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ORIGINAL RESEARCH PAPER

Optimal investment strategy for a DC pension fund plan in a finite horizon time: an optimal stochastic control approach

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Abstract

This paper obtains an optimal strategy in a finite horizon time for a portfolio of a defined contribution (DC) pension fund for an investor with the CRRA utility function. It employs the optimal stochastic control method in a financial market with two different asset markets, one risk-free and another one risky asset in which its jump follows either by a finite or infinite activity Lévy process. Sensitivity of jump parameters in an uncertainty financial market has been studied.

Keywords: Optimal strategy; Pension plans; Pension fund; Finite/Infinite activity Lévy processes

1. Introduction

There are several types of pension plans in the insurance market which differ in terms of advantages and financing. These plans categorise in two parts: (1) defined contribution (DC) pension plans and (2) defined benefit (DB) pension plans. In this paper, we focus on investment of the DC pension plans. The most original question that we face in these designs is: "what is the best investment strategy of these funds in a finite horizon time?"

In a defined contribution pension plan, each client opens a separate personal account and all benefits granting plus/minus income, and also returns and assailed related to this account will be recorded. During retirement, the retirement benefits of each member are made from their account. Of course at this point, account balances can be used to purchase an annual salary and to create a regular income for the individual. In this type of plan, the risk is fully realised by the plan's members. Defined contribution pension plans in recent years at the level of the world have been grown dramatically. In all these plans, especially the defined contribution pension plans, the market investment is a key element which needs a considerable attention from their executive managers. Another type of pension contract is the "TimePension" product. It assigns a high allocation to equities and high risky assets. Such kind of investment leads to a high expectations in both returns and pension's benefits. Since the TimePension plans are offered after retirement period in the form of an annuity, they also called smoothed investment-linked annuity pension schemes, see Jørgensen and Linnemann (2012) and Linnemann *et al.* (2015) for more details.

In this paper, we focus on investing (in a finite horizon time) for a person who purchased a DC retirement product. Moreover, we assumed that an investment portfolio is a combination of two risky and risk-free assets. Then, we derive an optimal strategy for such portfolio. Optimal allocation of capital among a set of financial assets under conditions of uncertainty and risk is a well-established research field in modern finance theory. In this respect and before Merton (1969) contribution to the field, most portfolio selection models have only considered one-period, static models based on Markowitz's mean variance (1952) modern portfolio theory. In (1969) and (1975), Merton studied "the combined problem of optimal portfolio selection and consumption rules for an individual in a continuous-time model." As a particular case, he examined in detail the two-asset model (a risk free asset and a risky one) with constant relative risk aversion or iso-elastic marginal utility.

This topic still receive a considerable attention from authors. For instance, Xu and Gao (2020) provided a closed-form solution for the optimal portfolio control problem of a DC pension. Dong and Zheng (2020) applied the concavification and dual control method to solve an optimal investment problem for a DC pension fund. Yao *et al.* (2020) considered the stochastic inflation rate which described by a discrete-time of the Ornstein-Uhlenbeck process to derive an analytical expressions for the efficient investment strategy for a DC pension fund. Dong and Zheng (2019) for a DC pension fund whose its manager is a loss averse person derived an optimal investment strategy in terms of the dual controlled process and the dual value function.

Merton's original works developed under the geometric Brownian motion assumption, for the risky asset's dynamics, reduced to a non-linear Hamilton-Jacobi-Bellman, say HJB, partial differential equation. To obtain an explicit solution for such equation, a wide range of intertemporal economic decision problems under uncertainty have been developed by authors, see Korn (1997) and many references thereinafter. In the case that asset returns are modelled by a general Lévy process, see Cox and Huang (1989), Kullsen (2000), Emmer and Klüppelberg (2004), Choulli and Hurd (2001) and Liu *et al.* (2003).

Using a stochastic differential equation with a jump to model stock market has been received several attention from authors. For instance, Øksendal and Sulem(2005) used a jump-diffusion process to model the financial market.

Moreover, many authors employed the optimal control strategy for a pension contract in a high-frequency market. For example, Liang and Ma (2015) computed the optimal asset allocation of a pension fund with mortality risk and salary risk. They considered two stochastic processes for the mortality and the salary risk. Gao (2008) considered a portfolio problem in the complete financial market with stochastic interest rate. He found an explicit solution for the optimal investment strategy under the logarithm utility function. Deelstra *et al.* (2003) defined the optimal guarantee as a solution of the contributor's optimisation programme and found the solution explicitly. They analysed the impact of the main parameters, and particularly the sharing rule between the contributor and the pension fund. We improve their findings in two directions: (1) we derive an exact solution for the value function with a better guess in the optimal structure problem, and (2) we study the impact of jump parameters in calculating the optimal strategy. To calculate an optimal strategy, similar to Pasin and Vargiolu (2010), we employ an exponential additive process to model risky assets.

The rest of this article is organised as follows. Section 2 collects some elements that play vital roles in the rest of this article. Using the stochastic optimal control method, section 3 calculates the optimal investment strategy for an asset which its stock dynamic has a jump process. In section 4, we define and price a pension contract for a customer who pays his/her total premium at issue time. Finally, in section 5 the numerical implementation of the results have been given.

2. Preliminaries

We start the model's description by assuming that an expected utility maximising, risk-averse economic agent makes investment decisions in a continuous-time setting in a finite time horizon $[0, T]$ in a market modelled by a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. All the processes in this paper are adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ which describes the flow of

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information in a given time period. Moreover, we assume that the market composes of two underling securities: (1) A safe and risk-free asset (e.g. a bond or a bank account) described by

$$\frac{dB(t)}{B(t)} = r(t)dt, \quad B(0) = 1 \tag{1}$$

for a locally deterministic interest rate process $r(t)$ and (2) A risky asset (e.g. a stock) specified by the stochastic dynamics for its return as

$$\frac{dS(t)}{S(t)} = \left(\mu(t) + \frac{1}{2}\sigma(t)^2\right) dt + \sigma(t)dZ(t) + \int_{-\infty}^{\infty} (e^x - 1)N(dt, dx), \quad S(0) = S_0 \tag{2}$$

where $\mu(t)$ and $\sigma(t)$ are two adapted processes, respectively, representing the drift and diffusion parts of the rate of return and $Z(t)$ is a standard Brownian motion and $N(\cdot, \cdot)$ is a Poisson random counting measure with the compensator $\Pi(\cdot, \cdot)$ both defined on $\mathbb{R}^+ \times \mathbb{R}$. Moreover, we assume that the compensator $\Pi(\cdot, \cdot)$ for any measurable random function $\Phi(t, x) := \Phi(\omega, t, x)$ satisfies

$$\mathbb{E}\left(\int_{\mathbb{R}} \Phi(t, x)N(dt, dx)\right) = \int_{\mathbb{R}} \Phi(t, x)\Pi(t, dx)dt \tag{3}$$

Hereafter now, we assume that the jump and the Brownian parts in the stock dynamic are independent. Note that the measure $\Pi(t, dx)$ specifies the intensity of the aggregate jump arrival rate, and for practical purposes, we could assume it to depend on a deterministic/stochastic state variable ν_t via $\Pi(t, dx) = \Pi(\nu_t, dx)$. We may think of this state variable as representing the current level of the market or business activity (e.g. trading volume or liquidity) or some other related micro-level indicator and assume that it follows a stochastic differential equation of the form

$$d\nu_t = m(\nu_t, t)dt + \sigma(\nu_t, t)dZ_t^\nu$$

where $m(\nu_t, t), \sigma(\nu_t, t)$ are the drift and diffusion parts of the state variable ν_t and Z_t^ν is a standard Brownian motion. In such a situation, we will assume that the drift μ, the diffusion σ and the interest rate r are all deterministic functions of time and the state variable. We should note that: in the special case that the jump arrival rate is proportional to the state according to

$$\Pi(\nu_t, dx) = \nu_t\Pi(dx) \tag{4}$$

Financial models with jumps can be decomposed as the jump-diffusion models and models with infinite number of jumps in every interval, say infinite activity models. The regular price for the jump-diffusion models can be obtained by a diffusion process, which its jumps punctuated at random intervals. Such the jumps represent rare events-crashes and large drawdowns, see Merton (1976) and the Kou (2002) for some examples on such approach. For the infinite activity models, since dynamics of jumps is already rich enough to generate non-trivial small time behaviour, one does not need to introduce a Brownian component. Moreover, Madan (2001), among other authors, has been argued that such infinite activity models give a more realistic description of the price process at various time scales. It is worthwhile mentioning that, many models from this class can be constructed via a Brownian subordination, which gives them additional analytical tractability compared to jump-diffusion models.

In this article, we consider two models, one in class of the finite activity models and one from class of the infinite activity models. The following model assumptions introduce such two models.

Model Assumption 1. *(Kou Model: Finite Activity Case) The double exponential jump-diffusion model or Kou model is a finite activity model, whose its jump's size distribution follows a two-sided exponential distribution with density function*

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$$f_X(x) = p\beta_1 e^{-\beta_1 x} I_{\{x>0\}} + q\beta_2 e^{\beta_2 x} I_{\{x<0\}} \tag{5}$$

where $\beta_1 > 1$, $\beta_2 > 0$ governing the decay of the tails for the distribution of positive and negative jump sizes and $p \in [0, 1]$, $p + q = 1$ representing the probability of an upward jump.

To calculate the optimal strategy, we need the following Kou Lévy measure, given by Kou (2004) as $\Pi(dx) = \lambda f_X(x)dx$.

Model Assumption 2. *(Variance Gamma Model: Infinite Activity Case) The variance gamma model could be considered as an extension of the Brownian motion process with drift which is obtained by a random time change specified by a gamma process as:*

$$X_t = \theta\tau_t + \rho Z(\tau_t)$$

where θ and ρ are some given constants and for fixed $l > 0$, and $\omega > 0$, the gamma process $\tau_t = \gamma_t(l, \omega)$ has mean rate $l\omega$ and variance rate $l^2\omega$.

We also note in passing that the last integral term in (2) indicates the presence of jumps in stock price dynamics, first considered by Merton (1975), where he assumed that the stock follows a jump-diffusion process with a Poissonian (slow) arrival rate (see also Liu *et al.* 2003 for a recent study about implications of jumps in pricing and volatility on investment strategies). However, we consider here the more realistic choice of Lévy processes with extremely fast (potentially infinite) jump rates. The specific example can be the variance gamma (VG) process which is a pure jump Lévy process with an infinite arrival rate of small jumps, and this process introduced to the literature by Madan and Seneta (1990). It is well-known that the VG process is a pure jump Lévy process. In other words, the volatility in the stock dynamic is zero ($\sigma = 0$).

In the VG model, the Lévy compensator measure could be represented as

$$\Pi(dx) = \frac{1}{x} e^{-\frac{x}{\lambda_u}} I_{\{x>0\}} - \frac{1}{x} e^{\frac{x}{\lambda_d}} I_{\{x<0\}}$$

where λ_u and λ_d are the positive solutions of equation $\lambda_u - \lambda_u = \theta l$ and $\lambda_u\lambda_u = \frac{1}{2}\rho^2 l$, which are $\lambda_u = \frac{1}{2}\left(\sqrt{\theta^2 l^2 + 2\rho^2 l} + \theta l\right)$ and $\lambda_d = \frac{1}{2}\left(\sqrt{\theta^2 l^2 + 2\rho^2 l} - \theta l\right)$. Under some deterministic measure $\Pi(\cdot)$, we may able to simplify several of our results, see below for more details.

Suppose $\{W(t)\}_{t\in[0,T]}$ denotes the wealth process of the investor representing the total accumulated wealth at time t. We need the following definitions before any further progress.

Definition 1. *A portfolio process (or portfolio strategy) is a real-valued progressively measurable process $\{\pi(t)\}_{t\in[0,T]}$, where almost surely, satisfies*

$$\int_0^T |\pi(t)W(t)|^2 dt < \infty$$

In the sequel, we assume that the investor maintains a self-financing portfolio by allocating his/her wealth among the two underlying assets in such a way that any wealth change is only due to consumption or gains/losses from the investment in the bond and the stock. In this respect, the wealth process $W^\pi := \{W^\pi(t)\}_{t\in[0,T]}$ corresponding to a self-financing portfolio strategy π will be the unique solution of the following stochastic differential equation

$$\begin{aligned} dW^\pi(t) &= \left(\left(r(t) + \pi(t)\left(\mu(t) + \frac{1}{2}\sigma(t)^2 - r(t)\right)\right) W^\pi(t)\right) dt + \pi(t)W^\pi(t)\sigma(t)dZ(t) \\ &+ \pi(t-)W^\pi(t-)\int_{-\infty}^{\infty} (e^x - 1)N(\nu_t, dx), \quad W^\pi(0) = W_0 \end{aligned} \tag{6}$$

A self-financing strategy π is said to be admissible if $W^\pi(t) \geq 0$, $\mathbb{P}-$a.s., for all $t \geq 0$. The set of all admissible strategies will be denoted by $\mathcal{A}$.

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Definition 2. *The CRRA utility function is given by*

$$U(x;\gamma) = \frac{x^{1-\gamma}}{1-\gamma} I_{(0,1)\cup(1,\infty)}(\gamma) + \log(x) I_{\{1\}}(\gamma) \tag{7}$$

The above definition provides two popular utility functions: the power utility ($U(x) = (x^{1-\gamma})/(1-\gamma)$) and the logarithm utility ($U(x) = \log(x)$).

3. Optimal Portfolio Allocation Rule in a Finite Horizon Time

In this section, we evaluate an optimal strategy, in a finite horizon $(0, T]$ time scale, with respect to two utility functions, given by Definition 2. Our idea for a finite horizon case has been inspired from Merton (1975). Moreover, we formulate the problem of choosing an optimal portfolio by selecting rules π^* from

$$J(W,t) = \sup_{\{\pi(s)\}_{0\le s\le t}} \mathbb{E}_t\left(U(W^{\pi}(T))\right) \tag{8}$$

where $\mathbb{E}_t$ denotes the conditional expectation operator with respect to σ-algebra $\mathcal{F}_t$ and $U(\cdot)$ is a given utility function.

In order to obtain the optimality equations, we employ the dynamic programming principle and stochastic optimal control theory. Such an approach derives us to the following non-linear Hamilton-Jacobi-Bellman, say HJB, partial differential equation

$$\begin{aligned} 0 = J_t + \sup_{\{\pi(t)\}} \Bigg\{ & \left(r + \pi\left(\mu + \frac{1}{2}\sigma^2 - r\right)\right) W_t J_W + \frac{1}{2}\pi^2\sigma^2 W_t^2 J_{WW} \\ & + \int_{-\infty}^{\infty} \left[J\big(W(1+\pi(e^x-1)), t\big) - J(W,t)\right] \Pi(\nu_t, dx) \end{aligned} \tag{9}$$

where J_W and J_t, respectively, denote the first partial derivatives of $J(W,t)$ with respect to W and t and similarly for higher derivatives.

Now set the right-hand side of equation (9) to be $K(W_t, \pi)$. A candidate for the optimal control π^* is obtained by taking partial derivatives of $K(W_t, \pi)$ with respect to π_t and equate to be 0. For convenience, we call such equations by the first-order conditions and note them by FOC.

Under the above first-order condition, say FOC, the HJB equation, given by equation (9), can be simplified by

$$\left(\mu + \frac{1}{2}\sigma^2 - r\right) W_t J_W + \pi^*\sigma^2 W_t^2 J_{WW} + \int_{-\infty}^{\infty} \frac{\partial}{\partial \pi} J(W(1+\pi^*(e^x-1)), t)\Pi(\nu_t, dx)\} = 0$$

Hereafter now, we calculate the optimal strategy under the CRRA utility function, given by Definition 2.

A general form of the logarithm utility function in a finite horizon time can be represented by

$$U(W_t) = e^{-\alpha T} \log(W_t) \tag{10}$$

Following Aït-Sahalia *et al.* (2009)'s approach, we solve generally equation (9) by assuming (and then verifying) that the indirect utility function J can be decomposed as

$$J(W,t) = U(W_t) f(t) + g(t) \tag{11}$$

where $f(t)$ and $g(t)$ are deterministic functions capturing the investment opportunity that depends on calendar time.

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In the following, we evaluate $f(t)$ and $g(t)$ under the logarithm utility function, given by equation (10), then we implicity determine an optimal portfolio in a finite horizon time $(0, T]$.

Theorem 1. *Suppose the coefficients μ, σ, and r given by equation (1) and $\nu_t \equiv \nu$. is a constant state. Moreover, suppose that there is a solution J for equation (9) and there is a deterministic function π^* that is solution of the following equation*

$$-\pi^*\sigma^2 + \left(\mu + \frac{1}{2}\sigma^2 - r\right) + \int_{-\infty}^{\infty} \frac{(e^x - 1)}{1 + \pi^*(e^x - 1)} \Pi(\nu_t, dx) = 0 \tag{12}$$

Then, under the logarithm utility function and following the Aït-Sahalia et al. (2009)'s approach, we have

$$f(t) = K, \tag{13}$$

$$g(t) = Ke^{-\alpha T}(T - t)\left(r + \pi^*\left(\mu + \frac{1}{2}\sigma^2 - r\right) - \frac{1}{2}\pi^{*2}\sigma^2 + \psi(\pi^*)\right), \tag{14}$$

where $\psi(\pi^*) = \int_{-\infty}^{\infty} \log(1 + \pi^*(e^x - 1))\Pi(\nu_t, dx)$

Proof. Under the FOC condition and by substituting the optimal strategy and (11) and the utility function from (10) into the HJB equation (9), we get $J(W_t, t) = e^{-\alpha T}\log(W_t)f(t) + g(t)$, $J_W = e^{-\alpha T}f(t)/W_t$, $J_{WW} = -e^{-\alpha T}f(t)/W_t^2$, $J_t = e^{-\alpha T}\log(W_t)f'(t) + g'(t)$. Therefore,

$$\begin{aligned} 0 = {} & e^{-\alpha T}\log(W_t)f'(t) + g'(t) + \left(r + \pi^*\left(\mu + \frac{1}{2}\sigma^2 - r\right)\right)e^{-\alpha T}f(t) - \frac{1}{2}\pi^{*2}\sigma^2 e^{-\alpha T}f(t) \\ & + e^{-\alpha T}f(t)\int_{-\infty}^{\infty}\log(1 + \pi^*(e^x - 1))\Pi(\nu_t, dx) \end{aligned} \tag{15}$$

Following Aït-Sahalia *et al.* (2009)'s approach, to remove W_t from equation (15), we assume $f'(t) = 0$, and consequently, $f(t) = K$ is constant function. Now, by substituting $f(t)$ in equation (15), we derive $g(t)$ with boundary condition $g(T) = 0$. $\square$

A general form of the power utility function is

$$U(W_t) = \mathrm{e}^{-\alpha T}\frac{{W_t}^{1-\gamma}}{1-\gamma} \tag{16}$$

Again following the Aït-Sahalia *et al.* (2009)'s approach, one may assume

$$J(W_t, t) = U(W_t)f(t) \tag{17}$$

where $f(t)$ is a deterministic function which captures the investment opportunity that depends on calendar time. In the special case that ν_t is a constant for all t, we could find the explicit solution of the HJB equation (9).

In the following, we evaluate $f(t)$ under the power utility function, given by equation (16), then we implicity determine an optimal portfolio in a finite horizon time $(0, T]$.

Theorem 2. *Suppose the coefficients μ, σ and r in equations (1) and (2) are driven by a constant state $\nu_t \equiv \nu$. Moreover, suppose that there is a solution J for equation (9) and there is a deterministic function π^* that is solution of the following equation*

$$-\gamma\sigma^2\pi^* + \left(\mu + \frac{1}{2}\sigma^2 - r\right) + \int_{-\infty}^{\infty}\left(1 + \pi^*\left(\mathrm{e}^x - 1\right)\right)^{-\gamma}(\mathrm{e}^x - 1)\Pi(\nu, dx) = 0 \tag{18}$$

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Then, under the power utility function and following the Aït-Sahalia et al. (2009)'s approach, we have

$$f(t) = \exp\left\{\left[(1-\gamma)\left(r+\pi^*\left(\mu+\frac{1}{2}\sigma^2-r\right)\right)-\frac{\gamma(1-\gamma)\pi^{*2}\sigma^2}{2}+\varphi(\pi^*)\right](T-t)\right\} \quad (19)$$

where $\phi(\pi^*) = \int_{-\infty}^{\infty}\left((1+\pi^*(e^x-1))^{1-\gamma}-1\right)\Pi(\nu_t, dx)$ *and boundary condition* $f(T)=1$.

Proof. Similar to the logarithm utility function (given by Theorem, 1), we use the FOC condition to compute strategy policy (π^*). Under the power utility function and following the Aït-Sahalia *et al.* (2009)'s J is $J(W_t,t)=e^{-\alpha T}{W_t}^{1-\gamma}f(t)/(1-\gamma)$, $J_W = e^{-\alpha T}f(t){W_t}^{-\gamma}$, $J_{WW} = -\gamma e^{-\alpha T}f(t){W_t}^{-\gamma-1}$ and $J_t = e^{-\alpha T}{W_t}^{1-\gamma}f'(t)/(1-\gamma)$.

Therefore, $f'(t)+\mathrm{K}(\pi^*)f(t)=0$. Now applying the boundary condition $f(T)=1$, the desired result will be arrived. □

The following lemma provides the optimal strategy under the Kou model (Model Assumption, 1) and the variance gamma model (Model Assumption, 2).

Lemma 1. *Under the Kou model (Model Assumption, 1) and the variance gamma model (Model Assumption, 2) the optimal strategy* π^* *given by Theorem (2) is, respectively, solution of the following equations*

$$0 = -\gamma\sigma^2\pi^* + \left(\mu+\frac{1}{2}\sigma^2-r\right)+V(\pi^*)$$
$$0 = [(\mu-r)+M(\pi^*)$$

where

$$V(\pi^*) = \int_0^1 \left[\lambda p\beta_1\left(1+\pi^*t-t\right)^{-\gamma}(1-t)^{\beta_1+\gamma-2}-\lambda q\beta_2\left(1-\pi^*t\right)^{-\gamma}(1-t)^{\beta_2-1}\right]tdt,$$

$$M(\pi^*) = \int_0^1\left[\left(1-\pi^*t\right)^{-\gamma}\left(\frac{t(1-t)^{\frac{1}{\lambda_d}}}{\ln(1-t)}\right)-\left(\frac{1-t(1-\pi^*)}{1-t}\right)^{-\gamma}\left(\frac{-t(1-t)^{\frac{1}{\lambda_u}}}{(1-t)\ln(1-t)}\right)\right]\frac{dt}{1-t}$$

Proof. To compute the integral part of equation (18), one may separate such integral into two parts and used the Kou Lévy measure to obtain

$$V(\pi^*) = \int_{-\infty}^0 \lambda q\beta_2\left(1+\pi^*(e^x-1)\right)^{-\gamma}(e^x-1)e^{\beta_2 x}dx$$
$$+\int_0^\infty \lambda p\beta_1\left(1+\pi^*(e^x-1)\right)^{-\gamma}(e^x-1)e^{-\beta_1 x}dx$$

Similarly, the desired result obtains for the variance gamma measure (with $\sigma=0$) as

$$M(\pi^*) = -\int_{-\infty}^0\left(1+\pi^*(e^x-1)\right)^{-\gamma}(e^x-1)\frac{1}{x}e^{\frac{x}{\lambda_d}}dx+\int_0^\infty\left(1+\pi^*(e^x-1)\right)^{-\gamma}(e^x-1)\frac{1}{x}e^{-\frac{x}{\lambda_u}}dx$$

In the next step, we change variable for positive part of integral to $x=-\ln(1-t)$ and for negative part of integral to $x=\ln(1-t)$ and get the desired result. □

Following by the Gaussian integration method, we can numerically solve the optimal strategy which reported in the numerical section.

To obtain result of Lemma 1, under the Logarithm utility function, one may set $\gamma=1$.

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4. Application to Insurance

Now as an application of the above findings, we consider a DC pension contract which its policyholders pay their constant premium P in full at the beginning of the contract. Moreover, we assume that at maturity time T, a given policyholder is alive. Following Iscanoglu-Cekic (2016)'s approach, we suppose that the value of pension fund W_t, at time t invests into two risky and risk-free markets, and their returns accumulates into policyholder's account after subtracting deducting participation costs. More precisely, we assume that (1) the risk-free market is described by equation (1) and (2) the risky market is described by equation (2) and modelled by either Model Assumption 1 or Model Assumption 2.

Therefore, the portfolio's value of such policyholder at time t is

$$\begin{aligned} dW_t^{\pi_t} &= \left(r_t + \pi_t\left(\mu_t + \frac{1}{2}\sigma_t^2 - r_t\right)\right) W_t^{\pi_t} dt + \pi_t\sigma_t W_t^{\pi_t} dZ_t \\ &\quad + \pi(t-)W^{\pi_t}(t-)\int_{-\infty}^{\infty}(e^x - 1)N(\nu_t, dx) \qquad , W_0^{\pi_t} = P \end{aligned} \tag{20}$$

Such contract consists of three accounts: (1) the investment wealth account $W_t^{\pi_t}$, with optimal strategy π_t^*, (2) the customer account C_t and (3) the reserve account R_t. In general, the relation between such three accounts is:

$$W_t^{\pi_t} = C_t + R_t \qquad t \in [0, T]$$

Now, we consider the following two scenarios.

Scenario 1. *Under this scenario the insurer guaranties interest rate g for the customer account. Therefore, the customer account and the reserve account at time t can be restated, respectively, as*

$$\begin{aligned} C_t &= PI_{\{0\}}(t) + \left[(1+g)C_{t-1} + \tau\left(W_t^{\pi_t^*} - (1+g)C_{t-1}\right)\right] I_{\{1,2,\cdots,T\}}(t), \\ R_t &= \left[W_t^{\pi_t^*} - (1+g)C_{t-1} - \tau\left(W_t^{\pi_t^*} - (1+g)C_{t-1}\right)\right] I_{\{1,2,\cdots,T\}}(t) \end{aligned}$$

where $\tau \in [0, 1]$ is the contributing rate that updated by the insurance company and $(1-\tau)$ of investing in the risky market is allocated in the reserve account.

It worthwhile mentioning that, under Scenario 1, the contract account updates at the beginning of each year, therefore, we should choose $t \in \{0, \cdots, T\}$.

Scenario 2. *Since under this scenario the policyholder just receives gain if the market's return exceeds the guaranteed interest rate g otherwise he/she receives zero. Therefore, the customer account and the reserve account at time t can be restated, respectively, as*

$$\begin{aligned} C_t{}^+ &= PI_{\{0\}}(t) + \left[(1+g)C_{t-1} + \tau\left[W_t^{\pi_t^*} - (1+g)C_{t-1}\right]^+\right] I_{\{1,2,\cdots,T\}}(t), \\ R_t{}^+ &= \left[W_t^{\pi_t^*} - (1+g)C_{t-1} - \tau\left[W_t^{\pi_t^*} - (1+g)C_{t-1}\right]^+\right] I_{\{1,2,\cdots,T\}}(t) \end{aligned}$$

where $[A]^+ = \max[A, 0]$ and $\tau \in [0, 1]$ is the contributing rate that updated by the insurance company and $(1-\tau)$ of investing in risky market is allocated in the reserve account.

When the market performance is bad and the market is very volatile, the reserve account helps the insurance company to deal with of its liabilities. It worthwhile to mention that such two accounts updated at the end of each year.

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4. Fair Pricing Contract

Now, we evaluate the fair price of such a DC pension contract under both Scenarios 1 and 2. Such a fair price arrives from the equivalence principle under risk-neutral probability measure. Because of the jumps, the risk-neutral probability measure is not unique. Following Kou and Wang (2004), we can choose a particular risk-neutral measure Q^*. Under this risk neutral probability measure, the asset price $S(t)$ still follows:

$$\frac{dS(t)}{S(t)} = \left(\alpha - \lambda^* \varsigma^*\right) dt + \sigma dZ^*(t) + \int_{-\infty}^{\infty} \left(e^{X^*} - 1\right) N^*(dt, dx), \quad S(0) = S_0 \tag{21}$$

where $Z^*(t)$ is the standard Brownian motion under Q^*, N^* is a Poisson random measure with intensity λ^* and random jump sizes $X_1^*, X_2^*, X_3^*, \cdots$ are independent and identically distributed random variables.

Moreover, note that parameter ς^* under in the Kou model is given by equation (22) and under the variance gamma model given by equation (23).

$$\varsigma^* := \mathbb{E}\left[e^{X^*}\right] - 1 = \frac{p^* \beta_1^*}{\beta_1^* - 1} + \frac{q^* \beta_2^*}{\beta_2^* + 1} - 1 \tag{22}$$

Since, we focus on a risk-neutral probability measure in this section, to simplify the notation (without causing much confusion), we shall drop the superscript $*$ in the parameters, i.e. using p, β_1, q, β_2 rather than $p^*, \beta_1^*, q^*, \beta_2^*$.

$$\varsigma^* := \mathbb{E}\left[e^{X^*}\right] - 1 = \frac{Ei\left(-\frac{\varepsilon(1-\lambda_u^*)}{\lambda_u^*}\right)}{Ei\left(-\frac{\varepsilon}{\lambda_u^*}\right)} + \frac{Ei\left(-\frac{\varepsilon(1-\lambda_d^*)}{\lambda_d^*}\right)}{Ei\left(-\frac{\varepsilon}{\lambda_d^*}\right)} - 1 \tag{23}$$

where $Ei(\cdot)$ represents the exponential integral. In this case, for computing ς^*, we used the jump size distribution that introduced in section 5.

We defined investment wealth account ${}^*W_t^{\pi_t}$, under risk-neutral probability measure:

$$\begin{aligned} d^*W_t^{\pi_t} &= \left(r_t + \pi_t \left(\alpha - \lambda^* \varsigma^* - r_t\right)\right) {}^*W_t^{\pi_t} dt + \pi_t \sigma_t {}^*W_t^{\pi_t} dZ_t^* \\ &\quad + \pi(t-)^* W^{\pi_t}(t-) \int_{-\infty}^{\infty} (e^{x^*} - 1) N^*(\nu_t, dx), \qquad {}^*W_0^{\pi_t} = P \end{aligned} \tag{24}$$

Under the equivalence principle, the premium in full, say P, which pays at issue time, is determined such that the expected value of discounted value of the terminal value in the customer account at issue time 0, say V_0^C, is equal to the premium P.

To do so, we write the customer account at the maturity time T as

$$C_T = (1+g)^T C_0 + \tau \sum_{t=1}^{T} \left[{}^*W_t^{\pi_t^*} - (1+g) C_{t-1}\right] (1+g)^{T-t} \tag{25}$$

Note that, before the maturity date T at the end of each year, there is no cash flow for the customer account. In the other, the only cash flow happens at the maturity time T, when the insurer pays the terminal value C_T to the policyholder. We are ready to support the model if we have cash flow in wealth account. In this way, if we call the cash flow CF_t, then we have:

$$\begin{aligned} d^*W_t^{\pi_t} &= \left(r_t + \pi_t \left(\alpha - \lambda^* \varsigma^* - r_t\right) + CF_t\right)^* W_t^{\pi_t} dt + \pi_t \sigma_t {}^*W_t^{\pi_t} dZ_t^* \\ &\quad + \pi(t-)^* W^{\pi_t}(t-) \int_{-\infty}^{\infty} (e^{x^*} - 1) N^*(\nu_t, dx), \qquad {}^*W_0^{\pi_t} = P \end{aligned}$$

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Suppose V_0^C denotes the market value of the customer account at issue time $t=0$ and $e_f=e^{\alpha}-1$ represents the discretely compounded annual risk-free interest rate. Discounting the terminal value C_T to issue time $t=0$, we get the initial price of this contract under both Scenarios 1 and 2, respectively, as following:

$$V_0^C=\left(\frac{1+g}{1+e_f}\right)^T C_0+\frac{\tau}{(1+e_f)^T}\mathbb{E}^{Q^*}\left(\sum_{t=1}^{T}\left[{}^*W_t^{\pi_t^*}-(1+g)C_{t-1}\right](1+g)^{T-t}\right) \tag{26}$$

$$V_0^{C+}=\left(\frac{1+g}{1+e_f}\right)^T C_0+\frac{\tau}{(1+e_f)^T}\mathbb{E}^{Q^*}\left(\sum_{t=1}^{T}\left[{}^*W_t^{\pi_t^*}-(1+g)C_{t-1}\right]^+(1+g)^{T-t}\right) \tag{27}$$

Now, for the fair price under both Scenarios 1 and 2 can be archived.

Under the Scenario 1, the fair price can be obtained in a closed form, while for the Scenario 2, such the fair price has to be found numerically. In this paper, we price the pension fund contract in two scenarios. In fact, we calculated both equations (26) and (27). In equation (26), due to the linearity of the expectation, the expectation passed through the summation, and therefore, we calculated $\mathbb{E}(W_t)$ as the following. But in equation (27), since the expectation cannot pass through the summation (because there is $[A]^+=\max[A,0]$), so it has to be solved numerically, which will be presented in numerical section. Since, we did not need $\mathbb{E}(R_t)$ and $\mathbb{E}(C_t)$ in our calculations, we did not calculate them. We may use from them in future work. The following theorem provide the exact fair price.

Theorem 3. *Assume that the coefficients μ,σ and r in equation (24) are given constants. Then, the stochastic differential equation (24) have the following exact solution*

$${}^*W_t^{\pi_t}={}^*W_0^{\pi_t}e^{\left(\eta-\frac{1}{2}\delta^2\right)t+\delta Z_t}exp\left\{\int_0^t\int_{-\infty}^{\infty}y(x)N(\nu_t,dx)\right\}$$

where $\eta=(r_t+\pi_t(\alpha-\lambda^*\varsigma^*-r_t))$, $\delta=\pi_t\sigma_t$ and $y(x)=\ln(1+\pi(e^x-1))$

Proof. Following Øksendal (2013) and Lamberton and Lapeyre (2007), we begin by calculating the stochastic differential equation (24) without any jump term. For this purpose, we define the following auxiliary factors $F_t=e^{-\delta Z_t+\frac{1}{2}\delta^2 t}$, $Y_t=F_t{}^*W_t^{\pi_t}$ and $f(t,{}^*W_t^{\pi_t})=\eta^*W_t^{\pi_t}$.

Using these auxiliary factors, we may conclude that

$$\frac{dY_t}{dt}=F_tf\left(t,F_t^{-1}Y_t\right)=\eta Y_t \tag{28}$$

Now, we add the jump factor to the above solution and get

$${}^*W_t^{\pi_t}=\left(e^{(\eta-\frac{1}{2}\delta^2)t+\delta Z_t*}W_0^{\pi_t}\right)\prod_{i=0}^{N(t)}(1+U_i)$$

where U_n is size of every jump and $N(t)$ is the number of jump. In our model, the size of every jump is $U_i=\pi(e^{x_i}-1)$.

By setting $y(x)=\ln(1+\pi(e^x-1))$, we may have

$$\begin{aligned}{}^*W_t^{\pi_t}&=\left(e^{(\eta-\frac{1}{2}\delta^2)t+\delta Z_t*}W_0^{\pi_t}\right)\exp\left\{\ln\left(\prod_{n=0}^{N(t)}(1+\pi(e^x-1))\right)\right\}\\&=\left(e^{(\eta-\frac{1}{2}\delta^2)t+\delta Z_t*}W_0^{\pi_t}\right)\exp\left\{\sum_{n=0}^{N(t)}\ln((1+\pi(e^x-1)))\right\}\end{aligned}$$

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Due to the continuity of time, ${}^*W_t^{\pi_t}$ can be restated as

$$ {}^*W_t^{\pi_t} = \left(e^{(\eta - \frac{1}{2}\delta^2)t + \delta Z_t}\, {}^*W_0^{\pi_t} \right) \exp\left\{ \int_0^t \int_{-\infty}^{\infty} y(x) N(\nu_t, dx) \right\} $$

This observation completes the desired result. □

Now, we derive the expectation of the investment account ${}^*W_t^{\pi_t}$ under the first scenario (Scenario, 1).

Lemma 2. *Under the Scenario 1, one may conclude that*

$$ \mathbb{E}\left({}^*W_t^{\pi_t}\right) = \left(e^{\eta t}\, {}^*W_0^{\pi_t}\right) \exp\left\{ \left(\int_{-\infty}^{\infty} \left(\pi\,(e^x - 1)\right) \Pi(\nu_t, dx) \right) t \right\} $$

Proof. Since the jump part and the Brownian part are independent, one may evaluate such an expectation as

$$ \mathbb{E}\left({}^*W_t^{\pi_t}\right) = \mathbb{E}\left(e^{(\eta - \frac{1}{2}\delta^2)t + \delta Z_t}\, {}^*W_0^{\pi_t} \right) \times \mathbb{E}\left(\exp\left\{ \int_0^t \int_{-\infty}^{\infty} y(x) N(\nu_t, dx) \right\} \right) $$

Using property of the expectation on the Brownian part, we have

$$ \mathbb{E}\left(e^{(\eta - \frac{1}{2}\delta^2)t + \delta Z_t}\, {}^*W_0^{\pi_t} \right) = e^{\eta t}\, {}^*W_0^{\pi_t} $$

Using the following exponential formula for Lévy processes (see Cont and Tankov, 2004, for more details),

$$ \mathbb{E}\left(\exp\left\{ \int_0^t \int_{-\infty}^{\infty} y(x) N(\nu_t, dx) \right\} \right) = \exp\left\{ \left(\int_{-\infty}^{\infty} \left(\pi\,(e^x - 1)\right) \Pi(\nu_t, dx) \right) t \right\} $$

the expectation of the jump part can be calculated. □

In the following, we determine the expectation of the jump part for Model Assumption 1 and Model Assumption 2.

Example 1. In this case, we computed $\int_{-\infty}^{\infty} \left(\pi\,(e^x - 1)\right) \Pi(\nu_t, dx)$, for the variance gamma model (Model Assumption, 2). For this purpose, we used the variance gamma Lévy measure

$$ \int_{-\infty}^{\infty} \left(\pi\,(e^x - 1)\right) \Pi(\nu_t, dx) = \nu_t \int_{-\infty}^{0} -\pi\,(e^x - 1) \frac{e^{\frac{x}{\lambda_d}}}{x} dx + \nu_t \int_0^{+\infty} \pi\,(e^x - 1) \frac{e^{-\frac{x}{\lambda_u}}}{x} dx $$

For the negative part of integral sets $x = \ln(1-t)$ and $x = -\ln(1-t)$ for the positive part of integral, we have

$$ \int_{-\infty}^{\infty} \left(\pi\,(e^x - 1)\right) \Pi(\nu_t, dx) = \int_0^1 \frac{\nu_t \pi\, t}{\ln(1-t)} \left((1-t)^{\frac{1}{\lambda_d} - 1} - (1-t)^{\frac{1}{\lambda_u} - 2} \right) dt $$

Following Gradshteyn and Ryzhik (2014)'s findings, we have

$$ \int_0^1 \frac{(1 - x^p)^n x^{q-1}}{(1-x)\ln(x)} dx = \sum_{k=0}^{n} (-1)^{k-1} \ln\left(\Gamma\left[(n-k)p + q\right]\right) \forall n > 1,\ \ q > 0,\ \ np > -q $$

For our case, set the parameters $n = 2$, $p = 1$, $x = 1 - t$ and $q = \frac{1}{\lambda_d}$ for the negative part and $q = \frac{1}{\lambda_u} - 1$ for the positive part of integral. Therefore, we may calculate the desired integral explicitly.

Now, we consider the Kou model, given by Model Assumption 1.

Example 2. Consider Model Assumption 1 with the Kou Lévy measure, the integral of the jump part can be restated and evaluated as

$$\begin{aligned}\int_{-\infty}^{\infty} \left(\pi(e^x-1)\right)\Pi(\nu_t,dx) &= \int_{-\infty}^{0} \kappa_2(e^x-1)e^{\beta_2 x}dx + \int_{0}^{+\infty} \kappa_1(e^x-1)e^{-\beta_1 x}dx\\ &= \frac{\kappa_2}{\beta_2} - \frac{\kappa_1}{\beta_1} + \frac{\kappa_1}{\beta_1-1} - \frac{\kappa_2}{\beta_2+1}\end{aligned}$$

where $\kappa_2 = \lambda q\beta_2\pi$ and $\kappa_1 = \lambda p\beta_1\pi$.

5. Numerical Results

To simulate W_t in Theorem 3, we employ Applebaum (2009)'s approach, which said that the integral of $\int_{-\infty}^{\infty} y(x)N(t,dx)$ in a wealth equation has the compound Poisson distribution. We should recall that every compound Poisson distribution has an intensity λ and jump size distribution $F(x)$.

For the case of an infinite activity model for stock price behaviour, such as the variance gamma Lévy process, we follow Cont and Tankov (2004)'s recommendation and approximate such an infinite activity variance gamma process by a compound Poisson process with capital $\Lambda(\varepsilon) = \int_{\varepsilon}^{\infty} \Pi(dx)$ and the jump size distribution $P^{\varepsilon}(x) = \frac{\Pi(dx)}{\Lambda(\varepsilon)} I_{(\varepsilon,\infty)}(x)$, where $\Pi(dx)$ is the VG Lévy measure given by

$$\Pi(dx) = \frac{1}{x}e^{-\frac{x}{\lambda_u}}1_{\{x>0\}} - \frac{1}{x}e^{\frac{x}{\lambda_d}}1_{\{x<0\}}$$

The intensity and jump size distribution in the VG model, given by Model Assumption 2, have the following two positive and negative jumps

$$\Lambda_1(\varepsilon) = \int_{\varepsilon}^{\infty} \frac{\exp\{-x/\lambda_u\}}{x}dx = -Ei\left(-\frac{\varepsilon}{\lambda_u}\right)$$

$$\Lambda_2(\varepsilon) = -\int_{-\infty}^{-\varepsilon} \frac{\exp\{x/\lambda_d\}}{x}dx = -Ei\left(-\frac{\varepsilon}{\lambda_d}\right)$$

where $Ei(\cdot)$ represents the exponential integral introduced by Gradshteyn and Ryzhik (2014). Moreover, the jump size distribution is

$$P_1^{\varepsilon}(x) = \frac{\exp\{-x/\lambda_u\}}{x\Lambda_1(\varepsilon)}1_{x\geq\varepsilon}$$

$$P_2^{\varepsilon}(x) = -\frac{\exp\{x/\lambda_d\}}{x\Lambda_2(\varepsilon)}1_{x\leq-\varepsilon}$$

To generate the jump size of this distribution, we employ the rejection method introduced by Cont and Tankov (2004) and Devroye (2006). For this purpose, for the positive jump distribution and for all $x \geq \varepsilon$, it is obvious that

$$P_1^{\varepsilon}(x) \leq f_1^{\varepsilon}(x)\frac{\lambda_u e^{-\varepsilon/\lambda_u}}{\varepsilon\Lambda_1(\varepsilon)}$$

where $f_1^{\varepsilon}(x) = \frac{e^{-\frac{(x-\varepsilon)}{\lambda_u}}}{\lambda_u}1_{x\geq\varepsilon}$ is a probability density function. Following Cont and Tankov (2004)'s results, $f_1^{\varepsilon}(x)$ has the survival function $\bar{F}_1^{\varepsilon}(x) = e^{-\frac{(x-\varepsilon)}{\lambda_u}}1_{x\geq\varepsilon}$.

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Algorithm 1: The algorithm to evaluate positive jump distribution.

Input: A sequence of parameters $\langle \lambda_u, \varepsilon, \Lambda_1(\varepsilon), f_1^{\varepsilon}, p_1^{\varepsilon} \rangle$.

Output: Random number for positive jump distribution

Set $N \sim Poisson(\Lambda_1(\varepsilon))$(Number of jump)

for $t \leftarrow 1$ ***to*** N **do**

  Set $V, Q \overset{\text{i.i.d.}}{\sim} U(0,1)$

  Set $X = \varepsilon - \lambda_u \ln(Q)$

  Set $T = \frac{f_1^{\varepsilon}(X)\lambda_u e^{-\varepsilon/\lambda_u}}{P_1^{\varepsilon}(X)\varepsilon\Lambda_1(\varepsilon)}$

  If $VT \leq 1$ then X is acceptable for distribution of $p_1^{\varepsilon}(x)$

return *A vector X as a random number for positive jump distribution.*

Random variables with distribution $P_1^{\varepsilon}(x)$ may be simulated using the rejection method as suggested by Devroye (2006). Similar to the positive jump distribution, we use this method for the negative jump distribution. Observe that for all $x \leq -\varepsilon$, we have

$$P_2^{\varepsilon}(x) \leq f_2^{\varepsilon}(x)\frac{\lambda_d e^{-\varepsilon/\lambda_u}}{\varepsilon\Lambda_2(\varepsilon)}$$

where $f_2^{\varepsilon}(x) = \frac{e^{\frac{(x-\varepsilon)}{\lambda_d}}}{\lambda_d} 1_{x\leq -\varepsilon}$ is a probability density function and its corresponding survival function is $F_2^{\varepsilon}(x) = e^{\frac{(x-\varepsilon)}{\lambda_d}} 1_{x\leq -\varepsilon}$.

The following algorithm represents our simulation method for A times replication to generate the random variable of these distributions. Also, the following algorithm represents our simulation method for A times replication to generate the random variable of these distributions.

Optimal allocation and fair price under the CRRA investor utility functions (the power utility and the logarithm utility) in a risky stock based on the moment data reported in Campbell (1997) and under Model Assumption 2 and Model Assumption 1 for both Scenarios 1 and 2 have been calculated and represented in Tables 1, 2, 3 and 4.

In this paper, we have designed two contracts to ensure that the premium received from the insurer is repaid with guarantee rate g. In addition, by investing in risky and risk-free markets, we share her in the benefits of this investment. We also priced the contracts without guarantee rate and showed it by $(NG)V_0^{C+}$ and $(NG)V_0^{C}$ for two scenarios. The results show us when we do not consider guarantee rate, the contract is out of fair mode. The significantly point of this paper is that our main contract is mentioned under the second scenario, and to compare this contract, we designed another contract under the first scenario. We know that the first scenario is not attractive to the customer at all. Numerical results indicate this fact. The comparison is based on the results

Table 1. Optimal allocation and fair price under the power utility function in a risky stock based on the moment data reported in Campbell (1997) with $T=20$, $r=0.04$, $P=0.7$, $\mu=0.28$, $\tau=0.12$, $g=0.02$ and $\gamma=3$ under Model Assumption 2 and both Scenarios 1 and 2. In the simulation, there are 200 steps in each year.

θ	l	ρ	π^*	V_0^{C+}	V_0^C	$(NG)V_0^{C+}$	$(NG)V_0^C$
−0.00799	0.6	0.864	0.3901	0.7081	0.5672	0.7215	0.6048
−0.0007	0.5	0.872	0.3947	0.7051	0.5685	0.7227	0.6121
−0.004282	0.5	0.794	0.3943	0.7094	0.5738	0.7222	0.6143
0.00245	0.4	0.671	0.4019	0.7162	0.5819	0.7526	0.6255
0.001514	0.4	0.632	0.4021	0.7146	0.5839	0.7450	0.6271

Table 2. Optimal allocation and fair price with logarithm utility function in a risky stock based on the moment data reported in Campbell (1997) with $T=20$, $r=0.04$, $P=0.7$, $\mu=0.28$, $\tau=0.12$, $g=0.02$ and $\gamma=1$ under Model Assumption 2 and both Scenarios 1 and 2. In the simulation, there are 200 steps in each year.

θ	l	ρ	π^*	V_0^{C+}	V_0^C	$(NG)V_0^{C+}$	$(NG)V_0^C$
−0.001328	0.6	0.632	0.7365	0.7214	0.5603	0.7517	0.5956
−0.00532	0.5	0.746	0.7299	0.7220	0.5595	0.7502	0.5939
−0.00799	0.5	0.861	0.7284	0.7163	0.5548	0.7466	0.5732
0.00305	0.4	0.872	0.7442	0.7184	0.5579	0.7389	0.5842
0.00245	0.4	0.883	0.7433	0.7106	0.5564	0.7380	0.5908

Algorithm 2: The algorithm to evaluate negative jump distribution.

Input: A sequence of parameters $\langle \lambda_d, \varepsilon, \Lambda_2(\varepsilon), f_2^{\varepsilon}, p_2^{\varepsilon} \rangle$.

Output: Random number for negative jump distribution

Set $N' \sim Poisson(\Lambda_1(\varepsilon))$ (Number of jump)

for $t \leftarrow 1$ *to* N' **do**

  Set $V', Q' \overset{\text{iid}}{\sim} U(0,1)$

  Set $X' = \lambda_d \ln(Q') - \varepsilon \qquad (X' \text{ has distribution } f_2^{\varepsilon}(x))$

  Set $T' = \frac{f_2^{\varepsilon}(X')\lambda_d e^{-\varepsilon/\lambda_d}}{P_2^{\varepsilon}(X')\varepsilon\Lambda_2(\varepsilon)}$

  If $V'T' \leq 1$ then X' is acceptable for distribution of $p_2^{\varepsilon}(x)$

return *A vector X' as a random number for negative jump distribution.*

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Table 3. Optimal allocation and fair price with power utility function in a risky stock based on different intensity of double exponential β and different risk aversion coefficient in utility function with $T=2$, $r=0.04$, $\sigma=0.16$, $p=0.5$, $\tau=0.12$, $g=0.02$ and $\mu=0.28$ under Model Assumption 1 and both Scenarios 1 and 2. In the simulation, there are 200 steps in each year.

γ	β	π^*	V_0^{C+}	V_0^C	$(NG)V_0^{C+}$	$(NG)V_0^C$
3	5.3	0.3878	0.7277	0.5597	0.7523	0.5904
3	6.1	0.3810	0.7366	0.4876	0.7654	0.5512
5	7.5	0.3213	0.7256	0.4843	0.7485	0.5707
5	8.2	0.3176	0.7135	0.4830	0.7431	0.5661

Table 4. Optimal allocation and fair price with the logarithm utility function in a risky stock based on different intensity of double exponential β with $T=20$, $r=0.04$, $\sigma=0.16$, $p=0.5$, $\tau=0.12$, $g=0.02$ and $\mu=0.28$ under Model Assumption 1 and both Scenarios 1 and 2. In the simulation, there are 200 steps in each year.

γ	β	π^*	V_0^{C+}	V_0^C	$(NG)V_0^{C+}$	$(NG)V_0^C$
1	6.3	0.6961	0.7217	0.5049	0.7541	0.5706
1	7.9	0.5917	0.7342	0.5138	0.7718	0.5829
1	8.4	0.5774	0.7400	0.5113	0.7865	0.5549
1	8.8	0.5670	0.7407	0.4934	0.7884	0.5425

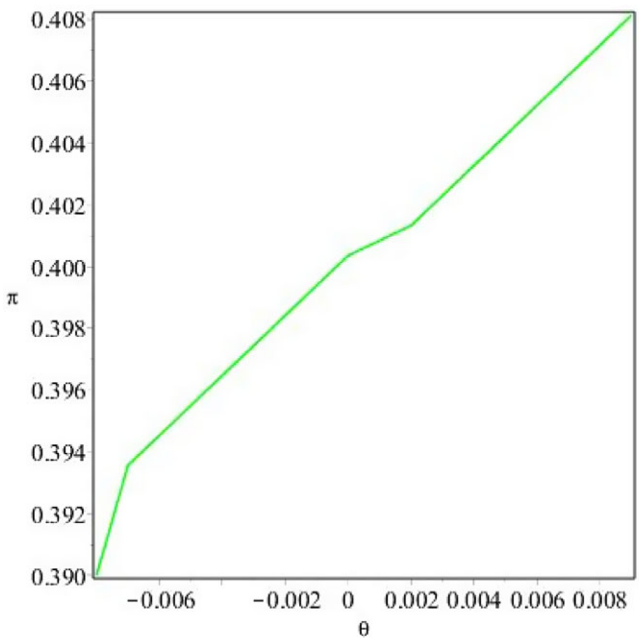

Figure 1. The behaviour of optimal strategy with $\rho=0.584$, $l=0.5$, $\mu=0.28$, $\gamma=3$ and different amount of θ in VG jump measure.

of Monte Carlo simulations with 200 time steps in each year. In Figures 1 and 2, we represent sensitivity of the investment strategy with respect to change of jump parameters in the variance gamma process. As one may observe from these two figures, the investment strategy does not change too much by changing such parameters.

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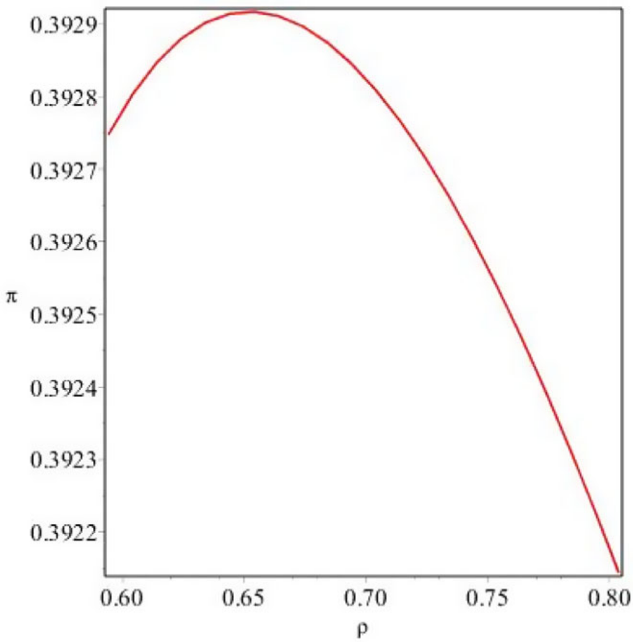

Figure 2. The behaviour of optimal strategy with $\theta = -0.00799$, $l = 0.5$, $\mu = 0.28$, $\gamma = 3$ and different amount of ρ in VG jump measure.

6. Conclusion and Suggestion

This article considers the problem of impact of two categories of jump processes from the Lévy family on the optimal investment strategy of a risk-averse investor in a finite horizon time. Similar to Cont and Tankov (2009), we employed jump parameters in our model to take into account both uncertainty and the risk of falling stock prices. Moreover to consider a wide range of possible processes, we considered both finite and infinite activity Lévy processes. To derive an application, we consider the problem of finding an optimal investment strategy in a finite horizon time for a defined contribution (DC) pension fund which its revenue has been invested into two risk-free and risky markets. The solution for such problem has been given under two different scenarios and two Lévy processes.

For a future possible study, we suggest the above optimality investment strategy in the presence of a consumption function in an infinite horizon time. Moreover, in Scenario 2, we assumed τ and g are two given constants. For a future work, one may consider the problem of calculating these two parameters based on a stock behaviour.

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References

Aït-Sahalia, **Y.**, **Cacho-Diaz**, **J.** & **Hurd**, **T.R.** (2009). Portfolio choice with jumps: a closed-form solution. *Annals of Applied Probability*, **19**(2), 556–584.

Applebaum, **D.** (2009). *Lévy Processes and Stochastic Calculus*. Cambridge University Press, New York.

Campbell, **J.Y.**, **Champbell**, **J.J.**, **Campbell**, **J.W.**, **Lo**, **A.W.**, **Lo**, **A.W.** & **MacKinlay**, **A.C.** (1997). *The Econometrics of Financial Markets*. princeton University Press.

https://doi.org/10.1017/S1748499521000270 Published online by Cambridge University Press

Choulli, **T.** & **Hurd**, **T.R.** (2001). The role of Hellinger processes in mathematical finance. *Entropy*, **3**(3), 150–161.
Cont, **R.** & **Tankov**, **P.** (2004). *Financial Modelling with Jump Processes*. Chapman and Hall/CRC.
Cont, **R.** & **Tankov**, **P.** (2009). Constant proportion portfolio insurance in the presence of jumps in asset prices. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, **19**(3), 379–401.
Cox, **J.C.** & **Huang**, **C.F.** (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of economic theory*, **49**(1), 33–83.
Deelstra, **G.**, **Grasselli**, **M.** & **Koehl**, **P.F.** (2004). Optimal design of the guarantee for defined contribution funds. *Journal of Economic Dynamics and Control*, **28**(11), 2239–2260.
Devroye, **L.** (2006). Nonuniform random variate generation. In *Handbooks in Operations Research and Management Science* (pp. 83–121), vol. **13**.
Dong, **Y.** & **Zheng**, **H.** (2019). Optimal investment of DC pension plan under short-selling constraints and portfolio insurance. *Insurance: Mathematics and Economics*, **85**, 47–59.
Dong, **Y.** & **Zheng**, **H.** (2020). Optimal investment with S-shaped utility and trading and Value at Risk constraints: an application to defined contribution pension plan. *European Journal of Operational Research*, **281**(2), 341–356.
Emmer, **S.** & **Klüppelberg**, **C.** (2004). Optimal portfolios when stock prices follow an exponential Lévy process. *Finance and Stochastics*, **8**(1), 17–44.
Gao, **J.** (2008). Stochastic optimal control of DC pension funds. *Insurance: Mathematics and Economics*, **42**(3), 1159–1164.
Gradshteyn, **I.S.** & **Ryzhik**, **I.M.** (2014). *Table of Integrals, Series, and Products*. Academic Press.
Iscanoglu-Cekic, **A.** (2016). An optimal Turkish private pension plan with a guarantee feature. *Risks*, **4**(3), 19.
Jørgensen, **P.L.** & **Linnemann**, **P.** (2012). A comparison of three different pension savings products with special emphasis on the payout phase. *Annals of Actuarial Science*, **6**(1), 137–152.
Kallsen, **J.** (2000). Optimal portfolios for exponential Lévy processes. *Mathematical Methods of Operations Research*, **51**(3), 357–374.
Korn, **R.** (1997). *Optimal Portfolios: Stochastic Models for Optimal Investment and Risk Management in Continuous Time*. World Scientific.
Kou, **S.G.** (2002). A jump-diffusion model for option pricing. *Management Science*, **48**(8), 1086–1101.
Kou, **S.G.** & **Wang**, **H.** (2004). Option pricing under a double exponential jump diffusion model. *Management Science*, **50**(9), 1178–1192.
Lamberton, **D.** & **Lapeyre**, **B.** (2007). *Introduction to Stochastic Calculus Applied to Finance*. CRC Press.
Linnemann, **P.**, **Bruhn**, **K.** & **Steffensen**, **M.** (2015). A comparison of modern investment-linked pension savings products. *Annals of Actuarial Science*, **9**(1), 72–84.
Liang, **Z.** & **Ma**, **M.** (2015). Optimal dynamic asset allocation of pension fund in mortality and salary risks framework. *Insurance: Mathematics and Economics*, **64**, 151–161.
Liu, **J.**, **Longstaff**, **F.A.** & **Pan**, **J.** (2003). Dynamic asset allocation with event risk. *The Journal of Finance*, **58**(1), 231–259.
Madan, **D.B.** & **Seneta**, **E.** (1990). The variance gamma (VG) model for share market returns. *Journal of Business*, **63**(4), 511–524.
Madan, **D.** (2001). *Financial Modeling with Discontinuous Price Processes. Lévy Processes-Theory and Applications*. Birkhauser, Boston.
Markowitz, **H.** (1952). Portfolio selection. *The Journal of Finance*, **7**(1), 77–91.
Merton, **R.C.** (1969). Lifetime portfolio selection under uncertainty: the continuous-time case. *The Review of Economics and Statistics*, **51**(3), 247–257.
Merton, **R.C.** (1975). Optimum consumption and portfolio rules in a continuous-time model. In *Stochastic Optimization Models in Finance*. Academic Press.
Merton, **R.C.** (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, **3**(1–2), 125–144.
Øksendal, **B.** (2013). *Stochastic Differential Equations: An Introduction with Applications*. Springer Science and Business Media.
Øksendal, **B.K.** & **Sulem**, **A.** (2005). *Applied Stochastic Control of Jump Diffusions*, vol. **498**. Springer, Berlin.
Pasin, **L.** & **Vargiolu**, **T.** (2010). Optimal portfolio for CRRA utility functions when risky assets are exponential additive processes. *Economic Notes*, **39**(1–2), 65–90.
Xu, **W.** & **Gao**, **J.** (2020). An optimal portfolio problem of DC pension with input-delay and jump-diffusion process. *Mathematical Problems in Engineering*, **2020**, 1–9.
Yao, **H.**, **Chen**, **P.**, **Zhang**, **M.** & **Li**, **X.** (2020). Dynamic discrete-time portfolio selection for defined contribution pension funds with inflation risk. *Journal of Industrial and Management Optimization*, **13**(5), 1–30.

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