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Bifurcation and stability of equilibria with asymptotically linear boundary conditions at infinity

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We consider an elliptic equation with a nonlinear boundary condition which is asymptotically linear at infinity and which depends on a parameter. As the parameter crosses some critical values, there appear certain resonances in the equation producing solutions that bifurcate from infinity. We study the bifurcation branches, characterize when they are sub- or supercritical and analyse the stability type of the solutions. Furthermore, we apply these results and techniques to obtain Landesman–Lazer-type conditions guaranteeing the existence of solutions in the resonant case and to obtain an anti-maximum principle.

1. Introduction

Over the last decade a lot of attention has been paid to problems with nonlinear boundary conditions. Hence, nowadays, the underlying mechanisms for dissipativeness or blow-up of solutions is fairly well understood (see, for example, [3,5,7,18,19]). Therefore, it is natural to analyse the dynamics and bifurcations induced by the nonlinear boundary conditions, and compare their effects in the case of an interior reaction term, which has been more widely studied. For example, in [6] the existence of patterns for such problems, i.e. a stable non-trivial equilibrium, was considered (see also the references therein for some previous and related results). In this work we consider the evolutionary equation of parabolic type,

$$\left.\begin{aligned} u_t - \Delta u + u &= 0 && \text{in } \Omega,\ t>0,\\ \frac{\partial u}{\partial n} &= \lambda u + g(\lambda, x, u) && \text{on } \partial\Omega,\ t>0,\\ u(0,x) &= u_0(x) && \text{in } \Omega, \end{aligned}\right\} \tag{1.1}$$

in a bounded and sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ with $N \geqslant 2$ and analyse the behaviour and stability properties of the equilibrium solutions. These equilibria are solutions of the following elliptic problem with nonlinear boundary conditions:

$$\left.\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega,\\ \frac{\partial u}{\partial n} &= \lambda u + g(\lambda, x, u) && \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.2}$$

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Our main goal here is to analyse some possible bifurcations of solutions as the parameter λ is varied, and to study the stability of such solutions. In particular, we are interested in the possibility of producing solutions that are large in Ω in a given sense. We are also interested in characterizing the super- or subcritical character of such bifurcations.

As we will show below, it is in fact possible to generate such large solutions; these will be obtained from a 'bifurcation from infinity' argument, even in the case in which the nonlinear boundary condition is sublinear at infinity. Such solutions will be generated by a resonant mechanism at the boundary.

We will also show that some stability or instability of such solutions can be derived.

Since we will also give conditions for either subcritical or supercritical bifurcations, we will obtain, as a by-product, the analogue to the well-known Landesman–Lazer conditions for the existence of equilibria in resonant cases [15]. Also, a form of the anti-maximum principle will also be derived [8]. A similar analysis for the case of an interior reaction term was first established in [2].

We now present our main results in a more precise way. The main hypothesis on the nonlinearity g is the sublinearity with respect to the variable u. Hence, we will assume a condition that, roughly speaking, will be of the type

$$|g(\lambda, x, u)| \leqslant C|u|^{\alpha} \quad \text{as } |u| \to \infty \text{ for some } \alpha < 1.$$

Observe that we do not exclude the case where α is negative. This condition means that, in the boundary condition, the dominant term for $|u|$ large is the linear term λu. In this respect we call this boundary condition asymptotically linear. This includes the case where $g(\lambda, x, u) = g(x)$ and it is well known that problem (1.2) will have a (unique) solution if λ is not an eigenvalue of the problem

$$\left.\begin{aligned} -\Delta\Phi + \Phi = 0 \quad & \text{in } \Omega, \\ \frac{\partial\Phi}{\partial n} = \sigma\Phi \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.3}$$

This eigenvalue problem is known as the Steklow eigenvalue problem and it is well known that (1.3) has a discrete set of eigenvalues $\{\sigma_i\}_{i=1}^{\infty}$. These numbers will play an essential role in the analysis below. In particular, for $\lambda \notin \{\sigma_i\}_{i=1}^{\infty}$, we consider the operator T_λ such that $T_\lambda b := v$, where v is the unique solution of

$$\left.\begin{aligned} -\Delta v + v = 0 \quad & \text{in } \Omega, \\ \frac{\partial v}{\partial n} - \lambda v = b \quad & \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.4}$$

for a function b given on $\partial\Omega$.

The fact that, for compact sets of λ far from the Steklov eigenvalues, the norm of the operator T_λ, in some appropriate spaces, is uniformly bounded, together with the sublinearity of the function g will allow us to show, by a fixed-point argument, the existence of at least one solution of (1.2) for any λ not a Steklov eigenvalue. Moreover, all solutions will be uniformly bounded for λ in compact intervals far from the Steklov eigenvalues (see theorem 2.7).

On the other hand, when the parameter λ approaches a Steklov eigenvalue, the norm of the operator T_λ diverges to ∞. This is the first hint of the possibility of finding unbounded branches of solutions and reveals the resonant mechanism at the boundary that produces such large solutions. For instance, when $g \equiv 0$, the structure of the solutions of the problem (1.2) is well known: if λ is not a Steklov eigenvalue, the only solution is the trivial solution and if λ is a Steklov eigenvalue, the whole space of eigenfunctions associated with that eigenvalue are solutions of the elliptic problem that can be regarded as unbounded branches of solutions. For the case in which g is sublinear at infinity, we will apply general techniques of bifurcation theory (see [9, 16, 17]) and prove the existence of unbounded branches of solutions whenever the parameter λ approaches a Steklov eigenvalue of odd multiplicity (see theorem 3.3). Moreover, since the first Steklov eigenvalue is simple, we will show the existence of unbounded branches of solutions bifurcating from the first eigenvalue. The fact that the first Steklov eigenfunction does not change sign will give us extra information that will permit us to analyse this branch of solutions in detail. In particular, we will show the existence of two branches of solutions: one consisting of positive solutions and the other of negative solutions (see theorem 3.4).

Once the existence of these bifurcation branches has been established, we pay attention to the type of bifurcation (i.e. sub- or supercritical) occurring. It is clear that a condition on the sublinearity of g is not sufficient to distinguish between the types of bifurcation, and to accomplish this we will need to specify the precise asymptotics of the function g at infinity. For instance, if we consider that the function g behaves like $a|u|^\alpha$ as $u \to +\infty$, we can easily see that the sign of a will determine whether the bifurcation of positive solutions emanating from the first eigenvalue is sub- or supercritical. To do this, if $0 < u_n \to \infty$ is a solution of (1.2) for $\lambda_n \to \sigma_1$, multiplying the equation by the first Steklov eigenfunction $\Phi_1 > 0$ and integrating by parts, we obtain

$$(\sigma_1 - \lambda_n) \int_{\partial\Omega} u_n \Phi_1 \,\mathrm{d}\varsigma = \int_{\partial\Omega} g(\lambda_n, x, u_n) \Phi_1 \,\mathrm{d}\varsigma.$$

But, since $u_n > 0$ and $u_n \to \infty$,

$$\int_{\partial\Omega} u_n \Phi_1 \,\mathrm{d}\varsigma > 0, \qquad \int_{\partial\Omega} g(\lambda_n, x, u_n) \Phi_1 \,\mathrm{d}\varsigma \approx a \int_{\partial\Omega} |u_n|^\alpha \Phi_1 \,\mathrm{d}\varsigma,$$

and the sign of $\sigma_1 - \lambda_n$ is the same as that of a. Hence, if $a > 0$, the bifurcation of positive solutions will be subcritical and if $a < 0$, it will be supercritical (see theorem 4.3, below, for a more general statement).

Moreover, we will also see that, typically, when a bifurcation from infinity occurs at the first eigenvalue, the branch of equilibria will be stable when the bifurcation is subcritical and unstable when the bifurcation is supercritical (see propositions 7.1 and 7.3, below).

Being able to give conditions which characterize when the bifurcation is sub- or supercritical will allow us to address two important issues for this problem.

On the one hand we will be able to give Landesman–Lazer-type conditions, guaranteeing that the nonlinear resonant problem (that is, when $\lambda = \sigma_i$ for some i) has at least a solution (see [15]). For this, imagine that for a value σ_i we can determine that all possible bifurcations occurring at this value of the parameter are, say, subcritical. This implies that, for $\lambda \in (\sigma_i, \sigma_i + \varepsilon)$, for some $\varepsilon > 0$ small, the solutions of (1.2) will have to be bounded in certain norms, uniformly for $\lambda \in (\sigma_i, \sigma_i + \varepsilon)$. Using elliptic regularity results will allow us to pass to the limit in a weak sense as $\lambda \to \sigma_i$ and show that the limit is a solution of the resonant problem (see theorem 5.1).

On the other hand, we will be able to prove anti-maximum principles for the problem (1.4). That means, in particular, that if b is such that $\int_{\partial\Omega} b\Phi_1 > 0$, then the bifurcation of negative solutions occurring at $\lambda = \sigma_1$ is supercritical and this implies that for $\lambda \in (\sigma_1, \sigma_1 + \varepsilon)$ the unique solution of (1.4) has to be strictly negative (see theorem 6.1). These types of result were first proved for elliptic problems of the form $-\Delta u = \lambda m(x)u + h(x)$ with Dirichlet boundary conditions in [8].

This paper is organized as follows. In § 2 we formulate the problem and show the existence of solutions for all values of λ that are different from the Steklov eigenvalues. To accomplish this, we analyse the linear problem (1.4), stating and proving several important regularity results. We then formulate the nonlinear problem (1.2) as a fixed-point problem in a certain function space on the boundary. Finally, the compactness results obtained through the regularity results and the Schaefer fixed-point theorem will show the existence of solutions.

In § 3 we apply bifurcation results, mainly from [16, 17], to show the existence of unbounded branches of solutions bifurcating from the Steklov eigenvalues (see theorem 3.3). We pay special attention to the bifurcations emanating from simple eigenvalues (see theorem 3.4).

In § 4 we give conditions on the behaviour of the nonlinearity g for $|u|$ large that allow us to determine when sub- or supercritical bifurcations occur.

In § 5 we apply the conditions from the previous section to obtain Landesman–Lazer-type conditions for the resonant problem.

In § 6 we state and prove the anti-maximum principle for (1.4) mentioned above.

In § 7 we analyse the stability properties of the solutions bifurcating from the first eigenvalue.

Finally, in § 8 we consider several important remarks and extensions. We study the conditions to be imposed on the nonlinearity g in order to obtain bifurcations from the trivial solution, instead of bifurcations from infinity. We also consider the case in which the boundary condition is of the type

$$\frac{\partial u}{\partial n} = \lambda m(x)u + g(\lambda, x, u),$$

where m is a potential that may change sign on $\partial\Omega$. We also consider the one-dimensional case, that is, where the equation (1.2) is posed in $\Omega = (0, 1) \subset \mathbb{R}$.

2. Setting the problem

In this section we rewrite equation (1.2) as a fixed-point problem in appropriate function spaces and analyse the existence of solutions for all $\lambda \in \mathbb{R}$ except for a discrete set. To accomplish this task we will use Schaefer's fixed-point theorem (see [11, p. 502]).

With respect to the nonlinearity g, we assume the following hypothesis.

(H1) $g : \mathbb{R}\times\partial\Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. $g = g(\lambda, x, s)$ is measurable in $x \in \Omega$, and continuous with respect to $(\lambda, s) \in \mathbb{R}\times\mathbb{R}$). Moreover, there exist $h \in L^r(\partial\Omega)$ with $r > N - 1$ and a continuous functions $\Lambda : \mathbb{R} \to \mathbb{R}^+$, $U : \mathbb{R} \to \mathbb{R}^+$, satisfying

$$|g(\lambda, x, s)| \leqslant \Lambda(\lambda)h(x)U(s) \quad \text{for all } (\lambda, x, s) \in \mathbb{R}\times\partial\Omega\times\mathbb{R}. \tag{2.1}$$

Moreover, we assume also the following condition on the function U:

(H2) $\lim_{|s|\to\infty} \dfrac{U(s)}{s} = 0.$

Observe that the sublinearity of g at infinity is given by condition (H2).

With respect to the linear problem, it is well known (see [1]) that the operator $A = -\Delta + I$, with homogeneous Neumann boundary conditions, defines an unbounded operator in $L^p(\Omega)$ for all $p > 1$ with domain $D(A) = \{u \in W^{2,p}(\Omega); \partial u/\partial n = 0 \text{ in } \partial\Omega\}$. Moreover, the operator A has an associated scale of interpolation–extrapolation spaces and, in particular, for each $p > 1$, we have that $A : W^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ is an isomorphism.

Hence, for any $q \geqslant 1$, since we have the embedding $L^q(\partial\Omega) \hookrightarrow W^{-1,p}(\Omega)$ continuous for $p = qN/(N-1)$ and compact if $p < qN/(N-1)$, for $b \in L^q(\partial\Omega)$ the unique solution of

$$\left.\begin{aligned} -\Delta v + v = 0 \quad &\text{in } \Omega, \\ \frac{\partial v}{\partial n} = b \quad &\text{on } \partial\Omega, \end{aligned}\right\} \tag{2.2}$$

is given by $v = A^{-1}(b) \in W^{1,p}(\Omega)$ and $\|v\|_{W^{1,p}(\Omega)} \leqslant C\|b\|_{L^q(\partial\Omega)}$. We will set $T_0(b) = v$ and $S_0(b) = \gamma T_0(b)$, where γ is the trace operator. The operator S_0 is known as the Neumann-to-Dirichlet operator. Hence, the operator T_0 takes functions defined on $\partial\Omega$ to functions defined in Ω and S_0 takes functions defined on $\partial\Omega$ to functions defined on $\partial\Omega$.

Our first task will be to show that any weak solution $u \in H^1(\Omega)$ of (1.2) lies in $C^\alpha(\bar{\Omega})$. To accomplish this, we will need several regularity results of the associated linear problems. As a matter of fact, as a consequence of the above and using embedding and trace theorems we can easily show the following regularity results.

LEMMA 2.1. *If $N \geqslant 2$ and $b \in L^q(\partial\Omega)$ with $q \geqslant 1$, then the solution $v = T_0 b$ of (2.2) satisfies $v \in W^{1,p}(\Omega)$ for $1 \leqslant p \leqslant qN/(N-1)$ with $\|v\|_{W^{1,p}(\Omega)} \leqslant C\|b\|_{L^q(\partial\Omega)}$.*

In particular, we have the following conditions.

(i) *If $1 \leqslant q < N-1$, then $\gamma v \in L^r(\partial\Omega)$ for all $1 \leqslant r \leqslant q(N-1)/(N-1-q)$ and the map $S_0 : L^q(\partial\Omega) \to L^r(\partial\Omega)$ is continuous for $1 \leqslant r \leqslant q(N-1)/(N-1-q)$ and compact for $1 \leqslant r < q(N-1)/(N-1-q)$.*

(ii) *If $q = N-1$, then $\gamma v \in L^r(\partial\Omega)$ for all $r \geqslant 1$ and the map $S_0 : L^q(\partial\Omega) \to L^r(\partial\Omega)$ is continuous and compact for $1 \leqslant r < \infty$.*

(iii) *If $q > N-1$, then $v \in C^\alpha(\bar{\Omega})$ with $\|v\|_{C^\alpha(\bar{\Omega})} \leqslant C\|b\|_{L^q(\partial\Omega)}$ for some $\alpha \in (0,1)$. Moreover, $\gamma v \in C^\alpha(\partial\Omega)$ and the map $S_0 : L^q(\partial\Omega) \to C^\alpha(\partial\Omega)$ is continuous and compact.*

As an immediate corollary, we have the following technical result.

COROLLARY 2.2.

(i) *For any* $q \geqslant 1$, *if* $b \in L^q(\partial\Omega)$, *then* $S_0 b \in L^{q+(1/N)}(\partial\Omega)$.

(ii) *If* b *satisfies* $|b(x)| \leqslant h(x)w(x)$, *where* $h \in L^r(\partial\Omega)$ *with* $r > N-1$, *then if we define*

$$\delta = \frac{N-1}{N-2} - r' > 0,$$

we find that if $w \in L^p(\partial\Omega)$ *with*

$$\frac{1}{N-1} \leqslant \frac{1}{p} + \frac{1}{r} \leqslant 1,$$

then $S_0 b := \gamma v \in L^{p+\delta}(\partial\Omega)$ *and* $\|S_0 b\|_{L^{p+\delta}(\partial\Omega)} \leqslant C\|w\|_{L^p(\partial\Omega)}$.

Proof. (i) Observe that if $q \geqslant N-1$, then, from the corollary above, $\gamma v \in L^r(\partial\Omega)$ for all $r \geqslant 1$. In the case when $1 \leqslant q < N-1$, $S_0 b \in L^r(\partial\Omega)$ for $r \leqslant q(N-1)/(N-1-q)$. A simple computation shows that

$$\frac{(N-1)q}{N-1-q} - q \geqslant \frac{1}{N} \quad \text{for } 1 \leqslant q < N-1.$$

(ii) Note that $hw \in L^{pr/(p+r)}(\partial\Omega)$ and $pr/(p+r) \geqslant 1$ because

$$\frac{1}{p} + \frac{1}{r} \leqslant 1.$$

Hence, by lemma 2.1, $\gamma v \in L^s(\partial\Omega)$ with

$$s = \frac{pr}{p+r}(N-1)\frac{1}{N-1-pr/(p+r)}.$$

If we set

$$y = \frac{pr}{p+r} = \left(\frac{1}{p} + \frac{1}{r}\right)^{-1},$$

then $1 \leqslant y \leqslant N-1$, $p = ry/(r-y)$ and

$$\min_{1/(N-1)\leqslant(1/p)+(1/r)\leqslant 1} \left\{\frac{pr}{p+r}(N-1)\frac{1}{N-1-pr/(p+r)} - p\right\} = \min_{1\leqslant y\leqslant N-1} \left\{\frac{y(N-1)}{N-1-y} - \frac{ry}{r-y}\right\}.$$

However, a simple computation shows that this last minimum is attained at $y = 1$. This concludes the proof of the corollary. □

These regularity results with a bootstrap argument will allow us to prove the following proposition.

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PROPOSITION 2.3. *Assume g satisfies (H1) and (H2). Then, for any $R > 0$, if $u \in H^1(\Omega)$ is a solution of (1.2) for some $|\lambda| \leqslant R$, we have*

$$\|u\|_{C^\alpha(\bar{\Omega})} \leqslant C(1 + \|u\|_{L^p(\partial\Omega)}) \tag{2.3}$$

for some positive α, where $C = C(R)$ and $p = 2(N-1)/(N-2)$.

Proof. Assume that $N \geqslant 3$ (the proof when $N = 2$ is simpler). Observe that the boundary condition satisfied by u is

$$\frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u)$$

and, by hypotheses (H1), (H2) and assuming that $|\lambda| \leqslant R$, we have $|g(\lambda, x, u)| \leqslant Ch(x)(1 + |u(x)|)$ for some constant $C = C(R)$. Hence,

$$\frac{\partial u}{\partial n} = b(x) \quad \text{with } |b(x)| \leqslant C(1 + h(x))(1 + |u(x)|).$$

Note also that $1 + h \in L^r(\partial\Omega)$ for some $r > N - 1$.

Now, if $u \in H^1(\Omega)$, then $\gamma u \in L^p(\partial\Omega)$ with $p = 2(N-1)/(N-2)$, which satisfies

$$\frac{1}{p} + \frac{1}{r} \leqslant 1 \quad \text{for any } r > N - 1.$$

Hence, $b \in L^s(\partial\Omega)$ with

$$\frac{1}{s} = \frac{1}{r} + \frac{1}{p}.$$

Thus, if $s > N - 1$, then lemma 2.1(iii) implies that $u \in C^\alpha(\bar{\Omega})$ and

$$\|u\|_{C^\alpha(\bar{\Omega})} \leqslant C\|b\|_{L^s(\partial\Omega)} \leqslant C(1 + \|u\|_{L^p(\partial\Omega)}).$$

On the other hand, if $s \leqslant N - 1$, applying the regularity result of corollary 2.2(ii), we find that $\gamma u \in L^{p+\delta}(\partial\Omega)$ and

$$\|u\|_{L^{p+\delta}(\partial\Omega)} \leqslant C(1 + \|u\|_{L^p(\partial\Omega)}). \tag{2.4}$$

Repeating this regularity argument k times, we get $\gamma u \in L^{p+k\delta}(\partial\Omega)$. Moreover, we also have

$$\|u\|_{L^{p+k\delta}(\partial\Omega)} \leqslant C(1 + \|u\|_{L^{p+(k-1)\delta}(\partial\Omega)}) \leqslant \cdots \leqslant C(1 + \|u\|_{L^p(\partial\Omega)}).$$

Certainly, since $r > N - 1$, after a finite number of iterations there exists k such that

$$\frac{1}{p + (k-1)\delta} + \frac{1}{r} \geqslant \frac{1}{N-1} \quad \text{and} \quad \frac{1}{p + k\delta} + \frac{1}{r} < \frac{1}{N-1}.$$

In particular, $b \in L^s(\partial\Omega)$ with

$$\frac{1}{s} = \frac{1}{r} + \frac{1}{p + k\delta}, \quad s > N - 1.$$

Hence, again applying lemma 2.1(iii), we finish the proof. □

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REMARK 2.4. The regularity result of the above proposition tells us that looking for solutions of problem (1.2) in $H^1(\Omega)$ is equivalent to looking for solutions in a more regular space like $C^\alpha(\bar{\Omega})$.

We now analyse the operator S_0 (the Neumann-to-Dirichlet operator). We have the following result.

LEMMA 2.5. *The operator $S_0 : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is a linear self-adjoint, positive and compact operator. If we denote its eigenvalues by $\{\tau_i\}_{i=1}^\infty$ and by $\sigma_i = 1/\tau_i$ we find that, for any $\lambda \in \mathbb{R}$, $\lambda \notin \{\sigma_i\}_{i=1}^\infty$, then the operator $S_\lambda : L^2(\partial\Omega) \to L^2(\partial\Omega)$ defined by $S_\lambda(g) = \gamma v$, where v is the unique solution of*

$$\left.\begin{aligned} -\Delta v + v &= 0 \quad \text{in } \Omega, \\ \frac{\partial v}{\partial n} - \lambda v &= g \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{2.5}$$

is self-adjoint, continuous and compact. Moreover, the first eigenvalue σ_1 is simple and its eigenfunction Φ_1 can be chosen to be strictly positive. Also, if $r > N - 1$ then, $S_\lambda : L^r(\partial\Omega) \to C^0(\partial\Omega)$ is continuous and compact and, for any compact set $K \subset \mathbb{R} \setminus \{\sigma_i\}_{i=1}^\infty$, the norm of $S_\lambda : L^r(\partial\Omega) \to C^0(\partial\Omega)$ is uniformly bounded for $\lambda \in K$. Also, $\|S_\lambda\| \to \infty$ as $\lambda \to \sigma_i$ for some i.

Proof. Observe that if $b_1, b_2 \in L^2(\partial\Omega)$ and if v_1, v_2 are the solutions of $-\Delta v_i + v_i = 0$ in Ω, $\partial v_i/\partial n = b_i$, $i = 1, 2$, then by the weak formulation of this problem we have

$$(S_0(b_1), b_2)_{L^2(\partial\Omega)} = \int_\Omega \nabla v_1 \nabla v_2 + \int_\Omega v_1 v_2 = (b_1, S_0(b_2))_{L^2(\partial\Omega)}. \tag{2.6}$$

From (2.6) it follows that S_0 is self-adjoint and positive. The fact that S_0 is compact follows from lemma 2.1, and the fact that the first eigenfunction can be chosen to be non-negative follows easily from the Rayleigh quotient for the first eigenvalue. Then, maximum principles imply that the first eigenfunction is actually strictly positive. In turn, this implies that the first eigenvalue is simple.

The rest of the proof follows merely by realizing that $S_\lambda = (I - \lambda S_0)^{-1} \circ S_0$ and applying the regularity results of corollary 2.2. □

It is now clear that we can set a fixed-point problem to obtain the solutions of (1.2). As a matter of fact, $u \in H^1(\Omega)$ is a solution of (1.2) if and only if its trace $v = \gamma u$ is a fixed point of

$$v = S_\lambda(g(\lambda, \cdot, v)) \quad (= (I - \lambda S_0)^{-1} \circ S_0(g(\lambda, \cdot, v))). \tag{2.7}$$

Note also that, once v is obtained, we recover u by solving $-\Delta u + u = 0$ in Ω with $u = v$ on the boundary.

Concerning the fixed-point problem (2.7), we have the following lemma.

LEMMA 2.6. *Under hypotheses (H1) and (H2), the map $C^0(\partial\Omega) \ni v \to g(\lambda, \cdot, v) \in L^r(\partial\Omega)$ is well defined and continuous. Moreover, for each $M > 0$, $\epsilon > 0$, there exists a constant $C = C(\epsilon, M)$ such that*

$$\|g(\lambda, \cdot, v)\|_{L^r(\partial\Omega)} \leqslant \epsilon \|v\|_{C^0(\partial\Omega)} + C \tag{2.8}$$

for all $v \in C^0(\partial\Omega)$, $|\lambda| \leqslant M$.

In particular, the map $C^0(\partial\Omega) \ni v \to S_\lambda(g(\lambda, \cdot, v)) \in C^0(\partial\Omega)$ is continuous and compact for all $\lambda \in \mathbb{R} \setminus \{\sigma_i\}_{i=1}^{\infty}$.

Proof. It follows from the bounds of g given by (H1) that this map is well defined. The continuity follows from the continuity of g with respect to the last variable, the bounds of g given by (H1) and the dominated convergence theorem. Statement (2.8) follows from the fact that, for each $\epsilon > 0$, we have the inequality $|U(s)| \leqslant \epsilon s + C$, for some constant $C = C(\epsilon)$, and the fact that the function $\Lambda(\lambda)$ is continuous.

The last part of the lemma follows easily. □

We are now in a position where we can show the existence of solutions of our original problem (1.2) for all $\lambda \in \mathbb{R} \setminus \{\sigma_i\}_{i=1}^{\infty}$. We have the following theorem.

THEOREM 2.7. *If g satisfies (H1) and (H2), then, for all $\lambda \in \mathbb{R} \setminus \{\sigma_i\}_{i=1}^{\infty}$ there exists at least one solution of problem (1.2). Moreover, for each compact set $K \subset \mathbb{R} \setminus \{\sigma_i\}_{i=1}^{\infty}$, we have the existence of a constant $C = C(K)$ such that any solution of problem (1.2) is bounded in $C^0(\Omega)$ by C.*

Proof. Consider the compact set $K \subset \mathbb{R} \setminus \{\sigma_i\}_{i=1}^{\infty}$ and observe that by lemma 2.5 we have that there exists a constant $C_1 = C_1(K)$ such that the norm of $S_\lambda : L^r(\partial\Omega) \to C^0(\Omega)$ is bounded by C_1 for all $\lambda \in K$.

We will apply Schaefer fixed-point argument to (2.7) (see [11]). For this we consider $\theta \in [0, 1]$ and let v be a fixed point of

$$v = \theta S_\lambda(g(\lambda, \cdot, v)) \tag{2.9}$$

for some $\lambda \in K$. Then $\|v\|_{C^0(\partial\Omega)} \leqslant C_1 \|g(\lambda, \cdot, v\|_{L^r(\partial\Omega)})$. But, by (2.8), we get

$$\|v\|_{C^0(\partial\Omega)} \leqslant C_1(\epsilon \|v\|_{C^0(\partial\Omega)} + C(\epsilon, K)).$$

Choosing ϵ to be small enough that $1 - C_1\epsilon \geqslant \frac{1}{2}$, we get $\|v\|_{C^0(\partial\Omega)} \leqslant 2C_1 C(\epsilon, K)$. Noticing that by lemma 2.6 we have that $v \to S_\lambda(g(\lambda, \cdot, v))$ is compact in $C^0(\partial\Omega)$ when $\lambda \notin \{\sigma_i\}_{i=1}^{\infty}$ and applying the Schaefer fixed-point argument, we prove the proposition. □

3. Unbounded branches of equilibria

From the results of the previous section it is clear that, when the value of the parameter λ is bounded away from the Steklov eigenvalues, the solutions of (1.2) are bounded uniformly in λ. On the other hand, since the norm of the operator S_λ blows up to infinity when λ approaches a Steklov eigenvalue (see lemma 2.5), it is natural to expect the existence of branches of solutions that diverge to infinity in certain norms when the parameter approaches a Steklov eigenvalue. For instance, if we consider the case in which $g \equiv 0$, then, for any $\lambda \notin \{\sigma_i\}_{i=1}^{\infty}$, the unique solution is $u \equiv 0$, while for $\lambda = \sigma_i$ we find that the whole finite-dimensional subspace given by the eigenfunctions associated with σ_i is a solution. This subspace constitutes an unbounded branch of solutions.

Let us start by analysing the behaviour of the solutions when we know explicitly that the solution blows up in a certain norm.

PROPOSITION 3.1. *Assume that* $\{\lambda_n\}_{n=1}^{\infty}$ *is a convergent sequence of real numbers for which there exist solutions* u_n *of (1.2) with* $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$ *as* $n \to \infty$. *Then necessarily* $\lambda_n \to \sigma_i$ *for certain* $i \in \mathbb{N}$ *and, for any subsequence of* u_n, *there exists another subsequence, which we denote by* $u_{n'}$, *and an eigenfunction* Φ_i *associated with* σ_i *with* $\|\Phi_i\|_{L^\infty(\partial\Omega)} = 1$ *such that*

$$\frac{u_{n'}}{\|u_{n'}\|_{L^\infty(\partial\Omega)}} \to \Phi_i \quad \text{in } C^\beta(\bar{\Omega})$$

for some $\beta > 0$.

Proof. Applying the Hölder estimate given by (2.3), we find that if

$$v_n = \frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}},$$

we obtain $\|v_n\|_{C^\alpha(\bar{\Omega})} \leqslant C$, for some C independent of n. Using the compact embedding $C^\alpha(\bar{\Omega}) \hookrightarrow C^\beta(\bar{\Omega})$ for $0 < \beta < \alpha$, we find that, for any subsequence of v_n, there exists another subsubsequence, $v_{n'}$, and a function $\Phi \in C^\beta(\bar{\Omega})$ such that $v_{n'} \to \Phi$ in $C^\beta(\bar{\Omega})$. Therefore, since $\|v_{n'}\|_{L^\infty(\partial\Omega)} = 1$ we find that $\|\Phi\|_{L^\infty(\partial\Omega)} = 1$ and, in particular, that Φ is not identically zero.

The equation satisfied by $v_{n'}$ is

$$\left.\begin{aligned} -\Delta v_{n'} + v_{n'} &= 0 && \text{in } \Omega, \\ \frac{\partial v_{n'}}{\partial n} &= \lambda_{n'} v_{n'} + \frac{g(\lambda, x, u_{n'})}{\|u_{n'}\|_{L^\infty(\partial\Omega)}} && \text{on } \partial\Omega. \end{aligned}\right\}$$

Passing to the limit in the weak formulation of this equation, taking into account the facts that

$$\frac{g(\lambda, x, u_{n'})}{\|u_{n'}\|_{L^\infty(\partial\Omega)}} \to 0 \text{ in } L^r(\partial\Omega) \quad \text{as } n' \to \infty$$

and $v_{n'} \to \Phi$, we find that Φ is a solution of

$$\left.\begin{aligned} -\Delta\Phi + \Phi &= 0 && \text{in } \Omega, \\ \frac{\partial\Phi}{\partial n} &= \sigma\Phi && \text{on } \partial\Omega, \end{aligned}\right\}$$

where $\sigma = \lim_{n' \to \infty} \lambda_{n'}$. Since $\|\Phi\|_{L^\infty(\partial\Omega)} = 1$, necessarily σ is a Steklov eigenvalue and Φ is a Steklov eigenfunction associated with σ. This proves the proposition. $\square$

We immediately have the following corollary.

COROLLARY 3.2. *With the same hypotheses as in proposition 3.1,*

(i) *the whole sequence satisfies* $\|u_n\|_{L^p(\partial\Omega)} \to \infty$ *for any* $1 \leqslant p \leqslant \infty$,

(ii) *if* $u_n \geqslant 0$ *for all* n, *then necessarily* $\lambda_n \to \sigma_1$ *and the whole sequence satisfies*

$$\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \to \Phi_1 \quad \text{in } C^\beta(\bar{\Omega}).$$

Proof. (i) Since $L^p(\partial\Omega) \hookrightarrow L^1(\partial\Omega)$, it will be sufficient to show the result for $p = 1$. If this is not the case, then there will exist a subsequence u_n bounded in $L^1(\partial\Omega)$. We can obtain from proposition 3.1 another subsequence $u_{n'}$ satisfying $u_{n'}/\|u_{n'}\|_{L^\infty(\partial\Omega)} \to \Phi_i$ and, in particular,

$$\frac{\|u_{n'}\|_{L^1(\partial\Omega)}}{\|u_{n'}\|_{L^\infty(\partial\Omega)}} \to \|\Phi_i\|_{L^1(\partial\Omega)} > 0.$$

This implies that $\|u_{n'}\|_{L^1(\partial\Omega)} \to \infty$, which is a contradiction.

(ii) From proposition 3.1, any possible convergent subsequence of $u_n/\|u_n\|_{L^\infty(\partial\Omega)}$ has to converge to a Steklov eigenfunction Φ_i with $\|\Phi_i\|_{L^\infty(\partial\Omega)} = 1$. Since in this case $u_n \geqslant 0$, we find that $\Phi_i \geqslant 0$. But σ_1 is the unique Steklov eigenvalue with a non-negative eigenfunction Φ_1 (see lemma 2.5). □

We will now show that any Steklov eigenvalue σ of odd multiplicity is a bifurcation point from infinity, that is, there exists a sequence λ_n with $\lambda_n \to \sigma$ and a sequence of solutions u_n of (1.2) for the value λ_n such that $\|u_n\|_{L^\infty(\Omega)} \to \infty$.

Before stating the result, consider the following notation. We will consider the solutions of (1.2) in $\mathbb{R} \times C(\bar{\Omega})$, where the first coordinate is the value of λ and the second is the function u, which is a solution of (1.2) for this value of λ. In this sense, we will denote the set of solutions by $\mathcal{S}$. Recall also that we have denoted the Steklov eigenvalues (eigenvalues of problem (1.3)) by $\{\sigma_i\}_{i=1}^{\infty}$.

We have the following result.

THEOREM 3.3. *Consider problem (1.2) and assume that the nonlinearity g satisfies conditions (H1) and (H2). If σ is a Steklov eigenvalue of odd multiplicity, then the set of solutions of (1.2), denoted by $\mathcal{S}$, possesses an unbounded component $\mathcal{D}$ which meets $(\sigma, \infty) \in \mathbb{R} \times C(\Omega)$.*

Moreover, if $[\lambda_-, \lambda_+] \subset \mathbb{R}$ is an interval such that $[\lambda_-, \lambda_+] \cap \{\sigma_i\}_{i=1}^{\infty} = \{\sigma\}$ and $\mathcal{M} = [\lambda_-, \lambda_+] \times \{u \in C(\bar{\Omega}) : \|u\|_{C(\bar{\Omega})} \geqslant 1\}$, then either

(i) *$\mathcal{D} \setminus \mathcal{M}$ is bounded in $\mathbb{R} \times C(\bar{\Omega})$ in which case $\mathcal{D} \setminus \mathcal{M}$ meets the set $\{(\lambda, 0), : \lambda \in \mathbb{R}\}$ at $(\lambda_0, 0)$ such that $g(\lambda_0, \cdot, 0) = 0$, or*

(ii) *$\mathcal{D} \setminus \mathcal{M}$ is unbounded in $\mathbb{R} \times C(\bar{\Omega})$.*

If $\mathcal{D} \setminus \mathcal{M}$ is unbounded, and it has a bounded projection on $\mathbb{R}$, then $\mathcal{D} \setminus \mathcal{M}$ meets $(\tilde{\sigma}, \infty) \in \mathbb{R} \times C(\bar{\Omega})$, with $\sigma \neq \tilde{\sigma} \in \{\sigma_i\}_{i=1}^{\infty}$, i.e. $\mathcal{D} \setminus \mathcal{M}$ meets another bifurcation point from infinity.

Proof. Observe first that the fixed-point problem (2.7) can be recast as

$$v = \lambda S_0 v + S_0(g(\lambda, \cdot, v)), \tag{3.1}$$

where S_0 is the Neumann-to-Dirichlet operator (see lemma 2.5).

We apply now the general techniques from [17] to the fixed-point problem (3.1) in the space $C(\partial\Omega)$. Thus, we have to prove that

(a) $S_0(g(\lambda, \cdot, v)) = o(\|v\|)$ at $v = \infty$ uniformly for λ in bounded intervals, and

(b) the map $(\lambda, v) \to \|v\|^2 S_0(g(\lambda, \cdot, v/\|v\|^2))$ is compact for λ in bounded intervals,

where for simplicity we denote by $\|v\| := \|v\|_{C(\partial\Omega)}$.

(a) For any $v \in C(\partial\Omega)$ we see from (H1) that $g(\lambda, \cdot, v) \in L^r(\partial\Omega)$. Therefore,

$$\frac{\|S_0(g(\lambda,\cdot,v))\|}{\|v\|} \leqslant C\frac{\|g(\lambda,\cdot,v)\|_{L^r(\partial\Omega)}}{\|v\|} \leqslant C\left(\varepsilon + \frac{C_\varepsilon}{\|v\|}\right), \tag{3.2}$$

where we have used lemma 2.1 for the first inequality and lemma 2.6 for the second one. From (3.2) we easily obtain (a).

(b) We must verify that $H : \mathbb{R} \times C(\partial\Omega) \to C(\partial\Omega)$ defined by

$$H(\lambda, v) := \|v\|^2 S_0\left(g\left(\lambda, x, \frac{v}{\|v\|^2}\right)\right)$$

is compact. Note first that the image of $\{(\lambda, v) \in [\underline{\lambda}, \bar{\lambda}] \times C(\partial\Omega) : \delta \leqslant \|v\|_{C(\partial\Omega)} \leqslant \rho\}$ under H is relatively compact for any $\underline{\lambda} < \bar{\lambda}$ and $0 < \delta \leqslant \rho < \infty$. This follows from the boundedness of g and the compactness of S_0. Thus, we need only to prove that the image of $[\underline{\lambda}, \bar{\lambda}] \times B_\delta$ under H is relatively compact in $C(\partial\Omega)$ for some $\delta > 0$ small enough, where $B_\delta := \{v \in C(\partial\Omega) : \|v\| \leqslant \delta\}$. Let us choose $v \in B_\delta$, and define $w = v/\|v\|^2$, which satisfies $\|w\| \geqslant 1/\delta$.

From (2.8) with $\varepsilon = 1$, we get

$$\frac{\|g(\lambda,\cdot,w)\|_{L^r(\partial\Omega)}}{\|w\|} \leqslant C, \tag{3.3}$$

with $C = C(\lambda, \|h\|_{L^r(\partial\Omega)}, \delta)$. Therefore,

$$\|v\|^2 \left\| g\left(\lambda, \cdot, \frac{v}{\|v\|^2}\right)\right\|_{L^r(\partial\Omega)} \leqslant C\|v\| \leqslant C\delta. \tag{3.4}$$

Now, the compactness of $S_0 : L^r(\partial\Omega) \to C(\partial\Omega)$ given by lemma 2.1 ends the proof. □

We now analyse the case where the eigenvalue σ is simple, and in particular the case of the first eigenvalue. We have the following theorem.

THEOREM 3.4. *Let σ denote a simple Steklov eigenvalue and Φ a corresponding eigenfunction. Assume g satisfies hypotheses (H1) and (H2). Then the set of solutions of (1.2) possesses two unbounded components $\mathcal{D}^+$ and $\mathcal{D}^-$ which meet $(\sigma, \infty) \in \mathbb{R} \times C(\bar{\Omega})$, satisfying the following conditions.*

(i) *There exists a neighbourhood $\mathcal{O}_1$ of (σ, ∞) such that $(\lambda, v) \in \mathcal{D}^+ \cap \mathcal{O}_1$, and $(\lambda, v) \neq (\sigma, \infty)$ implies that*

$$v = \alpha\Phi + w \quad \textit{where } \alpha > 0 \textit{ with } \|w\|_{L^\infty(\partial\Omega)} = o(|\alpha|) \textit{ at } |\alpha| = \infty.$$

(ii) *There exists a neighbourhood $\mathcal{O}_2$ of (σ, ∞) such that $(\lambda, v) \in \mathcal{D}^- \cap \mathcal{O}_2$, and $(\lambda, v) \neq (\sigma, \infty)$ implies that*

$$v = -\alpha\Phi + w \quad \textit{where } \alpha > 0 \textit{ with } \|w\|_{L^\infty(\partial\Omega)} = o(|\alpha|) \textit{ at } |\alpha| = \infty.$$

Proof. See [17, corollary 1.8] for the proof. □

Note, in particular, that if $\sigma = \sigma_1$, since the first eigenfunction can be chosen positive, this result implies the existence of branches of positive and negative solutions bifurcating from infinity.

4. Sufficient conditions for subcritical and supercritical bifurcations from infinity

In this section we give conditions on the nonlinearity g that allows us to characterize the different bifurcations that occur. Obviously, the type of bifurcation (sub- or supercritical) occurring at a bifurcation point will be dictated by the behaviour of the nonlinearity g for large values of s. For instance, assume that we have a sequence of solutions u_n for the value of the parameter λ_n and assume that $\lambda_n \to \sigma_1$, the first Steklov eigenvalue. From proposition 3.1 we find that the functions

$$v_n = \frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}},$$

possibly after taking a subsequence, converge in $L^\infty(\partial\Omega)$ to Φ_1 or $-\Phi_1$, where Φ_1 is the unique positive eigenfunction of σ_1 with $L^\infty(\partial\Omega)$-norm 1.

As an example, let us consider the case where $v_n \to \Phi_1$ and assume, for instance, that the function $g(\lambda, x, s)$ behaves for $s \to +\infty$ and $\lambda \to \sigma_1$ as

$$g(\lambda, x, s) \approx G(x)s^\alpha.$$

Then, considering equation (1.2) with $\lambda = \lambda_n$, multiplying it by Φ_1, integrating by parts and using the fact that Φ_1 is an eigenfunction, we get

$$(\sigma_1 - \lambda_n) \int_{\partial\Omega} u_n \Phi_1 = \int_{\partial\Omega} g(\lambda_n, x, u_n)\Phi_1.$$

Hence, since $u_n \to +\infty$ uniformly in $\partial\Omega$ and using the asymptotic expression of g, we can easily see that the sign of $\sigma_1 - \lambda_n$ is dictated, for n large enough, by the sign of

$$\int_{\partial\Omega} G(x)\Phi_1^{1+\alpha}.$$

In particular, if this latter integral is positive, the bifurcation is subcritical and if it is negative, the bifurcation is supercritical.

With this in mind, we define the following functions, which describe the behaviour of g for large values of s, at a given σ. Define, for some α, the following functions:

$$\left.\begin{aligned} \underline{G_+}(x) &:= \liminf_{(\lambda,s)\to(\sigma,+\infty)} \frac{g(\lambda,x,s)}{s^\alpha}, & \overline{G_+}(x) &:= \limsup_{(\lambda,s)\to(\sigma,+\infty)} \frac{g(\lambda,x,s)}{s^\alpha}, \\ \underline{G_-}(x) &:= \liminf_{(\lambda,s)\to(\sigma,-\infty)} \frac{g(\lambda,x,s)}{|s|^\alpha}, & \overline{G_-}(x) &:= \limsup_{(\lambda,s)\to(\sigma,-\infty)} \frac{g(\lambda,x,s)}{|s|^\alpha}. \end{aligned}\right\} \tag{4.1}$$

REMARK 4.1. (i) Observe that in fact G depends on σ and α. If we need to stress this dependence, we will write $\underline{G_+^{\alpha,\sigma}}(x)$, $\overline{G_+^{\alpha,\sigma}}(x)$, $\underline{G_-^{\alpha,\sigma}}(x)$ and $\overline{G_-^{\alpha,\sigma}}(x)$.

(ii) Observe that if g satisfies (H2) and $\alpha \geqslant 1$, then all the functions defined above are identically zero.

(iii) The way in which the functions defined in (4.1) describe the behaviour of the function g for large values of s can be expressed in the following way: for any $\varepsilon > 0$

small enough, we have

$$(\underline{G_+}(x) - \varepsilon)s^\alpha \leqslant g(\lambda, x, s) \leqslant (\overline{G_+}(x) + \varepsilon)s^\alpha, \quad s \to +\infty, \ \lambda \approx \sigma,$$

and similarly as $s \to -\infty$.

In order to establish conditions for sub- or supercritical bifurcations at the first eigenvalue, we prove first the following important result.

LEMMA 4.2. *Assume that the nonlinearity g satisfies hypotheses (H1) and (H2). Denote by σ_1 the first Steklov eigenvalue and by Φ_1 the first positive eigenfunction with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$. Consider a sequence of solutions u_n for the value of the parameter λ_n such that $\lambda_n \to \sigma_1$ and $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$. Then*

(i) *if $u_n > 0$, we have*

$$\frac{\int_{\partial\Omega} \underline{G_+}\Phi_1^{1+\alpha}}{\int_{\partial\Omega}\Phi_1^2} \leqslant \liminf_{n\to\infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leqslant \limsup_{n\to\infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leqslant \frac{\int_{\partial\Omega} \overline{G_+}\Phi_1^{1+\alpha}}{\int_{\partial\Omega}\Phi_1^2}; \quad (4.2)$$

(ii) *if $u_n < 0$, we have*

$$\frac{-\int_{\partial\Omega} \overline{G_-}\Phi_1^{1+\alpha}}{\int_{\partial\Omega}\Phi_1^2} \leqslant \liminf_{n\to\infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leqslant \limsup_{n\to\infty} \frac{\sigma_1 - \lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leqslant \frac{-\int_{\partial\Omega} \underline{G_-}\Phi_1^{1+\alpha}}{\int_{\partial\Omega}\Phi_1^2}. \quad (4.3)$$

Proof. Let us show (i) (proof of the other case follows similarly). Consider a family of solutions u_n of (1.2) for $\lambda = \lambda_n$ with $\lambda_n \to \sigma_1$ and $0 < u_n \to \infty$. Multiplying equation (1.2) by Φ_1 and integrating by parts, we get

$$(\sigma_1 - \lambda_n)\int_{\partial\Omega} u_n\Phi_1 = \int_{\partial\Omega} g(\lambda_n, x, u_n)\Phi_1. \quad (4.4)$$

However,

$$\int_{\partial\Omega} g(\lambda_n, x, u_n)\Phi_1 = \|u_n\|^\alpha_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \frac{g(\lambda_n, x, u_n)}{u_n^\alpha}\left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}}\right)^\alpha \Phi_1.$$

But, from Fatou's lemma,

$$\begin{aligned}\liminf_{n\to\infty} \int_{\partial\Omega} &\frac{g(\lambda_n, x, u_n)}{u_n^\alpha}\left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}}\right)^\alpha \Phi_1 \\ &\geqslant \int_{\partial\Omega} \liminf_{n\to\infty}\left[\frac{g(\lambda_n, x, u_n)}{u_n^\alpha}\left(\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}}\right)^\alpha \Phi_1\right] \\ &\geqslant \int_{\partial\Omega} \underline{G_+}(x)\Phi_1^{1+\alpha}, \end{aligned} \quad (4.5)$$

where we have used the definition of $\underline{G_+}(x)$, the facts that $\Phi_1 > 0$ for all x on $\partial\Omega$ and that

$$\frac{u_n}{\|u_n\|_{L^\infty(\partial\Omega)}} \to \Phi_1$$

uniformly in $\partial\Omega$ (see corollary 3.2).

Dividing by $\|u_n\|_{L^\infty(\partial\Omega)}$ in (4.4) and passing to the limit we obtain the first inequality of (4.2). The second inequality is trivial and the third is obtained in a similar manner to the first. □

Now, with respect to bifurcations from the first eigenvalue we can prove,

THEOREM 4.3 (bifurcation from the first eigenvalue). *Assume that the nonlinearity g satisfies hypotheses (H1) and (H2). Denote by σ_1 the first Steklov eigenvalue and by Φ_1 the first positive eigenfunction with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$.*

(i) (Subcritical bifurcations.) *Assume that there exists an $\alpha < 1$ such that $\underline{G_+} = \underline{G_+^{\alpha,\sigma_1}} \in L^1(\partial\Omega)$ (respectively, $\overline{G_-} = \overline{G_-^{\alpha,\sigma_1}} \in L^1(\partial\Omega)$). Then, if*

$$\int_{\partial\Omega} \underline{G_+}\Phi_1^{1+\alpha} > 0 \quad \left(\textit{respectively,} \int_{\partial\Omega} \overline{G_-}\Phi_1^{1+\alpha} < 0\right), \tag{4.6}$$

the bifurcation from infinity of positive (respectively, negative) solutions at $\lambda = \sigma_1$ is subcritical, i.e. $\lambda < \sigma_1$ for every positive (respectively, negative) solution (λ, v) of (1.2) with $(\lambda, \|v\|)$ in a neighbourhood of (σ_1, ∞).

(ii) (Supercritical bifurcations.) *Assume there exists an $\alpha < 1$ such that $\overline{G_+} = \overline{G_+^{\alpha,\sigma_1}} \in L^1(\partial\Omega)$ (respectively $\underline{G_-} = \underline{G_-^{\alpha,\sigma_1}} \in L^1(\partial\Omega)$). Then, if*

$$\int_{\partial\Omega} \overline{G_+}\Phi_1^{1+\alpha} < 0 \quad \left(\textit{respectively,} \int_{\partial\Omega} \underline{G_-}\Phi_1^{1+\alpha} > 0\right), \tag{4.7}$$

the bifurcation from infinity of positive (respectively, negative) solutions at $\lambda = \sigma_1$ is supercritical, i.e. $\lambda > \sigma_1$ for every positive (respectively, negative) solution (λ, v) of (1.2) with $(\lambda, \|v\|)$ in a neighbourhood of (σ_1, ∞).

Proof. The proof of this theorem follows directly from lemma 4.2. Observe that conditions (4.6) and (4.7) impose a definite sign on $\sigma_1 - \lambda_n$ in (4.2) and (4.3). □

As an example of this result we have the following corollary.

COROLLARY 4.4.

(i) *Assume the nonlinearity satisfies $g(\lambda, x, s) \approx a|s|^\alpha$ as $s \to +\infty$ for some $\alpha < 1$. Then, if $a > 0$, all bifurcations of positive solutions are subcritical, while if $a < 0$, all bifurcations of positive solutions are supercritical.*

(ii) *Assume the nonlinearity satisfies $g(\lambda, x, s) \approx a|s|^\alpha$ as $s \to -\infty$ for some $\alpha < 1$. Then, if $a > 0$, all bifurcations of negative solutions are supercritical, while if $a > 0$, all bifurcations of negative solutions are subcritical.*

We consider now the general case, that is, u_n are solutions of (1.2) for a sequence λ_n with $\lambda_n \to \sigma$ and $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$. Then, from proposition 3.1, we find that λ is an eigenvalue and, up to a subsequence, $u_n/\|u_n\|_{L^\infty(\partial\Omega)} \to \Phi$ uniformly for some eigenfunction Φ associated with the eigenvalue σ and with $\|\Phi\|_{L^\infty(\partial\Omega)} = 1$.

We have the following theorem.

THEOREM 4.5 (bifurcation from a general eigenvalue). *Assume that the nonlinearity g satisfies (H1) and (H2). Let σ be a Steklov eigenvalue for which a bifurcation from infinity of (1.2) occurs at $\lambda = \sigma$.*

(i) (Subcritical bifurcation.) *Assume that, for some $-1 \leqslant \alpha < 1$ and for this value of σ, we have $\underline{G_+}(x), \overline{G_-}(x) \in L^1(\partial\Omega)$. Then, if for any eigenfunction Φ associated with the eigenvalue σ, we have*

$$\int_{\partial\Omega} \underline{G_+}(x)|\Phi^+|^{1+\alpha} > \int_{\partial\Omega} \overline{G_-}(x)|\Phi^-|^{1+\alpha}, \tag{4.8}$$

then the bifurcation from infinity of solutions at $\lambda = \sigma$ is subcritical, i.e. $\lambda < \sigma$ for every solution (λ, v) of (1.2) with $(\lambda, \|v\|)$ in a neighbourhood of (σ, ∞).

(ii) (Supercritical bifurcation.) *Assume that, for some $-1 \leqslant \alpha < 1$ and for this value of σ, we have $\overline{G_+}(x), \underline{G_-}(x) \in L^1(\partial\Omega)$. Then, if for any eigenfunction Φ associated with the eigenvalue σ we have*

$$\int_{\partial\Omega} \overline{G_+}(x)|\Phi^+|^{1+\alpha} < \int_{\partial\Omega} \underline{G_-}(x)|\Phi^-|^{1+\alpha}, \tag{4.9}$$

then the bifurcation from infinity of solutions at $\lambda = \sigma$ is supercritical, i.e. $\lambda > \sigma$ for every solution (λ, v) of (1.2) with $(\lambda, \|v\|)$ in a neighbourhood of (σ, ∞)

Proof. We will show the first case. The supercritical case is proved in a similar way.

As in the proof of theorem 4.3, we need to study the sign of

$$\int_{\partial\Omega} g(\lambda_n, x, u_n)\Phi.$$

But, if we set $\partial\Omega^+ = \{x \in \partial\Omega : \Phi(x) > 0\}$ and $\partial\Omega^- = \{x \in \partial\Omega : \Phi(x) < 0\}$, we have

$$\begin{aligned}\int_{\partial\Omega} g(\lambda, x, u)\Phi &= \int_{\partial\Omega^+} g(\lambda, x, u)\Phi^+ - \int_{\partial\Omega^-} g(\lambda, x, u)|\Phi^-| \\ &= \|u\|^\alpha \int_{\partial\Omega^+} \frac{g(\lambda, x, u)}{(1+|u|)^\alpha}\Phi^+\left(\frac{1}{\|u\|} + \frac{|u|}{\|u\|}\right)^\alpha \\ &\quad - \|u\|^\alpha \int_{\partial\Omega^-} \frac{g(\lambda, x, u)}{(1+|u|)^\alpha}|\Phi^-|\left(\frac{1}{\|u\|} + \frac{|u|}{\|u\|}\right)^\alpha.\end{aligned} \tag{4.10}$$

Observe that, for any $\alpha \geqslant -1$,

$$\Phi^+\left(\frac{1}{\|u_n\|} + \frac{|u_n|}{\|u_n\|}\right)^\alpha \to |\Phi^+|^{1-\alpha} \text{ in } C(\partial\Omega^+) \quad \text{as } n \to \infty. \tag{4.11}$$

Now, passing to the limit in (4.10), using (4.11), hypothesis (4.8) and the Fatou lemma we conclude the proof. □

5. The resonant case

We are now concerned with the resonant problem, that is,

$$\left.\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= \sigma u + g(x, u) && \text{on } \partial\Omega, \end{aligned}\right\} \tag{5.1}$$

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where σ is a Steklov eigenvalue of (1.3). We are interested in giving conditions guaranteeing the existence of solutions in this case. As a matter of fact, we will see that if all possible bifurcations of the problem

$$\left.\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= \lambda u + g(x, u) && \text{on } \partial\Omega, \end{aligned}\right\} \tag{5.2}$$

with $\lambda \in \mathbb{R}$, $\lambda \approx \sigma$ are either subcritical or supercritical, then the resonant problem necessarily has at least one solution.

THEOREM 5.1. *Assume that every possible bifurcation from infinity at $\lambda = \sigma$ of problem (5.2) is subcritical, that is, condition (4.8) holds, or every possible bifurcation from infinity at $\lambda = \sigma$ of problem (5.2) is supercritical, that is, condition (4.9) holds. Then the resonant problem (5.1) has at least one solution.*

REMARK 5.2. Conditions (4.8) and (4.9) are known as Landesman–Lazer-type conditions.

Proof. Observe first that, from theorem 2.7, for $\varepsilon > 0$ sufficiently small, we find that problem (5.2) has at least one solution for all $\lambda \in (\sigma - \varepsilon, \sigma + \varepsilon) \setminus \{\sigma\}$. If, for instance, we assume that all possible bifurcations occurring at $\lambda = \sigma$ are subcritical, then necessarily there exists a constant M such that for any $\lambda \in (\sigma, \sigma + \varepsilon)$ all possible solutions of (5.2) satisfy $\|u\|_{L^\infty(\partial\Omega)} \leqslant M$. This allows us to take a sequence of $\lambda_n \to \sigma$ and solutions u_n of (5.2) with $\|u_n\|_{L^\infty(\partial\Omega)} \leqslant M$. Using the compactness given by elliptic regularity results applied to (5.2) and passing to the limit, we obtain a solution of (5.1). □

6. The anti-maximum principle for the Steklov problem

Let us consider the non-homogeneous linear Steklov problem (6.1)

$$\left.\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= \lambda u + g(x) && \text{on } \partial\Omega, \end{aligned}\right\} \tag{6.1}$$

and show an anti-maximum principle for this problem; see [2, 8] for the case where the nonlinear term is in Ω. As usual, we denote by σ_1 the first Steklov eigenvalue and by Φ_1 its positive eigenfunction.

THEOREM 6.1. *For every $g \in L^r(\partial\Omega)$ with $r > N - 1$, there exists $\epsilon = \epsilon(g)$ such that*

(i) *if $\int_{\partial\Omega} g\Phi_1 > 0$, then every solution (λ, u) of (6.1) satisfies the following:*

(a) $u > 0$ *if* $\sigma_1 - \epsilon < \lambda < \sigma_1$,

(b) $u < 0$ *if* $\sigma_1 < \lambda < \sigma_1 + \epsilon$,

(ii) *if $\int_{\partial\Omega} g\Phi_1 = 0$, then every solution (λ, u) of (6.1) with $\lambda \neq \sigma_1$ changes sign on $\partial\Omega$ and consequently in Ω.*

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Proof. Assume that $\int_{\partial\Omega} g\Phi_1 > 0$. The Fredholm alternative states that the linear problem (6.1) does not have solution if $\lambda = \sigma_1$ and has a unique solution if $\lambda \notin \sigma(S)$. Moreover, from theorem 3.3, $\lambda = \sigma_1$ is a bifurcation point from infinity, and, from theorem 4.3, the bifurcation from infinity of positive solutions is subcritical, i.e. there exists an $\epsilon = \epsilon(g)$ such that, for all (λ, u) solving (6.1) with $\lambda \to \sigma_1$, $\|u\| \approx \infty$ and $u > 0$, we have $\sigma_1 - \epsilon < \lambda < \sigma_1$.

Moreover, by theorem 4.3, the bifurcation from infinity of negative solutions is supercritical, i.e. there exists an $\epsilon = \epsilon(g)$ such that, for all (λ, u) solving (6.1) with $\lambda \to \sigma_1$, $\|u\| \approx \infty$ and $u < 0$, we have $\sigma_1 < \lambda < \sigma_1 + \epsilon$.

Assume now that $\int_{\partial\Omega} g\Phi_1 = 0$. Multiplying equation (6.1) with $\lambda \neq \sigma_1$, by Φ_1 and integrating by parts, we obtain that $\int_{\partial\Omega} u\Phi_1 = 0$. Since $\Phi_1 > 0$, u has to change sign in $\partial\Omega$ and the proof is concluded. □

7. Stability analysis

We analyse in this section the stability properties of the branches of solutions of (1.2) found in the previous section. We will regard these solutions as equilibrium points of the following parabolic evolutionary problem with nonlinear boundary condition:

$$\left.\begin{aligned} u_t - \Delta u + u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= \lambda u + g(\lambda, x, u) && \text{on } \partial\Omega, \\ u(0, x) &= u_0(x) && \text{in } \Omega. \end{aligned}\right\} \tag{7.1}$$

and will analyse their stability in relation to this problem.

We will also assume that the nonlinearity g, in addition to satisfying conditions (H1) and (H2), satisfies a locally Lipschitz condition in the variable u. By assuming this, we guarantee that for a given initial condition $u_0 \in C(\bar{\Omega})$ there exists a unique solution $u \in C([0, T], C(\bar{\Omega}))$ of problem (7.1) and that the solutions depend continuously on the initial data (see, for example, [4]).

From condition (H2) we easily find that

$$g(\lambda, x, u)u \leqslant \varepsilon|h(x)|u^2 + D_\varepsilon|h(x)|\,|u|$$

on bounded intervals of λ.

Hence, comparison arguments (see, for example, [5]) show that $|u(t, x)| \leqslant U(t, x)$, where u is the solution of (7.1) and U is the solution of the following linear problem:

$$\left.\begin{aligned} U_t - \Delta U + U &= 0 && \text{in } \Omega, \\ \frac{\partial U}{\partial n} &= (\lambda + \varepsilon|h(x)|)U + D_\varepsilon|h(x)| && \text{on } \partial\Omega, \\ U(0, x) &= |u_0(x)| && \text{in } \Omega. \end{aligned}\right\} \tag{7.2}$$

With this comparison we obtain the following information.

(1) Since problem (7.2) is linear and $h \in L^r(\partial\Omega)$ with $r > N - 1$, we find that the solutions of (7.2) are in $C(\bar{\Omega})$ and they are globally defined in time. This gives us estimates on the solution $u(t, x)$ of (7.1), which in turn imply that the solutions of (7.1) are global in time. Hence, for each $u_0 \in C(\bar{\Omega})$ we have a unique solution $u \in C([0, \infty), C(\bar{\Omega}))$.

(2) If we consider a fixed $\lambda < \sigma_1$, then, for ε sufficiently small, we have the existence of a unique solution $\varphi_\varepsilon \in C(\bar{\Omega})$ of the following elliptic problem:

$$\left.\begin{aligned} -\Delta\varphi + \varphi &= 0 && \text{in } \Omega, \\ \frac{\partial\varphi}{\partial n} &= (\lambda + \varepsilon|h(x)|)\varphi + D_\varepsilon|h(x)| && \text{on } \partial\Omega. \end{aligned}\right\} \tag{7.3}$$

To see this, we apply the Lax–Milgram theorem to the following bilinear form in $H^1(\Omega)$:

$$a_\varepsilon(u,v) = \int_\Omega (\nabla u \nabla v + uv) - \int_{\partial\Omega} (\lambda + \varepsilon|h(x)|)uv.$$

Observe that since $\lambda < \sigma_1$, the bilinear form above with $\varepsilon = 0$ is coercive. Now since, $h \in L^r(\partial\Omega)$ and $r > N-1$, for ε sufficiently small we can show, via Sobolev embeddings and trace theorems, that a_ε is also coercive and we obtain the existence and uniqueness of a weak solution. Using regularity results we get that the solution $\varphi_\varepsilon \in C(\bar{\Omega})$, since $r > N-1$.

(3) Now, the solution U of (7.2) is given by $U(t,x) = z(t,x) + \varphi_\varepsilon(x)$, where $z(t,x)$ is the solution of

$$\left.\begin{aligned} z_t - \Delta z + z &= 0 && \text{in } \Omega, \\ \frac{\partial z}{\partial n} &= (\lambda + \varepsilon|h(x)|)z && \text{on } \partial\Omega, \\ z(0,x) &= |u_0(x)| - \varphi_\varepsilon && \text{in } \Omega. \end{aligned}\right\} \tag{7.4}$$

But the coercitivity of the bilinear form a_ε and the smoothing properties of the solutions of (7.4) imply that

$$\|z(t,\cdot)\|_{C(\bar{\Omega})} \leqslant M_\varepsilon e^{-\gamma_\varepsilon t}\| |u_0| - \varphi\|_{C(\bar{\Omega})}$$

for some $M_\varepsilon, \gamma_\varepsilon > 0$. Hence, the solution u of (7.1) satisfies

$$\|u(t,\cdot)\|_{C(\bar{\Omega})} \leqslant \|U(t,\cdot)\|_{C(\bar{\Omega})} \leqslant M_\varepsilon e^{-\gamma_\varepsilon t}\||u_0| - \varphi\|_{C(\bar{\Omega})} + \|\varphi_\varepsilon\|_{C(\bar{\Omega})} \tag{7.5}$$

and also

$$\limsup_{t\to+\infty} |u(t,x)| \leqslant \varphi_\varepsilon(x) \quad \text{a.e. } x \in \Omega. \tag{7.6}$$

Estimate (7.5) implies that for $\lambda < \sigma_1$ the evolution of any initial condition for (7.1) is contained in a bounded set. Hence, this problem has an attractor (see [12]). Moreover, all the globally defined and bounded solutions are contained in the attractor. In particular, all the equilibria, connections between equilibria, etc., are contained in the attractor. Estimate (7.6) tells us that any point in the attractor is bounded pointwise by φ_ε. In particular, all equilibria are bounded by φ_ε.

With respect to the stability of the equilibria bifurcating from infinity at the first eigenvalue σ_1, when we have a subcritical bifurcation, we have the following.

PROPOSITION 7.1. *Assume the conditions of theorem 4.3 hold. We then have the following.*

(i) *If the bifurcation of positive solutions (respectively, negative solutions) at the first eigenvalue* $\lambda = \sigma_1$ *is subcritical, then there exists a* $\delta > 0$ *sufficiently small that, for* $\sigma_1 - \delta < \lambda < \sigma_1$*, the largest positive (respectively, smallest negative) solution bifurcating from infinity is globally asymptotically stable from above (respectively, from below). That is, if* $u_\lambda > 0$ *(respectively,* $u_\lambda < 0$*) is this solution then, for every initial condition* $w_0 > u_\lambda$ *(respectively,* $w_0 < u_\lambda$*), the solution* $u(t, x, w_0)$ *of (7.1) with this initial condition satisfies* $\lim_{t\to\infty} u(t, x, w_0) = u_\lambda$ *uniformly in* $x \in \bar{\Omega}$*, for* $\sigma_1 - \delta < \lambda < \sigma_1$.

(ii) *If in (4.1) we have* $\underline{G_+} \geqslant \varepsilon$ *(respectively,* $\overline{G_-} \leqslant -\varepsilon$*) for some* $\varepsilon > 0$*, then the bifurcation of positive (respectively, negative) solutions at* $\lambda = \sigma_1$ *is subcritical. Moreover, there exists a* $\beta_0 > 0$ *large enough such that if* $\tilde{u}_\lambda$ *is the smallest positive (respectively, largest negative) solution satisfying* $\tilde{u}_\lambda \geqslant \beta_0$ *(respectively,* $\tilde{u}_\lambda \leqslant -\beta_0$*), then there exists a* $\delta > 0$ *such that the equilibrium* $\tilde{u}_\lambda$ *is asymptotically stable from below (respectively, above) for* $\sigma_1 - \delta < \lambda < \sigma_1$.

In particular, if for some λ *in this range we have a unique positive (respectively, negative) equilibrium, that is,* $\tilde{u}_\lambda = u_\lambda$*, then this equilibrium is asymptotically stable.*

Proof. In order to prove this result we analyse the solution of (7.1) with initial condition $u_0 = \beta\Phi_1$, for $\beta \in \mathbb{R}$, where Φ_1 is the positive eigenfunction with $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$ associated with the first Steklov eigenvalue. Hence, if we denote this solution by $u(t)$, multiplying the equation (7.1) by a positive test function $\chi \in C^\infty(\mathbb{R}^N)$ and integrating by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\chi = -\int_\Omega (\nabla u(t)\nabla\chi + u(t)\chi) + \int_{\partial\Omega} \lambda u(t)\chi + g(\lambda, \cdot, u(t))\chi.$$

Evaluating this expression at $t = 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\chi|_{t=0} = -\beta\int_\Omega (\nabla\Phi_1\nabla\chi + \Phi_1\chi) + \int_{\partial\Omega} \lambda\beta\Phi_1\chi + g(\lambda, \cdot, \beta\Phi_1)\chi$$

and taking into account the fact that Φ_1 is the first Steklov eigenfunction, we get

$$\int_\Omega (\nabla\Phi_1\nabla\chi + \Phi_1\chi) = \sigma_1\int_{\partial\Omega} \Phi_1\chi,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\chi\bigg|_{t=0} = \int_{\partial\Omega}\left(\lambda - \sigma_1 + \frac{g(\lambda, \cdot, \beta\Phi_1)}{\beta\Phi_1}\right)\beta\phi_1\chi. \tag{7.7}$$

This is the basic equality with which we prove the result.

(i) Consider the case where we have a family of positive solutions, bifurcating from infinity and the bifurcation is subcritical. For fixed λ, denote by u_λ the largest positive solution.

We know from proposition 3.1 and corollary 3.2 that

$$\frac{u_\lambda}{\|u_\lambda\|_{L^\infty(\bar{\Omega})}} \to \Phi_1.$$

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For a fixed λ with $-\delta < \lambda - \sigma_1 < 0$, let β_λ be sufficiently large that $\beta_\lambda \Phi_1 > u_\lambda$ and

$$\left|\frac{g(\lambda, x, \beta_\lambda \Phi_1(x))}{\beta_\lambda \Phi_1(x)}\right| \leqslant \tfrac{1}{2}|\lambda - \sigma_1|.$$

This can be accomplished by condition (H2) and using $\inf_{x\in\partial\Omega} \Phi_1(x) > 0$. Hence, for $\beta \geqslant \beta_\lambda$ and $\chi > 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\chi\bigg|_{t=0} \leqslant \frac{1}{2}\int_{\partial\Omega}(\lambda - \sigma_1)\beta\phi_1\chi < 0. \tag{7.8}$$

Since $\chi > 0$ is arbitrary, this implies that the solution starting at $\beta\Phi_1$ for $\beta \geqslant \beta_\lambda$ is initially decreasing, that is, there exists a small t_0 such that $u(t, x, \beta\Phi_1) \leqslant \beta\Phi_1$ for $0 < t < t_0$. Since the flow generated by (7.1) is monotone, then we easily find that $u(t, x, \beta\Phi_1) \leqslant u(s, x, \beta\Phi_1) \leqslant \beta\Phi_1$ for all $0 < s \leqslant t$. Moreover, since we have chosen $\beta_\lambda \Phi_1 > u_\lambda$ and u_λ is an equilibrium, we get $u_\lambda \leqslant u(t, x, \beta\Phi_1) \leqslant \beta\Phi_1$ for all $t > 0$. Now, since the solution $u(t, x, \beta\Phi_1)$ is monotone decreasing in time and bounded below, and u_λ is the largest positive equilibrium solution, then, for each $\beta > \beta_\lambda$ necessarily $u(t, x, \beta\Phi_1) \to u_\lambda$ as $t \to \infty$ uniformly in $x \in \bar{\Omega}$.

Hence, for any initial condition $w_0 \in C(\bar{\Omega})$ with $w_0 > u_\lambda$, if we consider $\beta > \beta_\lambda$ such that $u_\lambda \leqslant w_0 \leqslant \beta\Phi_1$, by monotonicity of the flow we get

$$u_\lambda \leqslant \limsup_{t\to\infty} u(t, \cdot, w_0) \leqslant \lim_{t\to\infty} u(t, \cdot, \beta\Phi_1) = u_\lambda,$$

which proves the result.

(ii) If $\underline{G_+} \geqslant \varepsilon$ for some $\varepsilon > 0$, then, we know from theorem 4.3 that the bifurcation of positive solutions is subcritical.

Choose a $\beta_0 > 0$ sufficiently large and $\delta > 0$ sufficiently small that, from (4.1), we get

$$\frac{g(\lambda, x, \beta_0\Phi_1(x))}{(\beta_0\Phi_1(x))^\alpha} \geqslant \tfrac{1}{2}\varepsilon, \quad \sigma_1 - \delta < \lambda < \sigma_1, \quad x \in \partial\Omega.$$

This implies that, for this β_0 fixed, we have

$$\frac{g(\lambda, x, \beta_0\Phi_1(x))}{\beta_0\Phi_1(x)} \geqslant \frac{\varepsilon}{2(\beta_0\Phi_1(x))^{1-\alpha}} \geqslant \bar{\varepsilon}, \quad \sigma_1 - \delta < \lambda < \sigma_1, \quad x \in \partial\Omega,$$

where

$$\bar{\varepsilon} = \inf\left\{\frac{\varepsilon}{2(\beta_0\Phi_1(x))^{1-\alpha}} : x \in \partial\Omega\right\}.$$

Assuming that $\delta \leqslant \frac{1}{2}\bar{\varepsilon}$ (if this is not the case, we choose $\delta = \frac{1}{2}\bar{\varepsilon}$) from (7.7) with initial condition $\beta_0\Phi_1$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)\chi\bigg|_{t=0} \geqslant \tfrac{1}{4}\varepsilon\int_{\partial\Omega}\beta_0\phi_1\chi > 0, \tag{7.9}$$

which implies, as in (i), that the solution starting at $\beta_0\Phi_1$ is non-decreasing. Now, with similar monotonicity arguments to those in (i) we prove that the solution of (7.1) with initial condition w_0 and $\beta_0\Phi_1 \leqslant w_0 \leqslant u_\lambda$ has to converge to $\tilde{u}_\lambda$.

The case $\overline{G_-} < -\varepsilon$ is the same. □

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REMARK 7.2. A condition which guarantees that, for a fixed λ, there exists a unique sufficiently large positive (respectively, negative) solution is to assume that the function $s \to g(\lambda, x, s)/s$ is strictly monotone for $s > 0$ (respectively, $s < 0$) large enough and a.e. $x \in \partial\Omega$. To see this, assume that u_λ and $\tilde{u}_\lambda$ are two positive solutions with $u_\lambda(x), \tilde{u}_\lambda(x) \geqslant \beta$ and such that $s \to g(\lambda, x, s)/s$ is strictly monotone for $s \geqslant \beta$. Observe that without loss of generality we can assume that $\tilde{u}_\lambda < u_\lambda$. Then, u_λ is the solution of

$$\left.\begin{aligned} -\Delta u_\lambda + u_\lambda &= 0 && \text{in } \Omega, \\ \frac{\partial u_\lambda}{\partial n} &= \left(\lambda + \frac{g(\lambda, x, u_\lambda)}{u_\lambda}\right) u_\lambda && \text{on } \partial\Omega, \end{aligned}\right\}$$

that is u_λ is an eigenfunction associated with the eigenvalue $\mu = 0$, of the following eigenvalue problem:

$$\left.\begin{aligned} -\Delta \phi + \phi &= \mu\phi && \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} &= \left(\lambda + \frac{g(\lambda, x, u_\lambda)}{u_\lambda}\right) \phi, && \text{on } \partial\Omega, \end{aligned}\right\} \tag{7.10}$$

and, since $u_\lambda > 0$, 0 is the principal eigenfunction.

Similarly, we could argue that $\phi = \tilde{u}_\lambda > 0$ is the principal eigenfunction associated with the principal eigenvalue 0 of the following problem:

$$\left.\begin{aligned} -\Delta \phi + \phi &= \mu\phi && \text{in } \Omega, \\ \frac{\partial \phi}{\partial n} &= \left(\lambda + \frac{g(\lambda, x, \tilde{u}_\lambda)}{\tilde{u}_\lambda}\right) \phi && \text{on } \partial\Omega. \end{aligned}\right\} \tag{7.11}$$

But, since $\tilde{u}_\lambda < u_\lambda$, by the monotonicity of $s \to g(\lambda, x, s)/s$ we cannot have $\mu = 0$ as the first eigenvalue of both problems (7.10) and (7.11).

When the bifurcation at the first eigenvalue is supercritical we can make the following proposition.

PROPOSITION 7.3. *Assume that the function g is differentiable with respect to the last variable and consider the functions $\underline{G_+}$, $\overline{G_+}$, $\underline{G_-}$, $\overline{G_-}$ as defined in (4.1) for some $\alpha < 1$ and for $\sigma = \sigma_1$, the first Steklov eigenvalue. Hence, if we have*

$$\liminf_{(\lambda,s)\to(\sigma_1,+\infty)} \frac{g_u(\lambda, x, s)}{s^{\alpha-1}} \geqslant \alpha\overline{G_+} \quad \left(\text{respectively,} \quad \limsup_{(\lambda,s)\to(\sigma_1,-\infty)} \frac{g_u(\lambda, x, s)}{s^{\alpha-1}} \leqslant \alpha\underline{G_-}\right) \tag{7.12}$$

and if condition (4.7) holds, that is,

$$\int_{\partial\Omega} \overline{G_+}(x)\Phi_1^{1+\alpha} < 0 \quad \left(\text{respectively,} \int_{\partial\Omega} \underline{G_-}(x)\Phi_1^{1+\alpha} > 0\right),$$

then the bifurcation of positive (respectively, negative) solutions at the first eigenvalue is supercritical and any positive (respectively, negative) equilibrium solution bifurcating from infinity is unstable.

Proof. We consider only the case of bifurcation of positive solutions. The proof for that of negative solutions is similar.

Condition (4.7) guarantees that there exists a supercritical bifurcation of positive solutions from infinity at the first eigenvalue σ_1. Let us denote by u_λ a positive solution bifurcating from infinity. The eigenvalue problem associated with the linearization around u_λ is given by

$$\left.\begin{aligned} -\Delta w + w &= \mu w && \text{in } \Omega, \\ \frac{\partial w}{\partial n} &= \lambda w + g_u(\lambda, x, u_\lambda) w && \text{on } \partial\Omega. \end{aligned}\right\} \tag{7.13}$$

We will show that the first eigenvalue $\mu_1 = \mu_1(\lambda) < 0$ for $\lambda > \sigma_1$ sufficiently close to σ_1. This eigenvalue is given by

$$\begin{aligned} \mu_1 &= \min_{\phi \in H^1(\Omega)} \frac{\int_\Omega |\nabla\phi|^2 + |\phi|^2 - \int_{\partial\Omega} \lambda|\phi|^2 + g_u(\lambda, x, u_\lambda)|\phi|^2}{\int_\Omega |\phi|^2} \\ &\leqslant \frac{\int_\Omega |\nabla\Phi_1|^2 + |\Phi_1|^2 - \int_{\partial\Omega} \lambda|\Phi_1|^2 + g_u(\lambda, x, u_\lambda)|\Phi_1|^2}{\int_\Omega |\Phi_1|^2} \\ &= \frac{(\sigma_1 - \lambda)\int_{\partial\Omega} |\Phi_1|^2 - \int_{\partial\Omega} g_u(\lambda, x, u_\lambda)|\Phi_1|^2}{\int_\Omega |\Phi_1|^2}, \end{aligned} \tag{7.14}$$

where we have used the fact that Φ_1 is the first Steklov eigenfunction associated with the eigenvalue σ_1.

But observe that, from lemma 4.2, we have

$$\limsup_{\lambda\to\sigma_1} \frac{\sigma_1 - \lambda}{\|u_\lambda\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leqslant \frac{\int_{\partial\Omega} \overline{G_+}\Phi_1^{1+\alpha}}{\int_{\partial\Omega} \Phi_1^2}. \tag{7.15}$$

On the other hand, from (7.12) and corollary 3.2, we have

$$\liminf_{\lambda\to\sigma_1} \int_{\partial\Omega} \frac{g_u(\lambda, x, u_\lambda)}{u_\lambda^{\alpha-1}} |\Phi_1|^2 \geqslant \int_{\partial\Omega} \alpha\overline{G_+}(x)\Phi_1^{1+\alpha}. \tag{7.16}$$

Plugging expressions (7.15) and (7.16) into (7.14), we obtain

$$\limsup_{\lambda\to\sigma_1} \frac{\mu_1(\lambda)}{\|u_\lambda\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leqslant \frac{(1-\alpha)\int_{\partial\Omega} \overline{G_+}(x)\Phi_1^{1+\alpha}}{\int_\Omega \Phi_1^2}.$$

Now, since, by hypothesis, condition (4.7) holds and $\alpha < 1$, we find that $\mu_1 < 0$ for λ sufficiently close to σ_1 and the equilibrium is unstable. □

8. Remarks and extensions

We consider in this section several important remarks and extensions of the problem we are dealing with. These comments go in three directions.

First, in § 8.1, we will consider the case where bifurcations from the trivial solution may occur. For this, we will need to assume that the nonlinearity g is $g(\lambda, x, u) = o(u)$ as $u \to 0$.

Second, in § 8.2, we will consider the case where the nonlinear boundary conditions incorporate a potential with a possible non-definite sign, that is, the boundary conditions reads

$$\frac{\partial u}{\partial n} = \lambda m(x)u + g(\lambda, x, u).$$

Finally, in § 8.3, we analyse the simpler, but important and instructive case in which $N = 1$.

8.1. Bifurcation from the trivial solution

We consider problem (1.2) and assume that the nonlinearity g satisfies condition (H1) but, instead of specifying the behaviour of g for large values of u, we consider the behaviour of g for small values of u. That is, we assume that

(H3) $\displaystyle\lim_{|s|\to 0} \frac{U(s)}{s} = 0.$

We have the following result.

THEOREM 8.1. *Consider problem (1.2) and assume that the nonlinearity g satisfies conditions (H1) and (H3). If σ is a Steklov eigenvalue of odd multiplicity, then the set of solutions of (1.2) possesses a component emanating from the bifurcation point $(\sigma, 0) \in \mathbb{R} \times C(\bar{\Omega})$. Moreover, this component, is either bounded in $\mathbb{R} \times C(\bar{\Omega})$, in which case it meets another bifurcation point from zero (that is, another point $(\sigma', 0)$ for another Steklov eigenvalue σ'), or unbounded.*

Proof. The proof of this result follows the general results on bifurcations from the trivial solution given in [16]; see also [2] for similar results when the nonlinearity is in the interior. □

REMARK 8.2. Observe that it is possible to have nonlinearities in which both situations (that from theorem 8.1 and the one from theorem 3.3) hold. This is the case, for instance, where the nonlinearity $g(\lambda, x, u)$ is $o(u)$ at $u \to 0$ and at $u \to \infty$. In this situation, both theorems apply and if σ is a Steklov eigenvalue of odd multiplicity (for instance the first one) then both bifurcations, from zero and from infinity occurs at this value of the parameter.

8.2. Potential on the boundary

We now study the case in which the nonlinear elliptic problem contains a potential $m(x)$ in the boundary condition:

$$\left.\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= \lambda m(x)u + g(\lambda, x, u) && \text{on } \partial\Omega. \end{aligned}\right\} \tag{8.1}$$

For simplicity we may assume that $m \in L^\infty(\partial\Omega)$ and we will consider the important case in which the potential changes sign on $\partial\Omega$.

The role played in the whole analysis of the previous sections by the eigenvalues $\{\sigma_i\}_{i=1}^{\infty}$ of problem (1.3) is now played by the eigenvalues of the following problem:

$$\left.\begin{aligned} -\Delta\Phi + \Phi = 0 \qquad & \text{in } \Omega, \\ \frac{\partial\Phi}{\partial n} = \sigma m(x)\Phi \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{8.2}$$

We will still denote these values as Steklov eigenvalues. Hence, σ is a Steklov eigenvalue, if problem (8.2) has non-trivial solutions. Moreover, the multiplicity of σ is the number of linearly independent solutions of (8.2). Alternatively, $\sigma \in \mathbb{R}$ is a Steklov eigenvalue if and only if $\mu = 0$ is an eigenvalue of the following eigenvalue problem

$$\left.\begin{aligned} -\Delta\Phi + \Phi = \mu\Phi \qquad & \text{in } \Omega, \\ \frac{\partial\Phi}{\partial n} = \sigma m(x)\Phi \quad & \text{on } \partial\Omega, \end{aligned}\right\} \tag{8.3}$$

and the multiplicity of σ as a Steklov eigenvalue of (8.2) is the same as the multiplicity of the eigenvalue $\mu = 0$ of (8.3).

In terms of the structure of the Steklov eigenvalues we obtain the following result.

PROPOSITION 8.3. *Let the potential $m \in L^{\infty}(\partial\Omega)$, with $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ and let $\alpha > 0$. Then the following conditions hold.*

(i) *If $m \geqslant \alpha > 0$ in a subset $\Gamma_+ \subset \partial\Omega$ with $(N-1)$-dimensional measure $|\Gamma_+|_{N-1} > 0$, then there exists a sequence of Steklov eigenvalues $\{\sigma_i^+\}_{i=1}^{\infty}$, $0 < \sigma_1^+ < \sigma_2^+ \leqslant \cdots$, with the property that $\sigma_i^+ \to +\infty$ as $i \to +\infty$ and these are all the positive Steklov eigenvalues. Moreover, σ_1^+ is simple and the eigenfunction corresponding to the eigenvalue σ_1^+ does not change sign in $\bar{\Omega}$.*

(ii) *If $m \leqslant -\alpha < 0$ in $\Gamma_- \subset \partial\Omega$ with $|\Gamma_-|_{N-1} > 0$, then there exists a sequence of Steklov eigenvalues $\{\sigma_i^-\}_{i=1}^{\infty}$, $0 > \sigma_1^- > \sigma_2^- \geqslant \cdots$, with the property that $\sigma_i^- \to -\infty$ as $i \to +\infty$ and these are all the negative Steklov eigenvalues. Moreover, σ_1^- is simple and the eigenfunction corresponding to the eigenvalue σ_1^- does not change sign in $\bar{\Omega}$.*

Proof. We will sketch the proof; the reader may complete the details, since the arguments are similar to the case of potentials in Ω (see [10, 13]).

It is sufficient to show (i), since (ii) is obtained from (i) by observing that $\lambda m(x) = (-\lambda)(-m(x))$.

(i) Consider, for each fixed $\sigma \in \mathbb{R}$, the eigenvalues $\{\mu_k(\sigma)\}_{k=1}^{\infty}$ of problem (8.3)

Note that, for fixed $\sigma \in \mathbb{R}$, we find that the sequence $\{\mu_k(\sigma)\}_{k=1}^{\infty}$ corresponds to the eigenvalues of $-\Delta + I$ with the Robin boundary condition $\partial u/\partial n = \sigma m u$. Hence, $\mu_k(\sigma) \to +\infty$ as $k \to \infty$. In particular, if $\sigma = 0$, we recover the Neumann eigenvalues of $-\Delta + I$ and we know that $1 = \mu_1(0) < \mu_2(0) \leqslant \cdots \leqslant \mu_k(0) \to +\infty$ as $k \to \infty$. For fixed k we can consider the dependence of μ_k with respect to σ. These curves are continuous in σ (see [14]). Moreover, using the min–max characterization of the eigenvalues, we can see easily that, for $\sigma \geqslant 0$, we have $\tau_k(\sigma) \leqslant \mu_k(\sigma)$, where

$\tau_k(\sigma)$ are the eigenvalues of

$$\left.\begin{aligned} -\Delta\Phi + \Phi &= \tau\Phi && \text{in } \Omega, \\ \frac{\partial\Phi}{\partial n} &= \sigma m^+(x)\Phi && \text{on } \partial\Omega. \end{aligned}\right\} \tag{8.4}$$

Again using the min–max characterization of the eigenvalues and the fact that $m^+ \geqslant 0$, we can see easily that for $\sigma > 0$ the curves $\sigma \to \tau_k(\sigma)$ are non-increasing. Moreover, from the fact that $m \geqslant \alpha$ in Γ^+, it can be seen that both curves $\tau_k(\sigma), \mu_k(\sigma) \to -\infty$ as $\sigma \to +\infty$. The structure of these curves as $\sigma \to \infty$ and the characterization of the Steklov eigenvalues as the values $\sigma \geqslant 0$ for which some of these curves pass through zero easily prove the result. □

All the results of the previous sections can be easily adapted to the problem (8.1). In particular, the operator S_λ from lemma 2.5, which appears in the fixed-point problem (2.7), is obtained by using the trace of the solution of the following problem:

$$\left.\begin{aligned} -\Delta u + u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial n} - \lambda m(x)u &= g && \text{on } \partial\Omega, \end{aligned}\right\} \tag{8.5}$$

and the fixed-point problem (3.1) should be rewritten now as $v = \lambda S_0(mv) + S_0(g(\lambda,\cdot,v))$, where S_0 is as in lemma 2.5.

The existence of bifurcations from infinity at a Steklov eigenvalue σ_i^+ or σ_i^-, of odd multiplicity follows the same line of proof.

The characterization of the type of bifurcation (sub- or supercritical) when the parameter λ crosses one of the eigenvalues $\sigma_i^+ > 0$ for some $i = 1, 2, \dots$ is the same as in the case $m \equiv 1$, that is, theorems 4.3 and 4.5 apply directly to this case. For instance, if

$$\int_{\partial\Omega} \underline{G_+}^{\alpha,\sigma_1^+}(x)\Phi_{1,+}^{1+\alpha} > 0,$$

then the bifurcation of positive solutions at $\lambda = \sigma_1^+ > 0$ is subcritical. If the parameter λ crosses $\sigma_i^- < 0$, then the characterizations are exactly the opposite, that is, for instance if

$$\int_{\partial\Omega} \underline{G_+}^{\alpha,\sigma_1^-}(x)\Phi_{1,-}^{1+\alpha} > 0,$$

then the bifurcation of positive solutions at $\lambda = \sigma_1^- < 0$ is supercritical. The reversal of characterizations can easily be seen since analysing the behaviour of (8.1) for $\lambda < 0$ is the same as analysing the same problem for $\tau = -\lambda > 0$ for the potential $n = -m$, since $\lambda m = (-\lambda)(-m) = \tau n$.

In this same spirit, and for the case where the potential changes sign, for which we have two principal eigenvalues, $\sigma_1^- < 0 < \sigma_1^+$, with strictly positive eigenfunctions $\Phi_{1,-}$ and $\Phi_{1,+}$, respectively, the anti-maximum principle with a potential will be as follows.

THEOREM 8.4. *For every $g \in L^r(\partial\Omega)$ with $r > N-1$, there exists $\epsilon = \epsilon(g)$ such that*

(i) *if $\int_{\partial\Omega} g\Phi_{1,+} > 0$ (respectively, $\int_{\partial\Omega} g\Phi_{1,-} > 0$), then every solution (λ, u) of (8.5) satisfies*

   (a) $u > 0$ *if* $0 < \sigma_1^+ - \epsilon < \lambda < \sigma_1^+$ *(respectively,* $u < 0$ *if* $\sigma_1^- - \epsilon < \lambda < \sigma_1^- < 0$*)*,

   (b) $u < 0$ *if* $\sigma_1^+ < \lambda < \sigma_1^+ + \epsilon$ *(respectively,* $u > 0$ *if* $\sigma_1^- < \lambda < \sigma_1^- + \epsilon < 0$*)*,

(ii) *if $\int_{\partial\Omega} g\Phi_1 = 0$ then every solution (λ, u) of (6.1) with $\lambda \neq \sigma_1$ changes it sign on $\partial\Omega$ and consequently in Ω.*

8.3. The case $N = 1$

So far we have been treating the case where the equation is N-dimensional with $N \geqslant 2$. We give now some ideas on how to treat the one-dimensional case. We will see that the bifurcation problem is a two-parameter nonlinear problem that can be treated using finite-dimensional techniques.

Observe that, if we consider equation (1.2) (or, in a similar way, equation (8.1)), in the one-dimensional domain $\Omega = (0, 1)$ we can rewrite it as

$$\left.\begin{aligned} -u_{xx} + u &= 0 \quad \text{in } (0,1), \\ -u_x(0) &= \lambda u + g_0(\lambda, u(0)), \\ u_x(1) &= \lambda u + g_1(\lambda, u(1)). \end{aligned}\right\} \tag{8.6}$$

But in this case, the differential equation can be solved explicitly in terms of two constants a and b. The general solution is $u(x) = ae^x + be^{-x}$. By plugging this expression into the boundary conditions, we obtain the following two equations, which are the equivalent to equation (2.7):

$$\begin{aligned} -a + b &= \lambda(a+b) + g_0(\lambda, a+b), \quad x = 0, \\ a\mathrm{e} - b\mathrm{e}^{-1} &= \lambda(a\mathrm{e} + b\mathrm{e}^{-1}) + g_1(\lambda, a\mathrm{e} + b\mathrm{e}^{-1}), \quad x = 1. \end{aligned}$$

Observe that in this case we only have two Steklov eigenvalues, which are given by the values σ for which the following matrix has zero determinant:

$$\begin{pmatrix} -(1+\sigma) & (1-\sigma) \\ (1-\sigma)\mathrm{e} & -(1+\sigma)\mathrm{e}^{-1} \end{pmatrix}.$$

These two values are given by

$$\sigma_1 = \frac{\mathrm{e}-1}{\mathrm{e}+1} < \sigma_2 = \frac{1}{\sigma_1} = \frac{\mathrm{e}+1}{\mathrm{e}-1}.$$

The eigenfunctions Φ_1 and Φ_2 for this problem are given by

$$\Phi_1(x) = \frac{\mathrm{e}^x + \mathrm{e}^{1-x}}{1+\mathrm{e}}, \quad \Phi_2(x) = \frac{\mathrm{e}^x - \mathrm{e}^{1-x}}{1-\mathrm{e}}.$$

Observe that $\Phi_1(0) = \Phi_1(1) = 1$ and $\Phi_2(0) = 1 = -\Phi_2(1)$.

For any $\lambda \neq \sigma_1, \sigma_2$, the function $u = a\mathrm{e}^x + b\mathrm{e}^{-x}$ is a solution if (a, b) satisfy

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -(1+\lambda) & (1+\lambda) \\ (1-\lambda)\mathrm{e} & -(1+\lambda)\mathrm{e}^{-1} \end{pmatrix}^{-1} \begin{pmatrix} g_0(\lambda, a+b) \\ g_1(\lambda, a\mathrm{e} + b\mathrm{e}^{-1}) \end{pmatrix}.$$

The sublinearity of g_0 and g_1 as $u \to \infty$ allows to apply fixed-point arguments in $\mathbb{R}^2$ guaranteeing the existence of at least one solution for any $\lambda \neq \sigma_1, \sigma_2$. Moreover, the fact that both eigenvalues are simple, guarantee that under a sublinearity condition on g as $u \to \infty$, we have bifurcation curves from infinity.

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