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Boundary singularities on a wedge-like domain of a semilinear elliptic equation

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Let $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^{n+1}$ be a Lipschitz wedge-like domain. We construct positive weak solutions of the problem

$$\Delta u + u^p = 0 \quad \text{in } \Omega$$

that vanish in a suitable trace sense on $\partial\Omega$, but which are singular at a prescribed 'edge' of Ω if p is equal to or slightly above a certain exponent $p_0 > 1$ that depends on Ω. Moreover, for the case in which Ω is unbounded, the solutions have fast decay at infinity.

Keywords: prescribed boundary singularities; very weak solution; critical exponents; wedge-like domains

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with smooth boundary $\partial\Omega$. A model nonlinear elliptic boundary-value problem is the classical Lane–Emden–Fowler equation,

$$\left.\begin{aligned} -\Delta u &= |u|^p && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{in } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $p > 1$. Following Brezis and Turner [3] and Quittner and Souplet [13], we will say that a positive function u is a very weak solution of problem (1.1) if $u, \operatorname{dist}(x, \partial\Omega)u^p \in L^1(\Omega)$ and

$$\int_\Omega u\Delta v + |u|^p v \,\mathrm{d}x = 0 \quad \forall v \in C^2(\bar{\Omega}) \text{ with } v = 0 \text{ on } \partial\Omega.$$

From the results in [3, 13], it follows that if p satisfies the constraint

$$1 < p < \frac{n+1}{n-1}, \tag{1.2}$$

then $u \in C^2(\bar{\Omega})$, i.e. u is a classical solution of problem (1.1).

It is well known that if $1 < p < (n+2)/(n-2)$, one can use Sobolev's embedding theorem and standard variational techniques to prove the existence of a positive very weak solution of problem (1.1). However, if $(n+1)/(n-1) < p < (n+2)/(n-2)$, this very weak solution may not be bounded. A result in the understanding of very weak solutions was achieved by Souplet [14]. He constructed an example of a positive function $a \in L^\infty(\Omega)$ such that problem (1.1), with u^p replaced by $a(x)u^p$ for $p > (n+1)/(n-1)$, has a very weak solution that is unbounded, developing a point singularity on the boundary. This shows that the exponent $p = (n+1)/(n-1)$ is truly a critical exponent. Let us mention that the behaviour of any positive solution of (1.1) near an isolated boundary singularity when $p \geqslant (n+1)/(n-1)$ was studied by Bidaut-Véron *et al.* in [2]. Finally, del Pino *et al.* [5] showed the existence of $\varepsilon > 0$ such that for any $p \in [(n+1)/(n-1), (n+1)/(n-1) + \varepsilon)$, an unbounded positive very weak solution of (1.1) exists that blows up at a prescribed point of $\partial\Omega$. For the same problem with interior singularity see, for example, [4, 6, 11, 12].

Let us give some definitions for the convenience of the reader. Let $n \geqslant 2$ and let $(r, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$ be the spherical coordinates of $x \in \mathbb{R}^n$, abbreviated by $x = (r, \theta)$. Given an open Lipschitz spherical cap $\omega \subsetneq \mathbb{S}^{n-1}$, let

$$C_\omega = \{x = (r, \theta) \colon r > 0,\ \theta \in \omega\}$$

be the corresponding infinite cone. The set

$$C_\omega^R = C_\omega \cap B_R(0) \subset \mathbb{R}^n$$

is called a conical piece with spherical cap ω and radius R.

A bounded Lipschitz domain $\Omega \subset C_\omega$ is called a domain with a conical boundary piece if there exists a conical piece C_ω^R such that $\Omega \cap B_R(0) = C_\omega^R$.

We denote λ and $\phi_1(\theta)$ to be, respectively, the first eigenvalue and the corresponding eigenfunction of the problem

$$\left.\begin{aligned} -\Delta_{\mathbb{S}^{n-1}} u &= \lambda u \quad \text{in } \omega, \\ u &= 0 \qquad \text{on } \partial\omega, \end{aligned}\right\} \tag{1.3}$$

with $\int_\omega \phi_1^2 \, \mathrm{d}S_x = 1$.

Finally, we define the exponent

$$p^* = \frac{n+\gamma}{n+\gamma-2} \quad \text{with } \gamma = \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda} \tag{1.4}$$

and note that p^* depends on ω.

In the same spirit as above, McKennab and Reichel [9] generalized the results of Souplet [14] to domains with conical boundary piece, and they showed that the exponent p^* is a truly critical exponent in the sense that if $1 < p < p^*$, then every very weak solution of problem (1.1) is bounded (see also [1]). Finally, Horák *et al*. [8] considered a bounded Lipschitz domain Ω with a conical boundary piece of spherical cap $\omega \subset \mathbb{S}^{n-1}$ at $0 \in \partial\Omega$ and they proved the existence of $\varepsilon > 0$ such that for any $p \in (p^*, p^* + \varepsilon)$, an unbounded positive very weak solution of (1.1) exists that blows up at $0 \in \partial\Omega$.

Let us consider the following problem:

$$\left.\begin{aligned} \Delta_x u + u^p &= 0 \quad \text{in } C_\omega, \\ u &> 0 \quad \text{in } C_\omega, \\ u &= 0 \quad \text{on } \partial C_\omega \setminus 0. \end{aligned}\right\} \tag{1.5}$$

In [8] it was proved that problem (1.5) admits a positive solution of the form $w(\theta) = |x|^{-2/(p-1)}\phi_p(\theta)$, where ϕ_p solves the problem

$$\left.\begin{aligned} \Delta_{\mathbb{S}^{n-1}}\phi - \frac{2}{p-1}\left(-\frac{2}{p-1} + n - 2\right)\phi + \phi^p &= 0 \quad \text{in } \omega, \\ \phi &= 0 \quad \text{on } \partial\omega, \end{aligned}\right\} \tag{1.6}$$

for any $p \in (p^*, \infty)$ if $n = 2, 3$, and any $p \in (p^*, (n+1)/(n-3))$ if $n \geqslant 4$, but this solution does not have fast decay at infinity.

We note here that if $\omega = \mathbb{S}^{n-1}_+$, then $\gamma = 1$, and thus the critical exponent $p^* = (n+1)/(n-1)$ and $C_\omega = \mathbb{R}^n_+$. In [5], del Pino *et al*. constructed a solution of problem (1.5) in $\mathbb{R}^n_+$ with fast decay. More precisely, they showed that there exists $\varepsilon > 0$ such that for any $p \in ((n+1)/(n-1), (n+1)/(n-1)+\varepsilon)$, problem (1.5) in $\mathbb{R}^n_+$ admits a solution $u \in C^2(\mathbb{R}^n_+)$ satisfying

$$u(x) \approx |x|^{-2/(p-1)}\phi_p(\theta) \quad \text{as } |x| \to 0$$

and

$$u(x) \approx |x|^{-(n-1)}\phi_1(\theta) \qquad \text{as } |x| \to \infty.$$

The first result of this work is the construction of a singular solution at 0 with fast decay at infinity for problem (1.5). In particular, we prove the following theorem.

THEOREM 1.1. *There exists a number $p(n,\lambda) > p^*$ such that for any*

$$p \in (p^*, p(n,\lambda))$$

there exists a solution $u_1(x)$ to problem (1.5) such that

$$u_1(x) = |x|^{-2/(p-1)}\phi_p(\theta)(1 + o(1)) \quad \text{as } |x| \to 0,$$

where ϕ_p solves (1.6), and

$$u_1(x) = |x|^{2-\gamma-n}\phi_1(\theta)(1 + o(1)) \qquad \text{as } |x| \to \infty,$$

where γ is defined in (1.4). In addition, we have the pointwise estimate

$$|u_1(x)| \leqslant C|x|^{-2/(p-1)}\|\phi_p\|_{\mathcal{C}^2(\omega)},$$

for some constant $C > 0$ that does not depend on p.

To describe our main result, let us introduce some new notation.

Let $x \in \mathbb{R}^n$ with $n \geqslant 2$. Given $\tau \in \mathbb{R}$, we let $\omega(\tau) \subsetneq \mathbb{S}^{n-1}$ be the corresponding Lipschitz spherical cap. We set

$$r_{\sigma(\tau)} = |x - \sigma(\tau)|,$$

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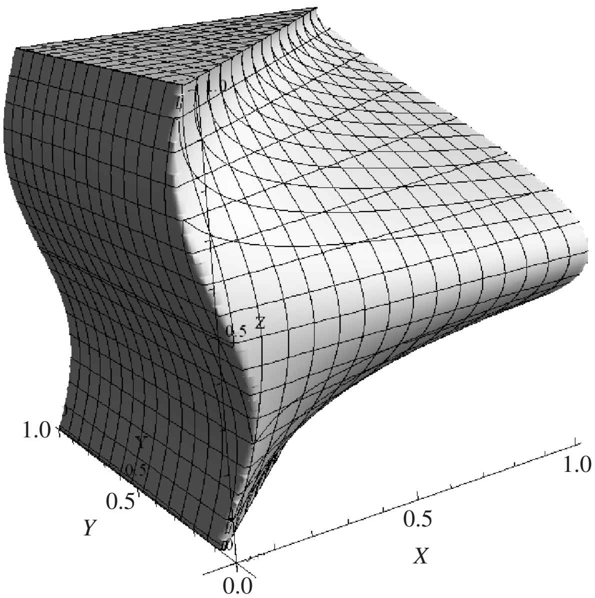

Figure 1. $\Omega_{0,1}$

where $\sigma\colon \mathbb{R} \to \mathbb{R}^n$ is a smooth curve such that

$$\sup_{\tau\in\mathbb{R}}\{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < C < \infty.$$

Now, given τ, we let $(r_{\sigma(\tau)}, \theta) \in [0,\infty) \times \mathbb{S}^{n-1}$ be the spherical coordinates of $x \in \mathbb{R}^n$ centred at $\sigma(\tau)$, abbreviated by $x = (r_{\sigma(\tau)}, \theta)$. We define

$$\tilde{C}_{\omega(\tau)} = \{x = (r_{\sigma(\tau)}, \theta)\colon r_{\sigma(\tau)} > 0,\ \theta \in \omega(\tau)\} \subset \mathbb{R}^n$$

and we set

$$\Omega_{\tau_1,\tau_2} = \{(\tau, x) \in (\tau_1,\tau_2) \times \mathbb{R}^n\colon x \in \tilde{C}_{\omega(\tau)}\} \subset \mathbb{R}^{n+1} \quad \text{(see figure 1)},$$
$$\Omega^R_{\tau_1,\tau_2} = \Omega_{\tau_1,\tau_2} \cap \{(\tau, x) \in (\tau_1,\tau_2) \times \mathbb{R}^n\colon x \in B_R(\sigma(\tau))\} \subset \mathbb{R}^{n+1}$$

and

$$S_{\tau_1,\tau_2} = \{(\tau, x) \in [\tau_1,\tau_2] \times \mathbb{R}^n\colon r_{\sigma(\tau)} = 0\}.$$

Finally, we define $\lambda^* = \inf_{\tau\in\mathbb{R}} \lambda(\tau)$ and $\gamma^* = \inf_{\tau\in\mathbb{R}} \gamma(\tau)$.

In this work we assume that $\omega(\tau)$ depends smoothly on τ, i.e. $\lambda(\tau)$ is a smooth bounded function with respect to τ with bounded derivatives. We also assume that $\inf_{\tau\in\mathbb{R}} \lambda(\tau) > 0$. Finally, we suppose that there exists $\varepsilon > 0$ such that for any $p \in (\sup_{\tau\in\mathbb{R}} p^*(\tau), \sup_{\tau\in\mathbb{R}} p^*(\tau) + \varepsilon)$ there exists a solution $u_1(\tau, x)$ of theorem 1.1. This means that $\operatorname{osc}_{\tau\in\mathbb{R}} \lambda(\tau)$ is small enough.

Theorem 1.2. *Let $\varepsilon > 0$ be small enough. There then exists a number $p_0 > \sup_{\tau\in\mathbb{R}} p^*$ such that, given $p \in (\sup_{\tau\in\mathbb{R}} p^*, p_0)$ and $2/(p-1) \leqslant -\rho < n + \gamma^* - 2$, the*

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problem

$$\begin{aligned} -\Delta u &= u^p \quad \text{in } \Omega_{-\infty,\infty}, \\ u &> 0 \quad \text{in } \Omega_{-\infty,\infty}, \\ u &= 0 \quad \text{on } \partial\Omega_{-\infty,\infty} \setminus S_{-\infty,\infty}, \end{aligned}$$

possesses very weak solutions u. In addition, we have that

$$u(\tau, x) \approx u_1\left(\tau, \frac{x - \sigma(\tau)}{\varepsilon}\right) \quad \text{as } r_{\sigma(\tau)} \to 0,$$

where u_1 is as in theorem 1.1, and

$$u(\tau, x) \leqslant C, r^{\rho}_{\sigma(\tau)} \quad \text{as } r_{\sigma(\tau)} \to \infty.$$

Our third and final result of this paper is the following theorem.

THEOREM 1.3. *Let $\alpha > 0$ be small enough and let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded Lipschitz domain such that*

$$\Omega \cap \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} = \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} \subset \mathbb{R}^{n+1}.$$

There exists a number $p_0 > \sup_{\tau\in\mathbb{R}} p^*$ such that, given $p \in (\sup_{\tau\in\mathbb{R}} p^*, p_0)$, there exist very weak solutions u to the problem

$$\begin{aligned} -\Delta u &= u^p \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \setminus S_{\tau_1-\alpha,\tau_2+\alpha}. \end{aligned}$$

Moreover, for all $(\tau, x) \in \Omega^R_{\tau_1-\alpha/4,\tau_2+\alpha/4}$,

$$u(\tau, x) \approx u_1\left(\tau, \frac{x - \sigma(\tau)}{\varepsilon}\right) \quad \text{as } r_{\sigma(\tau)} \to 0.$$

The paper is organized as follows. In § 3 we prove theorem 1.1. In § 3.1, we prove some regularity results with respect τ, for the function $u_1(\tau, x)$ in theorem 1.1. We devote § 4 to the proofs of theorems 1.2 and 1.3.

2. The eigenvalue problem on spherical caps

Let $n \geqslant 2$, let $\tau \in \mathbb{R}$ and let $\omega(\tau) \subsetneq \mathbb{S}^{n-1}$ be the corresponding open Lipschitz spherical cap. We denote $\lambda(\tau)$ and $\phi_1(\tau, \theta)$ to be, respectively, the first eigenvalue and eigenfunction of the eigenvalue problem

$$\left.\begin{aligned} -\Delta_{\mathbb{S}^{n-1}} u &= \lambda(\tau) u \quad \text{in } \omega(\tau), \\ u &= 0 \quad \text{on } \partial\omega, \end{aligned}\right\} \tag{2.1}$$

with $\int_{\omega(\tau)} \phi_1^2 \, \mathrm{d}S_x = 1$.

We assume that $\omega(\tau)$ depends smoothly on τ, i.e. $\lambda(\tau)$ is a smooth bounded function with respect to τ with bounded derivatives. In addition, we assume that $\inf_{\tau\in\mathbb{R}} \lambda(\tau) > 0$.

Now note that, without loss of generality, we can set $\theta_1 = \cos t$, with $0 < t < \beta(\tau)$, where $\beta(\tau)$ is a smooth function with bounded derivatives satisfying

$$0 < \inf_{\tau\in\mathbb{R}} \beta(\tau) < \sup_{\tau\in\mathbb{R}} \beta(\tau) < 2\pi \quad \text{for } n = 2$$

and

$$0 < \inf_{\tau\in\mathbb{R}} \beta(\tau) < \sup_{\tau\in\mathbb{R}} \beta(\tau) < \pi \qquad \text{for } n \geqslant 3.$$

Then problem (2.1) is equivalent to

$$\left.\begin{aligned} -\sin^{2-n} t\frac{\mathrm{d}}{\mathrm{d}t}\left(\sin^{n-2} t\frac{\mathrm{d}\phi_1}{\mathrm{d}t}\right) &= \lambda\phi_1 \quad \text{in } (0,\beta(\tau)),\\ \phi_1(\beta(\tau)) &= 0,\\ \frac{\mathrm{d}\phi_1}{\mathrm{d}t}(0) &= 0, \end{aligned}\right\} \tag{2.2}$$

with

$$C(n)\int_0^{\beta(\tau)} \sin^{n-2}(t)|u|^2\,\mathrm{d}t = \int_\omega |\phi_1|^2\,\mathrm{d}S = 1.$$

We note here that for $n = 2$ in problem (2.2), we may have $\phi_1(0) = 0$ instead of $(\mathrm{d}\phi_1/\mathrm{d}t)(0) = 0$.

We have the following lemma.

Lemma 2.1. *Let $\phi_1(\tau,\theta)$ be the first eigenfunction of the eigenvalue problem*

$$\left.\begin{aligned} -\Delta_{\mathbb{S}^{n-1}} u &= \lambda u \quad \text{in } \omega(\tau),\\ u &= 0 \quad\ \ \text{on } \partial\omega(\tau), \end{aligned}\right\} \tag{2.3}$$

with $\int_{\omega(\tau)} \phi_1^2\,\mathrm{d}S = 1$. There then exists a positive constant C such that

$$\sup_{\tau\in\mathbb{R}} \left\| |\phi_1| + \left|\frac{\partial\phi_1}{\partial\tau}\right| + \left|\frac{\partial^2\phi_1}{\partial\tau^2}\right| \right\|_{L^\infty(\omega(\tau))} < C. \tag{2.4}$$

We postpone the proof of this lemma until the appendix.

3. Positive singular solution in the cone

We keep the assumptions and notation of the previous section and we consider the cone

$$C_{\omega(\tau)} = \{(r,\theta) \colon r > 0,\ \theta\in\omega(\tau)\},$$

where $r = |x|$ and $\theta = x/|x|$. We define the critical exponent

$$p^*(\tau) = \frac{n+\gamma(\tau)}{n+\gamma(\tau)-2} \quad \text{with } \gamma(\tau) = \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda(\tau)}.$$

We consider the problem

$$\left.\begin{aligned} \Delta_x u + u^p &= 0 \quad \text{in } C_{\omega(\tau)},\\ u &> 0 \quad \text{in } C_{\omega(\tau)},\\ u &= 0 \quad \text{on } \partial C_{\omega(\tau)} \setminus 0. \end{aligned}\right\} \tag{3.1}$$

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If we set $w = |x|^{-2/(p-1)}\phi(\theta)$, we arrive at the problem

$$\left.\begin{aligned} \Delta_{\mathbb{S}^{n-1}}\phi - \frac{2}{p-1}\left(-\frac{2}{p-1} + n - 2\right)\phi + \phi^p = 0 \quad &\text{in } \omega(\tau), \\ \phi = 0 \quad &\text{on } \partial\omega(\tau). \end{aligned}\right\} \tag{3.2}$$

By [8, lemma 9], problem (3.2) has a positive solution $\phi_p \in H_1(\omega(\tau)) \cap L^\infty(\omega(\tau))$ for any $p \in (p^*, \infty)$ if $n = 2$ or 3, and for any $p \in (p^*(\tau), (n+1)/(n-3))$ if $n \geqslant 4$. Also, as $p \downarrow p^*(\tau)$ we have $-(2/(p-1))(-(2/(p-1)) + n - 2) \uparrow \lambda(\tau)$ and

$$\phi_p = \left(\frac{\lambda - (2/(p-1))(-(2/(p-1)) + n - 2)}{c_p}\right)^{1/(p-1)}(\phi_1 + o(1)),$$

where $c_p = \int_{\omega(\tau)} \phi_1^{p+1}\,\mathrm{d}\theta$.

In addition, for the same range of p, by [8, theorem 10], the function

$$w_p(\tau, r, \theta) = r^{-2/(p-1)}\phi_p(\tau, \theta)$$

is a positive solution of (3.1).

In the rest of this section, for convenience, we omit dependence on the parameter τ, writing $\lambda = \lambda(\tau)$, $\phi_1(\theta) = \phi_1(\tau, \theta)$ and so on.

Let $p \in (p^*, (n+2)/(n-2))$. We look for solutions of (3.1) of the form

$$u_1(x) = |x|^{-2/(p-1)}\phi(-\log|x|, \theta), \tag{3.3}$$

where $\theta = x/|x|$, so that the equation $\Delta u + u^p = 0$ reads, in terms of the function ϕ, defined for $t \in \mathbb{R}$ and $\theta \in \omega$, as

$$\partial_t^2\phi + A\phi_t - \varepsilon\phi + (\Delta_{\mathbb{S}^{n-1}}\phi + \lambda\phi) + \phi^p = 0, \tag{3.4}$$

where

$$t = -\log r, \quad A = -\left(n - 2\frac{p+1}{p-1}\right) \quad \text{and} \quad \varepsilon = \lambda + \frac{2}{p-1}\left(n - \frac{2p}{p-1}\right).$$

Letting $\mu = \int_\omega \phi_1^{p+1}\,\mathrm{d}\theta$, we define a_∞ by

$$\mu a_\infty^{p-1} = \varepsilon.$$

We look for a positive function a that is a solution of

$$a''(t) + Aa'(t) - \varepsilon a(t) + \mu a^p(t) = 0, \tag{3.5}$$

which converges to 0 as t tends to $-\infty$ and converges to a_∞ as t tends to $+\infty$. Observe that when $p \in (p^*, (n+2)/(n-2))$, the coefficients A and ε are positive, and therefore, in this range, classical ordinary differential equation (ODE) techniques yield the existence of a, a positive heteroclinic solution of (3.5) tending to 0 at $-\infty$ and tending to a_∞ at $+\infty$.

Observe that since (3.5) is autonomous, the function a is not unique and a can be normalized so that $a(0) = \frac{1}{2}a_\infty$. For more information about the function a, we refer the reader to lemmas 2.3–2.5 and the appendix in [5].

Proposition 3.1. *Let $0 \leqslant p_0 < \infty$ and ε be small enough. There then exists a unique operator*

$$G_{p_0}\colon a^{p_0}L^{\infty}(\mathbb{R}\times\omega) \mapsto a^{p_0}L^{\infty}(\mathbb{R}\times\omega)$$

such that for any $a^{-p_0}g \in L^{\infty}(\mathbb{R}\times\omega)$ the function $u = G_{p_0}(g)$ is the unique solution of

$$L_p u = (\partial_t^2 + A\partial_t - \varepsilon + (\Delta_{\mathbb{S}^{n-1}} + \lambda) + p\phi_0^{p-1})u = g, \quad \phi_0 = a(t)\phi_1(\theta),$$

with zero Dirichlet boundary data.

Furthermore,

$$\|d^{-1}a^{-p_0}(t)\psi\|_{L^{\infty}(\mathbb{R}\times\omega)} \leqslant \frac{C}{\varepsilon}\|a^{-p_0}(t)g\|_{L^{\infty}(\mathbb{R}\times\omega)}. \tag{3.6}$$

If, in addition, $g(t,\cdot)$ is L^2-orthogonal to ϕ_1 for almost every t, then we have

$$\|d^{-1}a^{-p_0}(t)\psi\|_{L^{\infty}(\mathbb{R}\times\omega)} \leqslant C\|a^{-p_0}(t)g\|_{L^{\infty}(\mathbb{R}\times\omega)},$$

where $d\colon \omega \to (0,\infty)$ denotes the distance function to $\partial\omega$.

Proof. The proof follows the same lines as in [5, lemma 2.6], so we will only focus on the differences. We first define ϕ_* to be the positive solution of

$$\left.\begin{aligned} \Delta_{\mathbb{S}^{n-1}}\phi_* + \lambda\phi_* + \delta(\delta - n - 2\gamma + 2)\phi_* &= -1 \quad \text{in } \omega,\\ \phi_* &= 0 \qquad \text{on } \partial\omega \end{aligned}\right\} \tag{3.7}$$

(see the proof of lemma 2.6 in [5] with obvious modifications). Using the function $(t,\theta) \to \mathrm{e}^{-\delta t}\phi_*(\theta)$ as a barrier, as in [5], we can show that, given any function g such that $a^{-p_0}g \in L^{\infty}(\mathbb{R}\times\omega)$ and given $t_1 < -1 < 1 < t_2$, we can solve the equation

$$L_p u = g$$

in $(t_1, t_2)\times\omega$ with 0 boundary conditions.

To prove estimate (3.6) we argue by contradiction, assuming that

$$\|a^{-p_0}\psi_i\|_{L^{\infty}} = 1$$

and

$$\lim_{i\to\infty}\|a^{-p_0}f_i\| = 0,$$

and we get a contradiction using a similar argument to that in [5, lemma 2.6]. The rest of the proof is the same as that in [5, lemma 2.6] with obvious modifications, so we omit it here. □

Proof of theorem 1.1. We look for a solution to problem (3.4) of the form

$$\phi = a(t)\phi_1(\theta) + \psi(t,\theta),$$

and we let G_p be the operator defined in proposition 3.1. To conclude the proof, it is enough to find a function ψ that is a solution of the fixed-point problem

$$\psi = -G_p(\mathcal{M}(\phi_0) + \mathcal{Q}(\psi)),$$

where

$$\begin{aligned}\phi_0(t,\theta) &= a(t)\phi_1(\theta),\\ \mathcal{M}(\phi_0) &= a^p(\phi_1^p - \mu\phi_1),\\ \mathcal{Q}(\psi) &= |\phi_0+\psi|^p - \phi_0^p - p\phi_0^{p-1}\psi.\end{aligned}$$

The rest of the proof is the same as in [5]. We recall here that $\psi \ll a\phi_1$. Also, in [5] they have proven that if ε is small enough, then there exists a t_0 such that for any $t \leqslant -t_0/\varepsilon$,

$$\tfrac{1}{2}\mathrm{e}^{\delta^- t} \leqslant a(t) \leqslant \mathrm{e}^{\delta^- t}$$

with $\delta^- = \frac{1}{2}(\sqrt{A^2+4\varepsilon} - A)$, and the result follows, since

$$\frac{1}{2}(\sqrt{A^2+4\varepsilon} - A) + \frac{2}{p-1} = n + \gamma - 2.$$

□

Remark 3.2. If $1 < p_0 < p$ is close enough to p, we can apply a fixed-point argument to the operator G_{p_0}, like in the proof of theorem 1.1.

In view of the proof of lemma 2.1, $\phi_* = \phi_*(t, \cos(s\beta(\tau)))$.

Thus, if the function g in proposition 3.1 is of the form $g = g(t, \cos(s\beta(\tau)))$, we have that the solution $u = G_{p_0}(g)$ is of the form $u = u(t, \cos(s\beta(\tau)))$. Hence, we obtain that the solution u_1 in theorem 1.1 is of the form

$$u_1 = r^{-2/(p-1)} u_1(r, \cos(s\beta(\tau))).$$

3.1. Regularity of the solution u_1 with respect to τ

We first recall some definitions and known results; see [7] for the proofs.

Let

$$Lu = a^{i,j}(x)D_{i,j}u + b^i(x)D_iu + c(x)u = g(x), \quad a^{i,j} = a^{j,i},$$

where the coefficients $a^{i,j}$, b^i, c and the function g are defined in an open bounded domain $\Omega \subset \mathbb{R}^n$ and

$$a^{i,j}\xi_i\xi_j \leqslant \mu|\xi|^2, \quad \mu > 0.$$

We assume that

$$\|a^{i,j}\|_{C^{2,a}}, \|b^i\|_{C^{2,a}}, \|c\|_{C^{2,a}} \leqslant \Lambda.$$

Definition 3.3. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ and its boundary $\partial\Omega$ are of class $C^{k,a}$, $0 \leqslant a \leqslant 1$, if at each point $x \in \partial\Omega$ there is a ball $B_r(x)$ and a one-to-one mapping ψ from $B_r(x)$ onto $D \subset \mathbb{R}^n$ such that

$$\psi(B_r(x)\cap\Omega) \subset \mathbb{R}^n_+, \qquad \psi(B_r(x)\cap\partial\Omega) \subset \partial\mathbb{R}^n_+, \qquad \psi \in C^{k,a}(B_r(x))$$

and

$$\psi^{-1} \in C^{k,a}(D).$$

A domain Ω will be said to have a boundary portion $T \subset \partial\Omega$ of class $C^{k,a}$ if at each point $x \in T$ there is a ball $B_r(x)$ in which the above conditions are satisfied and such that $B_r(x) \cap \partial\Omega \subset T$.

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Proposition 3.4 (Gilbarg and Trudinger [7, lemma 6.18]). *Let $0 < a \leqslant 1$, let Ω be a domain with a $C^{2,a}$ boundary portion T, and let $\phi \in C^{2,a}(\bar{\Omega})$. Suppose that u is a $C^2(\Omega) \cap C_0(\bar{\Omega})$ function satisfying $Lu = g$ in Ω, $u = \phi$ on T, where g and the coefficients of the strictly elliptic operator L belong to $C^a(\bar{\Omega})$. Then $u \in C^{2,a}(\Omega \cup T)$.*

Proposition 3.5 (Gilbarg and Trudinger [7, corollary 6.7]). *Let $0 < a \leqslant 1$, let Ω be a domain with a $C^{2,a}$ boundary portion T, and let $\phi \in C^{2,a}(\bar{\Omega})$. Suppose that u is a $C^{2,a}(\Omega \cup T)$ function satisfying $Lu = g$ in Ω, $u = \phi$ on T. Then, if $x \in T$ and $B = B_\rho(x)$ is a ball with radius $\rho < \operatorname{dist}(x, \partial\Omega - T)$, we have*

$$\|u\|_{C^{2,a}(B\cap\Omega)} \leqslant C(n, \mu, \Lambda, \Omega \cap B_\rho(x))(\|u\|_{C(\Omega)} + \|\phi\|_{C^{2,a}(\bar{\Omega})} + \|g\|_{C^a(\Omega)}).$$

We first prove the following result.

Lemma 3.6. *Let $\tau \in \mathbb{R}$ be fixed, let $x \in \mathbb{R}^n$, $n \geqslant 2$, let $g \in C^a(\overline{C_\omega} \setminus \{0\})$ and let $u = G_p(g)$ be the operator in proposition 3.1. Then,*

$$\left.\begin{aligned} |\nabla_x u(\tau, x)| &\leqslant C(n, p, \lambda, C_{\omega(\tau)}, g)|x|^{-1}, \\ |D_x^2 u(\tau, x)| &\leqslant C(n, p, \lambda, C_{\omega(\tau)}, g)|x|^{-2}. \end{aligned}\right\} \tag{3.8}$$

Proof. First we note that $\|u(\tau, \cdot)\|_{L^\infty(C_\omega(\tau))} \leqslant C\|g(t, \cdot)\|_{L^\infty(C_\omega(\tau))}$ and that u is a solution of

$$\begin{aligned} -\Delta_x u + \frac{4}{p-1}\frac{x \cdot \nabla_x u}{|x|^2} & \\ + \frac{2}{p-1}\left(n - \frac{2}{p-1} - 2\right)\frac{u}{|x|^2} - p\frac{\phi_0^{p-1}u}{|x|^2} &= -\frac{g}{|x|^2} \qquad \text{in } C_{\omega(\tau)} \\ u &= 0 \qquad \text{in } \partial C_{\omega(\tau)} \setminus 0. \end{aligned}$$

Set $R = |x|$, consider the domain

$$\Omega_R = \{y \in C_\omega \colon \tfrac{1}{4}R < |y| < 4R\},$$

let $y = x/R$ and define $v(y) = u(\tau, Ry)$. Then $y \in \Omega_1$ and v is a solution of

$$\begin{aligned} -\Delta v + \frac{4}{p-1}\frac{y \cdot \nabla v}{|y|^2} + \frac{2}{p-1}\left(n - \frac{2}{p-1} - 2\right)\frac{v}{|y|^2} - p\frac{\phi_0^{p-1}v}{|y|^2} &= -\frac{g}{|y|^2} \quad \text{in } \Omega_1, \\ v &= 0 \qquad \text{in } T, \end{aligned}$$

where we have set

$$T = \partial\Omega_1 \setminus \{y \in C_\omega \colon |y| = \tfrac{1}{4} \text{ or } |y| = 4\}.$$

Let $0 < \varepsilon < \frac{1}{4}\rho$ be small enough, where ρ is defined in proposition 3.5 with $\Omega = \Omega_1$. Let $y_0 \in \partial\Omega_1 \setminus \{y \in C_\omega \colon |y| = \frac{1}{6} \text{ or } |y| = \frac{8}{3}\}$. Then, by propositions 3.4 and 3.5, we have

$$\|v\|_{C^2(B_\rho(\psi_0)\cap\Omega_{2/3})} \leqslant C(n, \mu, \Lambda, \Omega_1 \cap B_\rho(y_0))\|g\|_{C^a(\overline{\Omega_1})},$$

where in the last inequality we have used the estimate in proposition 3.1.

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We note here that ρ depends only on Ω_1 and not on y_0. Thus, if we apply a covering argument and standard interior Schauder estimates, we have

$$\|v\|_{C^2(\Omega_{1/2})} \leqslant C(n, \mu, \Lambda, \Omega_1, \rho)\|g(x)\|_{C^\alpha(\overline{\Omega_1})}.$$

Using the facts that $x \in \Omega_{R/2}$, $\nabla v(y) = R\nabla u(x)$, $D_{i,j}v = R^2 D_{i,j}u$, $R = |x|$ and the above estimate, the result follows at once. □

In the rest of this paper we assume that the Lipschitz spherical cap $\omega(\tau)$ has the following property:

there exists $\tilde{\varepsilon} > 0$ such that for any $p \in (\sup_{\tau\in\mathbb{R}} p^*(\tau), \sup_{\tau\in\mathbb{R}} p^*(\tau) + \tilde{\varepsilon})$ there exists a solution u_1 of theorem 1.1. Thus, $\varepsilon(\tau)$ is a smooth bounded function with bounded derivatives and there exist $\varepsilon_0, \varepsilon_1 > 0$ such that $\varepsilon_0 \leqslant \varepsilon(\tau) \leqslant \varepsilon_1$ for all $\tau \in \mathbb{R}$.

We now recall some facts from the proof of theorem 1.1. Let $a(\tau, t)$ be the solution of the problem

$$\partial_t^2 a + A\partial_t a - \varepsilon(\tau)a + \mu(\tau)a^p = 0, \tag{3.9}$$

where

$$A = -\left(n - 2\frac{p+1}{p-1}\right), \qquad \varepsilon(\tau) = \lambda(\tau) + \frac{2}{p-1}\left(n - \frac{2p}{p-1}\right),$$
$$\mu(\tau) = \int_{\omega(\tau)} \phi_1^{p+1}(\tau, \theta)\, \mathrm{d}\theta$$

and $\mu(\tau)a_\infty^{p-1}(\tau) = \varepsilon(\tau)$. Recall also that we have chosen $a(\tau, t)$ such that

$$a(\tau, 0) = \tfrac{1}{2}a_\infty(\tau), \qquad \lim_{t\to\infty} a(\tau, t) = a_\infty(\tau) \quad \text{and} \quad \lim_{t\to-\infty} a(\tau, t) = 0.$$

We next prove the following lemma.

Lemma 3.7. *Let a be the solution of (3.9), let $\varepsilon_0 = \inf_{\tau\in\mathbb{R}} \varepsilon(\tau)$,*

$$\tilde{\delta}^+(\tau) = \frac{-A + \sqrt{A^2 - 4(p-1)\varepsilon(\tau)}}{2} \qquad \text{and} \qquad \delta^-(\tau) = \frac{-A + \sqrt{A^2 + 4\varepsilon(\tau)}}{2}.$$

There then exists a $\tilde{t} > 0$ such that

$$\left|\frac{\partial a}{\partial \tau}(\tau, t)\right| \leqslant C(\varepsilon_0, p, n)|t|\mathrm{e}^{\delta^-(\tau)t} \quad \forall(\tau, t) \in \mathbb{R} \times \left(-\infty, -\frac{\tilde{t}}{\varepsilon_0}\right),$$
$$\left|\frac{\partial^2 a}{\partial \tau^2}(\tau, t)\right| \leqslant C(\varepsilon_0, p, n)|t|^2\mathrm{e}^{\delta^-(\tau)t} \quad \forall(\tau, t) \in \mathbb{R} \times \left(-\infty, -\frac{\tilde{t}}{\varepsilon_0}\right),$$
$$\left|\frac{\partial a}{\partial \tau}(\tau, t)\right| \leqslant C(\varepsilon_0, p, n)|t|\mathrm{e}^{\tilde{\delta}^+(\tau)t} \quad \forall(\tau, t) \in \mathbb{R} \times \left(\frac{\tilde{t}}{\varepsilon_0}, \infty\right),$$
$$\left|\frac{\partial^2 a}{\partial \tau^2}(\tau, t)\right| \leqslant C(\varepsilon_0, p, n)|t|^2\mathrm{e}^{\tilde{\delta}^+(\tau)t} \quad \forall(\tau, t) \in \mathbb{R} \times \left(\frac{\tilde{t}}{\varepsilon_0}, \infty\right)$$

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and

$$\left|\frac{\partial a}{\partial \tau}(\tau,t)\right| \leqslant C(\varepsilon_0,p,n) \quad \forall(\tau,t)\in\mathbb{R}\times\left[-\frac{\tilde{t}}{\varepsilon_0},\frac{\tilde{t}}{\varepsilon_0}\right],$$
$$\left|\frac{\partial^2 a}{\partial \tau^2}(\tau,t)\right| \leqslant C(\varepsilon_0,p,n) \quad \forall(\tau,t)\in\mathbb{R}\times\left[-\frac{\tilde{t}}{\varepsilon_0},\frac{\tilde{t}}{\varepsilon_0}\right].$$

Proof. By our assumptions and [5, lemma 2.5], there exists a constant $\bar{t}<0$ (independent of p, μ and τ) such that

$$\tfrac{1}{2}\mathrm{e}^{\delta^-(\tau)t} \leqslant \frac{a(\tau,t)}{a_\infty(\tau)} \leqslant \mathrm{e}^{\delta^-(\tau)t} \quad \forall t\leqslant\frac{\bar{t}}{\varepsilon_0},$$

where

$$\delta^-(\tau)=\frac{-A+\sqrt{A^2+4\varepsilon(\tau)}}{2}.$$

Choose $\tau_0\in\mathbb{R}$ and set $a(\tau,t)=a_\infty(\tau)(\mathrm{e}^{\delta^-(\tau)t}+w(\tau,t))$. Then w is a solution of the fixed-point problem

$$w=-\varepsilon\mathrm{e}^{\delta^-(\tau)t}\int_{-\infty}^{t}\mathrm{e}^{-2\delta^-(\tau)\zeta-A\zeta}\left(\int_{-\infty}^{\zeta}\mathrm{e}^{\delta^-(\tau)s+As}(\mathrm{e}^{\delta^-(\tau)s}+w)^p\,\mathrm{d}s\right)\mathrm{d}\zeta := T[w]. \tag{3.10}$$

Indeed, let $1<p_0<p$ and let ρ be sufficiently small such that for any $\tau\in O_{\tau_0}=\{\tau\in\mathbb{R}\colon|\tau-\tau_0|<\rho\}$ we have

$$p\delta^-(\tau)\geqslant p_0\delta^-(\tau_0) \quad\text{and}\quad p\delta^-(\tau_0)\geqslant p_0\delta^-(\tau).$$

Thus, it is easy to find a fixed point in the set of functions defined in $(-\infty,\bar{t}/\varepsilon_0)$ and satisfying

$$|w|\leqslant\tfrac{1}{2}\mathrm{e}^{p_0\delta^-(\tau_0)t}$$

provided that $|\bar{t}|$ is fixed large enough (independent of p and τ).

Now let

$$G=\left\{g\colon\left(-\infty,\frac{\bar{t}}{\varepsilon_0}\right)\mapsto\mathbb{R}\colon\|\mathrm{e}^{-p_0\delta^-(\tau_0)t}g\|_{L^\infty(-\infty,\bar{t}/\varepsilon_0)}<C\right\}$$

and define $F(\tau,g)=g-T(g)$. By (3.10), we can apply the implicit function theorem in the domain $O_{\tau_0}\times G$ to obtain that there exists a unique function w such that

$$F(\tau,w(\tau,t))=0 \quad\text{for any } |\tau-\tau_0|<\rho_0<\rho$$

for some ρ_0 small enough. On the other hand, since $T(g)$ is smooth with respect to τ, we have that $w(\tau,t)$ is smooth with respect to τ.

Notice that

$$0=F_\tau(\tau,w(\tau,t))+F_g(\tau,w(\tau,t))\frac{\partial w}{\partial\tau},$$

and thus we have

$$\left|\frac{\partial w}{\partial\tau}(\tau,t)\right|\leqslant C(\varepsilon_0,p,n)|t|\mathrm{e}^{\delta^- t}, \tag{3.11}$$

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provided that $|\bar{t}|$ is fixed large enough. Similarly, we have

$$\left|\frac{\partial^2 w}{\partial \tau^2}(\tau,t)\right| \leqslant C(\varepsilon_0,p,n)|t|^2 \mathrm{e}^{\delta^- t}. \tag{3.12}$$

By (3.10) and the above inequalities, we have that the derivatives $\partial^2 w/\partial\tau\partial t$, $\partial^3 w/\partial^2\tau\partial t$ exist and are bounded.

Since the choice of τ_0 is abstract, we conclude that the functions $a, \partial_t a \in C^2$ with respect to τ for any $t \leqslant \bar{t}/\varepsilon_0$. We also have

$$\left.\begin{aligned} \left|\frac{\partial a}{\partial \tau}(\tau,t)\right| &\leqslant C(\varepsilon_0,p,n)|t|\mathrm{e}^{\delta^-(\tau)t} \quad \forall(\tau,t) \in \mathbb{R}\times\left(-\infty,-\frac{\tilde{t}}{\varepsilon_0}\right),\\ \left|\frac{\partial^2 a}{\partial \tau^2}(\tau,t)\right| &\leqslant C(\varepsilon_0,p,n)|t|^2\mathrm{e}^{\delta^-(\tau)t} \quad \forall(\tau,t) \in \mathbb{R}\times\left(-\infty,-\frac{\tilde{t}}{\varepsilon_0}\right).\end{aligned}\right\} \tag{3.13}$$

Let $t_0 \in (-\infty, \bar{t}/\varepsilon_0)$ such that $a(\tau,t_0), \partial a(\tau,t_0)/\partial t \in C^2$ with respect to τ. Using standard ODE techniques, we can prove that if $|h|$ is sufficiently small, then

$$|a(\tau,t) - a(\tau+h,t)| \leqslant C(t)h \quad \forall t \in \mathbb{R}, \tag{3.14}$$

where $C(t)$ is a positive smooth function such that $\lim_{t\to\infty} C(t) = \infty$.

Choose $|h|$ sufficiently small and set $v_h = (a(\tau+h,t) - a(\tau,t))/h$ and $a(\tau) = a(\tau,t)$. Then v_h satisfies

$$\left.\begin{aligned} &\frac{\partial^2 v_h}{\partial t^2} + A\frac{\partial v_h}{\partial t} - \varepsilon(\tau+h)v_h \\ &\qquad = -\mu(\tau+h)\frac{a^p(\tau+h)-a^p(\tau)}{h} - \frac{\mu(\tau+h)-\mu(\tau)}{h}a^p(\tau) \\ &\qquad\qquad + \frac{\varepsilon(\tau+h)-\varepsilon(\tau)}{h}a(\tau) \quad \text{in } (t_0,\infty), \\ &v_h(\tau,t_0) = \frac{a(\tau+h,t_0)-a(\tau,t_0)}{h}, \\ &\frac{\partial v_h(\tau,t_0)}{\partial t} = \frac{\partial a(\tau+h,t_0)/\partial t - \partial a(\tau,t_0)/\partial t}{h}. \end{aligned}\right\} \tag{3.15}$$

Using the expansion

$$\begin{aligned} a^p(\tau+h) &= a^p(\tau) + pa^{p-1}(\tau,t)(a(\tau+h)-a(\tau)) \\ &\quad + \frac{1}{2}\int_{a(\tau)}^{a(\tau+h)} p(p-1)t^{p-2}(a(\tau+h)-t)\,\mathrm{d}t, \end{aligned}$$

the properties of the initial data in (3.15), our assumptions on μ and ε, (3.14) and by using standard ODE techniques in (3.15), we can obtain that

$$|v_h|, \left|\frac{\partial v_h}{\partial t}\right| < C(t),$$

where $C(t)$ is a positive smooth function such that $\lim_{t\to\infty} C(t) = \infty$. Thus, by the Arzelà–Ascoli theorem, there exists a subsequence $\{v_{h_n}\}$ such that $v_{h_n} \to v$ locally

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uniformly and v satisfies

$$\frac{\partial^2 v}{\partial t^2} + A\frac{\partial v}{\partial t} - \varepsilon(\tau)v = -\mu(\tau)pa^{p-1}(\tau,t)v - \mu'(\tau)a^p(\tau) + \varepsilon'(\tau)a(\tau) \quad \text{in } (t_0,\infty),$$
$$v(\tau,t_0) = \frac{\partial a(\tau,t_0)}{\partial\tau},$$
$$\frac{\partial v(\tau,t_0)}{\partial t} = \frac{\partial^2 a(\tau,t_0)}{\partial\tau\partial t}.$$

By uniqueness of the above problem, we have that $\lim_{h\to 0} v_h = v$ for all $\tau\in\mathbb{R}$ and $t \geqslant t_0$. Thus, $(\partial/\partial\tau)a(\tau,t)$ exists for any $(\tau,t)\in\mathbb{R}^2$. Applying the same argument, we can obtain also that $(\partial^2/\partial\tau^2)a(\tau,t)$ exists for any $(\tau,t)\in\mathbb{R}^2$. The only difference is that we should use the fact that $a(\tau,t) > c > 0$ for any $(\tau,t)\in\mathbb{R}\times(t_0,\infty)$.

Set $a = a_\infty w$. Then w satisfies

$$\partial_t^2 w + A\partial_t w - \varepsilon(\tau)w + \varepsilon(\tau)w^p = 0. \tag{3.16}$$

Let us now recall some facts from [5, lemma 2.5]. Set

$$\tilde\delta^+(\tau) = \frac{-A+\sqrt{A^2-4(p-1)\varepsilon(\tau)}}{2} \quad \text{and} \quad \tilde\delta^-(\tau) = \frac{-A-\sqrt{A^2-4(p-1)\varepsilon(\tau)}}{2}.$$

There exists a $\hat t > 0$ (independent of p and τ) such that for all $t \geqslant \hat t/\varepsilon_0$,

$$\left.\begin{aligned} \tfrac12 e^{\tilde\delta^-(\tau)t} \leqslant 1 - w(\tau,t) \leqslant 2e^{\tilde\delta^-(\tau)t}, \\ \frac{1}{C(\varepsilon_0)}w(1-w) \leqslant \frac{\partial w}{\partial t} \leqslant C(\varepsilon_0)w(1-w). \end{aligned}\right\} \tag{3.17}$$

Notice that the function $\partial w/\partial\tau$ is a solution of

$$\frac{\partial^2 v}{\partial t^2} + A\frac{\partial v}{\partial t} - \varepsilon(\tau)v + pw^{p-1}(\tau,t)v = \varepsilon'(\tau)w^p(\tau) + \varepsilon'(\tau)w(\tau), \tag{3.18}$$

but the function $\partial a/\partial t$ is one solution of the corresponding homogeneous problem. For the other solution of the homogeneous problem ψ, we can easily prove by using (3.17) that

$$|\psi(t,\tau)| \leqslant C(\varepsilon_0)e^{\tilde\delta^-(\tau)t}.$$

Thus, by the representation formula and the properties of w, we can easily obtain

$$\left|\frac{\partial w}{\partial\tau}\right| \leqslant C(\varepsilon_0,p,n)|t|e^{\tilde\delta^+(\tau)t} \quad \forall t \geqslant \frac{\tilde t}{\varepsilon_0}.$$

Using (3.17) and the fact that w is a solution of (3.16), we can prove that

$$\left|\frac{\partial^2 w}{\partial t^2}\right| < C(\varepsilon_0,n,p)e^{\tilde\delta^+(\tau)t}.$$

Setting $w = (1 - e^{\tilde\delta^+(\tau)t} + v)$, v can be written as (see the appendix in [5])

$$v = \varepsilon e^{\tilde\delta^-(\tau)t}\int_{t_p}^{t} e^{-2\tilde\delta^-(\tau)\zeta - A\zeta}\left(\int_\zeta^\infty e^{\tilde\delta^-(\tau)s+As}\mathcal{Q}(-e^{\tilde\delta^+(\tau)s}+v)\,ds\right)d\zeta + \lambda_p e^{\tilde\delta^-(\tau)t}, \tag{3.19}$$

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where $Q(x) = |1+x|^p - 1 - px$, t_p is large enough and $\lambda_p(\tau)$ is a smooth bounded function. Thus, by (3.19) and the definition of v, we can prove that there exists a constant $C > 0$ such that

$$\frac{1}{C}\mathrm{e}^{\tilde{\delta}^+(\tau)t} \leqslant -\partial_t^2 w(\tau,t) \leqslant C\mathrm{e}^{\tilde{\delta}^+(\tau)t} \quad \forall t \geqslant t_p.$$

By the same argument, we can prove that

$$\left|\frac{\partial^2 w}{\partial \tau^2}(\tau,t)\right| \leqslant C(\varepsilon_0,p,n)|t|^2\mathrm{e}^{\tilde{\delta}^+(\tau)t} \quad \forall t \geqslant \frac{\tilde{t}}{\varepsilon_0}.$$

This completes the proof. □

LEMMA 3.8. *Let u_1 be the solution given by theorem 1.1. The estimates*

$$|\partial_\tau u_1(\tau,x)| \leqslant C|x|^{-2/(p-1)} \quad \text{and} \quad |\partial_\tau^2 u_1(\tau,x)| \leqslant C|x|^{-2/(p-1)}$$

then hold, where the constant C does not depend on τ and x.

Proof. In view of the proof of theorem 1.1,

$$u_1 = |x|^{-2/(p-1)}f(\tau,\theta) = |x|^{-2/(p-1)}(a(\tau,t)\phi_1(\tau,\theta) + \psi(\tau,\theta)),$$

where ψ is a solution of the fixed-point problem

$$\psi = -G_p(\mathcal{M}(\phi_0) + \mathcal{Q}(\psi)), \tag{3.20}$$

where $\phi_0(\tau,\theta) = a(\tau,t)\phi_1(\tau,\theta)$, $\mathcal{M}(\phi_0) = a^p(\phi_1^p - \mu\phi_1)$ and

$$\mathcal{Q}(\psi) = |\phi_0 + \psi|^p - \phi_0^p - p\phi_0^{p-1}\psi.$$

We recall here that $|\psi(t,\theta)| \ll a(\tau,t)\phi_1(\tau,\theta)$.

Here we will only treat the case in which $n \geqslant 3$. For $n = 2$ the proof is the same.

By uniqueness, our assumptions on $\omega(\tau)$ and remark 3.2, $\psi = \psi(t,\tilde{s})$, $\tilde{s} \in (0,\beta(\tau))$, and $\theta_1 = \cos\tilde{s}$, where $\beta(\tau)$ is a positive smooth function such that

$$0 < \inf_{\tau\in\mathbb{R}} \beta(\tau) \leqslant \sup_{\tau\in\mathbb{R}} \beta(\tau) < \pi.$$

Then ψ satisfies

$$\begin{aligned}(\partial_t^2 + A\partial_t - \varepsilon(\tau))\psi + \sin^{2-n}(\tilde{s})\partial_{\tilde{s}}(\sin^{n-2}(\tilde{s})\partial_{\tilde{s}}\psi) + \lambda(\tau)\psi + p\phi_0^{p-1}\psi \\ = -\mathcal{M}(\phi_0) - Q(\psi)\end{aligned}$$

for any $(t,\tilde{s}) \in \mathbb{R}\times(0,\beta(\tau))$, and $\psi(t,\beta(\tau)) = 0$.

Now setting $s = \tilde{s}/\beta(\tau)$, we have that $\psi(\tau,t,s)$ satisfies

$$\begin{aligned}\tilde{L}_p\psi &:= (\partial_t^2 + A\partial_t - \varepsilon(\tau))\psi + \frac{1}{\beta^2(\tau)}\partial_s^2\psi + (n-2)\frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\partial_s\psi \\ &\quad + \lambda\psi + p\phi_0^{p-1}\psi \\ &= -\mathcal{M}(\phi_0) - Q(\psi)\end{aligned} \tag{3.21}$$

for any $(t,s) \in \mathbb{R}\times(0,1)$, and $\psi(\tau,t,1) = 0$.

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Let $1 < p_0 < p$ such that $p - p_0$ is small enough and let $g\colon \mathbb{R}\times(0,1)\to\mathbb{R}$ such that $g \in C^a(\mathbb{R}\times[0,1])$ for some $0 < a \leqslant 1$ and

$$\sup_{\tau\in\mathbb{R}} \sup_{(t,s)\in\mathbb{R}\times(0,1)} |a^{-p}(\tau,t)g(t,s)| < \infty.$$

Let $u(\tau,t,s) = -\tilde{G}_p(\mathcal{M}(\phi_0)+\mathcal{Q}(g))$ be the solution of (3.21). This solution exists since problem (3.21) is equivalent to (3.20). In addition, by proposition 3.1, we have the estimate

$$\sup_{(t,s)\in\mathbb{R}\times(0,1)} |d^{-1}a^{-p_0}(\tau,t)u(\tau,t)| \leqslant C \sup_{(t,s)\in\mathbb{R}\times(0,1)} |a^{-p_0}(\tau,t)\mathcal{M}(\phi_0)(\tau,t,s)| + \frac{C}{\varepsilon} \sup_{(t,s)\in\mathbb{R}\times(0,1)} |a^{-p_0}(\tau,t)\mathcal{Q}(g)(\tau,t)| \tag{3.22}$$

for some constant $C > 0$ that does not depend on τ.

We can easily prove that

$$\lim_{h\to 0} \sup_{(t,s)\in\mathbb{R}\times(0,1)} |u(\tau+h,t,s) - u(\tau,t,s)| = 0.$$

Recall the definitions

$$u_h(\tau,t,s) = \frac{u(\tau+h,t,s)-u(\tau,t,s)}{h}, \quad u(\tau) = u(\tau,t,s), \ldots .$$

Clearly, u_h satisfies

$$\begin{aligned}
(\partial_t^2 + A\partial_t - \varepsilon(\tau+h))u_h(\tau) &+ \frac{1}{\beta^2(\tau+h)}\partial_s^2 u_h(\tau) \\
&+ \frac{(n-2)\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)}\partial_s u_h(\tau) + \lambda(\tau+h)u_h + p\phi_0^{p-1}(\tau+h)u_h(\tau) \\
&= -\frac{1/\beta^2(\tau+h) - 1/\beta^2(\tau)}{h}\partial_s^2 u(\tau) + \frac{\varepsilon(\tau+h)-\varepsilon(\tau)}{h}u(\tau) \\
&\quad - \frac{\lambda(\tau+h)-\lambda(\tau)}{h}u(\tau) \\
&\quad - \frac{n-2}{h}\left(\frac{\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\right)\partial_s u(\tau) \\
&\quad - p\frac{\phi_0^{p-1}(\tau+h)-\phi_0^{p-1}(\tau)}{h}u(\tau) \\
&\quad - \frac{\mathcal{M}(\phi_0)(\tau+h)-\mathcal{M}(\phi_0)(\tau)}{h} - \frac{\mathcal{Q}(g)(\tau+h)-\mathcal{Q}(g)(\tau)}{h}.
\end{aligned}$$

Now notice that $u(\tau,t,s) = w(t,\cos(s\beta(\tau))) = v(\tau,x)$, where $x_1 = |x|\cos(s\beta(\tau))$. In addition, $v(\tau,x)$ satisfies

$$\begin{aligned}
-\Delta_x v + \frac{4}{p-1}\frac{x\cdot\nabla_x v}{|x|^2} + \frac{2}{p-1}\left(n - \frac{2}{p-1} - 2\right)\frac{v}{|x|^2} - p\frac{\phi_0^{p-1}v}{|x|^2} &= -\frac{g}{|x|^2} \quad \text{in } C_\omega(\tau), \\
v &= 0 \quad \text{in } \partial C_\omega(\tau)\setminus 0.
\end{aligned}$$

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Thus, by lemma 3.6 we have

$$\left|\frac{1}{\sin s\beta(\tau)}\frac{\partial u}{\partial s}\right| \leqslant \frac{1}{\inf_{\tau\in\mathbb{R}}\beta(\tau)}|x|\,|v_{x_1}| < C.$$

Similarly, we can obtain $|\partial^2 u/\partial s^2| < C$ for some constant $C > 0$ that does not depend on τ.

Thus, we have

$$\left.\begin{aligned} \sup_{(t,s)\in\mathbb{R}\times(0,1)}\left|\frac{1}{\sin s\beta(\tau)}\frac{\partial u}{\partial s}(\tau,t,s)\right| < C,\\ \sup_{(t,s)\in\mathbb{R}\times(0,1)}\left|\frac{\partial^2 u}{\partial s}(\tau,t,s)\right| < C,\end{aligned}\right\} \tag{3.23}$$

where the constant $C > 0$ does not depend on τ. Now we have

$$\begin{aligned} \lim_{h\to 0}\sup_{\tau\in\mathbb{R}}&\left|\frac{1}{h}\left(\frac{\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\right)\partial_s u(\tau)\right| \\ &= \sup_{\tau\in\mathbb{R}}\left|\left(-\frac{\beta'(\tau)}{\beta^2(\tau)}\cot(\beta(\tau)s) - \frac{s\beta'(\tau)}{\sin^2\beta(\tau)s}\right)\partial_s u(\tau)\right| \\ &< C,\end{aligned}$$

where in the last inequality we have used the fact that

$$0 < \inf_{\tau\in\mathbb{R}}\beta(\tau) \leqslant \sup_{\tau\in\mathbb{R}}\beta(\tau) < \pi$$

and (3.23). Using the fact that

$$\begin{aligned} &a^p(\tau+h)\phi_1^p(\tau+h) - a^p(\tau)\phi_1^p(\tau) \\ &\qquad = (a^p(\tau+h) - a^p(\tau))\phi_1^p(\tau+h) + a^p(\tau)(\phi_1^p(\tau+h) - \phi_1^p(\tau))\end{aligned}$$

and

$$\begin{aligned} a^p(\tau+h) = a^p(\tau) &+ pa^{p-1}(\tau)(a^p(\tau+h) - a^p(\tau)) \\ &+ \frac{p(p-1)}{2}\int_{a^p(\tau)}^{a^p(\tau+h)} t^{p-2}(a^p(\tau+h) - t)\,\mathrm{d}t\end{aligned}$$

(the same for ϕ_1), and lemmas 2.1 and 3.7, we have that

$$\left|\lim_{h\to 0}\frac{\mathcal{M}(\phi_0)(\tau+h) - \mathcal{M}(\phi_0)(\tau)}{h}\right| = \left|\frac{\partial\mathcal{M}(\phi_0)}{\partial\tau}\right| < C.$$

Similarly, we have that

$$\left|\lim_{h\to 0}\frac{\mathcal{Q}(g)(\tau+h) - \mathcal{Q}(g)(\tau)}{h}\right| = \left|\frac{\partial\mathcal{Q}(g)}{\partial\tau}\right| < C.$$

By proposition 3.1, we have

$$\sup_{\tau\in\mathbb{R}}\sup_{(t,s)\in\mathbb{R}\times(0,1)}|u_h| < C,$$

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and thus, by the Arzelà–Ascoli theorem, there exists a subsequence $\{u_{h_n}\}$ such that $u_{h_n} \to v$ locally uniformly, and $v(\tau, t, s)$ satisfies

$$(\partial_t^2 + A\partial_t - \varepsilon(\tau))v + \frac{1}{\beta^2(\tau)}\partial_s^2 v + \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\partial_s v + \lambda(\tau)u + p\phi_0^{p-1}(\tau)v = H(\phi_1, a, g),$$

with $v(\tau, t, 1) = 0$. Notice that

$$\sup_{\tau\in\mathbb{R}} \sup_{(t,s)\in\mathbb{R}\times(0,1)} |H(\tau, t, s)| < C,$$

and thus, by proposition 3.1, v is a unique solution. Furthermore,

$$\lim_{h\to 0} u_h = v = \frac{\partial u}{\partial \tau}$$

and

$$\sup_{\tau\in\mathbb{R}} \sup_{(t,s)\in\mathbb{R}\times(0,1)} \left|\frac{\partial u}{\partial \tau}(\tau, s, t)\right| < C \tag{3.24}$$

for some constant C independent of g.

Similarly to (3.23), we can prove that

$$\sup_{\tau\in\mathbb{R}} \sup_{(t,s)\in\mathbb{R}\times(0,1)} \left|\frac{1}{\sin s\beta(\tau)} \frac{\partial^2 u}{\partial\tau\partial s}(\tau, t, s)\right| < C,$$

$$\sup_{\tau\in\mathbb{R}} \sup_{(t,s)\in\mathbb{R}\times(0,1)} \left|\frac{\partial^3 u}{\partial\tau\partial s\partial s}(\tau, t, s)\right| < C$$

and, by the same argument as above,

$$\sup_{\tau\in\mathbb{R}} \sup_{(t,s)\in\mathbb{R}\times(0,1)} \left|\frac{\partial^2 u}{\partial\tau\partial\tau}(\tau, t, s)\right| < C, \tag{3.25}$$

where C is a constant that depends on g.

Now we consider the fixed-point problem (3.21). Let $\tau_0 \in \mathbb{R}$ and let ρ be small enough such that for any $\tau \in O_{\tau_0} = \{\tau \in \mathbb{R}\colon |\tau - \tau_0| < \rho\}$ we have $p\delta^-(\tau) \geqslant p_0\delta^-(\tau_0)$, where

$$\delta^-(\tau) = \frac{-A + \sqrt{A^2 + 4\varepsilon(\tau)}}{2}.$$

We can easily show that $a^p(\tau, t) \leqslant Ca^{p_0}(\tau_0, t)$ for all $\tau \in O_{\tau_0}$ for some positive constant C independent of τ and t.

Now, since $0 < p - p_0$ is small enough, we can use a fixed-point argument like that in [5] (see remark 3.2) in the Banach space

$$X = \Big\{g \in L^\infty(\mathbb{R}\times(0,1))\colon \sup_{(t,s)\in\mathbb{R}\times(0,1)} |a^{-p_0}(\tau_0, t)g(t, s)| < \infty\Big\}$$

to prove that there exists a unique solution

$$\psi(\tau, t, s) = -\tilde{G}_p(\mathcal{M}(\phi_0) + \mathcal{Q}(\psi(\tau, t, s))) \quad \forall \tau \in O_{\tau_0}.$$

Now, let $(\tau, g) \in O_{\tau_0} \times X$ and define the bounded operator

$$T(\tau, g) = g + \tilde{G}_p(g).$$

We can apply the implicit function theorem to $O_{\tau_0} \times X$ to obtain the following.

Let $0 < \rho_0 \leqslant \rho$ be small enough. Then, for any $\tau \in \{\tau \in \mathbb{R} \colon |\tau - \tau_0| < \rho_0\} \subset O_{\tau_0}$, there exists a function $\psi(\tau, t, s)$ such that

$$T(\tau, \psi(\tau, t, s)) = 0.$$

Using (3.24), (3.25) and again the implicit function theorem, we can also prove that $\partial_\tau \psi$, $\partial_\tau^2 \psi$ exist. Furthermore, using the fact that

$$0 = T_\tau(\tau, \psi(\tau)) + T_g(\tau, \psi(\tau))\partial_\tau \psi$$

and the estimate (3.24), we have that

$$\sup_{\tau \in (\tau_0 - \rho_0, t_0 + \rho_0)} \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{\partial u}{\partial \tau}(\tau, t, s) \right| < C.$$

Similarly, we have

$$\sup_{\tau \in (\tau_0 - \rho_0, t_0 + \rho_0)} \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{\partial^2 u}{\partial \tau \partial \tau}(\tau, t, s) \right| < C.$$

And the result follows since τ_0 is abstract. □

4. The proof of theorems 1.2 and 1.3

Let $x \in \mathbb{R}^n$, $n \geqslant 2$, let $R > 0$, let $B_R(0) \subset \mathbb{R}^n$ and let

$$r_{\sigma(\tau)} = |x - \sigma(\tau)|,$$

where $\sigma \colon \mathbb{R} \to \mathbb{R}^n$ is a smooth curve such that

$$\sup_{\tau \in \mathbb{R}} \{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < C < \infty.$$

Define

$$\tilde{r}^2 = \sum_{i=1}^{n} |(x_i - \sup |\sigma(\tau)|)^2.$$

Given τ, let $(r_{\sigma(\tau)}, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$ be the spherical coordinates of $x \in \mathbb{R}^n$ centred at $\sigma(\tau)$, abbreviated by $x = (r_{\sigma(\tau)}, \theta)$. We define the cone

$$\tilde{C}_{\omega(\tau)} = \{x = (r_{\sigma(\tau)}, \theta) \colon r_{\sigma(\tau)} > 0, \theta \in \omega(\tau)\} \subset \mathbb{R}^n$$

and we define

$$\Omega_{\tau_1,\tau_2} = \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n \colon x \in \tilde{C}_{\omega(\tau)}\} \subset \mathbb{R}^{n+1},$$
$$\Omega^R_{\tau_1,\tau_2} = \Omega_{\tau_1,\tau_2} \cap \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n \colon x \in B_R(\sigma(\tau))\} \subset \mathbb{R}^{n+1}$$

and

$$S_{\tau_1,\tau_2} = \{(\tau, x) \in [\tau_1, \tau_2] \times \mathbb{R}^n \colon r_{\sigma(\tau)} = 0\}.$$

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Let $C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})$ be the set of continuous functions $f \in C(\Omega^R_{\tau_1,\tau_2})$ with norm

$$\|f\|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})} := \sup_{(\tau,x)\in\Omega^R_{\tau_1,\tau_2}} (\chi_{[0,1]}(r_{\sigma(\tau)})r_{\sigma(\tau)}^{-\delta}|f| + \chi_{[1,\infty)}(r_{\sigma(\tau)})\tilde{r}^{-\rho}|f|).$$

Let $\delta \in (-n-\gamma+2,\gamma)$. We define $\phi_\delta(\tau,\theta)$ to be the unique positive solution of

$$\begin{aligned} \Delta_{\mathbb{S}^{n-1}}\phi_\delta + \lambda\phi_\delta + (\delta(\delta+n-2)-\lambda)\phi_\delta &= -1 \quad \text{in } \omega(\tau), \\ \phi_\delta &= 0 \quad \text{on } \partial\omega(\tau). \end{aligned}$$

Notice here that $\lambda = \gamma^2 + \gamma(n-2)$, and thus $\delta(\delta+n-2) - \lambda < 0$ if and only if $\delta \in (-n-\gamma+2,\gamma)$. A direct computation shows that

$$-\Delta_x(|x|^\delta\phi_\delta) = |x|^{\delta-2}.$$

In view of lemma 2.1, we have that $\phi_\delta = \phi_\delta(t)$, where $t \in (0,\beta(\tau))$, and satisfies

$$\begin{aligned} \sin^{2-n} t\frac{\mathrm{d}}{\mathrm{d}t}\left(\sin^{n-2} t\frac{\mathrm{d}\phi_\delta}{\mathrm{d}t}\right) + \lambda\phi_\delta + (\delta(\delta+n-2)-\lambda)\phi_\delta &= -1 \quad \text{in } (0,\beta(\tau)), \\ \phi_\delta(\beta(\tau)) &= 0. \end{aligned}$$

We next set $\beta^* = \sup_{\tau\in\mathbb{R}}\beta(\tau)$, $\lambda^* = \inf_{\tau\in\mathbb{R}}\lambda(\tau)$, $\gamma^* = \inf_{\tau\in\mathbb{R}}\gamma(\tau)$ and we let ϕ^*_δ be the solution of

$$\begin{aligned} \sin^{2-n} t\frac{\mathrm{d}}{\mathrm{d}t}\left(\sin^{n-2} t\frac{\mathrm{d}\phi^*_\delta}{\mathrm{d}t}\right) + \lambda^*\phi^*_\delta + (\delta(\delta+n-2)-\lambda^*)\phi^*_\delta &= -1 \quad \text{in } (0,\beta^*), \\ \phi_\delta(\beta^*) &= 0, \end{aligned}$$

with $\gamma \in (-n-\gamma^*+2,\gamma^*)$.

Thus, ϕ^*_δ is the unique solution of the problem

$$\begin{aligned} \Delta_{\mathbb{S}^{n-1}}\phi^*_\delta + \lambda^*\phi^*_\delta + (\delta(\delta+n-2)-\lambda^*)\phi^*_\delta &= -1 \quad \text{in } \omega^*, \\ \phi_\delta &= 0 \quad \text{on } \partial\omega^*, \end{aligned}$$

where $\omega^* = \bigcup_\tau \omega(\tau)$ and by assumption we have that $\omega^* \subsetneq \mathbb{S}^{n-1}$.

PROPOSITION 4.1. *Assume that* $\delta,\rho \in (-n-\gamma^*+2,0]$ *and that*

$$\sup_{\tau\in\mathbb{R}}\{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < \varepsilon, \tag{4.1}$$

where $\varepsilon > 0$ *is small enough. Then, for all* $\tau_1 < \tau_2 \in \mathbb{R}$ *and* $R > 0$, *there exists a unique operator*

$$G_{\delta,\rho,R,\tau_1,\tau_2}\colon C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2}) \to C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})$$

such that, for each $f \in C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})$, *the function* $G_{\delta,\rho,R,\tau_1,\tau_2}(f)$ *is a solution of problem*

$$\left.\begin{aligned} \Delta u &= \frac{1}{r^2_{\sigma(\tau)}}f \quad \text{in } \Omega^R_{\tau_1,\tau_2}, \\ u &= 0 \quad \text{on } \partial\Omega^R_{\tau_1,\tau_2}\setminus S_{\tau_1,\tau_2}. \end{aligned}\right\} \tag{4.2}$$

Moreover, the norm of $G_{\delta,\rho,R,\tau_1,\tau_2}$ *is bounded by a constant* $c > 0$ *that does not depend on* R, τ_1 *and* τ_2.

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Proof. Without loss of generality we can assume that $R > 4$.

We first solve, for each $r \in (0, \frac{1}{4})$, the problem

$$\left.\begin{aligned} \Delta u &= \frac{1}{|x-\sigma(\tau)|^2} f \quad \text{in } \Omega^R_{\tau_1,\tau_2} \setminus \Omega^r_{\tau_1,\tau_2}, \\ u &= 0 \qquad\qquad\qquad \text{on } \partial(\Omega^R_{\tau_1,\tau_2} \setminus \Omega^r_{\tau_1,\tau_2}), \end{aligned}\right\} \tag{4.3}$$

and call u_r its unique solution.

A straightforward calculation shows that

$$-\Delta(r^{\delta}_{\sigma(\tau)}\phi^*_\delta) \geqslant r^{\delta-2}_{\sigma(\tau)}(1 - |\delta|(|\delta|+1)|\sigma'|) - |\delta|\,|\sigma''|r^{\delta-1}_{\sigma(\tau)}.$$

We choose ε small enough such that

$$-\Delta(r^{\delta}_{\sigma(\tau)}\phi^*_\delta) \geqslant \tfrac{1}{2}(r^{\delta-2}_{\sigma(\tau)} - r^{\delta-1}_{\sigma(\tau)}).$$

Let ψ be the solution of

$$\left.\begin{aligned} \Delta_{\mathbb{S}^{n-1}}\psi &= -C\|f\|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})} \quad \text{in } \omega^*, \\ \psi &= 0 \qquad\qquad\qquad\quad \text{on } \partial\omega*, \end{aligned}\right\}$$

for some constant $C > 0$ and we define the cut-off function $\eta\colon \mathbb{R}^n \to [0,1]$ by $\eta = 1$ in $B_{1/2}(0) \subset \mathbb{R}^n$ and $\eta \in C^\infty_0(B_1(0))$.

We next set

$$\Phi(\tau,x) = C\|f\|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\eta(x)r^{\delta}_{\sigma(\tau)}\phi^*_\delta + \psi.$$

If we choose the uniform constant $C > 0$ large enough, we have, by the maximum principle,

$$\begin{aligned} |u_r(\tau,x)| &\leqslant \Phi(\tau,x) \leqslant C\|f\|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\phi^*_\delta|x|^\delta + \psi \\ &\leqslant C\|f\|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\phi^*_\delta(\theta)(|x|^\delta + 1) \quad \forall(\tau,x) \in \Omega^R_{\tau_1,\tau_2} \setminus \Omega^r_{\tau_1,\tau_2}, \end{aligned} \tag{4.4}$$

where in the last inequality we have used the fact that

$$\psi(\theta) \leqslant C\|f\|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\phi^*_\delta(\theta) \quad \forall\theta \in \omega^*.$$

Using (4.4) and again the maximum principle, we obtain

$$|u_r(\tau,x)| \leqslant C\|f\|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\phi^*_\delta(\theta)|x|^\delta \quad \forall(\tau,x) \in \Omega^{1/2}_{\tau_1,\tau_2} \setminus \Omega^r_{\tau_1,\tau_2}. \tag{4.5}$$

Now set $\psi_0 = \tilde{r}^\rho\phi^*_\rho$. Then,

$$\Delta_{\mathbb{S}^{n-1}}\psi_0 = -\tilde{r}^{\rho-2}.$$

Thus, using (4.5) and the maximum principle, we obtain

$$|u_r| \leqslant C\Big(\sup_{\tau\in\mathbb{R}}|\sigma|\Big)\|f\|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\|\phi^*_\rho\|_{L^\infty(\omega)}|x|^\rho \quad \forall r_{\sigma(\tau)} > \tfrac{1}{2}. \tag{4.6}$$

By standard interior elliptic estimates and Arzelà–Ascoli theorem, there exists a subsequence $\{u_{r_j}\}$ such that $r_j \downarrow 0$ and $u_{r_j} \to u$ locally uniformly. By standard elliptic theory, (4.5) and (4.6), we have that $u \in C^2(\Omega^R_{\tau_1,\tau_2})$ and is unique. □

Proof of theorem 1.2. We choose $\delta = -2/(p-1)$ and we set

$$u_\varepsilon(x,\tau) = \eta(x)\varepsilon^{-2/(p-1)}u_1\left(\frac{x-\sigma}{\varepsilon}\right),$$

where u_1 is the function given in theorem 1.1 and $\eta\colon \mathbb{R}^n \to [0,1]$ is a cut-off function such that $\eta = 1$ in $B_{1/2}(0) \subset \mathbb{R}^n$ and $\eta \in C_0^\infty(B_1(0))$.

By construction of $u_1(x)$ and lemma 3.6, we have

$$\left.\begin{aligned} |\nabla_x u_1(\tau,x)| &\leqslant C(n,p,\lambda,C_{\omega(\tau)})|x|^{-1},\\ |D_x^2 u(\tau,x)| &\leqslant C(n,p,\lambda,C_{\omega(\tau)})|x|^{-2}.\end{aligned}\right\} \tag{4.7}$$

First we assume that

$$\sup_{\tau\in\mathbb{R}}\{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < \tilde{\varepsilon}, \tag{4.8}$$

where $\tilde{\varepsilon} > 0$ is small enough. Then, by (4.7), (4.8) and lemma 3.8, we have

$$|\partial_\tau^2 u_\varepsilon(x,\tau)| \leqslant Cr^{-2/(p-1)}(\tau) + C(n,\gamma^*)\tilde{\varepsilon}\big(r_{\sigma(\tau)}^{-(2/(p-1))-2} + r_{\sigma(\tau)}^{-(2/(p-2))-1}\big). \tag{4.9}$$

Now, let $R > 4$, let $\tau_1 < \tau_2 \in \mathbb{R}$ and define the following problem:

$$\left.\begin{aligned} -\Delta u &= u^p \quad \text{in } \Omega^R_{\tau_1,\tau_2},\\ u &> 0 \quad \text{in } \Omega^R_{\tau_1,\tau_2},\\ u &= 0 \quad \text{on } \partial\Omega^R_{\tau_1,\tau_2} \setminus S_{\tau_1,\tau_2}.\end{aligned}\right\} \tag{4.10}$$

We then look for a solution of the form $u = u_\varepsilon + v$. By virtue of proposition 4.1, we can rewrite this equation as the fixed-point problem

$$\left.\begin{aligned} v &= -G_{\delta,\rho,R,\tau_1,\tau_2}(|x|^2(\Delta u_\varepsilon + |u_\varepsilon + v|^p)),\\ \Delta v &= -|u_\varepsilon + v|^p - \Delta u_\varepsilon.\end{aligned}\right\} \tag{4.11}$$

We assume that ε is small enough. Then, by (4.9), we have for some constant $C_0(n,\gamma) > 0$,

$$\begin{aligned} \||u_\varepsilon|^p + \Delta u_\varepsilon\|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})} &\leqslant C_0(\varepsilon^{n+\gamma-2-(p-3)/(p-1)} + \varepsilon^2 + \varepsilon + \tilde{\varepsilon})\\ &\leqslant C_0(\varepsilon + \tilde{\varepsilon}),\end{aligned}$$

and we recall here that $\delta = -2/(p-1)$.

Then, using theorem 1.1 one can easily see that

$$\begin{aligned} ||x|^2|v_\varepsilon + v_1|^p - |v_\varepsilon + v_2|^p|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})} &\\ \leqslant C_1(n,\gamma^*,p)\Big(\sup_{\tau\in\mathbb{R}}||\phi_p||_{L^\infty(\omega)} + \tilde{\varepsilon}\Big)^{p-1}&\|v_1 - v_2\|_{C_{\delta,\rho}(\Omega^1_{\tau_1,\tau_2})}\\ + C(n,\gamma^*,p)(\varepsilon + \tilde{\varepsilon})^{p-1}&\|v_1 - v_2\|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2}\setminus\Omega^1_{\tau_1,\tau_2})}\end{aligned} \tag{4.12}$$

for all $v_1, v_2 \in C_{\delta,\beta}(C_\omega^R \setminus \{0\} \times (\tau_1,\tau_2))$ such that

$$\|v_i\|_{C_{\delta,\beta}(C_\omega^R\setminus 0\times(\tau_1,\tau_2))} \leqslant 2C_0(\varepsilon + \tilde{\varepsilon}).$$

We recall that the constants above do not depend on R, t_1, t_2, ε and $\tilde{\varepsilon}$. To obtain a contraction mapping, it is enough to take ε, $\tilde{\varepsilon}$ small enough and p close enough to $\sup_{\tau\in\mathbb{R}} p^*$ to ensure that $\sup_{\tau\in\mathbb{R}} \|\phi_p(\tau,\cdot)\|_{L^\infty(\omega(\tau))}$ is as small as we need. The above estimates allow an application of the contraction mapping principle in the ball of radius $2C_0(\varepsilon+\tilde{\varepsilon})$ in $\Omega^R_{\tau_1,\tau_2}$ to obtain a solution to problem (4.11), which we denote by

$$u_{R,\tau_1,\tau_2} = u_\varepsilon + v_{R,\tau_1,\tau_2}.$$

In view of the fixed-point argument, we have that $|v_{R,t_1,t_2}| \leqslant u_\varepsilon/4$ near S_{τ_1,τ_2}, and thus the solution u_{R,t_1,t_2} is singular along S_{τ_1,τ_2} and positive near S_{τ_1,τ_2}. The maximum principle then implies that

$$u_{R,t_1,t_2} > 0 \quad \text{in } \Omega^R_{\tau_1,\tau_2}.$$

Moreover, we have that

$$\|v_{R,\tau_1,\tau_2}\|_{C_{\delta,\beta}(\Omega^R_{\tau_1,\tau_2})} \leqslant 2C_0(\varepsilon+\tilde{\varepsilon}).$$

That is, v_{R,τ_1,τ_2} is uniformly bounded by a constant that depends only on n, γ^*, p. By standard interior elliptic estimates and the Arzelà–Ascoli theorem, there exists a subsequence $\{u_{R_j,-\tau_j,\tau_j}\}$ such that $R_j \uparrow \infty$, $\tau_j \uparrow \infty$ and $u_{R_j,-\tau_j,\tau_j} \to u$ locally uniformly. Again, standard elliptic theory yields $u \in C^2(\Omega_{-\infty,\infty})$.

For the general case in which

$$\sup_{\tau\in\mathbb{R}}\{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < C,$$

set $\tilde{\sigma} = \sigma/k$, where $k > 0$ is large enough such that

$$\sup_{\tau\in\mathbb{R}}\{|\tilde{\sigma}(\tau)| + |\tilde{\sigma}'(\tau)| + |\tilde{\sigma}''(\tau)|\} < \tilde{\varepsilon}.$$

As before, we can find a solution $u(x)$ of the problem with a singularity along $\{(\tau,x)\in\mathbb{R}\times\mathbb{R}^n\colon |x-\tilde{\sigma}(\tau)| = 0\}$. But the function $v(y) = k^{2/(p-1)}u(ky)$, where $y = kx$, is a singular solution of the problem and has singularity along $S_{-\infty,\infty}$, and the result follows. □

Let $\alpha > 0$ and let Ω be a bounded Lipschitz domain such that

$$\Omega \cap \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} = \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} \subset \mathbb{R}^{n+1}.$$

Let $C_\delta(\Omega^R_{\tau_1,\tau_2})$ be the set of continuous functions $f \in C(\Omega^R_{\tau_1,\tau_2})$ with norm

$$\|f\|_{C_\delta(\Omega^R_{\tau_1,\tau_2})} = \sup_{(\tau,x)\in\Omega^R_{\tau_1,\tau_2}} (r^{-\delta}(\tau)|f|).$$

We define $C_\delta(\Omega)$ to be the space of the continuous functions in Ω with the norm

$$\|f\|_{C_\delta(\Omega)} = \|f\|_{C_\delta(\Omega^R_{\tau_1-\alpha,\tau_2+\alpha})} + \|f\|_{L^\infty(\bar{\Omega}\setminus\Omega^{R/2}_{\tau_1-\alpha/4,\tau_2+\alpha/4})}.$$

We consider a smooth, positive bounded function $\nu\colon \bar{\Omega} \to (0,\infty)$, which is equal to $r_{\sigma(\tau)}$ in $\Omega^{R/2}_{\tau_1-\alpha/4,\tau_2+\alpha/4}$ and satisfies

$$0 < \sup_{x\in\bar{\Omega}\setminus\Omega^R_{\tau_1-\alpha/2,\tau_2+\alpha/2}} \nu < C.$$

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We obtain the following proposition.

Proposition 4.2. *Let $\tau_1 < \tau_2 \in \mathbb{R}$ and let $\alpha > 0$ be small enough. Assume that Ω is a bounded Lipschitz domain such that*

$$\Omega \cap \Omega^R_{\tau_1-2\alpha,\tau_2+2\alpha} = \Omega^R_{\tau_1-2\alpha,\tau_2+2\alpha} \subset \mathbb{R}^{n+1},$$

$\delta \in (-n-\gamma^* + 2, 0]$ and

$$\sup_{\tau\in\mathbb{R}}\{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < \varepsilon \tag{4.13}$$

for some $\varepsilon > 0$ small enough. There then exists a unique operator

$$G_{\delta,\tau_1,\tau_2} \colon C_\delta(\Omega) \to C_\delta(\Omega)$$

such that, for each $f \in C_\delta(\Omega)$, the function $G_{\delta,\tau_1,\tau_2}(f)$ is a solution of the problem

$$\left.\begin{aligned} \Delta u &= \frac{1}{\nu^2} f \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \setminus S_{\tau_1-\alpha,\tau_2+\alpha}. \end{aligned}\right\} \tag{4.14}$$

Moreover, the norm of G_{δ,τ_1,τ_2} is bounded by a constant $c > 0$ that does not depend on R, τ_1 and τ_2.

Proof. Let $\hat{\sigma}(t)$ be a bounded smooth curve such that

$$\begin{aligned} &\sup_{\tau\in\mathbb{R}}\{|\hat{\sigma}(\tau)| + |\hat{\sigma}'(\tau)| + |\hat{\sigma}''(\tau)|\} < 2\varepsilon, \\ &r_{\hat{\sigma}(\tau)} = r_{\sigma(\tau)} \quad \forall(\tau,x) \in \Omega^R_{\tau_1-\alpha/4,\tau_2+\alpha/4}, \\ &r_{\hat{\sigma}(\tau)} \geqslant r_{\sigma(\tau)} \quad \forall(\tau,x) \in \Omega, \end{aligned}$$

and

$$r_{\hat{\sigma}(\tau)} > c > 0 \quad \forall(\tau,x) \in \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} \setminus \overline{\Omega^R_{\tau_1-\alpha/2,\tau_2+\alpha/2}}.$$

Given τ, we let $\hat{\omega}(\tau) \subsetneq \mathbb{S}^{n-1}$ be the corresponding Lipschitz spherical cap and let $(r_{\hat{\sigma}(\tau)}, \theta) \in [0,\infty) \times \mathbb{S}^{n-1}$ be the spherical coordinates of $x \in \mathbb{R}^n$ centred at $\hat{\sigma}(\tau)$, abbreviated by $x = (r_{\hat{\sigma}(\tau)}, \theta)$.

We set

$$\begin{aligned} \hat{C}_{\hat{\omega}(\tau)} &= \{(r_{\hat{\sigma}(\tau)}, \theta) \colon \hat{r}(\tau) > 0,\ \theta \in \hat{\omega}(\tau)\}, \\ \hat{\Omega}_{\tau_1,\tau_2} &= \{(\tau,x) \in (\tau_1,\tau_2) \times \mathbb{R}^n \colon x \in \hat{C}_{\hat{\omega}(\tau)}\} \end{aligned}$$

and $\hat{\Omega}^R_{\tau_1,\tau_2} = \hat{\Omega}_{\tau_1,\tau_2} \cap \{(\tau,x) \in (\tau_1,\tau_2) \times \mathbb{R}^n \colon x \in B_R(\hat{\sigma}(\tau))\} \subset \mathbb{R}^{n+1}$. We construct $\hat{\omega}(\tau)$ such that

$$\begin{aligned} \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} &\subsetneq \hat{\Omega}^{2R}_{\tau_1-\alpha,\tau_2+\alpha}, \\ \hat{\Omega}^R_{\tau_1-\alpha/4,\tau_2+\alpha/4} &= \Omega^R_{\tau_1-\alpha/4,\tau_2+\alpha/4}. \end{aligned}$$

We next define η be a cut-off function satisfying $\eta = 1$ in $\Omega^{R/2}_{\tau_1-\alpha/2,\tau_2+\alpha/2}$ and $\eta = 0$ in $\Omega \setminus \Omega^R_{\tau_1-\alpha,\tau_2+\alpha}$. We write $\hat{f} = \eta f$ and we let $u_1 = G_{\delta,\rho,R,\tau_1,\tau_2}(\hat{f})$ be the function given by proposition 4.1 in $\hat{\Omega}^{2R}_{\tau_1-\alpha,\tau_2+\alpha}$.

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Set

$$\tilde{f} = f - \nu\Delta(\eta u_1).$$

Then $\tilde{f}$ has support in $\Omega \setminus \Omega^{R/2}_{\tau_1-\alpha/4,\tau_2+\alpha/4}$ and $\tilde{f} \in C(\Omega)$. Furthermore, we have

$$\|\tilde{f}\|_{C_\delta(\Omega)} \leqslant C\|f\|_{C_\delta(\Omega)}$$

for some positive constant $C > 0$.

Finally, let u_2 be a solution of

$$\begin{aligned} \Delta u &= \frac{1}{\nu^2}\tilde{f} \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

which clearly satisfies the bound

$$\|u_2\|_{L^\infty(\Omega)} \leqslant C\|\tilde{f}\|_{C_\delta(\Omega)} \leqslant C\|f\|_{C_\delta(\Omega)}.$$

The desired result then follows by looking for a solution of (4.14) of the form $u = \eta u_1 + u_2$. □

Proof of theorem 1.3. We choose $\delta = -2/(p-1)$ and set

$$u_\varepsilon(x,\tau) = \eta(x)\varepsilon^{-2/(p-1)}u_1\left(\frac{x-\sigma}{\varepsilon}\right),$$

where u_1 is the function given by theorem 1.1 and $\eta\colon \mathbb{R}^n \to [0,1]$ is a cut-off function such that $\eta = 1$ in $\Omega^{R/2}_{\tau_1-\alpha/2,\tau_2+\alpha/2}$ and $\eta = 0$ in $\Omega \setminus \Omega^{R}_{\tau_1-\alpha,\tau_2+\alpha}$.

The rest of the proof is the same as in theorem 1.2, the only difference being that we use proposition 4.2 instead of proposition 4.1. □

Appendix A. Proof of lemma 2.1

To prove lemma 2.1, we need the following inequality, whose the proof can be found in [10, p. 43, theorem 2].

LEMMA A.1. *Let $A(r)$, $B(r)$ be non-negative functions such that $1/A(r)$, $B(r)$ are integrable in (r,∞) and $(0,r)$, respectively, for all positive $r < \infty$. Then, for $q \geqslant 2$ the Sobolev inequality*

$$\left[\int_0^s B(t)|u(t)|^q\,\mathrm{d}t\right]^{1/q} \leqslant C\left[\int_0^s A(t)|u'(t)|^2\,\mathrm{d}t\right]^{1/2} \tag{A 1}$$

is valid for all $u \in C^1[0,s]$ such that $u(s) = 0$ (or vanish near infinity, if $s = \infty$) if and only if

$$K = \sup_{r\in(0,s)}\left[\int_0^r B(t)\,\mathrm{d}t\right]^{1/q}\left[\int_r^s (A(t))^{-1}\,\mathrm{d}t\right]^{1/2}$$

is finite. The best constant in (A 1) satisfies the following inequality:

$$K \leqslant C \leqslant K\left(\frac{q}{q-1}\right)^{1/2} q^{1/q}.$$

Proof of lemma 2.1. Let $n \geqslant 3$ (for $n = 2$ the proof is easy and we omit it). By our assumptions on $\omega(\tau)$ and without loss of generality, we can set $\theta_1 = \cos t$, with $0 < t < \beta(\tau)$, where $\beta(\tau)$ is a smooth function with bounded derivatives such that

$$0 < \inf_{\tau \in \mathbb{R}} \beta(\tau) < \sup_{\tau \in \mathbb{R}} \beta(\tau) < \pi.$$

Then problem (2.1) is clearly equivalent to

$$\left.\begin{aligned} -\sin^{2-n} t \frac{\mathrm{d}}{\mathrm{d}t}\left(\sin^{n-2} t \frac{\mathrm{d}\phi_1}{\mathrm{d}t}\right) = \lambda\phi_1 \quad \text{in } (0, \beta(\tau)),\\ \phi_1(\beta(\tau)) = 0,\\ \partial_t \phi_1(0) = 0. \end{aligned}\right\} \tag{A 2}$$

We denote by $\mathcal{H}((0, \beta(\tau)))$ the completion of $C^\infty([0, \beta(\tau)])$ under the norm

$$\|v\|^2_{\mathcal{H}((0,\beta(\tau)))} = \int_0^{\beta(\tau)} \sin^{n-2}(t)|\partial_t v|^2 \,\mathrm{d}t < \infty$$

with the property $v(\beta(\tau)) = \partial_t v(0) = 0$.

The space $\mathcal{H}(\omega(\tau))$ is a Hilbert space with inner product

$$(u, v) = \int_0^{\beta(\tau)} \sin^{n-2}(t)\partial_t u \partial_t v \,\mathrm{d}t.$$

Indeed, by lemma A.1 and our assumptions on $\beta(\tau)$, we can easily obtain that

$$\int_0^{\beta(\tau)} v^2 \sin^{n-3} t \,\mathrm{d}t \leqslant C(n) \int_0^{\beta(\tau)} \sin^{n-2}(t)|\partial_t v|^2 \,\mathrm{d}t. \tag{A 3}$$

By the above inequality, we can prove that the space $\mathcal{H}(\omega(\tau))$ is compactly embedded in

$$L^2_{\sin t}((0, \beta(\tau))) := \left\{u \colon (0, \beta(\tau)) \to \mathbb{R} \colon \int_0^{\beta(\tau)} u^2 \sin^{n-2}(t) \,\mathrm{d}t < \infty\right\}.$$

Thus, using standard arguments we can prove that the eigenvalue problem

$$0 < \lambda(\tau) = \inf_{u \in \mathcal{H}((0,\beta(\tau)))} \frac{\int_0^{\beta(\tau)} \sin^{n-2}(t)|\mathrm{d}u/\mathrm{d}t|^2 \,\mathrm{d}t}{\int_0^{\beta(\tau)} u^2 \sin^{n-2}(t) \,\mathrm{d}t}$$

has a positive minimizer $\phi_1(\tau, t) \in \mathcal{H}(0, \beta(\tau))$.

But

$$\left.\begin{aligned} C(n) \int_0^{\beta(\tau)} \sin^{n-2}(t)|\partial_t \phi_1|^2 \,\mathrm{d}t = \int_\omega |\nabla\phi_1|^2 \,\mathrm{d}S,\\ C(n) \int_0^{\beta(\tau)} \sin^{n-2}(t)|u|^2 \,\mathrm{d}t = \int_\omega |\phi_1|^2 \,\mathrm{d}S = 1, \end{aligned}\right\} \tag{A 4}$$

and thus $\phi_1 \in H^1_0(\omega(\tau))$ and is a weak solution of the eigenvalue problem (2.1). Hence, by standard elliptic arguments we can prove that $\phi_1 \in L^\infty(\omega(\tau))$. In addition, by our assumption we have that

$$\sup_{\tau \in \mathbb{R}} \sup_{t \in (0,\beta(\tau))} |\phi_1(\tau, t)| < C. \tag{A 5}$$

By the ODE (A 2) and the estimate (A 5), we can write

$$\phi_1(\tau,t) = \lambda \int_t^{\beta(\tau)} \frac{1}{\sin^{n-2} s} \int_0^s \sin^{n-2}(r)\phi_1(\tau,r)\,\mathrm{d}r\,\mathrm{d}s. \tag{A 6}$$

Thus, we have the following estimates:

$$\left.\begin{aligned} \sup_{\tau\in\mathbb{R}} \sup_{t\in(0,\beta(\tau))} \left|\frac{1}{\sin t}\partial_t\phi_1(\tau,t)\right| &\leqslant C \sup_{\tau\in\mathbb{R}} \sup_{t\in(0,\beta(\tau))} |\phi_1(\tau,t)|, \\ \sup_{\tau\in\mathbb{R}} \sup_{t\in(0,\beta(\tau))} |\partial_t^2\phi_1(\tau,t)| &\leqslant C \sup_{\tau\in\mathbb{R}} \sup_{t\in(0,\beta(\tau))} |\phi_1(\tau,t)|. \end{aligned}\right\} \tag{A 7}$$

Setting now $s = t/\beta(\tau)$, we have that $\phi_1 = \phi_1(\tau,s)$ satisfies

$$\left.\begin{aligned} \frac{1}{\beta^2(\tau)}\partial_s^2\phi_1(\tau,s) + \frac{(n-2)\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\partial_s\phi_1(\tau,s) + \lambda(\tau)\phi_1(\tau,s) = 0 \quad &\text{in } (0,1), \\ \phi_1(1) = 0,& \\ \partial_t\phi_1(0) = 0.& \end{aligned}\right\} \tag{A 8}$$

It is easy to see that $\lim_{h\to 0}\phi_1(\tau+h,s) = \phi_1(\tau,s)$ in $L^\infty(\mathbb{R}\times(0,1))$. We set

$$u_h(\tau) = \frac{\phi_1(\tau+h,s) - \phi_1(h,s)}{h}, \quad \phi_1(\tau) = \phi_1(\tau,t),$$

and then u_h satisfies

$$\begin{aligned} &\frac{1}{\beta^2(\tau+h)}\partial_s^2 u_h(\tau) + \frac{(n-2)\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)}\partial_s u_h(\tau) + \lambda(\tau+h)u_h(\tau) \\ &\quad = -\frac{1}{h}\left(\frac{1}{\beta^2(\tau+h)} - \frac{1}{\beta^2(\tau)}\right)\partial_s^2\phi_1(\tau) - \frac{\lambda(\tau+h)-\lambda(\tau)}{h}\phi_1(\tau) \\ &\quad\quad - \frac{n-2}{h}\left(\frac{\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\right)\partial_s\phi_1(\tau) \\ &\quad = F_h(\tau,s) \end{aligned} \tag{A 9}$$

with $u_h(\tau,1) = \partial_s u_h(\tau,0) = 0$. On the other hand, notice that

$$\begin{aligned} &\sup_{\tau\in\mathbb{R}}\left|\frac{n-2}{h}\left(\frac{\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\right)\partial_s\phi_1(\tau,s)\right| \\ &\quad \leqslant \sup_{\tau\in\mathbb{R}}\left|(n-2)\left(-\frac{\beta'(\tau)}{\beta^2(\tau)}\cot(\beta(\tau)s) - \frac{s\beta'(\tau)}{\sin^2\beta(\tau)s}\right)\partial_s\phi_1(\tau,s)\right| \\ &\quad < C\Big(n, \inf_{\tau\in\mathbb{R}}\beta(\tau)\Big), \end{aligned} \tag{A 10}$$

where in the last inequality we have used (A 7) and our assumptions on β. Also, using our assumption on λ we have that

$$\sup_{h\in\mathbb{R}}\sup_{\tau\in\mathbb{R}} F_h(\tau,s) < C\Big(n, \inf_{\tau\in\mathbb{R}}\beta(\tau)\Big). \tag{A 11}$$

Finally, combining (A 9)–(A 11), we have

$$\lim_{h\to 0}\sup_{\tau\in\mathbb{R}}\int_0^1 u_h^2(\tau,s)\sin^{n-2}(\beta(\tau)s)\,\mathrm{d}s < C < \infty. \tag{A 12}$$

By (A 12), we can prove that

$$\sup_{\tau\in\mathbb{R}}\sup_{\tau\in\omega(\tau)}|u_h| < C$$

and we have the following representation formula

$$\begin{aligned}\frac{u_h(\tau,s)}{\beta^2(\tau+h)} &= \lambda(\tau+h)\int_s^1 \frac{1}{\sin^{n-2}(\beta(\tau+h)\xi)}\int_0^\xi \sin^{n-2}(\beta(\tau+h)r)u_h(\tau,r)\,\mathrm{d}r\,\mathrm{d}\xi \\ &\quad - \int_s^1 \frac{1}{\sin^{n-2}(\beta(\tau+h)\xi)}\int_0^\xi \sin^{n-2}(\beta(\tau+h)r)F_h(\tau,r)\,\mathrm{d}r\,\mathrm{d}\xi.\end{aligned}$$

The rest of the proof is standard and we omit it. □

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