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SUBGROUPS WITH NO ABELIAN COMPOSITION FACTORS ARE NOT DISTINGUISHED

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Abstract

Given a finite group G, define the minimal degree $\mu(G)$ of G to be the least n such that G embeds into S_n. We call G exceptional if there is some $N \trianglelefteq G$ with $\mu(G/N) > \mu(G)$, in which case we call N distinguished. We prove here that a subgroup with no abelian composition factors is not distinguished.

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1. Related results

Perhaps counter-intuitively, it is possible to find an integer n and a finite group G with a normal subgroup $N \trianglelefteq G$ such that G embeds into S_n and G/N does not. Such a group is called exceptional. This is equivalent to $\mu(G) < \mu(G/N)$, where $\mu(G)$ denotes the minimal degree of G which is the least n such that G embeds into S_n.

An early example of this was given by Neumann [11] and described in more generality in [7]. There G is the direct product of $k > 1$ copies of D_8, the dihedral group of order eight. One can show that $\mu(G) = 4k$ and that there is a central subgroup N of G of order 2^{k-1} such that $\mu(G/N) = 2^{k+1}$.

It is in this sense that $\mu(G/N)$ can be exponential in $\mu(G)$. It was shown in [7] that $\mu(G/N) \leq 4.5^{\mu(G)}$. Further examples of exceptional p-groups can be found for example in [2, 5, 10]. These examples have led to the suggestion that exceptionality of a group somehow comes from its abelian composition factors, leading to the main result of this paper. An analogous result, Theorem 1 in [9], states that if G/N has no abelian normal subgroup, then N is not distinguished. A corollary of this result, or of the main theorem in this paper, is that a group with no abelian composition factors is not exceptional. In fact, if N is distinguished in G, then both N and G/N must contain an abelian composition factor.

This also adds to a list of results suggesting a connection between minimal normal subgroups of a group and minimal degree. It was shown for example in [1] that if G

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and H have central socles, then $\mu(G \times H) = \mu(G) + \mu(H)$. Also, a method was given in [3] to calculate the minimal degree of a group with no abelian minimal normal subgroups.

2. Main result and proof

Throughout we assume that each group G is finite and that $G \leq S_{\mu(G)}$. We call the group G D-minimal if G is of least order such that there exists some distinguished $N \trianglelefteq G$ with no abelian composition factors.

Proposition 2.1. *Let $N_0 \trianglelefteq G$ be distinguished, $N \trianglelefteq G$ and $N \leq N_0$; then either N is distinguished or N_0/N is distinguished in G/N.*

Proof. If N_0/N is not distinguished in G/N, then

$$\mu(G) < \mu(G/N_0) = \mu\Big(\frac{G/N}{N_0/N}\Big) \leq \mu(G/N).$$

Hence, N is distinguished. □

Lemma 2.2. *Let N, L, K be normal subgroups in G with N minimal and nonabelian. Then $N(K \cap L) = NK \cap NL$.*

Proof. Clearly $N(K \cap L) \subseteq NK \cap NL$.

If $N \leq L$ or $N \leq K$, then the result is the modular law for groups, so assume that $N \cap K = N \cap L = 1$. We first consider orders:

$$\begin{aligned} |N(K \cap L)| &= |N||K \cap L| \\ &= |N||K||L|/|KL|, \\ |NK \cap NL| &= |NK||NL|/|NKL| \\ &= |N||K||L||N \cap KL|/|KL|. \end{aligned}$$

So, if $N(K \cap L) \neq NK \cap NL$, then $|N \cap KL| > 1$ and therefore $N \subseteq KL$. However, as N and K are normal subgroups in G with $N \cap K = 1$, it follows that $N \subseteq C_G(K)$. Similarly $N \subseteq C_G(L)$. So, $N \subseteq C_G(KL) \leq C_G(N)$, contradicting the assumption that N is nonabelian. Hence, $N(K \cap L) = NK \cap NL$. □

For the next proposition, we use notation given in [8, Section 1]. Specifically we use the correspondence between permutation representations of a group G and multi-sets of subgroups of G.

Proposition 2.3. *If G is D-minimal with nonabelian distinguished minimal normal subgroup N, then G is transitive.*

Proof. Let $\{H_1, \ldots, H_k\}$ define a minimal permutation representation of G of degree $\mu(G)$. Denote $K_i = \mathrm{core}_G(H_i)$, so $\cap_{i=1}^k K_i = 1$. The action of G/K_i on the right cosets of H_i then defines a minimal representation of G/K_i (if $\{H_{i0}/K_i, \ldots, H_{ik_i}/K_i\}$ defines a representation of smaller degree, then replacing H_i with $H_{i0}, \ldots, H_{ik_i}$ defines a representation of degree less than $\mu(G)$).

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Suppose that $k > 1$, so that $|K_i| > 1$ for each i. Since G is D-minimal, $\mu(G/NK_i) \leq \mu(G/K_i)$, so there is some $\{H_{i0}, \ldots, H_{ik_i}\}$ with $\sum_{j=1}^{k_i}[G : H_{ii_j}] \leq [G : H_i]$ and $\mathrm{core}_G(\cap_{j=1}^{k_i}(H_{ii_j})) = NK_i$. In particular, $\sum_{i=1}^{k}\sum_{j=1}^{k_i}[G : H_{ii_j}] \leq \sum_{i=1}^{k}[G : H_i] = d$ and $\mathrm{core}_G(\cap_{i=1}^{k}\cap_{j=1}^{k_i}(H_{ii_j})) = \cap_{i=1}^{k} NK_i$. Using Lemma 2.2 inductively then gives $\mathrm{core}_G(\cap_{i=1}^{k}\cap_{j=1}^{k_i}(H_{ii_j})) = N \cap_{i=1}^{k} K_i = N$, so $\{H_{ii_j}\}$ defines a faithful representation of G/N of degree at most $\mu(G)$, contradicting the assumption that N is distinguished. Hence, $k = 1$ and G is transitive. □

Proposition 2.4. *If G has a nonabelian distinguished minimal normal subgroup N, then $C_G(N)$ is nontrivial.*

Proof. As N is a minimal normal subgroup, $N = S^k$ for some simple group S. If $C_G(N) = 1$, then the action of G on N by conjugation gives an embedding of G/N into $\mathrm{Out}(N) \cong \mathrm{Out}(S) \wr S_k$. Hence, $\mu(G/N) \leq \mu(\mathrm{Out}(S) \wr S_k) \leq k\mu(\mathrm{Out}(S))$. For each simple group S, $\mathrm{Out}(S)$ and $\mu(S)$ are known (see, for example, [4]) and one can check that $\mu(\mathrm{Out}(S)) \leq \mu(S)$. It was also shown in [5] that if $T_1, \ldots, T_r$ are simple groups, then $\mu(T_1 \times \cdots \times T_r) = \mu(T_1) + \cdots + \mu(T_r)$. So,

$$\mu(G/N) \leq k\mu(\mathrm{Out}(S)) \leq k\mu(S) = \mu(N) \leq \mu(G),$$

contradicting the assumption that N is distinguished. Hence, $C_G(N)$ is nontrivial. □

We will use the following result (see, for example, [12, Proposition 12.1]) without further reference.

Proposition 2.5. *Suppose that G is transitive and $B_\Gamma = \{\Gamma_1, \ldots, \Gamma_r\}$ forms a block system for G. Then G embeds into $(G_{\Gamma_1})^{\Gamma_1} \wr G^{B_\Gamma}$.*

Proposition 2.6. *If G is D-minimal and has a nonabelian distinguished minimal normal subgroup N, then N is transitive.*

Proof. By Proposition 2.3, G is transitive. Suppose that N is intransitive. The orbits of N form a block system $B_\Gamma = \{\Gamma_1, \ldots, \Gamma_r\}$ of G in Ω. We may therefore embed $\phi : G \hookrightarrow (G_{\Gamma_1})^{\Gamma_1} \wr G^{B_\Gamma}$.

Let $N_1 = N^{\Gamma_1} \trianglelefteq (G_{\Gamma_1})^{\Gamma_1}$ and $M = N_1^r \trianglelefteq (G_{\Gamma_1})^{\Gamma_1} \wr G^{B_\Gamma}$. Now, N is a direct product of isomorphic simple groups, so $M \cap \phi(G)$ is a direct product of isomorphic simple groups. Also, $\phi(N)$ is normal in $M \cap \phi(G)$ and a subdirect product of $M \cap \phi(G)$. Hence, $\phi(N) = M \cap \phi(G)$. Therefore, $G/N \cong \phi(G)/\phi(N)$ embeds into $(G_{\Gamma_1})^{\Gamma_1} \wr G^{B_\Gamma}/M \cong (G_{\Gamma_1})^{\Gamma_1}/N_1 \wr G^{B_\Gamma}$. This gives $\mu(G/N) \leq \mu((G_{\Gamma_1})^{\Gamma_1}/N^{\Gamma_1})\mu(G)/|\Gamma_1|$.

If $\mu((G_{\Gamma_1})^{\Gamma_1}) < |\Gamma_1|$, then $\mu(G) \leq \mu((G_{\Gamma_1})^{\Gamma_1} \wr G^{B_\Gamma}) < |\Gamma_1||B_\Gamma| = \mu(G)$, which is absurd. So, $\mu((G_{\Gamma_1})^{\Gamma_1}) = |\Gamma_1|$. If N^{Γ_1} is not distinguished in $(G_{\Gamma_1})^{\Gamma_1}$, then $\mu((G_{\Gamma_1})^{\Gamma_1}/N_1) \leq |\Gamma_1|$. Therefore, $\mu(G/N) \leq \mu((G_{\Gamma_1})^{\Gamma_1}/N_1 \wr G^{B_\Gamma}) \leq |\Gamma_1||B_\Gamma| = \mu(G)$, so N is not distinguished.

Hence, N^{Γ_1} distinguished in $(G_{\Gamma_1})^{\Gamma_1}$. This contradicts the assumption that G is D-minimal. Hence, N must be transitive. □

Lemma 2.7. *If S is a nonabelian simple group, then $|\mathrm{Out}(S)| \leq \mu(S)$.*

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Proof. This is a systematic check, so we omit the proof. The only challenging cases here are the simple groups of Lie type. A full list of the minimal degrees of these groups can be found in [6]. □

We use the following result (see, for example, [13, Proposition 4.3]) without further reference.

Proposition 2.8. *Suppose that $N \leq G$ is transitive. Then $C_G(N)$ is semiregular.*

Proposition 2.9. *Suppose that G is D-minimal with nonabelian distinguished minimal normal subgroup N. Then N is not simple.*

Proof. Suppose that such an N is simple. By Propositions 2.3 and 2.6, G and N are transitive. Let H be the stabiliser of some point in Ω, so $G = HN$. In particular, $G/N \cong H/(H \cap N)$, so $H \cap N$ is distinguished in H. Also, $\mu(G) = [G : H] = [N : H \cap N]$.

As $C = C_G(N)$ is semiregular, $H \cap C = 1$. In particular, H embeds into G/C, which in turn embeds into $\mathrm{Aut}(N)$ via conjugation. Let $H_{\mathrm{Inn}(N)}$ be the elements of H which act on N via inner automorphisms. This gives $H \cap N \trianglelefteq H_{\mathrm{Inn}(N)}$.

We note that the image of $H_{\mathrm{Inn}(N)}$ in $\mathrm{Aut}(N)$ is strictly contained in $\mathrm{Inn}(N)$. Indeed, by assumption, if $H \cap N$ is trivial, then $\mu(G/N) = \mu(H/(H \cap N)) = \mu(H) \leq \mu(G)$, contrary to assumption. And, if $H \cap N$ is nontrivial and the image of $H_{\mathrm{Inn}(N)}$ in $\mathrm{Aut}(N)$ is $\mathrm{Inn}(N)$, then simplicity of N implies that $H \cap N = N$, contradicting the fact that H is core-free. Hence, the image of $H_{\mathrm{Inn}(N)}$ in $\mathrm{Aut}(N)$ is strictly contained in $\mathrm{Inn}(N)$.

This means that $H_{\mathrm{Inn}(N)}$ is isomorphic to a core-free subgroup of N. Hence, $|H_{\mathrm{Inn}(N)}| \leq |N|/\mu(N)$. We also have, by definition of $H_{\mathrm{Inn}(N)}$, that $H/H_{\mathrm{Inn}(N)}$ embeds into $\mathrm{Out}(N)$. By Lemma 2.7, $|\mathrm{Out}(N)| < \mu(N)$. This gives

$$|H/(H \cap N)| = \frac{|H|}{|H_{\mathrm{Inn}(N)}|} \frac{|H_{\mathrm{Inn}(N)}|}{|H \cap N|} \leq \frac{|\mathrm{Out}(N)|}{\mu(N)} \frac{|N|}{|H \cap N|} < \mu(G).$$

This means that $\mu(G/N) = \mu(H/(H \cap N)) < \mu(G)$, contrary to assumption. Therefore, N is not simple. □

Lemma 2.10. *Suppose that $G = HN$ such that $\{H\}$ defines a minimal representation of G, $N \trianglelefteq G$ and $Z(N) = 1$. Denote $C = C_G(N)$ and $H_{\mathrm{Inn}(N)}$ the subgroup of H which acts on N under conjugation by inner automorphisms of N. Then $C \cong H_{\mathrm{Inn}(N)}/(H \cap N)$.*

If, in addition, N is a distinguished minimal normal subgroup of G, then $\mu(G) = |C|\mu(G/C)$.

Proof. Define a group homomorphism $\phi : H_{\mathrm{Inn}(N)} \to C$ as follows. If $h \in H_{Inn(N)}$, then, as $Z(N) = 1$, there is a unique $n_h \in N$ such that h acts on N identically under conjugation to n_h. Let $c_h = hn_h^{-1} \in C$ and $\phi(h) = c_h$. To see that ϕ is a homomorphism, notice that

$$c_{h_1}c_{h_2} = h_1n_1^{-1}h_2n_2^{-1} = h_1h_2(n_1^{-1})^{h_2}n_2^{-1} = c_{h_1h_2}.$$

To see that ϕ is surjective, suppose that $c \in C$. As $G = HN$, we have $c = hn$ for some $h \in H$, $n \in N$. In particular, h acts on N identically under conjugation to n^{-1},

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so $h \in H_{\mathrm{Inn}(N)}$ and $c = \phi(h)$. Finally, $h \in \ker(\phi)$ if and only if $hn_h^{-1} = 1$ if and only if $h = n_h$ if and only if $h \in H \cap N$. This gives $C \cong H_{\mathrm{Inn}(N)}/(H \cap N)$.

Now suppose further that N is a distinguished minimal normal subgroup of G.

Let Γ be the orbit of C under the representation defined by $\{H\}$. As N is transitive, C is semiregular, so $H \cap C = 1$ and $|\Gamma| = |C|$. The orbit Γ forms a block for the action of G, so G embeds into $(G_\Gamma)^\Gamma \wr G^{\mathcal{B}_\Gamma}$. This gives

$$\mu(G) \leq \mu((G_\Gamma)^\Gamma \wr G^{\mathcal{B}_\Gamma}) \leq \mu((G_\Gamma)^\Gamma)\mu(G^{\mathcal{B}_\Gamma}) \leq |\Gamma|\frac{\mu(G)}{|\Gamma|} = \mu(G).$$

Hence, $\mu((G_\Gamma)^\Gamma) = |\Gamma| = |C|$ and $\mu(G^{\mathcal{B}_\Gamma}) = \mu(G)/|C|$. It suffices then to show that $G^{\mathcal{B}_\Gamma} \cong G/C$. The action $G^{\mathcal{B}_\Gamma}$ is defined by $\{HC\}$, so it suffices to show that $\mathrm{core}_G(HC) = C$. Immediately $C \leq \mathrm{core}_G(HC)$. Suppose that $K \leq HC$ with $K \trianglelefteq G$. If $K \cap N = N$, then K is transitive, so HC and therefore C is transitive. But then N is contained in the centre of a transitive normal subgroup C, so $N \cap H = 1$ and $\mu(G/N) = \mu(H) \leq \mu(G)$, contrary to assumption. Hence, $K \cap N = 1$ and $K \leq C$. This gives $\mathrm{core}_G(HC) = C$ and completes the proof. □

THEOREM 2.11. *Given a finite group G and distinguished normal subgroup $N \trianglelefteq G$, N must have an abelian chief factor.*

PROOF. We consider a counterexample (G, N) such that G is of least order. In particular, G is D-minimal and N has no abelian composition factors. Let N_0 be a minimal normal subgroup of G contained in N. As G is D-minimal, N/N_0 is not distinguished in G/N_0, so by Proposition 2.1 N_0 is distinguished in G. Replacing N with N_0 if necessary, we may assume that N is minimal.

By Propositions 2.9, 2.3 and 2.6, N is not simple and G and N are transitive. In particular, we may denote $N = T_1 \times \cdots \times T_k$ with $k > 1$, where for some simple T we have $T_i \cong T$ for each i.

Let H be the stabiliser of some point in Ω, so that $G = HN$. In particular, $\mu(G) = [G : H] = [N : H \cap N]$ and $G/N \cong H/(H \cap N)$, so $H \cap N$ is distinguished in H. Also, by Lemma 2.10, $C \cong H_{\mathrm{Inn}(N)}/(H \cap N)$ and $\mu(G) = |C|\mu(G/C)$.

Let $C = C_G(N)$, so $H \cap C = 1$. In particular, H embeds into G/C, which in turn embeds into $\mathrm{Aut}(N) \cong \mathrm{Aut}(T) \wr S_k$ via conjugation. Let $\phi : G \to S_k$ be the natural map on G through $\mathrm{Aut}(N)$. Together this gives

$$\begin{aligned}|N| &= |N \cap H|\mu(G)\\ &= |N \cap H||C|\mu(G/C)\\ &\leq |N \cap H||C|k\mu(\mathrm{Aut}(T))\\ &= |H_{\mathrm{Inn}(N)}|k\mu(\mathrm{Aut}(T)).\end{aligned}$$

Define $\psi : H_{\mathrm{Inn}(N)} \to \mathrm{Aut}(T)$ by the conjugation of T_1 by $H_{\mathrm{Inn}(N)}$. Since N is minimal, $\phi(G)$ and therefore $\phi(H)$ is transitive. This means that the action of $H_{\mathrm{Inn}(N)}$

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on each T_i by conjugation has isomorphic image in $\mathrm{Aut}(T)$. Hence, $|H_{\mathrm{Inn}(N)}| \leq |\psi(H_{\mathrm{Inn}(N)})|^k$. This gives

$$\frac{|T|}{|\psi(H_{\mathrm{Inn}(N)})|} \leq \left(\frac{|N|}{|H_{\mathrm{Inn}(N)}|}\right)^{1/k} \leq k^{1/k}\mu(\mathrm{Aut}(T))^{1/k}.$$

We show here that $|T|/|\psi(H_{\mathrm{Inn}(N)})| < \mu(T)$ and therefore that $\psi(H_{\mathrm{Inn}(N)}) \cong T$. The values for $\mu(\mathrm{Aut}(T))$ and $\mu(T)$ are known for all simple groups T. We use [3, Proposition 2.2], a corollary of which is that $\mu(\mathrm{Aut}(T))/\mu(T) \leq 28/9$. We begin with the small cases, $T = A_5, A_6$.

If $T = A_5$, then $k^{1/k}\mu(\mathrm{Aut}(T))^{1/k} = k^{1/k}5^{1/k} < 5$.

If $T = A_6$, then $k^{1/k}\mu(\mathrm{Aut}(T))^{1/k} = k^{1/k}10^{1/k} < 6$.

For all other simple groups, $\mu(T) \geq 7$. We use [3, Proposition 2.2], a corollary of which is that $\mu(\mathrm{Aut}(T))/\mu(T) \leq 28/9$, so $\mu(\mathrm{Aut}(T)) \leq (28/9)\mu(T)$.

Let $f(x) = x^k - (28/9)kx$, so $f(x) > 0$ if and only if $(28/9)^{1/k}k^{1/k}x^{1/k} < x$. For $x \geq 7$, $f'(x) = kx^{k-1} - (28/9)k > 0$, so if $f(7) > 0$, then $f(x) > 0$ for $x \geq 7$. One can check that $f(7) > 0$. Hence, $|T|/|\psi(H_{\mathrm{Inn}(N)})| \leq (28/9)^{1/k}k^{1/k}\mu(T)^{1/k} < \mu(T)$. This completes the proof that $\psi(H_{\mathrm{Inn}(N)}) \cong T$.

This means that $H_{\mathrm{Inn}(N)}$ is a subdirect product of $N \cong T^k$, so it is isomorphic to T^r for some r. Also, $H \cap N \trianglelefteq H_{\mathrm{Inn}(N)}$, so it has no abelian chief factors. But $H \cap N$ is distinguished in H, contradicting the fact that G is D-minimal and completing the proof. □

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