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IMPROVED INEQUALITIES FOR THE NUMERICAL RADIUS: WHEN INVERSE COMMUTES WITH THE NORM

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Abstract

New inequalities relating the norm $n(X)$ and the numerical radius $w(X)$ of invertible bounded linear Hilbert space operators were announced by Hosseini and Omidvar ['Some inequalities for the numerical radius for Hilbert space operators', *Bull. Aust. Math. Soc.* **94** (2016), 489–496]. For example, they asserted that $w(AB) \leq 2w(A)w(B)$ for invertible bounded linear Hilbert space operators A and B. We identify implicit hypotheses used in their discovery. The inequalities and their proofs can be made good by adding the extra hypotheses which take the form $n(X^{-1}) = n(X)^{-1}$. We give counterexamples in the absence of such additional hypotheses. Finally, we show that these hypotheses yield even stronger conclusions, for example, $w(AB) = w(A)w(B)$.

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1. Introduction

Let X be a bounded linear operator on a complex d-dimensional Hilbert space. Then $Sp(X)$ will denote its spectrum, $r(X)$ its spectral radius, $n(X)$ its norm and $w(X)$ its numerical radius. Let F (respectively G, H) be the set of solutions to $f(X^{-1}) = f(X)^{-1}$, that is, 'f commutes with inverse' when $f = r$ (respectively n, w).

This note clarifies some issues surrounding the results announced in [3] by doing four main things.

(I) We present counterexamples to the claims in [3] that if T, A, B are invertible bounded linear operators then

THEOREM 2.12: $n(X)^2 \leq 2w(X)^2$ for $X = T$ or T^{-1};
COROLLARY 2.13: $\max\{n(T), n(T^{-1})\} \leq \sqrt{2}\max\{w(T), w(T^{-1})\}$;
THEOREM 2.14: $n(T) \leq \sqrt{2}w(T)$;
COROLLARY 2.15: $w(AB) \leq 2w(A)w(B)$.

We shall refer to these claims simply as (2.12), ..., (2.15).

(II) We identify the hypotheses $T, A, B \in G$ which are used in [3] but not stated. Adding these hypotheses salvages the proofs and the conclusions (2.12), . . . , (2.15).

(III) We show that G is the multiplicative group of all nonzero scalar multiples of all unitary operators. It follows that the corrected versions of (2.12), . . . , (2.15) can be improved. For example, assuming $T, A, B \in G$, then (2.14) becomes $n(T) = w(T)$ and (2.15) becomes $w(AB) = w(A)w(B)$.

(IV) We show that the hypotheses $T, A, B \in H$, although apparently weaker than $T, A, B \in G$ because $H \supseteq G$, lead to nothing new, because in fact $H = G$.

2. The counterexamples

Let $M(x, y) = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$ and set $I = M(1, 0)$, $E = M(0, 1)$ and $T = aI + E$ for $a > 0$. Then $T^{-1} = a^{-2}(aI - E)$ and so

$$n(T^{-1}) = a^{-2} n(aI - E) = a^{-2} n(T),$$

because $aI - E = V^* T V$ where V is a diagonal unitary matrix with diagonal entries $1, -1$. This unitary similarity also allows us to conclude that

$$w(T^{-1}) = a^{-2} w(T) = a^{-2}(a + w(E)) = a^{-2}(a + \tfrac{1}{2})$$

since, as is rather well known and not hard to compute, the numerical range of E is the closed origin-centred disc of radius $\frac{1}{2}$.

When $a > 0$ is small enough, $n(T)$ will be near $n(E) = 1$ and $w(T)$ will be near $w(E) = \frac{1}{2}$. So $n(T)/w(T)$ will be near 2. Since our calculations above show that this quotient is the same, that is, near 2, with T^{-1} in place of T, we know (2.12), which says $\sqrt{2}$ will not be exceeded, cannot be correct.

When $a > 0$ is small, the maxima in (2.13) will be $n(T^{-1})$ and $w(T^{-1})$ and again their quotient will be near $2 > \sqrt{2}$, contradicting (2.13).

Yet again $n(T)/w(T)$ for small $a > 0$ will exceed $\sqrt{2}$, faulting (2.14).

Finally, if $A = T$ and B is its transpose then for $a > 0$ small, $w(AB)$ will be near 1 while $w(A) = w(B)$ will be near $\frac{1}{2}$, so (2.15) cannot hold.

The original assertions are clearly valid when $d = 1$. Counterexamples for higher dimensions can be constructed by forming the direct sum of the counterexamples just given for $d = 2$ with an identity operator of the appropriate size.

These examples all have invertible matrices approaching singular ones and rely on the continuity of n and w. In fact an inequality involving n and w which is valid for all invertible matrices, must be valid in the limit for all matrices, with strict inequalities becoming nonstrict, because all singular matrices are limits of invertible ones.

3. The extra hypothesis which makes the proofs correct

In 'proving' (2.12), the authors of [3] derived an inequality involving T assuming $n(T^{-1}) \le n(T)$ and then replaced T in that inequality with $L = n(T)^{-1} T$. This is acceptable exactly when $T \in G$ because

$$n(L^{-1}) \le n(L) \Leftrightarrow n(T^{-1}) n(T) = 1.$$

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To see this, observe that

$$\Rightarrow: 1 = n(TT^{-1}) \le n(T)n(T^{-1}) = n(n(T)T^{-1}) = n(L^{-1}) \le n(L) = 1;$$
$$\Leftarrow: n(L) = 1 = n(T^{-1})n(T) = n(n(T)T^{-1}) = n(L^{-1}).$$

Since (2.13)–(2.15) rely on (2.12) directly or indirectly, for the given proofs to be valid, the hypotheses $T, A, B \in G$ are needed.

4. Characterising F and G and improving the results

Since Sp 'commutes with inverse', that is, $Sp(X^{-1}) = (Sp(X))^{-1}$, the set F defined above is the set of all invertible operators X such that $Sp(X)$ is a subset of an origin-centred circle of positive radius $r(X)$. Since $1 \le r(X^{-1})r(X)$ for every invertible X, to conclude that $X \in F$ it suffices to show that $r(X^{-1})r(X) \le 1$.

Since $r \le w \le n$, for every invertible X,

$$1 \le r(X^{-1})r(X) \le w(X^{-1})w(X) \le n(X^{-1})n(X),$$

so $F \supseteq H \supseteq G$. If $w(T^{-1})w(T) = 1$, that is, $T \in H$, then $r(X) = w(X)$ for $X = T, T^{-1}$; and if $n(T^{-1})n(T) = 1$, that is, $T \in G$, then $r(X) = w(X) = n(X)$ for $X = T, T^{-1}$.

Let $T \in G$. Then

$$n((T^*T)^{-1}) = n(T^{-1}(T^{-1})^*) = n((T^{-1})^*)^2 = n(T^{-1})^2 = n(T)^{-2} = n(T^*T)^{-1},$$

so $T^*T \in G$ and therefore $T^*T \in F$. Hence $Sp(T^*T)$ lies on the origin-centred circle of radius $r(T^*T)$ and, since T^*T is positive definite, $r(T^*T)$ is the only point in its spectrum. Thus the polar decomposition of T is $\sqrt{r(T^*T)}I$ times some unitary operator and therefore $\sqrt{r(T^*T)} = r(T) = w(T) = n(T)$.

Conversely, every nonzero scalar multiple of a unitary operator is clearly in G. These multiples form a group on which r, w, n, agree and define a positive-valued homomorphism with kernel the unitary operators. The improved simplified versions of (2.14) and (2.15) stated in (III) above follow and the reader may wish to work out the (uninteresting) improved simplified versions of (2.12) and (2.13).

REMARK 4.1. The unilateral shift S has a left inverse S^* and $n(S^*)n(S) = 1$, which is 'almost' the equation we are studying, but S is not invertible and its spectrum is the entire origin-centred unit disc.

REMARK 4.2. Of course $r(T), w(T), n(T)$ will all be equal if the biggest equals the smallest: $r(T) = n(T)$. Problem 173 in [2] shows they will all be equal if $w(T) = n(T)$. But $r(T) = w(T)$ does not entail equality of all three if $d > 2$. For example, take $T = \mathrm{Diag}(2, I + 2E)$.

5. What about assuming instead that $T, S, B \in H \supseteq G$?

Let $X \to X'$ denote the faithful $*$-representation described in [1].

Suppose $T \in H$. Then $Sp(T') = Sp(T)$, which by the above is a subset of the origin-centred circle of radius $r(T) = w(T)$. Thus every point of $Sp(T)$ is a boundary point

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and hence is in the approximate point spectrum of T which is the point spectrum of T' (see [1]). Let z be any one of those eigenvalues and let x be any associated unit eigenvector. Then, since $(T^*)' = (T')^*$,

$$\langle (T^*T - w(T)^2 I)'x, x\rangle = \langle (T')^* T'x, x\rangle - w(T)^2 = \langle T'x, T'x\rangle - w(T)^2$$
$$= z^* z\langle x, x\rangle - w(T)^2 = 0.$$

Were the closed linear span of the eigenvectors of T' not the entire space, there would be more to the spectrum of T'. Hence $(T^*T - w(T)^2 I)'$ is 0 and so is $T^*T - w(T)^2 I$. Thus the polar decomposition of T is $w(T)I$ times a unitary operator. This shows that $H \subseteq G \subseteq H$.

We are indebted to C.-K. Li for pointing out the usefulness here of [1] (see also [4, Item 7]).

REMARK 5.1. If we weaken our hypotheses to $T, A, B \in F \supseteq H \supseteq G$ the classical inequalities $n(T) \leq 2w(T)$ and $w(AB) \leq 4w(A)w(B)$ cannot be improved, as the counterexamples given above to (2.14) and (2.15) show. These all satisfy $r(X) = a > 0$ with a small enough. These can be rescaled to $M(1, 1/a)$ giving examples satisfying $r(X) = 1$, if that is preferred.

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