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The Gelfond–Schnirelman Method in Prime Number Theory

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Abstract. The original Gelfond–Schnirelman method, proposed in 1936, uses polynomials with integer coefficients and small norms on $[0,1]$ to give a Chebyshev-type lower bound in prime number theory. We study a generalization of this method for polynomials in many variables. Our main result is a lower bound for the integral of Chebyshev's ψ-function, expressed in terms of the weighted capacity. This extends previous work of Nair and Chudnovsky, and connects the subject to the potential theory with external fields generated by polynomial-type weights. We also solve the corresponding potential theoretic problem, by finding the extremal measure and its support.

1 Lower Bounds for Arithmetic Functions

Let $\pi(x)$ be the number of primes not exceeding x. The celebrated Prime Number Theorem (PNT), suggested by Legendre and Gauss, states that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty. \tag{1.1}$$

We include a very brief sketch of its history, referring for details to many excellent books and surveys available on this subject (see, *e.g.*, [8, 10, 17, 29]). Chebyshev [6] made the first important step towards the PNT in 1852, by proving the bounds

$$0.921\frac{x}{\log x} \le \pi(x) \le 1.106\frac{x}{\log x} \quad \text{as } x \to \infty. \tag{1.2}$$

Riemann's famous paper [24], published in 1859, gave a strong impulse to the study of complex analytic methods related to the zeta function. Thus, Hadamard and de la Vallée Poussin independently proved the Prime Number Theorem in 1896, via establishing that $\zeta(s)$ does not have zeros on the line $\{1+it,\ t\in\mathbb{R}\}$. But the "elementary" approaches to the PNT, which do not use complex analysis and the zeta function, still remained attractive. Selberg [26] and Erdős [11] found the first elementary proof of the Prime Number Theorem in 1949. A survey of elementary methods, with detailed history, may be found in Diamond [10]. The subject of this paper is the elementary

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method of Gelfond and Schnirelman (see Gelfond's comments in [6, pp. 285–288]), proposed in 1936. Consider the Chebyshev function

$$\psi(x) := \sum_{p^m \le x} \log p, \tag{1.3}$$

where the summation extends over the primes p. Note that $\psi(x) = \log \operatorname{lcm}(1, \dots, x)$ for $x \in \mathbb{N}$. It is well known that the PNT is equivalent to

$$\psi(x) \sim x \quad \text{as } x \to +\infty \tag{1.4}$$

(see [8, 10, 17] and [19, Ch. 10]). The idea of Gelfond and Schnirelman was based on a clever use of polynomials with integer coefficients $p_n(x) = \sum_{k=0}^n a_k x^k$ and their integrals

$$\int_0^1 p_n(x)\,dx = \sum_{k=0}^n \frac{a_k}{k+1}.$$

Observe that multiplying the above integral by the least common multiple $\operatorname{lcm}(1, \dots, n+1)$ gives an integer, so that

$$\operatorname{lcm}(1, \dots, n+1) \left| \int_0^1 p_n(x)\,dx \right| \ge 1, \tag{1.5}$$

provided $\int_0^1 p_n(x)\,dx \ne 0$. Taking the log of (1.5), we have

$$\psi(n+1) \ge -\log \left| \int_0^1 p_n(x)\,dx \right| \ge -\log \max_{x \in [0,1]} |p_n(x)|.$$

Hence

$$\liminf_{n \to \infty} \frac{\psi(n+1)}{n} \ge -\log \limsup_{n \to \infty} \Big(\max_{x \in [0,1]} |p_n(x)| \Big)^{1/n}. \tag{1.6}$$

If one could find a sequence of polynomials p_n with sufficiently small sup norms $\|p_n\|_{[0,1]}$, so that

$$\lim_{n \to \infty} \|p_n\|_{[0,1]}^{1/n} \stackrel{?}{=} 1/e, \tag{1.7}$$

then the PNT would follow from (1.6). A nice account of the original Gelfond–Schnirelman attempt is contained in Montgomery [19, Ch. 10] (also see Chudnovsky [7]). We are led by this method to the so-called integer Chebyshev problem on polynomials with integer coefficients minimizing the sup norm (see, *e.g.*, Borwein [5]). Let $\mathbb{Z}_n[x]$ be the set of polynomials over integers, of degree at most n. In view of (1.6)–(1.7), we are interested in the integer Chebyshev constant

$$t_{\mathbb{Z}}([0,1]) := \lim_{n \to \infty} \left(\inf_{0 \not\equiv p_n \in \mathbb{Z}_n[x]} \|p_n\|_{[0,1]} \right)^{1/n}. \tag{1.8}$$

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It was found by Gorshkov [15] in 1956 that (1.7) can never be achieved. In fact, $0.4213 < t_{\mathbb{Z}}([0,1]) < 0.4232$ (see [22] for a survey of recent results on this problem). Thus the Gelfond–Schnirelman method failed in its original form, but one can generalize it for polynomials in many variables. Such an idea apparently had first appeared in Trigub [31], and was independently implemented by Nair [21] and Chudnovsky [7]. The basis of their argument lies in another equivalent form of the Prime Number Theorem [17]:

$$\int_1^x \psi(t)\,dt \sim \frac{x^2}{2} \quad \text{as } x \to +\infty. \tag{1.9}$$

Both Nair and Chudnovsky used the following weighted version of the Vandermonde determinant:

$$\begin{aligned} V_n^w(x_1,\dots,x_n) &:= \prod_{1\le i<j\le n} (x_i - x_j) w(x_i) w(x_j) \\ &= \prod_{i=1}^n w^{n-1}(x_i) \prod_{1\le i<j\le n} (x_i - x_j), \end{aligned} \tag{1.10}$$

where $x_i \in [0,1]$ and $w(x) = (x(1-x))^{\alpha_1}$, $\alpha_1 > 0$, to generate multivariate polynomials with small sup norms on the cube $[0,1]^n$. They obtained the numerical bound

$$\int_1^x \psi(t)\,dt \ge 0.99035\,\frac{x^2}{2} \quad \text{as } x \to +\infty, \tag{1.11}$$

produced by the optimal choice $\alpha_1 \approx 0.195$ (our notations differ from those of [7, 21]). Chudnovsky [7] also indicated how this approach can be generalized for the weights of the form

$$w(x) = \prod_{i=1}^k |Q_{m_i}(x)|^{\alpha_i}, \tag{1.12}$$

where $Q_{m_i} \in \mathbb{Z}_{m_i}[x]$ and $\alpha_i > 0$, $i = 1,\dots,k$. We develop the ideas of [7, 21], and establish a connection with the weighted potential theory (or potential theory with external fields) that originated in the work of Gauss [14] and Frostman [13] (see [25] for a modern account of this theory). An important part of the method is the analysis of the asymptotic behavior for the supremum norms of the weighted Vandermonde determinants (1.10), which is governed by the weighted capacity c_w of $[0,1]$ corresponding to the weight w (*cf.* Section 2 below and [25]). This method leads to the following lower bound for the integral of ψ-function via c_w.

Theorem 1.1 *Let $w(x)$ be as in* (1.12) *and let $\alpha := \sum_{i=1}^k \alpha_i m_i$. Then*

$$\int_1^x \psi(t)\,dt \ge \frac{-2\log c_w}{4\alpha+3}\,\frac{x^2}{2} + O(x\log^2 x) \quad \textit{as } x \to +\infty. \tag{1.13}$$

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We recover the results of Nair and Chudnovsky as a special case of Theorem 1.1.

Corollary 1.2 *If* $w(x) = x^{\alpha_1}(1-x)^{\alpha_2}$, $x \in [0,1]$, $\alpha_1 = \alpha_2 = 0.195$, *then* $c_w \approx 0.1045575588$ *and* (1.11) *holds true.*

It is natural to try improving the bound (1.11) by choosing a weight with a proper combination of factors $Q_{m_i}(x)$ and exponents α_i. The most interesting question is, of course, can one find a weight $w(x)$ of the form (1.12) such that

$$\frac{-2\log c_w}{4\alpha+3} = 1?$$

It turns out this is impossible to achieve for any *fixed* weight of the type (1.12). The reason for such a conclusion transpires from the error term in (1.13), which is "too good." Indeed, it is known from Littlewood's theorem that the difference $\int_1^x \psi(t)\,dt - x^2/2$ takes both positive and negative values of the amplitude $cx^{3/2}$, $c > 0$, infinitely often as $x \to +\infty$. This is conveniently written in the notation

$$\int_1^x \psi(t)\,dt - \frac{x^2}{2} = \Omega_{\pm}(x^{3/2}) \quad \text{as } x \to +\infty$$

(*cf.* [17, pp. 91–92]). Hence the correct error term should be of the order $O(x^{3/2})$. Relating this to (1.9) and (1.13), we obtain in such an indirect way the following.

Proposition 1.3 *Given a weight* $w(x)$ *of the form* (1.12), *we have*

$$B(w) := \frac{-2\log c_w}{4\alpha+3} < 1, \tag{1.14}$$

where $\alpha = \sum_{i=1}^k \alpha_i m_i$.

We should also note that if the Riemann hypothesis is true, then

$$\int_1^x \psi(t)\,dt - \frac{x^2}{2} = O(x^{3/2}) \quad \text{as } x \to +\infty$$

(see [17, Theorem 30, p. 83]). It would be very interesting to find a direct potential-theoretic argument explaining (1.14). Although (1.13) cannot provide a proof of the PNT for a fixed weight w, this does not preclude the possibility that such a proof can be obtained by finding a *sequence* of weights w_n with $B(w_n) \to 1$, as $n \to \infty$. On the other hand, we did not observe a numerical improvement of the estimate (1.11) when using further factors of the one-dimensional integer Chebyshev polynomials for the weight w, beyond the factors x and $1-x$ (see [7, 19, 22]). Thus one needs a better insight into the arithmetic nature of such factors to address the problem stated below.

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Problem 1.4 *For $w(x)$ as in* (1.12) *and $\alpha = \sum_{i=1}^{k} \alpha_i m_i$, find*

$$B := \sup_w \frac{-2\log c_w}{4\alpha + 3}. \tag{1.15}$$

If $B = 1$, then find a sequence of weights that gives this value. If $B < 1$, then investigate whether B is attained for a weight of the form (1.12).

The solution of this problem also requires a detailed knowledge of the potential theory with external fields generated by the weights (1.12), which is discussed in the following section.

We remark that the scope of the multivariate Gelfond–Schnirelman method still remains much wider than the approach proposed by Nair and Chudnovsky. Indeed, the weighted Vandermonde determinant $V_n^w(x_1, \ldots, x_n)$ of (1.10) is a very special case of a multivariate polynomial with small norm, which is unlikely to be best possible. It is of great interest to design other sequences of polynomials providing good bounds for the arithmetic functions, along the lines discussed. Among the natural candidates are the multivariate Vandermonde determinants (see [3, 32]) and other sequences of minimal polynomials [4]. This subject is closely related to pluripotential theory [18].

2 Potential Theory With External Fields

We consider a special case of the weighted energy problem on a segment of the real line $[a, b]$ which is associated with the "polynomial-type" weights (1.12). A comprehensive treatment of the potential theory with external fields, or weighted potential theory, is contained in the book by Saff and Totik [25], together with historical remarks and numerous references. It is convenient to rewrite the weight function in the following more general form:

$$w(x) = A \prod_{i=1}^{K} |x - z_i|^{p_i}, \quad x \in [a, b], \tag{2.1}$$

where $A > 0$, $p_i > 0$ and $z_i \in \mathbb{C}$. Let $\mathcal{M}([a, b])$ be the set of positive unit Borel measures supported on $[a, b]$. For any measure $\mu \in \mathcal{M}([a, b])$ and weight w of (2.1), we define the energy functional

$$\begin{aligned} I_w(\mu) &:= \iint \log \frac{1}{|z - t| w(z) w(t)} \, d\mu(z) d\mu(t) \\ &= \iint \log \frac{1}{|z - t|} \, d\mu(z) d\mu(t) - 2 \int \log w(t) \, d\mu(t), \end{aligned} \tag{2.2}$$

and consider the minimum energy problem

$$V_w := \inf_{\mu \in \mathcal{M}([a,b])} I_w(\mu). \tag{2.3}$$

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It follows from [25, Theorem I.1.3] that V_w is finite, and there exists a unique equilibrium measure $\mu_w \in \mathcal{M}([a,b])$ such that $I_w(\mu_w) = V_w$. Thus μ_w minimizes the energy functional (2.2) in the presence of the external field generated by the weight w. Furthermore, we have for the potential of μ_w that

$$U^{\mu_w}(x) - \log w(x) \geq F_w, \quad x \in [a,b], \tag{2.4}$$

and

$$U^{\mu_w}(x) - \log w(x) = F_w, \quad x \in S_w, \tag{2.5}$$

where $U^{\mu_w}(x) := -\int \log|x-t|\, d\mu_w(t)$, $F_w := V_w + \int \log w(t)\, d\mu_w(t)$ and $S_w :=$ supp μ_w (see [25, Theorems I.1.3 and I.5.1]). The weighted capacity of $[a,b]$ is defined by

$$\operatorname{cap}([a,b],w) := e^{-V_w}. \tag{2.6}$$

In agreement with the notation of Section 1, we set

$$c_w := \operatorname{cap}([0,1],w).$$

If $w \equiv 1$ on $[a,b]$, then we obtain the classical logarithmic capacity $\operatorname{cap}([a,b],1) = (b-a)/4$ (*cf.* [23]).

The support S_w plays a crucial role in determining the equilibrium measure μ_w itself, as well as other components of this weighted energy problem. Indeed, if S_w is known, then μ_w can be found as a solution of the singular integral equation

$$\int \log \frac{1}{|x-t|}\, d\mu(t) - \log w(x) = F, \quad x \in S_w,$$

where F is a constant (*cf.* (2.5) and [25, Ch. IV]). For w given by (2.1) or (1.12), this equation can be solved by potential theoretical methods, using balayage techniques, so that μ_w is expressed as a linear combination of harmonic measures (see Lemma 3.3 and [22]). We follow another path here, via the methods of singular integral equations, which gives a more explicit solution. This approach for polynomial-type weights was suggested by Chudnovsky [7] and developed further by Amoroso [1]. For more general weights, one should consult [25, Ch. IV] and the paper of Deift, Kreicherbauer and McLaughlin [9]. We give an explicit form of the equilibrium measure and describe its support in the following result.

Theorem 2.1 *Let $Z := \bigcup_{i=1}^{K}\{z_i\} \subset [a,b]$ be the set of zeros for w of* (2.1), *where $z_1 = a < z_2 < \cdots < z_K = b$. There exist an integer L, $1 \leq L \leq K-1$, a polynomial $P(x) = x^{K-L-1} + \cdots \in \mathbb{R}_{K-L-1}[x]$, and L intervals $[a_l, b_l] \subset [a,b] \setminus Z$, with $a < a_1 < b_1 < a_2 < b_2 < \cdots < a_L < b_L < b$, such that*

$$S_w = \bigcup_{l=1}^{L} [a_l, b_l], \tag{2.7}$$

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and the equilibrium measure μ_w is given by

$$d\mu_w(x) = (-1)^{L+l+1} \frac{(1+p)\sqrt{|R(x)|}\,P(x)}{\pi \prod_{j=1}^{K}(x-z_j)}\,dx, \quad x \in [a_l, b_l], \tag{2.8}$$

where $l = 1, \dots, L$, $p := \sum_{j=1}^{K} p_j$ and $R(x) := \prod_{l=1}^{L}(x-a_l)(x-b_l)$.

Furthermore, the polynomial $P(x)$ and the endpoints of S_w satisfy the equations

$$P(z_j) = (-1)^{L+l} \frac{p_j \prod_{m\neq j}(z_j - z_m)}{(1+p)\sqrt{|R(z_j)|}}, \quad z_j \in (b_l, a_{l+1}),\ j = 1, \dots, K, \tag{2.9}$$

where we set $b_0 = -\infty$, $a_{L+1} = +\infty$, and the equations

$$\int_{b_l}^{a_{l+1}} \frac{\sqrt{|R(x)|}\,P(x)}{\prod_{j=1}^{K}(x-z_j)}\,dx = 0, \quad l = 1, \dots, L-1, \tag{2.10}$$

where the integrals are understood as Cauchy principal values.

Recall that we need the quantity $-\log \operatorname{cap}([a,b],w) = V_w$ for Theorem 1.1. This can be found from (2.5) as

$$V_w = U^{\mu_w}(x) - \log w(x) - \int \log w\, d\mu_w,$$

for any $x \in S_w$.

The assumption that the weight w vanishes at the endpoints of $[a,b]$ seems appropriate in this case, due to the role of factors x and $1-x$, for $w(x)$ on $[0,1]$, in the work of Nair and Chudnovsky. Other cases of weights (2.1) with real zeros can be handled similarly, along the lines of this paper. Perhaps a more interesting problem is to consider weights with complex zeros, when $-\log w(x)$ is not piecewise convex on $[a,b]$.

If the support S_w consists of $L = K-1$ intervals, then $P(x) \equiv 1$ in Theorem 2.1, and we obtain the following result.

Corollary 2.2 *If $L = K-1$ in Theorem 2.1, then*

$$d\mu_w(x) = \frac{(1+p)\sqrt{|R(x)|}}{\pi \prod_{j=1}^{K}|x-z_j|}\,dx, \quad x \in S_w. \tag{2.11}$$

Furthermore, the following equations hold true:

$$\sqrt{|R(z_j)|} = \frac{p_j}{1+p} \prod_{m\neq j} |z_j - z_m|, \quad j = 1, \dots, K, \tag{2.12}$$

and

$$\int_{b_l}^{a_{l+1}} \frac{\sqrt{|R(x)|}}{\prod_{j=1}^{K}(x-z_j)}\,dx = 0, \quad l = 1, \dots, K-2. \tag{2.13}$$

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Remark 2.3 Amoroso [1] studied the weighted energy problem for the weights of Theorem 2.1 (expressed in slightly different terms). In particular, [1, Theorem 2.2] states that the equilibrium measure always has the form (2.11), which is not true, as shown in the next section. The main shortcoming is in the assumption of [1] that the support S_w *always* consists of $K-1$ intervals, one less than the number of zeros of w. Also, [1] defines the endpoints of the support as solutions of (2.12)–(2.13), without analyzing that these solutions exist and fall within the needed range. Assuming that (2.12)–(2.13) hold true, the representation (2.11) is then deduced from those equations in [1]. We emphasize that the results of [1] on the irrationality measures of logarithms, stated in Theorems 4.2 and 4.3, are correct. In those applications, the parameters are selected so that (2.12)–(2.13) are valid, and the support indeed has $K-1$ intervals.

Note that equations (2.9)–(2.10) (correspondingly, (2.12)–(2.13)) may be used to find the coefficients of $P(x)$ and the endpoints of S_w. Thus we have $K-L-1$ coefficients and $2L$ endpoints to find, which gives $K+L-1$ unknowns and the same number of equations (2.9)–(2.10). For example, if $K=2$, *i.e.,* we have the so-called Jacobi-type weight w on $[a,b]$, then $L=1$ and (2.12) gives just two equations for the endpoints of $S_w=[a_1,b_1]$. It is easy to solve them explicitly, and find the well-known representation for S_w and μ_w from Corollary 2.2 (see, *e.g.,* [25, Examples IV.1.17 and IV.5.2]).

Corollary 2.4 *Suppose $w(x)=x^{p_1}(1-x)^{p_2}, x\in[0,1]$, where $p_1,p_2>0$. Then*

$$d\mu_w(x)=\frac{(1+p_1+p_2)\sqrt{(x-a_1)(b_1-x)}}{\pi x(1-x)}dx,\quad x\in[a_1,b_1]. \tag{2.14}$$

The endpoints of the support are

$$a_1=\frac{1+r_1^2-r_2^2-\sqrt{(1+r_1^2-r_2^2)^2-4r_1^2}}{2} \tag{2.15}$$

and

$$b_1=\frac{1+r_1^2-r_2^2+\sqrt{(1+r_1^2-r_2^2)^2-4r_1^2}}{2}, \tag{2.16}$$

where we set

$$r_1:=\frac{p_1}{1+p_1+p_2}\quad \textit{and}\quad r_2:=\frac{p_2}{1+p_1+p_2}.$$

The connection between the potential theory with external fields and this version of the Gelfond–Schnirelman method arises in the need for asymptotics of the weighted Vandermonde determinant (1.10). It is known that

$$\lim_{n\to\infty}\Big(\max_{x_1,\dots,x_n\in[a,b]}|V_n^w(x_1,\dots,x_n)|\Big)^{\frac{2}{n(n-1)}}=\operatorname{cap}([a,b],w) \tag{2.17}$$

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(see [25, Theorem III.1.3]). The quantity on the left-hand side of (2.17) is called the weighted transfinite diameter of $[a, b]$. In the case $w \equiv 1$, it was introduced by Fekete [12] for arbitrary compact sets in the plane. Szegő [28] showed that the transfinite diameter coincides with the logarithmic capacity, so that (2.17) is a generalization of his result. We quantify the rate of convergence in (2.17).

Lemma 2.5 *Let w be as in* (2.1). *There exist constants $d = d(w) > 1$ and $D = D(w) > 0$ such that*

$$(2.18) \qquad (\operatorname{cap}([a,b],w))^{n(n-1)} \le \max_{[a,b]^n} (V_n^w)^2 \le D\, d^{n\log^2 n}\, (\operatorname{cap}([a,b],w))^{n(n-1)}.$$

Equation (2.18) is the only fact from potential theory needed in the proof of Theorem 1.1. It is very likely that $\log^2 n$ can be replaced by $\log n$, matching the classical case (see [2, Theorem 1.3.3]). This would give a corresponding improvement in the error term of (1.13), but we do not pursue this direction.

3 Proofs

Proof of Theorem 1.1 The proof is based on an argument similar to the original Gelfond–Schnirelman idea (*cf.* [7, 21]). We consider the integrals of small polynomials with integer coefficients over the cube $[0, 1]^n$, $n \in \mathbb{N}$. It is important that the integrals are non-zero, so that we work with the square of the weighted Vandermonde determinant (1.10), instead:

$$(3.1) \qquad \Delta_n^w(x_1,\dots,x_n) := \left(V_n^w\right)^2 = \prod_{i=1}^{n} w^{2(n-1)}(x_i) \prod_{1\le i<j\le n} (x_i - x_j)^2.$$

If $w(x) \equiv 1$ then Δ_n^w is the classical discriminant. In general, Δ_n^w is not a polynomial in the x_i's because of the real exponents α_i in the weight (1.12). Hence we modify it further into

$$(3.2) \qquad \tilde{\Delta}_n^w(x_1,\dots,x_n) := \prod_{j=1}^{n}\prod_{i=1}^{k} \left(Q_{m_i}(x_j)\right)^{2\lceil \alpha_i(n-1)\rceil} \prod_{1\le i<j\le n} (x_i - x_j)^2,$$

where $\lceil a \rceil$ denotes the ceiling function: the smallest integer at least a. It is now clear that $\tilde{\Delta}_n^w(x_1,\dots,x_n)$ is a positive polynomial with integer coefficients that has the following form

$$(3.3) \qquad \tilde{\Delta}_n^w(x_1,\dots,x_n) = \sum a_{i_1\dots i_n} x_1^{i_1}\dots x_n^{i_n}.$$

Recall the definition of the classical Vandermonde determinant

$$V_n := \begin{vmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n} (x_i - x_j).$$

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Using the standard expansion of V_n, we observe that each term of this expansion is a (signed) product of all powers of x_i's from 0 to $n-1$. The expression in (3.2) is equal to V_n^2 times the weight part. Thus if we arrange the powers $i_1, \dots, i_n$ in every term of (3.3) in the increasing order, we have that

$$i_j \le n + j - 2 + N, \quad j = 1, \dots, n, \tag{3.4}$$

where $N := 2\sum_{i=1}^{k} m_i \lceil \alpha_i(n-1) \rceil$ is the contribution of the weight part in (3.2). Hence

$$\int_0^1 \dots \int_0^1 \tilde{\Delta}_n^w(x_1, \dots, x_n)\, dx_1 \dots dx_n = \sum \frac{a_{i_1 \dots i_n}}{(i_1+1)\dots(i_n+1)} \ne 0$$

is a rational number whose denominator divides $\prod_{l=n+N}^{2n-1+N} \operatorname{lcm}(1, \dots, l)$ by (3.4). It follows as in (1.5) that

$$\prod_{l=n+N}^{2n-1+N} \operatorname{lcm}(1, \dots, l) \int_{[0,1]^n} \tilde{\Delta}_n^w \ge 1.$$

On taking the logarithm, we obtain that

$$\sum_{l=n+N}^{2n-1+N} \psi(l) \ge -\log \int_{[0,1]^n} \tilde{\Delta}_n^w \ge -\log \max_{[0,1]^n} \tilde{\Delta}_n^w.$$

It is clear from (3.1) and (3.2) that

$$\tilde{\Delta}_n^w = \Delta_n^w \prod_{j=1}^{n} \prod_{i=1}^{k} |Q_{m_i}(x_j)|^{2(\lceil \alpha_i(n-1) \rceil - \alpha_i(n-1))},$$

which gives

$$\max_{[0,1]^n} \tilde{\Delta}_n^w \le \max_{[0,1]^n} \Delta_n^w \prod_{i=1}^{k} \left(\max(1, \|Q_{m_i}\|_{[0,1]})\right)^{2n}.$$

Since $\psi(x)$ is constant between integers, we arrive at the estimate

$$\int_{n+N}^{2n+N} \psi(y)\, dy \ge -\log \max_{[0,1]^n} \Delta_n^w + O(n) \quad \text{as } n \to \infty. \tag{3.5}$$

We now need the following consequence of Lemma 2.5:

$$\log \max_{[0,1]^n} \Delta_n^w = n^2 \log c_w + O(n \log^2 n) \quad \text{as } n \to \infty.$$

Applying this in (3.5), we have

$$\int_{n+N}^{2n+N} \psi(y)\, dy \ge -n^2 \log c_w + O(n \log^2 n) \quad \text{as } n \to \infty.$$

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Note that $2\alpha(n-1) \le N \le 2\alpha(n-1) + 2\sum_{i=1}^{k} m_i$. If we set $2n(\alpha+1) = x$, then

$$\int_{\frac{2\alpha+1}{2\alpha+2}x}^{x} \psi(y)\,dy \ge -\frac{\log c_w}{4(\alpha+1)^2}\,x^2 + O(x\log^2 x) \quad \text{as } x \to \infty. \tag{3.6}$$

Using the substitution $x \to \frac{2\alpha+1}{2\alpha+2}x$ iteratively and summing up the results, we obtain

$$\int_1^x \psi(y)\,dy \ge \frac{-\log c_w}{4\alpha+3}x^2 + O(x\log^2 x) \quad \text{as } x \to \infty. \qquad \blacksquare$$

Proof of Corollary 1.2 Corollary 2.4 gives the weighted equilibrium measure μ_w for such weights in (2.14)–(2.16). Hence we have by (2.5) that the numerical value of c_w can be computed from

$$-\log c_w = U^{\mu_w}(a_1) - \log w(a_1) - \int \log w\, d\mu_w,$$

where a_1 is defined in (2.15). The same equation yields (1.11), as a consequence of Theorem 1.1. $\blacksquare$

We now start preparations for the proof of Theorem 2.1. Recall the function $R(z) = \prod_{l=1}^{L}(z-a_l)(z-b_l)$, where $a_1 < b_1 < a_2 < b_2 < \cdots < a_L < b_L$ are real numbers. The branch of $\sqrt{R(z)}$ satisfying $\lim_{z\to\infty}\sqrt{R(z)}/z^L = 1$ is analytic in $\mathbb{C} \setminus \bigcup_{l=1}^{L}[a_l, b_l]$. For further reference, we describe the values of $\sqrt{R(z)}$ on the real line:

$$\sqrt{R(x)} = \begin{cases} \sqrt{|R(x)|} & x \ge b_L, \\ (-1)^{L+l} i\sqrt{|R(x)|} & a_l \le x \le b_l, \quad l = 1, \ldots, L, \\ (-1)^{L+l}\sqrt{|R(x)|} & b_l \le x \le a_{l+1}, \quad l = 1, \ldots, L-1, \\ (-1)^L\sqrt{|R(x)|} & x \le a_1. \end{cases} \tag{3.7}$$

Here and throughout, the values of $\sqrt{R(x)}$ for $x \in \bigcup_{l=1}^{L}[a_l, b_l]$ are understood as the upper limiting values of $\sqrt{R(z)}$, when $\Im z \to 0^+$.

Lemma 3.1 *Let $S := \bigcup_{l=1}^{L}[a_l, b_l]$. For any $T_{L-1} \in \mathbb{R}_{L-1}[x]$, we have*

$$\frac{1}{\pi i}\int_S \frac{T_{L-1}(t)\,dt}{(t-z)\sqrt{R(t)}} = \begin{cases} 0 & z \in S, \\ T_{L-1}(z)/\sqrt{R(z)} & z \in \mathbb{C} \setminus S. \end{cases} \tag{3.8}$$

Proof A detailed proof of this known fact may be found in [20] (*cf.* Chapter 11). We give a sketch of argument based on Cauchy integral formula. Consider a contour Γ that consists of L simple closed curves around each of the intervals $[a_l, b_l]$, located close to those intervals. Then

$$\frac{1}{2\pi i}\oint_\Gamma \frac{T_{L-1}(t)\,dt}{(t-z)\sqrt{R(t)}} = \begin{cases} 0 & z \in S, \\ T_{L-1}(z)/\sqrt{R(z)} & z \in \operatorname{Ext}(\Gamma). \end{cases}$$

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Note that the limiting values of $\sqrt{R(t)}$ on S, from above and below, are opposite in sign. Letting the contour Γ shrink to S, we obtain that

$$\lim_{\Gamma\to S}\frac{1}{2\pi i}\oint_{\Gamma}\frac{T_{L-1}(t)\,dt}{(t-z)\sqrt{R(t)}}=\frac{1}{\pi i}\int_S\frac{T_{L-1}(t)\,dt}{(t-z)\sqrt{R(t)}}. \qquad \blacksquare$$

Let $\omega(z,\,\cdot\,,\Omega)$ be the harmonic measure at $z\in\Omega$ with respect to a domain $\Omega\subset\overline{\mathbb{C}}$, which is a positive unit Borel measure supported on $\partial\Omega$. The background information on harmonic measures may be found in [23, §4.3] and [25, Appendix A.3]. In particular, $\omega(z,\,\cdot\,,\Omega)$ is equal to the balayage of the unit point mass δ_z to $\partial\Omega$. We give the following explicit representations for these measures.

Lemma 3.2 *Let $\Omega:=\overline{\mathbb{C}}\setminus\bigcup_{l=1}^{L}[a_l,b_l]$. There exist polynomials $T_\infty\in\mathbb{R}_{L-1}[x]$ and $T_{z_j}\in\mathbb{R}_{L-1}[x]$, $j=1,\dots,K$, such that*

$$d\omega(\infty,x,\Omega)=\frac{T_\infty(x)dx}{\pi i\sqrt{R(x)}},\quad x\in\bigcup_{l=1}^{L}[a_l,b_l], \tag{3.9}$$

and

$$d\omega(z_j,x,\Omega)=\frac{T_{z_j}(x)dx}{\pi i(x-z_j)\sqrt{R(x)}},\quad x\in\bigcup_{l=1}^{L}[a_l,b_l], \tag{3.10}$$

where $z_j\in\Omega\setminus\{\infty\}$, $j=1,\dots,K$.

Proof These formulas are essentially known (see, *e.g.,* [27, Lemma 4.4.1] and [30, Lemma 2.3]). It is possible to deduce (3.9)–(3.10) from Lemma 3.1, which is done below.

We select $T_\infty(t)=\sum_{j=1}^{L-1}c_jt^j\in\mathbb{R}_{L-1}[t]$ so that it satisfies the following equations:

$$\int_{b_l}^{a_{l+1}}\frac{T_\infty(t)\,dt}{\sqrt{R(t)}}=\sum_{j=1}^{L-1}c_j\int_{b_l}^{a_{l+1}}\frac{t^j\,dt}{\sqrt{R(t)}}=0,\quad l=1,\dots,L-1, \tag{3.11}$$

and

$$\frac{1}{\pi i}\int_S\frac{T_\infty(t)\,dt}{\sqrt{R(t)}}=\sum_{j=1}^{L-1}\frac{c_j}{\pi i}\int_S\frac{t^j\,dt}{\sqrt{R(t)}}=1, \tag{3.12}$$

where $S=\bigcup_{l=1}^{L}[a_l,b_l]$. The polynomial $T_\infty(t)$ is uniquely defined by these equations, because the corresponding homogeneous system of linear equations (with zero on the right of (3.12)), in the coefficients c_j of $T_\infty(t)$, has only the trivial solution. Indeed, let $T_h(t)$ be a nontrivial solution of this homogeneous system. Since the sign of $\sqrt{R(t)}$ is constant on each (b_l,a_{l+1}), by (3.7), $T_h(t)$ must change sign on

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each $[b_l, a_{l+1}]$, $l = 1, \dots, L-1$, by (3.11). Hence $T_h(t)$ has a simple zero in each (b_l, a_{l+1}), $l = 1, \dots, L-1$, and it alternates sign on the intervals $[a_l, b_l]$, $l = 1, \dots, L$. (Note that the same is true for $T_\infty(t)$.) It follows from (3.7) that $T_h(t)/(\pi i \sqrt{R(t)})$ does not change sign on S, contradicting

$$\frac{1}{\pi i} \int_S \frac{T_h(t)\, dt}{\sqrt{R(t)}} = 0.$$

Thus $T_\infty(t)$ exists and is unique. In addition, the above argument and (3.12) show that $T_\infty(t)/(\pi i \sqrt{R(t)})$ keeps positive sign on S, *i.e.,* (3.9) actually defines a positive unit Borel measure on S. Let

$$h(x) := \frac{1}{\pi i} \int_S \frac{T_\infty(t)\, dt}{(t-x)\sqrt{R(t)}}, \quad x \in \mathbb{R}.$$

We have from (3.8) that $h \in L_p([a_1, b_L])$, $1 \le p < 2$, and $h \in C^\infty([a_1, b_L] \setminus \{a_l, b_l\}_{l=1}^L)$. Note that $h(x)$ is the derivative of potential for the measure on the right of (3.9). The fundamental theorem of calculus gives that

$$\frac{1}{\pi i} \int_S \log \frac{1}{|x-t|} \frac{T_\infty(t)\, dt}{\sqrt{R(t)}} = \int_{a_1}^{x} h(t)\, dt + C, \quad x \in [a_1, b_L],$$

where C is a constant. Using (3.8) and (3.11), we obtain that

$$\frac{1}{\pi i} \int_S \log \frac{1}{|x-t|} \frac{T_\infty(t)\, dt}{\sqrt{R(t)}} = C, \quad x \in S.$$

Frostman's theorem (see [23, Theorem 3.3.4]) and the uniqueness of the equilibrium measure (*cf.* [23, Theorem 3.7.6]) imply that $T_\infty(x)\, dx/(\pi i \sqrt{R(x)})$ is the classical (not weighted) equilibrium measure for S. The latter is known to be equal to $\omega(\infty, x, \Omega)$ (see [23, Theorem 4.3.14]), which proves (3.9). Equations (3.10) follow from (3.9) by using the transformations

$$f_j(z) := \frac{1}{z - z_j}, \quad z \in \overline{\mathbb{C}},\ j = 1, \dots, K,$$

and the conformal invariance of harmonic measures (*cf.* [23, Theorem 4.3.8]). Indeed, if $\Omega_j := f_j(\Omega)$ and $S_j := f_j(S) = \partial\Omega_j$, then

$$\omega(z_j, x, \Omega) = \omega(\infty, (x - z_j)^{-1}, \Omega_j), \quad j = 1, \dots, K.$$

We can use (3.9) on the right side of the above equation, because S_j is also a union of L segments of the real line, which gives (3.10). ∎

In [22], we showed that the weighted energy problem discussed in Section 2 can be solved in terms of a linear combination of harmonic measures.

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Lemma 3.3 *Let $w(x)$ be as in (2.1), $x \in [a,b]$. Then $S_w \subset [a,b] \setminus \{z_j\}_{j=1}^K$, and the extremal measure for the weighted energy problem (2.2)–(2.3) is given by*

$$\mu_w = (1+p)\omega(\infty,\cdot,\Omega) - \sum_{j=1}^{K} p_j \omega(z_j,\cdot,\Omega), \tag{3.13}$$

where $\Omega := \overline{\mathbb{C}} \setminus S_w$ and $p = \sum_{j=1}^K p_j$.

Proof The existence of a weighted equilibrium measure μ_w, whose support is a compact set $S_w \subset [a,b] \setminus \{z_j\}_{j=1}^K$, follows from [25, Theorem I.1.3]. Let δ_z be a unit point mass at z. Observe that

$$\log w(z) = \log A - U^\nu(z), \quad z \in \mathbb{C}, \tag{3.14}$$

where U^ν is the logarithmic potential of the measure

$$\nu := \sum_{j=1}^{K} p_j \delta_{z_j}.$$

It is clear that ν is a positive Borel measure of total mass $\nu(\mathbb{C}) = p$. Let $\hat{\nu}$ be the balayage of ν from Ω onto S_w (see, *e.g.*, [25, §II.4]). Then $\hat{\nu}$ is a positive Borel measure of the same mass as ν, which is supported on S_w. Furthermore, we can express $\hat{\nu}$ via harmonic measures

$$\hat{\nu} = \sum_{j=1}^{K} p_j \omega(z_j,\cdot,\Omega)$$

(*cf.* [25, Appendix A.3]). The potentials of ν and $\hat{\nu}$ are related by the equation

$$U^{\hat{\nu}}(x) = U^\nu(x) + C_1, \quad \text{for q.e. } x \in S_w, \tag{3.15}$$

where C_1 is a constant. This equation holds quasi everywhere on S_w, *i.e.*, with the exception of a set of zero logarithmic capacity (see [25, Theorem II.4.4]). Using (2.5), (3.14) and (3.15), we obtain for quasi every $x \in S_w$ that

$$\begin{aligned} F_w = U^{\mu_w}(x) - \log w(x) &= U^{\mu_w}(x) + U^\nu(x) - \log A \\ &= U^{\mu_w}(x) + U^{\hat{\nu}}(x) - C_1 - \log A. \end{aligned}$$

Recall that the potential $U^{\omega(\infty,\cdot,\Omega)}(x)$ is constant q.e. on S_w, by Frostman's theorem. Thus we have

$$U^{\mu_w+\hat{\nu}}(x) = U^{(1+p)\omega(\infty,\cdot,\Omega)}(x) + C_2, \quad \text{for q.e. } x \in S_w, \tag{3.16}$$

where C_2 is another constant. Note that all measures in the above equation have finite logarithmic energy. Since the mass of $\mu_w + \hat{\nu}$ is equal to that of $(1+p)\omega(\infty,\cdot,\Omega)$, we can apply the Principle of Domination (*cf.* [25, Theorem II.3.2]) to conclude that (3.16) holds for every $x \in \mathbb{C}$. The Unicity Theorem (see [25, Theorem I.3.3]) now shows that

$$\mu_w + \hat{\nu} = (1+p)\omega(\infty,\cdot,\Omega). \qquad \blacksquare$$

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Proof of Theorem 2.1 It is clear from (2.5) that $S_w \subset [a,b] \setminus Z$. Since $-\log w(x)$ is convex on each interval (z_j, z_{j+1}), $j = 1, \dots, K-1$, the intersection of S_w with (z_j, z_{j+1}) is either a closed interval or an empty set (*cf.* [25, Theorem IV.1.10(b)]). Hence $S_w = \bigcup_{l=1}^{L} [a_l, b_l]$, $L \le K-1$, where we have at most one interval $[a_l, b_l]$ between any neighboring zeros z_j and z_{j+1} of w.

Theorem 1.38 of [9] states that for the upper limiting values of

$$F(z) := \frac{\sqrt{R(z)}}{\pi i} \int_{S_w} \frac{-i(\log w(t))'\, dt}{\pi(t-z)\sqrt{R(t)}}, \quad z \in \mathbb{C} \setminus S_w, \tag{3.17}$$

we have

$$d\mu_w(x) = \Re F(x)\, dx, \quad x \in S_w. \tag{3.18}$$

We remark that the statement of this theorem in [9] requires $\log w(x)$ be real analytic in a neighborhood of $[a,b]$. But this is only used to conclude that the support of μ_w consists of finitely many intervals, which we already have anyway. For their analysis leading to (3.17)–(3.18), it is sufficient that $\log w \in C^2([a,b])$. In fact, (3.17)–(3.18) are obtained by solving the singular integral equation with Cauchy kernel (by the methods similar to those of [20, Ch. 11]):

$$\int \frac{d\mu(t)}{t-x} = (\log w(x))', \quad x \in S_w,$$

which arises via differentiation of (2.5). In order to achieve that $\log w \in C^2([a,b])$, we can modify w in the small neighborhoods of zeros, outside the compact set

$$S_w^* := \{x \in [a,b] : U^{\mu_w}(x) - \log w(x) \le F_w\} \subset [a,b] \setminus Z, \tag{3.19}$$

so that the new weight $\tilde{w}$ satisfies

$$U^{\mu_w}(x) - \log \tilde{w}(x) > F_w, \quad x \in [a,b] \setminus S_w^*. \tag{3.20}$$

Theorem I.3.3 of [25] and (3.19)–(3.20) then give that the equilibrium measure $\mu_{\tilde{w}} = \mu_w$. Hence the result (3.17)–(3.18) of [9, Theorem 1.38] is valid in our case.

With the help of Lemma 3.1, we obtain for $z \in \mathbb{C} \setminus S_w$ that

$$\begin{aligned}
F(z) &= \frac{\sqrt{R(z)}}{\pi i} \sum_{j=1}^{K} \frac{p_j}{\pi i} \int_{S_w} \frac{dt}{(t-z_j)(t-z)\sqrt{R(t)}} \\
&= \frac{\sqrt{R(z)}}{\pi i} \sum_{j=1}^{K} \frac{p_j}{\pi i (z-z_j)} \int_{S_w} \Big(\frac{1}{t-z} - \frac{1}{t-z_j} \Big) \frac{dt}{\sqrt{R(t)}} \\
&= \frac{\sqrt{R(z)}}{\pi i} \sum_{j=1}^{K} \frac{p_j}{\pi i (z-z_j)} \Big(\int_{S_w} \frac{dt}{(t-z)\sqrt{R(t)}} - \int_{S_w} \frac{dt}{(t-z_j)\sqrt{R(t)}} \Big)
\end{aligned}$$

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$$= \frac{\sqrt{R(z)}}{\pi i} \sum_{j=1}^{K} \left(\frac{p_j}{(z-z_j)\sqrt{R(z)}} - \frac{p_j}{(z-z_j)\sqrt{R(z_j)}} \right)$$

$$= \frac{1}{\pi i} \sum_{j=1}^{K} \frac{p_j}{z-z_j} - \frac{\sqrt{R(z)}}{\pi i} \sum_{j=1}^{K} \frac{p_j}{(z-z_j)\sqrt{R(z_j)}}.$$

Recall that $\sqrt{R(x)}$ is pure imaginary for $x \in S_w$, and $\sqrt{R(z_j)}$ is real, $j = 1, \dots, K$, by (3.7). Thus we have that

$$\Re F(x) = -\frac{\sqrt{R(x)}}{\pi i} \sum_{j=1}^{K} \frac{p_j}{(x-z_j)\sqrt{R(z_j)}}, \quad x \in S_w.$$

If we set

$$P(x) := \frac{1}{1+p} \sum_{j=1}^{K} \frac{p_j}{\sqrt{R(z_j)}} \prod_{l \neq j} (x - z_l), \tag{3.21}$$

then

$$d\mu_w(x) = -\frac{(1+p)\sqrt{R(x)}\,P(x)}{\pi i \prod_{j=1}^{K}(x-z_j)}\,dx, \quad x \in S_w, \tag{3.22}$$

by (3.18). Equation (2.8) now follows from (3.7). Also, (3.21) and (3.7) give

$$P(z_j) = \frac{p_j \prod_{m\neq j}(z_j - z_m)}{(1+p)\sqrt{R(z_j)}} = (-1)^{L+l} \frac{p_j \prod_{m\neq j}(z_j - z_m)}{(1+p)\sqrt{|R(z_j)|}}, \tag{3.23}$$

where $z_j \in (b_l, a_{l+1})$, $j = 1, \dots, K$, which proves (2.9). On the other hand, Lemmas 3.2 and 3.3 yield another representation for μ_w, of the form

$$d\mu_w(x) = \frac{T(x)\,dx}{\pi i \sqrt{R(x)} \prod_{j=1}^{K}(x-z_j)}, \quad x \in S_w,$$

where $T \in \mathbb{R}_{K+L-1}[x]$. Comparing this with (3.22), we conclude that

$$-(1+p)R(x)P(x) = T(x)$$

and $2L + \deg(P) = \deg(T) \le K + L - 1$. Thus the actual degree of P satisfies

$$\deg(P) \le K - L - 1.$$

We now prove that P has leading coefficient 1, by using the following identity:

$$\lim_{x\to\infty} (-x) \int_{S_w} \frac{d\mu_w(t)}{t-x} = \mu_w(\mathbb{C}) = 1. \tag{3.24}$$

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Observe that $\sqrt{R(t)}P(t)/\big((t-x)\prod_{j=1}^{K}(t-z_j)\big)$ is an analytic function of t in $\overline{\mathbb{C}}\setminus S_w$, except for the simple poles at x and z_j, $j=1,\dots,K$. Applying Cauchy's integral theorem, we have for a small $r>0$:

$$\begin{aligned}\int_{S_w}\frac{d\mu_w(t)}{t-x} &= -\frac{1+p}{\pi i}\int_{S_w}\frac{\sqrt{R(t)}\,P(t)\,dt}{(t-x)\prod_{j=1}^{K}(t-z_j)}\\ &= -\frac{1+p}{2\pi i}\oint_{S_w}\frac{\sqrt{R(t)}\,P(t)\,dt}{(t-x)\prod_{j=1}^{K}(t-z_j)}\\ &= -\frac{1+p}{2\pi i}\Big(\oint_{|t-x|=r}+\sum_{l=1}^{K}\oint_{|t-z_l|=r}\Big)\frac{\sqrt{R(t)}\,P(t)\,dt}{(t-x)\prod_{j=1}^{K}(t-z_j)}\\ &= -(1+p)\frac{\sqrt{R(x)}\,P(x)}{\prod_{j=1}^{K}(x-z_j)}-(1+p)\sum_{l=1}^{K}\frac{\sqrt{R(z_l)}\,P(z_l)}{(z_l-x)\prod_{j\neq l}(z_l-z_j)}.\end{aligned}$$

Taking into account (3.23), we obtain that

$$\int_{S_w}\frac{d\mu_w(t)}{t-x}=-(1+p)\frac{\sqrt{R(x)}\,P(x)}{\prod_{j=1}^{K}(x-z_j)}-\sum_{l=1}^{K}\frac{p_l}{z_l-x},\quad x\in\mathbb{C}\setminus(S_w\cup Z).\tag{3.25}$$

If c_{K-L-1} is the leading coefficient of P, then

$$\lim_{x\to\infty}(-x)\int_{S_w}\frac{d\mu_w(t)}{t-x}=(1+p)c_{K-L-1}-\sum_{l=1}^{K}p_l=1,$$

by (3.24). Thus $(1+p)c_{K-L-1}=1+p$ and

$$c_{K-L-1}=1.$$

It only remains to prove (2.10) now. We find from (2.5) that

$$U^{\mu_w}(a_{l+1})-U^{\mu_w}(b_l)=\log w(a_{l+1})-\log w(b_l),\quad l=1,\dots,L-1.\tag{3.26}$$

The potential $U^{\mu_w}(x)$ is continuous in $\mathbb{C}$ by [25, Theorem I.4.8], and it is infinitely differentiable on (b_l,a_{l+1}). Hence we obtain by the fundamental theorem of calculus and (3.25) that

$$\begin{aligned}U^{\mu_w}(a_{l+1})-U^{\mu_w}(b_l) &= \int_{b_l}^{a_{l+1}}\frac{d}{dx}\left(U^{\mu_w}(x)\right)\,dx=\int_{b_l}^{a_{l+1}}\int_{S_w}\frac{d\mu_w(t)}{t-x}\,dx\\ &= \int_{b_l}^{a_{l+1}}\left(-(1+p)\frac{\sqrt{R(x)}\,P(x)}{\prod_{j=1}^{K}(x-z_j)}+\sum_{j=1}^{K}\frac{p_j}{x-z_j}\right)dx\\ &= -(1+p)\int_{b_l}^{a_{l+1}}\frac{\sqrt{R(x)}\,P(x)\,dx}{\prod_{j=1}^{K}(x-z_j)}\\ &\qquad+\sum_{j=1}^{K}p_j\left(\log|a_{l+1}-z_j|-\log|b_l-z_j|\right),\end{aligned}$$

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where the last equality holds in the principal value sense. Therefore, we have by (3.26) that

$$-(1+p)\int_{b_l}^{a_{l+1}} \frac{\sqrt{R(x)}\,P(x)\,dx}{\prod_{j=1}^{K}(x-z_j)} + \log\frac{w(a_{l+1})}{w(b_l)} = \log\frac{w(a_{l+1})}{w(b_l)},$$

which proves (2.10). ■

Proof of Corollary 2.2 If $L = K-1$ then obviously $P(x) \equiv 1$ and $z_j \in (b_{j-1}, a_j)$, $j = 1, \dots, K$. Hence

$$\prod_{j=1}^{K}(x-z_j) = (-1)^{L+1-l}\prod_{j=1}^{K}|x-z_j|, \quad x \in [a_l, b_l],$$

and (2.11) follows from (2.8). Also, we have

$$\prod_{m\neq j}(z_j - z_m) = (-1)^{L+1-j}\prod_{m\neq j}|z_j - z_m|,$$

which gives (2.12) by (2.9). Equation (2.13) is an immediate consequence of (2.10). ■

Proof of Remark 2.3 Consider the weight $w(x) = (1+x)^{p_1}|x|^{p_2}(1-x)^{p_3}$ on $[-1,1]$, where we take $p_2 = p_3 = 1$. We show that the support S_w, for large p_1, consists of only one interval. Assume to the contrary that $S_w = [a_1, b_1] \cup [a_2, b_2]$. Then Corollary 2.2 applies here, and (2.12)–(2.13) hold true. Note that these equations are identical to the equations of Amoroso [1, p. 1184], used to define the endpoints a_1, b_1, a_2, b_2. As it turns out, they do not have a solution for large p_1. Indeed, (2.13) gives

$$\int_{b_1}^{a_2} \frac{\sqrt{(x-a_1)(x-b_1)(a_2-x)(b_2-x)}}{x(x^2-1)}\,dx = 0, \tag{3.27}$$

and (2.12) gives for $z_1 = -1$ that

$$\sqrt{(1+a_1)(1+b_1)(1+a_2)(1+b_2)} = \frac{2p_1}{p_1+3}.$$

Since $-1 < a_1 < b_1 < 0 < a_2 < b_2 < 1$, we have from the latter equation that

$$\sqrt{(1+a_2)(1+b_2)} > \frac{2p_1}{p_1+3},$$

and

$$\lim_{p_1\to+\infty} a_2 = \lim_{p_1\to+\infty} b_2 = 1.$$

Similarly,

$$2\sqrt{(1+a_1)(1+b_1)} > \frac{2p_1}{p_1+3}$$

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implies that

$$\lim_{p_1\to+\infty} a_1 = \lim_{p_1\to+\infty} b_1 = 0.$$

Therefore $\sqrt{(x-a_1)(x-b_1)(a_2-x)(b_2-x)}$ is strictly increasing on $[b_1, -b_1]$, for all sufficiently large p_1, and

$$\int_{b_1}^{-b_1} \frac{\sqrt{(x-a_1)(x-b_1)(a_2-x)(b_2-x)}}{x(x^2-1)}\,dx < 0.$$

Coupling this with the obvious inequality

$$\int_{-b_1}^{a_2} \frac{\sqrt{(x-a_1)(x-b_1)(a_2-x)(b_2-x)}}{x(x^2-1)}\,dx < 0,$$

we obtain a contradiction to (3.27).

It is possible to show that the number of intervals of the support for μ_w in Theorem 2.1 can indeed take any value between 1 and $K-1$. ■

Proof of Corollary 2.4 This result follows from Corollary 2.2. The representation (2.14) for μ_w is an immediate consequence of (2.11). We also obtain from (2.12) that the endpoints of the support must satisfy the equations

$$a_1 b_1 = r_1^2 \quad \text{and} \quad (1-a_1)(1-b_1) = r_2^2.$$

Solving those equations, we obtain (2.15) and (2.16). ■

Proof of Lemma 2.5 Consider the weighted Fekete points $\{\zeta_i^{(n)}\}_{i=1}^n \subset [a,b]$, $n \in \mathbb{N}$, that maximize the absolute value of the weighted Vandermonde determinant (1.10). The relation between the problems of minimizing energy (2.2)–(2.3) and maximizing (1.10) becomes transparent if we consider $-\frac{2}{n(n-1)} \log |V_n^w|$, which is essentially a discrete version of the weighted energy functional (2.2). Indeed, the normalized counting measures

$$\nu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{\zeta_i^{(n)}}$$

converge weakly to μ_w, the extremal measure for (2.2)–(2.3), and (2.17) holds true (*cf.* [25, §III.1]). Thus the discrete problem is a good approximation of the continuous one.

We deduce (2.18) from the results of Götz and Saff [16]. They require that $\log w(x)$ be Hölder continuous on $[a,b]$, which is not true if w of (2.1) has zeros on $[a,b]$. But we can modify w in the small neighborhoods of those zeros, outside the compact set

$$S_w^* = \{x \in [a,b] : U^{\mu_w}(x) - \log w(x) \le F_w\} \subset [a,b] \setminus Z, \tag{3.28}$$

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so that the new weight $\tilde{w}$ satisfies

$$U^{\mu_w}(x) - \log \tilde{w}(x) > F_w, \quad x \in [a,b] \setminus S_w^*, \tag{3.29}$$

and $\log \tilde{w}(x)$ is Hölder continuous on $[a,b]$. It follows from Theorem I.3.3 of [25] and (3.28)–(3.29) that the equilibrium measure $\mu_{\tilde{w}} = \mu_w$, and from Theorem III.1.2 of [25] and (3.28) that the weighted Fekete points for $\tilde{w}$ are identical to those for w. Thus all results of [16] are applicable here. Set

$$F_n(x) := \prod_{i=1}^{n} (x - \zeta_i^{(n)}), \quad n \in \mathbb{N}.$$

Then $\frac{1}{n} \log |F_n(x)| = -U^{\nu_n}(x)$ and

$$U^{\mu_w}(x) + \frac{1}{n} \log |F_n(x)| \le c_0 \frac{\log n}{n}, \quad x \in \mathbb{C},$$

where $c_0 > 0$ depends only on w, by [16, Theorem 1]. Hence

$$|F_n(x)| \le n^{c_0} \, e^{-n\, U^{\mu_w}(x)}, \quad x \in \mathbb{C}. \tag{3.30}$$

For a small $r > 0$, write

$$F_n'(x) = \frac{1}{2\pi i} \int_{|x-z|=r} \frac{F_n(z)\, dz}{(z-x)^2}, \quad x \in S_w^*,$$

and estimate

$$w^{n-1}(x) |F_n'(x)| \le \frac{w^{n-1}(x)}{r} \max_{|x-z|=r} |F_n(z)| = O(n^{c_0}) \frac{w^n(x)}{r} \max_{|x-z|=r} e^{-nU^{\mu_w}(z)},$$

as $n \to \infty$, by (3.30). Note that $U^{\mu_w}(x)$ is Hölder continuous in $\mathbb{C}$, because it is a harmonic function in $\mathbb{C} \setminus S_w$, with smooth boundary values $\log w(x) + F_w$ on S_w (see [25, Theorem I.4.7] and [16, Lemma 2]). If $\lambda \in (0,1]$ is the Hölder exponent for $U^{\mu_w}(x)$, then we choose $r = n^{-1/\lambda}$ and obtain

$$\max_{|x-z|=n^{-1/\lambda}} e^{-n\, U^{\mu_w}(z)} = O(1) e^{-nU^{\mu_w}(x)}, \quad x \in S_w^*.$$

Hence

$$\begin{aligned} w^{n-1}(x)|F_n'(x)| &= O(n^{c_0+1/\lambda})\, e^{n(\log w(x) - U^{\mu_w}(x))} \\ &= O(n^{c_0+1/\lambda})\, e^{-n\, F_w}, \quad x \in S_w^*, \end{aligned} \tag{3.31}$$

by (3.28) and (2.4). Recall that the weighted Fekete points are contained in the compact set $S_w^* \subset [a,b] \setminus Z$ (*cf.* [25, Theorem III.1.2]), where $\log w(x)$ is continuous. Therefore, we have from Theorem 3 of [16] that

$$\int \log w \, d\nu_n - \int \log w \, d\mu_w = O\left(\frac{\log^2 n}{n}\right) \quad \text{as } n \to \infty.$$

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This implies

$$\prod_{i=1}^{n} w(\zeta_i^{(n)}) = O(e^{\log^2 n})\, e^{n\int \log w\, d\mu_w} \quad \text{as } n \to \infty. \tag{3.32}$$

Observe that

$$(V_n^w(\zeta_1^{(n)}, \dots, \zeta_n^{(n)}))^2 = \prod_{i=1}^{n} w^{2(n-1)}(\zeta_i^{(n)}) \prod_{i=1}^{n} F_n'(\zeta_i^{(n)}).$$

We now use (3.31) and (3.32) to estimate

$$\begin{aligned}\max_{[a,b]^n}(V_n^w)^2 &= \prod_{i=1}^{n} w^{n-1}(\zeta_i^{(n)}) \prod_{i=1}^{n} w^{n-1}(\zeta_i^{(n)})|F_n'(\zeta_i^{(n)})| \\ &= O(d^{n\ \log^2 n})\, e^{n(n-1)(\int \log w\, d\mu_w - F_w)} \\ &= O(d^{n\ \log^2 n})\, e^{-n(n-1)V_w} = O(d^{n\ \log^2 n})\ (\operatorname{cap}([a,b],w))^{n(n-1)},\end{aligned}$$

where $d > e$, as $n \to \infty$. Thus the upper bound in (2.18) is proved. The lower bound of (2.18) is a well-known consequence of extremal properties for the weighted Fekete points and Vandermonde determinants, see [25, Theorem III.1.1], which states that the sequence $|V_n^w(\zeta_1^{(n)}, \dots, \zeta_n^{(n)})|^{\frac{2}{n(n-1)}}$ decreases to $\operatorname{cap}([a,b],w)$ as $n \to \infty$. ∎

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