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AN OSCILLATION RESULT FOR SINGULAR NEUTRAL EQUATIONS

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ABSTRACT. In this paper, extending the results in [1], we establish a necessary and sufficient condition for oscillation in a large class of singular (*i.e.*, the difference operator is nonatomic) neutral equations.

1. **Introduction.** Oscillation for various classes of neutral functional differential equations has been studied very extensively in recent years (see *e.g.*, the monograph [8] and the references therein). We also mention the recent articles [2] and [17] for equations with distributed delays. In all of the above papers the standard assumption is that the difference operator in the neutral equation has an atom at zero. Here we relax the atomicity condition and consider a class of scalar singular (*i.e.*, the difference operator is nonatomic at zero) neutral functional differential equations (SNFDE's). Such equations arise for example in aeroelastic modeling and include many singular integro-differential equations. Motivated by aeroelastic control applications, the questions of well-posed state-space formulations, approximation and stabilization of systems governed by SNFDEs have received considerable attention in recent years (see *e.g.*, [4], [5], [6], [7], [10], [11], [12], [13], [14] and [15]).

Another interesting question concerning SNFDEs (since they have been proposed to model aeroelastic flutter) is to provide a characterization of oscillatory, nonoscillatory behavior of their solutions.

In this paper we develop an oscillation theory for SNFDEs. In particular, extending the results in [1], we give a necessary and sufficient condition for oscillation for a large class of SNFDEs via their characteristic equations.

In Section 2, for the convenience of the reader, we summarize some results concerning the well-posedness of SNFDEs and give conditions (in terms of the equation in hand) guaranteeing exponential boundedness of their solutions. In Section 3 we state and prove our oscillation theorem.

2. **Preliminaries.** In this section we give sufficient conditions guaranteeing exponential boundedness of the solution of the scalar linear neutral equation

$$\frac{d}{dt}Dx_t + Lx_t = 0, \quad t \geq 0, \tag{2.1}$$

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with initial data

$$x_0(s) = \psi(s), \quad s \in [-r, 0], \tag{2.2}$$

where r is a positive constant, $\psi(\cdot)$ denotes the initial function; $\psi(\cdot) \in C[-r, 0]$, x_t stands for a solution segment, *i.e.*, $x_t(s) = x(t+s)$, $s \in [-r, 0]$, and the linear operators D and L are assumed to belong $\mathcal{B}(C[-r, 0]\ ;\ R)$, where as usual $\mathcal{B}(X\ ;\ Y)$ denotes the space of bounded operators from the Banach space X to the Banach space Y.

We shall assume that for $\psi(\cdot) \in C[-r, 0]$ the operators D and L have the representations

$$D\psi = \int_{-r}^{0} \psi(s)\, d\nu(s) \tag{2.3}$$

and

$$L\psi = \int_{-r}^{0} \psi(s)\, d\mu(s), \tag{2.4}$$

where ν and μ denote scalar valued functions of bounded variation.

REMARK 2.1. It is well known (see *e.g.* [9]) that under the additional assumption that the linear operator D in (2.1) has an atom at $s = 0$ (*i.e.* $\nu(\cdot)$ in (2.3) is atomic at $s = 0$) the initial value problem (2.1)–(2.2) leads to a linear dynamical system on $C[-r, 0]$, or equivalently we have

$$x_t = T(t)\psi, \quad t \geq 0, \tag{2.5}$$

where the family, $\{T(t)\}_{t \geq 0}$, is a strongly continuous (or C_0) semigroup of bounded linear operators, satisfying $\|T(t)\| < Me^{\omega t}$ for some real constants, $M \geq 1$ and ω.

DEFINITION 2.2. The neutral-equation (2.1) is called *singular* if $\nu(\cdot)$ in (2.3) is nonatomic at $s = 0$.

REMARK 2.3. Recall that $D \in \mathcal{B}([-r, 0])$ is atomic at $s = 0$ if there exists a nonzero constant, a, such that for $\nu(\cdot)$ in (2.3) we can write

$$\nu(s) = a\rho(s) + \tilde{\nu}(s), \tag{2.6}$$

where

$$\rho = \begin{cases} 1 & -r \leq s < 0 \\ 0 & s = 0 \end{cases} \tag{2.7}$$

and $\tilde{\nu}(\cdot)$ is of bounded variation on $[-r, 0]$ and such that

$$\lim_{\varepsilon \to 0+} \int_{-\varepsilon}^{0} |d\tilde{\nu}(s)| = 0. \tag{2.8}$$

Then for $\psi(\cdot) \in C[-r, 0]$ we have the representation

$$D\psi(\cdot) = a\psi(0) + \int_{-r}^{0} \psi(s)\, d\tilde{\nu}(s). \tag{2.9}$$

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The "singular" or nonatomic case corresponds to $a = 0$ in (2.6). An example for SNFDE's is (2.1) with $D\psi(\cdot) = \int_{-r}^{0} \psi(s)|s|^{-\alpha}\, ds$, which has been studied quite extensively (see *e.g.* [5], [6], [11], [14]).

In the remaining part of this section we develop sufficient conditions for the well-posedness (on C) of certain classes of SNFDEs. In particular, we make the following assumption (see also [11], [13]):

(H) There exists an integrable function $g(\cdot)$, such that $d\nu(s) = g(s)\, ds$ for $s \in [-r, 0)$. Moreover, the function $g(\cdot)$ satisfies:
 (i) $g > 0$ on $[-r, 0)$ and $g(s) \longrightarrow \infty$ as $s \longrightarrow 0^-$
 (ii) $g \in H^1_{\mathrm{loc}}[-r, 0)$ and $g' \geq 0$ on $[-r, 0)$.

REMARK 2.4. Hypothesis (H) is motivitated and satisfied by the "aeroelastic" kernel function

$$k(s) = \sqrt{1 - \frac{c}{s}}, \quad s \in [-r, 0),$$

where c is a positive constant (see *e.g.*, [4], [6]). Note also that hypothesis (H) implies

$$\lim_{\omega \longrightarrow \infty} \omega \int_{-r}^{0} e^{\omega s} g(s)\, ds = \infty,$$

which guarantees weak atomicity of $\nu(\cdot)$ at zero (*i.e.*, that

$$\lim_{\lambda \longrightarrow +\infty,\, \lambda \in R} \left| \lambda \int_{-r}^{0} e^{\lambda s}\, d\nu(s) \right| = \infty,$$

see also [14]). It was shown in [14] that weak atomicity of $\nu(\cdot)$ is necessary for the well-posedness of the initial value problem (2.1)–(2.2) on $C[-r, 0]$.

With assumption (H) the initial value problem (2.1)–(2.2) can be rewritten as

$$\frac{d}{dt} \int_{-r}^{0} x(t+s) g(s)\, ds + \int_{-r}^{0} x(t+s)\, d\mu(s) = 0, \quad t \geq 0 \tag{2.10}$$

$$x(s) = \psi(s), \quad s \in [-r, 0]. \tag{2.11}$$

For $\omega \in R^+$ we define the function $y(\cdot)$ by

$$y(t) = e^{-\omega t} x(t), \quad t \geq -r, \tag{2.12}$$

and consider the neutral equation

$$\frac{d}{dt} \int_{-r}^{0} e^{\omega(t+s)} y(t+s) g(s)\, ds + \int_{-r}^{0} e^{\omega(t+s)} y(t+s)\, d\mu(s) = 0, \quad t \geq 0 \tag{2.13}$$

$$y(s) = e^{-\omega s} \psi(s) \equiv \varphi(s), \quad s \in [-r, 0]. \tag{2.14}$$

Differentiation with respect to t and simplification by $e^{\omega t}$ in (2.13) provide a more convenient form for (2.13)–(2.14), *i.e.*

$$\frac{d}{dt} \int_{-r}^{0} e^{\omega s} y(t+s) g(s)\, ds + \omega \int_{-r}^{0} e^{\omega s} y(t+s) g(s)\, ds + \int_{-r}^{0} e^{\omega s} y(t+s)\, d\mu(s) = 0, \tag{2.15}$$

$$y(s) = \varphi(s), \quad s \in [-r, 0]. \tag{2.16}$$

Clearly, initial value problems (2.10)–(2.11) and (2.15)–(2.16) are equivalent in the sense that $x(\cdot)$ satisfies (2.10)–(2.11) if and only it satisfies (2.15)–(2.16).

Define the linear operator $\mathcal{A}$ on

$$\mathcal{D}(\mathcal{A}) = \Big\{\varphi \in C([-r,0]) : \dot{\varphi} \in C([-r,0]) \text{ and } \int_{-r}^{0} e^{\omega s}\dot{\varphi}(s)g(s)\,ds + \omega\int_{-r}^{0} e^{\omega s}\varphi(s)g(s)\,ds + \int_{-r}^{0} e^{\omega s}\varphi(s)\,d\mu(s) = 0\Big\} \tag{2.17}$$

by

$$\mathcal{A}\varphi = \dot{\varphi}, \quad \varphi \in \mathcal{D}(\mathcal{A}). \tag{2.18}$$

For the sake of completeness we include the following well-posedness result which is a slight modification of related developments in [7], [11] and [13].

THEOREM 2.5. *Pick $\omega \in R^{+}$ such that*

$$\omega\int_{-r}^{0} e^{\omega s}g(s)\,ds + \int_{-r}^{0} e^{\omega s}\,d\mu(s) \geq k > 0, \tag{2.19}$$

assume that (H) is satisfied, and consider the initial value problem (2.15)–(2.16). Then the unique solution to (2.15)–(2.16), $y(t)$, $t \geq -r$, can be represented as

$$y(t+s) = \big(S(t)\varphi\big)(s), \quad t > 0, \quad s \in [-r,0], \tag{2.20}$$

$\{S(t)\}_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on the Banach-space $C[-r,0]$ whose infinitesimal operator, $\mathcal{A}$, is given by (2.17)–(2.18).

PROOF. We establish that the operator $\mathcal{A}$ defined by (2.17)–(2.18) satisfies the following:

(i) $\mathcal{A}$ is dissipative on $C[-r,0]$, *i.e.*,

$$\|(\lambda I - \mathcal{A})\| \geq \lambda(\varphi), \quad \text{for } \varphi \in \mathcal{D}(\mathcal{A}), \quad \lambda > 0, \quad \lambda \in R \tag{2.21}$$

(ii) For $\lambda > 0$, $\lambda \in R$ the equation

$$(\lambda I - \mathcal{A})\varphi = \xi \tag{2.22}$$

is solvable for any $\xi(\cdot) \in C[-r,0]$ with solution $\varphi(\cdot) \in \mathcal{D}(\mathcal{A})$.

(iii) $\mathcal{D}(\mathcal{A})$ is dense in $C[-r,0]$.

Then by the Lumer-Phillips theorem ([16]) $\mathcal{A}$ generates a contraction semigroup, $\{S(t)\}_{t\geq 0}$, on $C[-r,0]$, *i.e.*, the abstract Cauchy problem

$$\frac{d}{dt}(y_t) = \mathcal{A}y_t, \tag{2.23}$$

with the initial data

$$y_0(\cdot) = \varphi(\cdot) \tag{2.24}$$

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is well-posed and via the equivalence between the solutions of (2.15)–(2.16) and (2.23)–(2.24) (see [6], [7]) the claim of the theorem follows.

According to the above comments we show the validity of (i), (ii) and (iii) now to complete the proof.

(i) DISSIPATIVENESS. The only nontrivial case is when $\|\varphi\| = \sup_{s\in[-r,0]} |\varphi(s)| = |\varphi(0)| \neq 0$. Without loss of generality we can assume $\varphi(0) > 0$. Integrating by parts and using hypothesis (H) yield that

$$\begin{aligned}\int_{-r}^{0} e^{\omega s}\dot\varphi(s)g(s)\,ds &= \int_{-r}^{0} e^{\omega s}\frac{d}{ds}\big(\varphi(s)\varphi(0)\big)g(s)\,ds\\ &= -e^{-\omega r}\big(\varphi(-r)-\varphi(0)\big)g(-r) - \int_{-r}^{0} e^{\omega s}\big(\varphi(s)-\varphi(0)\big)\dot g(s)\,ds\\ &\quad -\omega\int_{-r}^{0} e^{\omega s}\varphi(s)g(s)\,ds + \varphi(0)\Big(\omega\int_{-r}^{0} e^{\omega s}g(s)\,ds\Big).\end{aligned}$$

Substituting this last expression into the domain condition, *i.e.* into (2.17), we get

$$\begin{aligned}&\int_{-r}^{0} e^{\omega s}\dot\varphi(s)g(s)\,ds + \omega\int_{-r}^{0} e^{\omega s}\varphi(s)g(s)\,ds + \int_{-r}^{0} e^{\omega s}\varphi(s)\,d\mu(s)\\ &\quad= e^{-\omega r}\big(\varphi(0)-\varphi(-r)\big)g(-r) + \int_{-r}^{0} e^{\omega s}\big(\varphi(0)-\varphi(s)\big)\dot g(s)\,ds\\ &\qquad+\varphi(0)\Big(\omega\int_{-r}^{0} e^{\omega s}g(s)\,ds\Big) + \int_{-r}^{0} e^{\omega s}\varphi(s)\,d\mu(s) = 0.\end{aligned} \tag{2.25}$$

Hypothesis (H) and the selection of ω imply that the left hand side of (2.25) is positive which is a contradiction. Therefore $\varphi \in \mathcal{D}(\mathcal{A})$, $\|\varphi\| = \varphi(0) > 0$ is not possible.

If $\varphi \in \mathcal{D}(\mathcal{A})$, $\|\varphi\| = |\varphi(s)|$, $s \in (-r, 0)$, then $\dot\varphi(s) = 0$ and $\|\lambda\varphi - \dot\varphi\| \geq |\lambda\varphi(s)| = \lambda\|\varphi\|$ for $\lambda > 0$. The case $\|\varphi\| = |\phi(-r)|$ is also trivial because then $\varphi(-r)\dot\varphi(-r) \leq 0$. Dissipativeness of $\mathcal{A}$ follows.

(ii) UNIQUE SOLVABILITY. Consider for $\xi \in C[-r, 0]$ the equation

$$(\lambda I - \mathcal{A})\varphi = \lambda\varphi - \dot\varphi = \xi. \tag{2.26}$$

The solution of (2.26) is given by

$$\varphi(s) = e^{\lambda s}\varphi(0) + \int_{s}^{0} e^{\lambda(s-u)}\xi(u)\,du, \tag{2.27}$$

where $\varphi(0)$ is to be determined. Substituting (2.27) into the domain condition we have

$$\begin{aligned}&\int_{-r}^{0} e^{\omega s}\Big(\lambda e^{\lambda s}\varphi(0) - \xi(s) + \int_{s}^{0} e^{\lambda(s-u)}\xi(u)\,du\Big)g(s)ds\\ &\quad+\omega\int_{-r}^{0} e^{\omega s}\Big(e^{\lambda s}\varphi(0) + \int_{s}^{0} e^{\lambda(s-u)}\xi(u)\,du\Big)g(s)\,ds\\ &\quad+\int_{-r}^{0} e^{\omega s}\Big(e^{\lambda s}\varphi(0) + \int_{s}^{0} e^{\lambda(s-u)}\xi(u)\,du\Big)\,d\mu(s) = 0,\end{aligned}$$

which can be solved for $\varphi(0)$ for $\lambda > 0$ because the selection of ω and hypothesis (H) guarantee that

$$(\omega+\lambda)\int_{-r}^{0} e^{(\omega+\lambda)s}g(s)\,ds + \int_{-r}^{0} e^{(\omega+\lambda)s}\,d\mu(s) \geq k > 0.$$

(iii) DENSITY OF $\mathcal{D}(\mathcal{A})$. Following [11], for $\varphi \in C^1[-r,0]$ we define the sequence $\{\varphi_n\}$, $\varphi_n \in C^1[-r,0]$, $n = 1,2,\ldots$ as follows

$$\varphi_n(s) = \begin{cases} \varphi(s) & s \in [-r, -\frac{1}{n}] \\ \varphi(s) + c_n \frac{n}{2}(s+\frac{1}{n})^2 & s \in [-\frac{1}{n}, 0], \end{cases}$$

where

(2.28)
$$c_n = \frac{\int_{-r}^0 e^{\omega s}\dot{\varphi}(s)g(s)\,ds + \omega\int_{-r}^0 e^{\omega s}\varphi(s)g(s)\,ds + \int_{-r}^0 e^{\omega s}\varphi(s)\,d\mu(s)}{n\int_{-\frac{1}{n}}^0 e^{\omega s}(s+\frac{1}{n})g(s)\,ds + \frac{n}{2}\omega\int_{-\frac{1}{n}}^0 e^{\omega s}(s+\frac{1}{n})^2 g(s)\,ds + \frac{n}{2}\int_{-\frac{1}{n}}^0 e^{\omega s}(s+\frac{1}{n})^2\,d\mu(s)}.$$

Note that hypothesis (H) and the selection of ω guarantee that the denominator of (2.28) is not equal to zero. Then $\varphi_n(\cdot) \in \mathcal{D}(\mathcal{A})$. Furthermore,

$$\|\varphi - \varphi_n\| = \frac{|c_n|}{2n} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,$$

because

$$\begin{aligned} 2n\cdot n\cdot \int_{-\frac{1}{n}}^0 e^{\omega s}\Big(s+\frac{1}{n}\Big)g(s)\,ds &\geq 2n^2\int_{-\frac{1}{2n}}^0 e^{-\frac{\omega}{2n}}\frac{1}{2n}g\Big(-\frac{1}{2n}\Big)\,ds \\ &= \frac{1}{2}e^{-\frac{\omega}{2n}}g\Big(-\frac{1}{2n}\Big) \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \end{aligned}$$

Using the fact that $C^1[-r,0]$ is dense in $C[-r,0]$, the density of $\mathcal{D}(\mathcal{A})$ in $C[-r,0]$ follows. The proof of the theorem is complete.

We conclude this section with the following result on the solutions of the initial value problem (2.10)–(2.11).

COROLLARY 2.6. *Let $x(t)$, $t \geq -r$, be the unique solution of (2.10)–(2.11). Then there exist constants M and α such that*

(2.29)
$$|x(t)| \leq Me^{\alpha t}, \quad t \geq 0.$$

PROOF. By Theorem 2.5 we have

$$\begin{aligned} |x(t)| = |e^{\omega t}y(t)| &\leq e^{\omega t}\|y(t)\| \\ &\leq e^{\omega t}\|S(t)\|\,\|\varphi\| \\ &\leq \tilde{M}e^{(\omega+\tilde{\omega})t}\|e^{-\omega\cdot}\psi(\cdot)\| \\ &\leq \tilde{M}e^{(\omega+\tilde{\omega})t}e^{\omega r}\|\psi\| \\ &= Me^{\alpha t}, \end{aligned}$$

where $\|S(t)\| \leq \tilde{M}e^{\tilde{\omega}t}$, $M = \tilde{M}e^{\omega r}\|\psi\|$ and $\alpha = \omega + \tilde{\omega}$.

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3. **Oscillation in singular** NFDEs. Consider the neutral functional-differential equation with distributed delays

$$\frac{d}{dt}\Big(\int_{-\tau}^{0} x(t+s)\,d\nu(s)\Big) + \int_{-\sigma}^{0} x(t+s)\,d\mu(s) = 0, \quad t \geq 0 \tag{3.1}$$

where $\tau > 0$ and $\sigma > 0$ are constants and the functions $\nu\colon [-\tau, 0] \longrightarrow R$ and $\mu\colon [-\sigma, 0] \longrightarrow R$ are of bounded variation.

Without loss of generality we can assume that μ and ν are defined on R and are normalized in the following sense:

$$\mu(u) = \nu(u) = 0 \quad \text{for } u \geq 0$$
$$\mu(u) = \mu(-\sigma), \quad u \leq -\sigma \text{ and } \nu(u) = \nu(-\tau), \quad u \leq -\tau,$$

where $\mu(\cdot)$ and $\nu(\cdot)$ are left-hand continuous functions.

REMARK 3.1. The class of SNFDEs we have considered in the previous section represents a subset of the problems investigated the remaining part of this paper, *i.e.*, when $\nu(\cdot)$ in (3.1) satisfies hypothesis (H) and

$$r \equiv \max\{\tau, \sigma\}. \tag{3.2}$$

DEFINITION 3.2. A function $x\colon [-r, \infty) \longrightarrow R$ is a solution of (3.1) if $x(\cdot)$ is continuous on $[-r, \infty)$, $\int_{-r}^{0} x(t+s)\,d\nu(s)$ is differentiable and $x(\cdot)$ satisfies (3.1) on $[0, \infty)$.

We shall assume throughout this section that:

(A) Every eventually positive/negative solution, $x(\cdot)\colon [-r, \infty) \longrightarrow R$, of (3.1) is exponentially bounded, *i.e.*, satisfies an estimate of the type (2.29) (see also (3.4) below).

REMARK 3.3. The solutions of the class of SNFDEs discussed in Section 2 satisfy assumption (A). On the other hand, since hypothesis (H) is only a sufficient condition for exponential boundedness, it is not known at this time what the largest class of SNFDEs is for which assumption (A) holds.

In the sequel we shall use the (so called) characteristic equation associated with (3.1), *i.e.*,

$$\lambda \int_{-\tau}^{0} e^{\lambda s}\,d\nu(s) + \int_{-\sigma}^{0} e^{\lambda s}\,d\mu(s) = 0. \tag{3.3}$$

Note that if λ_0 is a real root of (3.3), then $x(t) = e^{\lambda_0 t}$ is a nonoscillatory, exponentially bounded solution of (3.1). (In Theorem 3.6 below we show that if (3.1) has an exponentially bounded nonosciallatory solution, then (3.3) has a real root.)

DEFINITION 3.4. We say a function $x\colon [-r, \infty) \longrightarrow R$ is *oscillatory* if for all $\tilde{t} \geq 0$ there exists a $t_1 > \tilde{t}$ such that $x(t_1) = 0$. Otherwise $x(t)$ is called *nonoscillatory*.

REMARK 3.5. Since (3.1) is autonomous one can easily verify that if $x\colon [-r, \infty) \longrightarrow R$ is a solution of (3.1), then for all fixed $c > 0$ and $i = 0, 1$, $(-1)^i x(t+c)$ is also a solution of (3.1). Thus it can be easily seen that (3.1) has a nonoscillatory solution if and only if it has a positive solution on $[-r,\infty)$. Accordingly, after a possible translation, we can modify (2.29) as follows: $x\colon [-r,\infty)\longrightarrow R$ is an exponentially bounded positive solution of (3.1) if there exist constants $M = M(x) > 0$ and $\alpha = \alpha(x) \in R$ such that

$$x(t) \leq Me^{\alpha t}, \quad t \geq -r. \tag{3.4}$$

THEOREM 3.6. *Suppose that assumption (A) is satisfied. Then the following two statements are equivalent:*

(i) The neutral equation (3.1) has a nonoscillatory solution.

(ii) The characteristic equation (3.3) has a real root.

To prove this theorem we need the following lemma which is related to the Laplace-transform

$$X(s) = \int_0^\infty e^{-st}x(t)\,dt \tag{3.5}$$

for an exponentially bounded solution $x(t)$ of (3.1).

LEMMA 3.7. *If $x\colon [-r,\infty)\longrightarrow R$ is an exponentially bounded solution of (3.1), then the spectral abscissa δ,*

$$\delta = \inf\{\operatorname{Re} s : X(s) \text{ exists}\} \tag{3.6}$$

satisfies $\delta \in (-\infty,\infty)$, and for all $\operatorname{Re} s > \delta$ one has

$$X(s)F(s) = \Phi(s), \tag{3.7}$$

where $F(s)$ is defined as follows:

$$F(s) = s\int_{-\tau}^0 e^{su}\,d\nu(u) + \int_{-\sigma}^0 e^{su}\,d\mu(u) \tag{3.8}$$

and

$$\begin{aligned}\Phi(s) = \int_{-\tau}^0 x(u)\,d\nu(u) - s\int_{-\tau}^0\Big(\int_\theta^0 e^{s(\theta-u)}x(u)\,du\Big)\,d\nu(\theta)\\ -\int_{-\sigma}^0\Big(\int_\theta^0 e^{s(\theta-u)}x(u)\,du\Big)\,du(\theta).\end{aligned} \tag{3.9}$$

PROOF. Since $x(t)$ is an exponentially bounded solution, there exist $M = M(x) > 0$ and $\alpha = \alpha(x) \in R$ such that $|x(t)| \leq Me^{\alpha t}$, $t \geq -r$. It follows that

$$\begin{aligned}\Big|\int_{-\tau}^0 x(t+u)\,d\nu(u)\Big| &\leq \int_{-\tau}^0 Me^{\alpha(t+u)}\,d|\nu(u)|\\ &= M_1 e^{\alpha t}, \quad t \geq -r.\end{aligned} \tag{3.10}$$

and similarly,

$$\begin{aligned}\Big|\int_{-\sigma}^0 x(t+u)\,d\mu(u)\Big| &\leq \int_{-\sigma}^0 Me^{\alpha(t+u)}\,d|\mu(u)|\\ &\leq M_2 e^{\alpha t}, \quad t \geq -r,\end{aligned} \tag{3.11}$$

where $M_1 = M\int_{-\tau}^{0} e^{\alpha u}\, d|\nu(u)|$ and $M_2 = M\int_{-\sigma}^{0} e^{\alpha u}\, d|\mu(u)|$.

Inequalities (3.10)–(3.11) imply the existence of the Laplace transforms

$$\int_0^\infty e^{-st}\frac{d}{dt}\Big(\int_{-\tau}^0 x(t+u)\, d\nu(u)\Big)\, dt$$

and

$$\int_0^\infty e^{-st}\Big(\int_{-\sigma}^0 x(t+u)\, d\mu(u)\Big)\, dt.$$

Moreover, we have

$$\begin{aligned}\int_0^\infty e^{-st}\frac{d}{dt}\Big(\int_{-\tau}^0 x(t+u)\, d\nu(u)\Big)\, dt &= -\int_{-\tau}^0 x(u)\, d\nu(u) + s\int_0^\infty \Big(e^{-st}\int_{-\tau}^0 x(t+u)\, d\nu(u)\Big)\, dt\\ &= s\int_{-\tau}^0\Big(\int_0^\infty e^{-st}x(t+u)\, dt\Big)\, d\nu(u) - \int_{-\tau}^0 x(u)\, d\nu(u)\\ &= s\int_{-\tau}^0\Big(\int_u^\infty e^{-s(v-u)}x(v)\, dv\Big)\, d\nu(u) - \int_{-\tau}^0 x(u)\, d\nu(u)\\ &= s\int_{-\tau}^0\Big(\int_u^0 e^{s(v-u)}x(v)\, dv\Big)\, d\nu(u)\\ &\quad + s\int_{-\tau}^0 e^{su}\, d\nu(u)\int_0^\infty e^{-st}x(v)\, dv - \int_{-\tau}^0 x(u)\, d\nu.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_0^\infty\Big(e^{-st}\int_{-\sigma}^0 x(t+u)\, d\mu(u)\Big)\, dt &= \int_{-\sigma}^0\Big(\int_0^\infty e^{-st}x(t+u)\, dt\Big)\, d\mu(u)\\ &= \int_{-\sigma}^0\Big(\int_u^\infty e^{-s(v-u)}x(v)\, dv\Big)\, d\mu(u)\\ &= \int_{-\sigma}^0\Big(\int_u^u e^{-s(v-u)}x(v)\, dv\Big)\, d\mu(u)\\ &\quad + \int_{-\sigma}^0 e^{su}\, d\mu(u)\int_0^\infty e^{-st}x(v)\, dv.\end{aligned}$$

Summarizing the above relations, from equation (3.1) we have that (3.7) is valid for $\operatorname{Re} s > \alpha$.

Now, we show that (3.7) is valid for all $\operatorname{Re} s > \delta$, where $\delta = \delta(x)$ is defined in (3.6). First, it can be seen easily that the functions $F(s)$ and $\Phi(s)$ are analytic in the whole complex plane. On the other hand, $X(s)$ is analytic for all $\operatorname{Re} s > \delta$ and (3.7) is satisfied for all $\operatorname{Re} s > \alpha$. Thus it is obvious, that (3.7) is satisfied for all $\operatorname{Re} s > \delta$. The proof of the lemma is complete.

Proof of Theorem 3.6. (ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (ii) is proved indirectly, that is, for the sake of contradiction we assume that equation (3.1) has a non-oscillatory solution, say $x(t)$, and at the same time $F(s) \neq 0$ for all real s. Since equation (3.1) is autonomous and homogeneous we can (and will) assume that $x(t)$ is exponentially bounded on $[-r, \infty)$. Thus by Lemma 3.7, we obtain that the Laplace-transform $X(s)$ of $x(t)$ satisfies (3.7) for all $\operatorname{Re} s > \delta$, where the spectral abscissa δ of $x(t)$ is defined in (3.6).

Now we show that $X(s)$ exists for all real s, that is $\delta = -\infty$. Otherwise $\delta > -\infty$ and by Theorem 5b on page 58 in [18] we have that δ is a singularity point of $X(s)$. This means that there is no analytic extension of $X(s)$ for any neighborhood on δ. But the function $F(s)$ does not have real roots and hence $F(\delta) \neq 0$. Thus there exists an $\varepsilon > 0$ such that for all s such that $|s-\delta| < \varepsilon$ the $F(s)$ is not zero and hence $\frac{\Phi(s)}{F(s)}$ is analytic for all s with $|s-\delta| < \varepsilon$. Moreover (3.7) yields

$$X(\delta_n) = \frac{\Phi(s_n)}{F(s_n)}$$

for all s with $\operatorname{Re} s > \delta$ and this means that $X(s)$ has an analytic extension for a neighborhood of δ, contradicting the definition of δ. It follows that $\delta = -\infty$. Consequently, (3.7) is valid for all real s since $F(s) \neq 0$ for $s \in R$, we conclude that

$$X(s) = \frac{\Phi(s)}{F(s)}, \quad s \in R. \tag{3.12}$$

Using the definition of $\Phi(s)$ and $F(s)$ we obtain the estimates:

$$|F(s)| \leq |s| \int_{-T}^{0} |e^{su}|\, d|\nu(u)| + \int_{-\sigma}^{0} |e^{su}|\, d\mu(u)| \leq Ke^{\beta|s|} \tag{3.13}$$

and

$$|\Phi(s)| \leq \int_{-T}^{0} |x(u)|\, d|\mu(u)| + |s| \int_{-T}^{0} \Big(\int_{-\theta}^{0} |e^{s(\theta-u)}|\, d\mu \Big)\, d|\nu(\theta)|$$
$$+ \int_{-\sigma}^{0} \Big(\int_{-\theta}^{0} |e^{s(\theta-u)}|\, du \Big)\, d|\mu(\theta)| \leq Ke^{\beta|s|}$$

for all s, where K and β are positive reals. By a result of Cartwright (see Theorem 3.3.1 on page 43 in [3]), for $\varepsilon > 0$, the function $F(s)$ satisfies the relation

$$\limsup_{r\to\infty} \Big(\min_{|s|=r} |F(s)|\Big)\Big(\max_{|s|=r} |F(s)|\Big)^{1+\varepsilon} = \infty. \tag{3.15}$$

Therefore, taking $\varepsilon = 1$ in (3.15) and using (3.13) we obtain

$$\frac{1}{|F(-r_k)|} \leq \frac{1}{\min_{|s|=r_k} |F(s)|} \leq \Big(\max_{|s|=r_k} |F(s)|\Big)^2 \leq K^2 e^{2\beta r_K} \tag{3.16}$$

for a sequence $\{r_k\}_{k=1}^{\infty}$ such that $r_k \to \infty$, as $k \to \infty$. Combining inequalities (3.16) and (3.14), we have

$$X(-r_k) = \frac{\Phi(-r_k)}{F(-r_k)} \leq K^3 e^{3\beta r_k}$$

for all $k \geq 1$. On the other hand, for all $T \geq 0$, the estimate

$$e^{-r_k T} \int_T^{\infty} x(t)\, dt \leq \int_T^{\infty} e^{-r_k t} x(t)\, dt \leq X(-r_k)$$
$$= \int_0^{\infty} e^{-r_k t} x(t)\, dt \leq K^3 e^{3\beta r_k}, \quad k \geq 1$$

yields that

$$\int_T^{\infty} x(t)\, dt \leq K^3 e^{(3\beta - r) r_k}, \quad k \geq 1,$$

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and for $T > 3\beta$

$$\int_T^\infty x(t)\,dt \le K^3 e^{(3\beta - r)r_k} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{3.17}$$

On the other hand, $x(t) > 0$, $t \ge 0$ which implies $\int_T^\infty x(t)\,dt > 0$. This is a contradiction with (3.17). The proof of the theorem is complete.

EXAMPLE 3.8. Consider the singular neutral equation

$$\frac{d}{dt}\Big(\int_{-1}^0 (-s)^{-\alpha} x(t+s)\,ds\Big) + b\int_{-1}^0 (-s)^{-\beta} x(t+s-\tau)\,ds = 0, \tag{3.18}$$

where $\alpha, \beta \in (0,1)$, $b > 0$ and $\tau > 0$ are given constants. Now, we show that the characteristic equation

$$\lambda \int_{-1}^0 (-s)^{-\alpha} e^{\lambda s}\,ds + b\int_{-1}^0 (-s)^{-\beta} e^{\lambda(s-\tau)}\,ds = 0 \tag{3.19}$$

does not have a real root in the case when

$$\beta \ge \alpha \text{ and } b\tau e > 1. \tag{3.20}$$

Otherwise, there is a real root λ_0 of (3.19), which has to be negative. Thus $\mu_0 = -\lambda_0 > 0$, satisfies

$$\mu_0 = b\frac{\int_0^1 v^{-\beta} e^{\mu_0 v}\,dv}{\int_0^1 v^{-\alpha} e^{\mu_0 v}\,dv} e^{\mu_0 \tau} \ge b e^{\mu_0 \tau} > b\mu_0 \tau e,$$

which contradicts our assumption $b\tau e > 1$. Thus, under condition (3.20), the characteristic equation (3.19) does not have a real root, and hence, by Theorem 3.6, we have that all of the solutions of (3.18) are oscillatory. If $\alpha = \beta$ the it can be easily seen that equation (3.19) has a real root if and only if the equation

$$\mu = b e^{\mu\tau}$$

has a positive root, or equivalently $b\tau e \le 1$. This means that if $\alpha = \beta$ then condition (3.20) is sharp.

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