.2] since the bipartite graph H whose edge ideal is equal to $\mathrm{in}_<(J_G)$ has at least one leaf, namely the vertex x_{n-1}. ∎

As we have seen in Section 2, for every closed graph G, we have $J_G^i = J_G^{(i)}$ for $i \ge 1$. This equalities imply that $\mathrm{Ass}(J_G^i) = \mathrm{Ass}(J_G^{i+1})$ for $i \ge 1$, thus J_G has the *persistence property*, that is $\mathrm{Ass}(J_G^i) \subseteq \mathrm{Ass}(J_G^{i+1})$ for $i \ge 1$. But we can prove even more, namely, that J_G has the *strong persistence property*. Let us recall that an ideal I in a polynomial ring satisfies the strong persistence property if and only if $I^{k+1} : I = I^k$ for all k; see [24]. We will derive this property from a slightly more general statement.

PROPOSITION 3.13. *Let $I \subset R = K[x_1, x_2, \ldots, x_n]$ be a homogeneous ideal and assume that there exists a monomial order $<$ on R such that the following conditions hold:*

(a) $\mathrm{in}_<(I)$ *has the strong persistence property and*
(b) $\mathrm{in}_<(I^j) = (\mathrm{in}_<(I))^j$ *for every $j \geq 1$.*

Then the ideal I has the strong persistence property. In particular, I has the persistence property.

Proof. We have to prove that $I^{j+1} : I = I^j$ for $j \geq 1$. Since $I^j \subseteq I^{j+1} : I$, it is enough to show that $\mathrm{in}_<(I^j) = \mathrm{in}_<(I^{j+1} : I)$. The inclusion $\mathrm{in}_<(I^j) \subseteq \mathrm{in}_<(I^{j+1} : I)$ is obvious. For the other inclusion, let us consider a monomial $w \in \mathrm{in}_<(I^{j+1} : I)$. Then there exists a polynomial $g \in I^{j+1} : I$ such that $w = \mathrm{in}_<(g)$. As $gI \subseteq I^{j+1}$, we get

$$w\,\mathrm{in}_<(I) \subseteq \mathrm{in}_<(I^{j+1}) = (\mathrm{in}_<(I))^{j+1},$$

which yields

$$w \in (\mathrm{in}_<(I))^{j+1} : \mathrm{in}_<(I) = (\mathrm{in}_<(I))^j = \mathrm{in}_<(I^j).$$

∎

COROLLARY 3.14. *Let G be a closed graph. Then J_G has the strong persistence property.*

Proof. Let $<$ be the lexicographic order on S. Then $\mathrm{in}_<(J_G) = (x_i y_j : \{i,j\} \in E(G))$ is an edge ideal. Therefore, by [32, Lem. 2.12], it follows that $\mathrm{in}_<(J_G)$ has the strong persistence property. Moreover, by (3), we also have $\mathrm{in}_<(J_G^i) = (\mathrm{in}_<(J_G))^i$ for every $i \geq 1$. The claim follows by Proposition 3.13. ∎

§4. Regularity

In this section, we compute the regularity of the powers of binomial edge ideals of closed graphs and of their initial ideals. First, we recall some notions and results from Graph Theory.

A graph G is called *co-chordal* if its complement graph G^c is chordal. The *co-chordal cover number* of G, denoted co-chord(G), is the smallest number m for which there exist some co-chordal subgraphs $G_1, \ldots, G_m$ of G such that $E(G) = \cup_{i=1}^m E(G_i)$.

A graph G is *weakly chordal* if every induced cycle in G and in G^c has length at most 4. For a graph G, we denote by im(G) the number of edges in a largest induced matching of G. By an *induced matching* we mean an induced subgraph of G which consists of pairwise disjoint edges. In other words, im(G) is the monomial grade of the edge ideal $I(G)$, that is, the maximum length of a regular sequence of monomials in $I(G)$. In [5, Prop. 3] it is proved that if G is weakly chordal, then im$(G) =$ co-chord(G).

On the other hand, we will use [30, Th. 3.6] which states that if H is a bipartite graph and $I(H)$ is its edge ideal, then, for $i \geq 1$, we have

$$\mathrm{reg}(I(H)^i) \leq \text{co-chord}(H) + 2i - 1. \tag{7}$$

THEOREM 4.1. *Let G be a connected closed graph. Then, for every $i \geq 1$, we have*

$$\mathrm{reg}\,\frac{S}{J_G^i} = \mathrm{reg}\,\frac{S}{\mathrm{in}_<(J_G^i)} = \ell + 2(i-1),$$

where ℓ is the length of the longest induced path in G.

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

Proof. The inequality $\operatorname{reg} S/J_G^i \geq \ell + 2(i-1)$ follows by [28, Cor. 3.4]. Hence, we have

$$\operatorname{reg} \frac{S}{\operatorname{in}_<(J_G^i)} \geq \operatorname{reg} \frac{S}{J_G^i} \geq \ell + 2(i-1).$$

Thus, it is enough to prove that $\operatorname{reg} S/\operatorname{in}_<(J_G^i) \leq \ell + 2(i-1)$. Since G is closed, by (3), we have $\operatorname{in}_<(J_G^i) = (\operatorname{in}_<(J_G))^i$. Therefore, we get

$$\operatorname{reg} \frac{S}{\operatorname{in}_<(J_G^i)} = \operatorname{reg} \frac{S}{(\operatorname{in}_<(J_G))^i}.$$

As we have already mentioned in Section 2, the monomial ideal $\operatorname{in}_<(J_G) = (x_i y_j : \{i,j\} \in E(G))$ is the edge ideal $I(H)$ of a bipartite graph on $\{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\}$. Then inequality (7) implies that

$$\operatorname{reg} \frac{S}{(\operatorname{in}_<(J_G))^i} \leq \operatorname{co-chord}(H) + 2(i-1).$$

In [16, Lem. 3.3] it was proved that H is a weakly chordal graph. This implies that $\operatorname{co-chord}(H) = \operatorname{im}(H)$. On the other hand, by [16, Prop. 3.5], it follows that $\operatorname{im}(H) = \ell$, which completes the proof. ∎

The arguments of the above proof can be extended to disconnected closed graphs.

PROPOSITION 4.2. *Let G be a closed graph with connected components $G_1, \ldots, G_c$. Let ℓ_i be the length of the longest induced path in the component G_i for $1 \leq i \leq c$. Then, for all $i \geq 1$, we have*

$$\operatorname{reg} \frac{S}{J_G^i} = \operatorname{reg} \frac{S}{\operatorname{in}_<(J_G^i)} = \ell_1 + \ell_2 + \cdots + \ell_c + 2(i-1).$$

Proof. The inequality $\operatorname{reg} S/J_G^i \geq \ell_1 + \ell_2 + \cdots + \ell_c + 2(i-1)$ follows from [28, Prop. 3.3] and [28, Obser. 3.2] since the union of the longest induced paths in $G_j, 1 \leq j \leq c$, form an induced subgraph in G, and the inequality $\operatorname{reg} S/\operatorname{in}_<(J_G^i) \leq \ell_1 + \ell_2 + \cdots + \ell_c + 2(i-1)$ holds since, obviously, in the bipartite graph H such that $\operatorname{in}_<(J_G) = I(H)$ we have $\operatorname{im}(H) = \ell_1 + \ell_2 + \cdots + \ell_c$. ∎

§5. Powers of binomial edge ideals of block graphs

In this section, we discuss powers of binomial edge ideals of block graphs. We recall that a graph G is called a *block graph*, if each block of G is a clique. A block of G is a connected subgraph of G that has no cutpoint and is maximal with respect to this property. A vertex v is a cutpoint of a graph H if the induced subgraph obtained by removing the vertex v has more connected components than H. The block graphs whose binomial edge ideal is Cohen–Macaulay are classified in [14, Th. 1.1]. It is shown that for a block graph G, the following conditions are equivalent:

(a) J_G is unmixed.
(b) J_G is Cohen–Macaulay.
(c) Each vertex of G is the intersection of at most two maximal cliques.

The following theorem shows that the equality between symbolic and ordinary powers does not hold, in general, for binomial edge ideals of block graphs.

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

THEOREM 5.1. *Let G be a block graph such that J_G is Cohen–Macaulay. Then, the following statements are equivalent:*

(a) *G is closed,*
(b) *$J_G^i = J_G^{(i)}$ for all $i \geq 2$,*
(c) *$J_G^i = J_G^{(i)}$ for some $i \geq 2$,*
(d) *$J_G^2 = J_G^{(2)}$, and*
(e) *G is net-free, that is, it does not contain a net as an induced subgraph (see Figure 2).*

Proof. (a) ⇒(b) follows by (4). The implications (b) ⇒(c) and (b) ⇒(d) are trivial.

Next, we prove (d) ⇒(e). Let us assume that G contains as an induced subgraph the net N with the edge set $E(N) = \{\{1,2\},\{3,4\},\{5,6\},\{2,3\},\{3,5\},\{2,5\}\}$.

Set $g = x_3x_5x_6y_1y_2y_4 - x_1x_5x_6y_2y_3y_4 - x_3x_4x_5y_1y_2y_6 + x_1x_2x_5y_3y_4y_6 + x_1x_3x_4y_2y_5y_6 - x_1x_2x_3y_4y_5y_6$. We show that $g \in J_G^{(2)} \setminus J_G^2$.

Since $J_G = \bigcap_{P_W(G)\in \mathrm{Ass}(J_G)} P_W(G)$, we show that $g \in (P_W(G))^2$, for all W with the property that $P_W(G) \in \mathrm{Ass}(J_G)$. Then it follows that $g \in J_G^{(2)}$, because, as it was observed in [13], $P_W(G)^{(i)} = P_W(G)^i$ for all $i \geq 1$ and all $W \subset [n]$. We consider the following cases. Let $W \subseteq [n]$.

Case 1. If $W \cap [6] \in \{\emptyset, \{1\}, \{4\}, \{6\}\}$, then $g = x_5y_4(x_6y_2 - x_2y_6)(x_3y_1 - x_1y_3) + x_3y_6(x_4y_2 - x_2y_4)(x_1y_5 - x_5y_1) \in (P_W(G))^2$.

Case 2. $W \cap [6] = \{2\}$. Then $g = x_1x_2y_4y_6(x_5y_3 - x_3y_5) + x_5x_6y_1y_2(x_3y_4 - x_4y_3) + x_3x_4y_1y_2(x_6y_5 - x_5y_6) + x_4x_6y_1y_2(x_5y_3 - x_3y_5) + x_1x_5y_2y_3(x_4y_6 - x_6y_4) + x_1x_4y_2y_6(x_3y_5 - x_5y_3) \in (P_W(G))^2$.

Case 3. $W \cap [6] = \{3\}$. Then $g = x_1x_5y_3y_4(x_2y_6 - x_6y_2) - x_3x_4y_1y_2(x_5y_6 - x_6y_5) + x_1x_3y_4y_6(x_5y_2 - x_2y_5) - x_3x_5y_2y_4(x_1y_6 - x_6y_1) + x_3x_4y_2y_5(x_1y_6 - x_6y_1) \in (P_W(G))^2$.

Case 4. $W \cap [6] = \{5\}$. Then $g = x_1x_3y_5y_6(x_4y_2 - x_2y_4) + x_5x_6y_2y_4(x_3y_1 - x_1y_3) + x_2x_5y_3y_6(x_1y_4 - x_4y_1) + x_4x_5y_1y_6(x_2y_3 - x_3y_2) \in (P_W(G))^2$.

Next, we show that, if G contains N as an induced subgraph, then $g \notin J_G^2$. Since N is an induced subgraph of G, by the proof of [28, Prop. 3.3], it follows that $J_N^i = J_G^i \cap K[x_1,\ldots,x_6,y_1,\ldots,y_6]$ for all $i \geq 1$. Therefore, it suffices to show that $g \notin J_N^2$.

Suppose $g \in J_N^2$. Then we have $x_1x_2x_5y_3y_4y_6 \in J_N^2 + (x_3,y_2)$. Since $\{x_1,y_4\}$ is a regular sequence on $S/(J_N^2 + (x_3,y_2))$, we have $x_2x_5y_3y_6 \in J_N^2 + (x_3,y_2)$. Since any monomial of degree 4 in $J_N^2 + (x_3,y_2)$ which is not divided by neither x_3 nor y_2 is not divided by y_6, it follows $x_2x_5y_3y_6 \notin J_N^2 + (x_3,y_2)$, contradiction.

For (c) ⇒(e), we show that $g(x_2y_3 - x_3y_2)^{i-2} \in J_G^{(i)} \setminus J_G^i$. Taking into account the above arguments, it is obvious that $g(x_2y_3 - x_3y_2)^{i-2} \in (P_W(G))^i$, for all W with the property that $P_W(G) \in \mathrm{Ass}(J_G)$, thus $g(x_2y_3 - x_3y_2)^{i-2} \in J_G^{(i)}$. We show that $g(x_2y_3 - x_3y_2)^{i-2} \notin J_N^i$. Suppose $g(x_2y_3 - x_3y_2)^{i-2} \in J_N^i$. Then we have $x_1x_2x_5y_3y_4y_6(x_2y_3)^{i-2} \in J_N^i + (x_3,y_2)$. Since $\{x_1,y_4\}$ is a regular sequence on $S/(J_N^i + (x_3,y_2))$, we have $x_2x_5y_3y_6(x_2y_3)^{i-2} \in J_N^i + (x_3,y_2)$. Since any monomial of degree $2i$ in $J_N^i + (x_3,y_2)$ which is not divided by neither x_3 nor y_2 is not divided by y_6, it follows that $x_2x_5y_3y_6(x_2y_3)^{i-2} \notin J_N^i + (x_3,y_2)$, contradiction.

Finally, we show that (e) ⇒(a). Since G is a block graph, it follows that G is chordal and tent-free (see Figure 2). On the other hand, as J_G is Cohen–Macaulay and, in particular, unmixed, it follows that G is claw-free (see Figure 1). Therefore, the hypothesis implies that G is closed by Theorem 2.3 (v). ∎

PROPOSITION 5.2. *Let G be a connected block graph which is not a path. Then J_G^i is not Cohen–Macaulay for every $i \geq 2$.*

Proof. We analyze the following cases.

Case 1. Suppose that G is a net-free (see Figure 2) block graph which is not a path and J_G is Cohen–Macaulay. Then, by using Theorem 2.3 (v), it follows that G is a closed graph. Then, Proposition 3.7 implies that J_G^i is not Cohen–Macaulay for every $i \geq 2$.

Case 2. Let G be a block graph which contains a net as an induced subgraph and such that J_G is Cohen–Macaulay. Then, by Theorem 5.1, we have $J_G^i \subsetneq J_G^{(i)}$, for every $i \geq 2$ and, in particular, it follows that J_G^i has embedded components. Consequently, J_G^i is not unmixed, and, therefore, J_G^i is not Cohen–Macaulay for $i \geq 2$.

Case 3. Suppose that G is a block graph and J_G is not Cohen–Macaulay. Then J_G is not unmixed. It follows that J_G^i is not unmixed for all i, thus J_G^i is not Cohen–Macaulay as well. ∎

§6. Open problems

As we have seen in Section 3, the depth function of Cohen–Macaulay binomial edge ideals of closed graphs is non-increasing. The depth function of $\text{in}_<(J_G)$ is also nonincreasing for every closed graph G. Therefore it is natural to ask the following.

QUESTION 6.1. *Is it true that the depth function of J_G is nonincreasing for every closed graph G?*

Of course, taking into account Proposition 3.12, we can answer positively this question by showing that if G is closed, then $\operatorname{depth} S/J_G^i = \operatorname{depth} S/\text{in}_<(J_G^i)$ for every $i \geq 1$.

A partial positive answer to this question is the following. Let G be a closed graph with maximal cliques $F_i = [a_i, b_i], 1 \leq i \leq r$, ordered as in Theorem 2.3 (iii). Assume that $F_1 = [1,2]$, in other words, the vertex 1 is a leaf of G. We claim that

$$\operatorname{depth} \frac{S}{J_G^{k+1}} \leq \operatorname{depth} \frac{S}{J_G^k},$$

for every $k \geq 1$. In order to prove this inequality, we first observe that $J_G^{k+1} : f_{12} = J_G^k$ for all k. Indeed, since $J_G^k \subseteq J_G^{k+1} : f_{12}$, it is enough to show that $\text{in}_<(J_G^{k+1} : f_{12}) = \text{in}_<(J_G^k)$. Let us assume that $\text{in}_<(J_G^{k+1} : f_{12}) \supsetneq \text{in}_<(J_G^k)$. Then there exists a monomial $w \in \text{in}_<(J_G^{k+1} : f_{12}) \setminus \text{in}_<(J_G^k)$. Let $h \in J_G^{k+1} : f_{12}$ such that $\text{in}_<(h) = w$. We can write $h = w + h_1$, where $\text{in}_<(h_1) < w$. Then, $f_{12}h = f_{12}(w + h_1) \in J_G^{k+1}$, which implies that

$$\begin{aligned}\text{in}_<(f_{12})w = x_1y_2w \in \text{in}_<(J_G^{k+1}) = (\text{in}_<(J_G))^{k+1} = ((x_1y_2) + \text{in}_<(J_{G-\{1\}}))^{k+1} \\ = (x_1y_2)(\text{in}_<(J_G))^k + (\text{in}_<(J_{G-\{1\}}))^{k+1}.\end{aligned}$$

Since x_1 and y_2 do not divide any of the minimal monomial generators of $\text{in}_<(J_{G-\{1\}})$ and $w \notin (\text{in}_<(J_G))^k$, thus $w \notin (\text{in}_<(J_{G-\{1\}}))^{k+1}$, we get $\text{in}_<(f_{12})w \in (x_1y_2)(\text{in}_<(J_G))^k$ which yields $w \in (\text{in}_<(J_G))^k$, a contradiction. Therefore, we have proved the equality $J_G^{k+1} : f_{12} = J_G^k$. We consider the exact sequence of S-modules

$$0 \to \frac{S}{J_G^{k+1} : f_{12}} = \frac{S}{J_G^k} \to \frac{S}{J_G^{k+1}} \to \frac{S}{(J_G^{k+1}, f_{12})} \to 0. \tag{8}$$

https://doi.org/10.1017/nmj.2021.1 Published online by Cambridge University Press

Since $J_G^{k+1} = (J_{G-\{1\}} + f_{12})^{k+1} = J_{G-\{1\}}^{k+1} + f_{12}J_G^k$, it follows that $(J_G^{k+1}, f_{12}) = (J_{G-\{1\}}^{k+1}, f_{12})$. But f_{12} is obviously regular on $S/J_{G-\{1\}}^{k+1}$, thus

$$\operatorname{depth} \frac{S}{(J_G^{k+1}, f_{12})} = \operatorname{depth} \frac{S}{(J_{G-\{1\}}^{k+1}, f_{12})} = \operatorname{depth} \frac{S}{J_{G-\{1\}}^{k+1}} - 1.$$

As $G - \{1\}$ is an induced subgraph of G, by [28, Prop. 3.3] it follows that $\operatorname{depth} S/J_{G-\{1\}}^{k+1} \geq \operatorname{depth} S/J_G^{k+1}$. Therefore, we obtain

$$\operatorname{depth} \frac{S}{(J_G^{k+1}, f_{12})} \geq \operatorname{depth} \frac{S}{J_G^{k+1}} - 1.$$

The Depth Lemma applied to sequence (8) gives the desired inequality.

The following conjecture, which is still open, was formulated in [14]: if G is a closed graph, then J_G and $\operatorname{in}_<(J_G)$ have the same graded Betti numbers. On the other hand, from several computer experiments, we noticed that the same equality holds for small powers of J_G. Therefore, we suggest the following conjecture which extends the one in [14].

CONJECTURE 6.2. *Let G be a closed graph. Then, for every $i \geq 1, J_G^i$ and $(\operatorname{in}_<(J_G))^i = \operatorname{in}_<(J_G^i)$ have the same graded Betti numbers.*

Let us remark in support of our conjecture that in the previous sections we proved that $\operatorname{reg} J_G^i = \operatorname{reg}(\operatorname{in}_<(J_G))^i$ for G closed and $\operatorname{depth} J_G^i = \operatorname{depth}(\operatorname{in}_<(J_G))^i$ for G closed and with J_G Cohen–Macaulay. Of course, if the above conjecture is true, then it also answers Question 6.1.

In addition, we note that Conjecture 6.2 is true in two "extremal" cases, namely when G is a complete graph or a path.

Indeed, if $G = K_n$, then $(\operatorname{in}_<(J_G))^i$ has a linear resolution for every $i \geq 1$. Since $(\operatorname{in}_<(J_G))^i = \operatorname{in}_<(J_G^i)$, it follows that J_G^i has a linear resolution for $i \geq 1$. As the Hilbert functions of J_G^i and $\operatorname{in}_<(J_G^i)$ coincide, we derive that the conjecture is true when $G = K_n$. On the other hand, if $G = P_n$ with the edges $\{i, i+1\}, 1 \leq i \leq n-1$, then $\operatorname{in}_<(J_G)$ and J_G are complete intersections generated in degree 2 and the conjecture is true by [18, Cor. 1.3].

When G is a closed graph such that J_G is Cohen–Macaulay, we have $\beta_{ij}(S/J_G) = \beta_{ij}(S/\operatorname{in}_<(J_G))$ for all i, j by [14, Prop. 3.2]. A possible strategy to prove Conjecture 6.2 in this case is the following. We begin with the following nice consequence of Lemma 3.4.

COROLLARY 6.3. *With the same notation of Lemma 3.4, we have*

$$\beta_{ij}^S\left(\frac{S}{J_G^k}\right) = \beta_{ij}^{S(G')}\left(\frac{S(G')}{J_{G'}^k}\right)$$

and

$$\beta_{ij}^S\left(\frac{S}{\operatorname{in}_<(J_G^k)}\right) = \beta_{ij}^{S(G')}\left(\frac{S(G')}{\operatorname{in}_<(J_{G'}^k)}\right),$$

for all i, j, and $k \geq 1$.

This corollary implies that Conjecture 6.2 holds for the closed graphs with Cohen–Macaulay binomial edge ideal once we show that for every graph H whose connected components are complete graphs we have

$$\beta_{ij}^{S(H)}\left(\frac{S(H)}{J_H^k}\right) = \beta_{ij}^{S(H)}\left(\frac{S(H)}{\mathrm{in}_<(J_H^k)}\right)$$

for all i, j, and $k \geq 1$.

In Proposition 5.2, we proved that if G is a connected block graph, then J_G^i is Cohen–Macaulay for every $i \geq 2$ if and only if G is a path. Then we may ask the following.

Question 6.4. *Let G be a connected (chordal) graph. Is it true that J_G^i is Cohen–Macaulay for every $i \geq 2$ if and only if G is a path?*

The net graph N (see Figure 2) which plays an important role in Theorem 5.1 has the nice property that $J_N^{(2)}$ is Cohen–Macaulay. On the other hand, J_N^2 is not Cohen–Macaulay (see proof of Proposition 5.2, Case 2). This naturally yields the following.

Problem 6.5. *Classify all the block graphs with the property that the second symbolic power of the associated binomial edge ideal is Cohen–Macaulay.*

The last question is inspired by Theorem 5.1.

Question 6.6. *Let G be a graph. Is it true that the following conditions are equivalent?*

(a) $J_G^i = J_G^{(i)}$ *for all* $i \geq 2$,
(b) $J_G^i = J_G^{(i)}$ *for some* $i \geq 2$,
(c) $J_G^2 = J_G^{(2)}$, *and*
(d) G *is net-free.*

Computer experiments showed that for every graph G with at most eight vertices the equivalence (c) $\Leftrightarrow$ (d) holds. Moreover, the implications (a) $\Rightarrow$(c) $\Rightarrow$(b) $\Rightarrow$(d) hold and they can be shown similarly to the proof of Theorem 5.1.

Acknowledgment. We thank Amin Fakhari and Arvind Kumar for their valuable comments on Section 6. We acknowledge the use of Macaulay2 [17] for our computations. We are grateful to the anonymous referee for valuable comments.

References

[1] R. Achilles and W. Vogel, *Über vollständige Durchschnitte in lokalen Ringen*, Math. Nachr. **89** (1979), 285–298.
[2] A. Almousa, K. N. Lin, and W. Liske, *Rees algebras of closed determinantal ideals*, preprint, arXiv:2008.10950.
[3] H. Baskoroputro, V. Ene, and C. Ion, *Koszul binomial edge ideals of pairs of graphs*, J. Algebra **515** (2018), 344–359.
[4] M. Brodmann, *The asymptotic nature of the analytic spread*, Math. Proc. Camb. Philos. Soc. **86** (1979), 35–39.
[5] A. H. Busch, F. F. Dragan, and R. Shritharan, “New min–max theorems for weakly Chordal and dually Chordal graphs” In *Combinatorial Optimization and Applications. Part II*, Lecture Notes in Comput. Sci. **6509**, Springer, Berlin, Heidelberg, 2010, 207–218.
[6] A. Conca, E. De Negri, and E. Gorla, *Cartwright-Sturmfels ideals associated to graphs and linear spaces*, J. Comb. Algebra **2** (2018), no. 3, 231–257.
[7] A. Conca, J. Herzog, and G. Valla, *Sagbi bases with applications to blow–up algebras*, J. Reine Angew. Math. **474** (1996), 113–138.

[8] R. C. Cowsik and M. V. Nori, *On the fibers of blowing up*, J. Indian Math. Soc. (N.S.) **40** (1976), 217–222.

[9] M. Crupi, and G. Rinaldo, *Binomial edge ideals with quadratic Gröbner bases*, Electron. J. Comb. **18** (2011), no. 1, # P211, 1–13.

[10] M. Crupi and G. Rinaldo, *Closed graphs are proper interval graphs*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **22** (2014), no. 3, 37–44.

[11] D. Eisenbud and C. Huneke, *Cohen–Macaulay Rees algebras and their specializations*, J. Algebra **81** (1983), 202–224.

[12] V. Ene and J. Herzog, *Gröbner bases in Commutative Algebra*, Grad. Stud. Math. **130**, Amer. Math. Soc., Providence, 2011.

[13] V. Ene and J. Herzog, "On the symbolic powers of binomial edge ideals", in (D. Stamate and T. Szemberg, Eds.), *Combinatorial Structures in Algebra and Geometry*, Springer Proceedings in Mathematics & Statistics **331**, Springer, Berlin, Heidelberg, 2020, 43–50.

[14] V. Ene, J. Herzog, and T. Hibi, *Cohen-Macaulay binomial edge ideals*, Nagoya Math. J. **204** (2011), 57–68.

[15] V. Ene, J. Herzog, and T. Hibi, F. Mohammadi, *Determinantal facet ideals*, Mich. Math. J. **62** (2013), 39–57.

[16] V. Ene and A. Zarojanu, *On the regularity of binomial edge ideals*, Math. Nachr. **288** (2015), no. 1, 19–24.

[17] D. Grayson and M. Stillman, Macaulay2, a software system for research in algebraic geometry, 2002, http://www.math.uiuc.edu/Macaulay2/

[18] E. Guardo and A. Van Tuyl, *Powers of complete intersections: Graded Betti numbers and applications*, Illinois J. Math. **49** (2005), no. 1, 265–279.

[19] H. T. Hà, N. V. Trung, and T. N. Trung, *Depth and regularity of powers of sums of ideals*, Math. Z. **282** (2016), 819–838.

[20] J. Herzog and T. Hibi, *The depth of powers of an ideal*, J. Algebra **291** (2005), 534–550.

[21] J. Herzog and T. Hibi, *Monomial Ideals*, Grad. Texts in Math. **260**, Springer, London, 2010.

[22] J. Herzog, T. Hibi, F. Hreinsdóttir, T. Kahle, and J. Rauh, *Binomial edge ideals and conditional independence statements*, Adv. Appl. Math. **45** (2010), 317–333.

[23] J. Herzog, T. Hibi, and H. Ohsugi, *Binomial Ideals*, Grad. Texts in Math. **279**, Springer, London, 2018.

[24] J. Herzog, A. Asloob Qureshi, *Persistence and stability properties of powers of ideals*, J. Pure Appl. Algebra **219** (2015), 530–542.

[25] M. Hochster, *Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes*, Ann. Math. **96** (1972), 318–337.

[26] C. Huneke, *Powers of ideals generated by weakd-sequences*, J. Algebra **68** (1981), 471–509.

[27] C. Huneke, *On the associated graded ring of an ideal*, Illinois. J. Math. **26** (1982), 121–137.

[28] A. V. Jayanthan, A. Kumar, and R. Sarkar, *Regularity of powers of quadratic sequences with applications to binomial edge ideals*, J. Algebra **564** (2020), 98–118.

[29] A. V. Jayanthan, A. Kumar, and R. Sarkar, *Almost complete intersection binomial edge ideals and their Rees algebras*, J. Pure Appl. Algebra **225** (2021), no. 6, 106628.

[30] A. V. Jayanthan, N. Narayanam, and S. Selvaraja, *Regularity of powers of bipartite graphs*, J. Algebr. Combin. **47** (2018), 17–38.

[31] K. Kimura, N. Terai, and S. Yassemi, *The projective dimension of the edge ideal of a very well-covered graph*, Nagoya Math. J. **230** (2018), 160–179.

[32] J. Martinez-Bernal, S. Morey, and R. H. Villarreal, *Associated primes of powers of edge ideals*, Collect. Math. **63** (2012), 361–374.

[33] H. Ohsugi and T. Hibi, *Koszul bipartite graphs*, Adv. Appl. Math. **22** (1999), 25–28.

[34] M. Ohtani, *Graphs and ideals generated by some2–minors*, Comm. Algebra **39** (2011), 905–917.

[35] A. Rauf and G. Rinaldo, *Construction of Cohen-Macaulay binomial edge ideals*, Comm. Algebra **42** (2014), no. 1, 238–2252.

[36] A. Simis, W. V. Vasconcelos, and R. H. Villarreal, *On the ideal theory of graphs*, J. Algebra **167** (1994), no. 2, 389–416.

[37] T. N. Trung, *Stability of depths of powers of edge ideals*, J. Algebra **452** (2016), 157–187.

[38] R. Waldi, *Vollständige durchschnitte in Cohen-Macaulay Ringen*, Arch. Math. (Basel) **31** (1978), 439–442.

Viviana Ene
Faculty of Mathematics and Computer Science,
Ovidius University,
Bd. Mamaia 124, 900527 Constanta,
Romania

vivian@univ-ovidius.ro

Giancarlo Rinaldo
Department of Mathematics,
University of Trento,
Via Sommarive, 14 38123 Povo (Trento),
Italy

giancarlo.rinaldo@unitn.it

Naoki Terai
Department of Mathematics,
Okayama University,
3-1-1, Tsushima-naka, Kita-ku,
Okayama, 700-8530,
Japan
terai@okayama-u.ac.jp