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PARALLEL TRANSLATION IN VECTOR BUNDLES WITH ABELIAN STRUCTURE GROUP AND THE GAUSS-BONNET FORMULA

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Introduction. Most proofs for the classical Gauss-Bonnet formula use special coordinates, or other non-trivial preparations. Here, a simple proof is given, based on the fact that the structure group $SO(2)$ of the tangent bundle of an oriented 2-dimensional Riemannian manifold is abelian. Since only this hypothesis is used, we prove a slightly more general result (Theorem 1).

Let us recall the classical formula

$$\int_A Kdo + \int_{\partial A} \kappa ds = 2\pi. \tag{1}$$

Here, A denotes a compact contractible disc with smooth boundary ∂A in a 2-dimensional Riemannian manifold (of class C^∞). K is the Gaussian curvature, κ the geodesic curvature of the boundary ∂A, which is oriented in the usual way. do and ds are the area and arc-length measure, respectively.

If we shrink A to a point, the first integral converges to 0, while the second one converges to 2π. Thus (1) is equivalent to

$$\int_A Kdo + \int_{\partial A} \kappa ds \equiv 0 \mod (2\pi). \tag{2'}$$

Let us identify $SO(2)$ with the unit circle in $\mathbf{C}$, and its Lie algebra with $\mathbf{R}$, the exponential map being $\exp(r) = e^{ir}$. Then (2′) becomes

$$\exp\left(-\int_{\partial A} \kappa ds\right) = \exp\left(\int_A Kdo\right). \tag{2}$$

If we fix a point x_0 on ∂A, we may identify the group of orientation-preserving isometric automorphisms of the tangent space T_{x_0} of the manifold with $SO(2)$. In particular, the parallel translation along the closed curve ∂A is an element of $SO(2)$. By the definition of geodesic curvature, this parallel translation is precisely the left-hand side of (2). Thus, we have to prove that it is also equal to the right-hand side of (2). We shall deal with a slightly more general problem in § 1.

The result is applied in § 2 to prove the equivalence of two different definitions for the Chern-classes of a splitting complex vector bundle.

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1. Parallel translation in bundles with abelian structure group. In this article, G denotes a connected abelian Lie group with Lie algebra $\mathfrak{g}$. Let M be a smooth manifold and $\xi = (E, p, M)$ a smooth vector bundle over M with fibre F and structure group G. If $\mathfrak{U} = (U_i)_{i\in I}$ is an open cover of M, such that ξ is trivial over U_i, the bundle ξ is determined by the cocycle of its transition functions $g_{ij} : U_i \cap U_j \to G$, $i, j \in I$.

A connection on ξ may be expressed by $\mathfrak{g}$-valued 1-forms θ_i on U_i, subjected to the transformation law

$$(3') \qquad \theta_j = g_{ji} \cdot (g_{ij}' + \theta_i \cdot g_{ij})$$

(see also [**2**]). In this formula, for $g_1, g_2 \in G$ and $h \in T_{g_2}G$, $g_1 \cdot h$ denotes the image of h in $T_{g_1g_2}G$ under the differential of the diffeomorphism $g_1 : G \to G$ (multiplication by g_1). G being abelian, (3′) reduces to

$$(3) \qquad \theta_j = g_{ji} \cdot g_{ij}' + \theta_i.$$

Over U_i, the curvature tensor of the connection on ξ is given by

$$(4') \qquad H_i = d\theta_i + \tfrac{1}{2}[\theta_i, \theta_i].$$

Again this becomes

$$(4) \qquad H_i = d\theta_i,$$

since $\mathfrak{g}$ is abelian.

Lemma 1. *For $h, k \in \mathfrak{g}$, $\exp'(h; k) = (\exp h) \cdot k = k \cdot \exp h$.*

Proof. For $t \in \mathbf{R}$, $\exp(h + tk) = (\exp h) \cdot (\exp tk)$, whence $\exp'(h; k) = (\exp h) \cdot \exp'(0; k) = (\exp h) \cdot k = k \cdot \exp h$.

We may assume that the cover $\mathfrak{U}$ is 1-simple, i.e. the open sets U_i and the intersections $U_{ij} = U_i \cap U_j$ are simply connected. In this case, we can write $g_{ij} = \exp h_{ij}$ for differentiable $h_{ij} : U_{ij} \to \mathfrak{g}$, and by Lemma 1 we have $g_{ji} \cdot g_{ij}' = (\exp h_{ji}) \cdot (\exp h_{ij}) \cdot h_{ij}' = h_{ij}' = dh_{ij}$, and hence over U_{ij}

$$(5) \qquad \theta_j - \theta_i = dh_{ij}.$$

In particular, H_j coincides with H_i over U_{ij}. Thus the curvature tensor can be regarded as a $\mathfrak{g}$-valued 2-form H, defined on all of M. In the non-abelian case, this is not true!

Proposition 1. *Let $k : \mathbf{R} \to \mathfrak{g}$ be a differentiable function. Then the differential equation*

$$\dot{g}(t) = g(t) \cdot k(t)$$

with initial condition

$$g(0) = e$$

has the solution

$$g(t) = \exp\left(\int_0^t k(\tau)d\tau \right).$$

Proof. This follows by differentiation and Lemma 1.

Now let $\xi = (\mathbf{R}^2 \times F, p, \mathbf{R}^2)$ be a trivial bundle over $\mathbf{R}^2$, and let $\gamma(t)$, $0 \leqq t \leqq 1$, be a closed smooth Jordan curve in $\mathbf{R}^2$ with interior A, oriented in the usual way as boundary of A.

Regard ξ as a bundle with structure group G, and fix a connection for this G-bundle, given by its $\mathfrak{g}$-valued connection form θ, and denote its curvature form by H. Then the parallel translation P_γ along γ is an automorphism of the fibre $F_{\gamma(0)}$ and even an element of G, acting on $F_{\gamma(0)}$. (As element in G, P_γ is uniquely determined only in the case that G acts effectively on F; but we do not need the uniqueness of P_γ.)

THEOREM 1. *With the above notations and hypotheses,*

$$P_\gamma = \exp\left(-\int_A H\right).$$

Proof. With respect to the given trivialization of the bundle ξ, the parallel translation along γ from $\gamma(0)$ to $\gamma(t)$ may be regarded as an element $g(t)$ of G. For fixed $X \in F_{\gamma(0)}$, $X(t) = g(t) \cdot X$ is determined by the differential equation $D_{\dot\gamma(t)}X = 0$ with initial condition $X(0) = X$. Here, D denotes the covariant derivative with respect to the given connection. In terms of the connection form θ,

$$D_{\dot\gamma(t)}X = dX(\dot\gamma(t)) + \theta(\gamma(t);\dot\gamma(t)) \cdot X(t),$$

i.e. the differential equation for X becomes

$$\dot X(t) + \theta(\gamma(t);\dot\gamma(t)) \cdot X(t) = 0. \tag{6'}$$

So, if $g(t) \in G$ verifies the differential equation

$$\begin{aligned} \dot g(t) &= -\theta(\gamma(t);\dot\gamma(t)) \cdot g(t), \\ g(0) &= e, \end{aligned} \tag{6}$$

it describes the parallel translation along γ as an element of the group G.

According to Proposition 1, a solution is

$$g(t) = \exp\left(-\int_0^t \theta(\gamma(\tau);\dot\gamma(\tau))d\tau\right),$$

and in particular for $t = 1$:

$$P_\gamma = g(1) = \exp\left(-\int_\gamma \theta\right) = \exp\left(-\int_A d\theta\right) = \exp\left(-\int_A H\right),$$

by Stokes' theorem and (4).

In the classical case, where ξ is the tangent bundle of a 2-dimensional Riemannian manifold, the connection form θ is an expression in the Christoffel symbols, while $H = -Kdo$, where K is the Gaussian curvature and do the area element. Thus, the theorem reduces to (2). The proof can be simplified, too, such that it does not involve the concept of Lie group, but only the very special group $SO(2)$.

In the non-abelian case, (4), Lemma 1 and Proposition 1 are not true, and the above proof does not work. Also, the theorem is false, as can be shown by simple counterexamples with $G = SO(3)$. One can prove, however, a relation between the curvature tensor and the limit of the parallel translations along a sequence of closed curves, converging to a point. (See [**1**].)

2. Curvature and characteristic classes for bundles with abelian structure group. Let G, $\mathfrak{g}$ be as above, and let $\Gamma = \ker(\exp)$ be the *integer-lattice* in $\mathfrak{g}$. If $\xi = (E, p, M)$ is a vector bundle over M with structure group G and a connection, the curvature form H of this connection is closed, as follows from (4). By de Rham's theorem, it represents a cohomology class $[H] \in H^2(M; \mathfrak{g})$.

PROPOSITION 2. *The element $[H] \in H^2(M; \mathfrak{g})$ is an integer class, i.e. it is the image of an element in $H^2(M; \Gamma)$ under the canonical map*

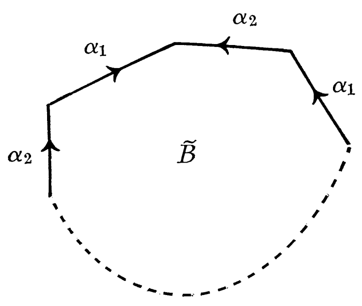
$$j_\# : H^2(M; \Gamma) \to H^2(M; \mathfrak{g}).$$

Proof. It must be shown that the integral of H over any integer 2-cycle in M is in Γ. It is sufficient, to regard smooth 2-cycles of the form $f : B \to M$, where B is an oriented 2-dimensional closed manifold and f a smooth map. In this case, we must show

$$\int_B f^*H \in \Gamma.$$

Now f^*H is the curvature form of the induced bundle $f^*\xi$ over B, with respect to the induced connection. Thus we may restrict ourselves to the case $B = M$, $f = \mathrm{id}$.

Let us regard B as the quotient of a disc $\tilde{B}$ with boundary $\partial\tilde{B}$ the Poincaré polygon of B, the parts of which are identified in the usual way:



Then, by Theorem 1, the parallel translation along the image of $\partial\tilde{B}$ in B is given by

$$P_{\partial\tilde{B}} = \exp\left(-\int_B H\right).$$

On the other hand, this parallel translation is the identity, since G is abelian:

$$P_{\partial \tilde{B}} = P_{\alpha_1} \cdot P_{\alpha_2} \cdot P_{\alpha_1}{}^{-1} \cdot P_{\alpha_2}{}^{-1} \cdot \ldots = \mathrm{id}.$$

Therefore

$$\int_B H \in \Gamma.$$

An immediate consequence of this result is

PROPOSITION 3. *$[H]$ does not depend on the choice of connection for the bundle ξ.*

Proof. For two connections D_0, D_1 on ξ, there is a family D_t, $0 \leqq t \leqq 1$, of connections, depending continuously on t. In particular the integral of the curvature form $\int_B H_t$ depends continuously on t. Since $\int_B H_t \in \Gamma$, it must be constant.

Hence there must be another way to describe the "characteristic class" $[H]$. For simplicity let us assume M connected. We have two exact sequences:

$$0 \longrightarrow \Gamma \xrightarrow{j} \mathfrak{g} \xrightarrow{\exp} G \longrightarrow 0 \tag{7}$$

and

$$0 \longrightarrow \bar{\Gamma} \xrightarrow{\bar{\gamma}} \bar{\mathfrak{g}} \xrightarrow{\exp} \bar{G} \longrightarrow 0. \tag{8}$$

The first one is an exact sequence of abelian groups, while in the second sequence $\bar{G}$, $\bar{\mathfrak{g}}$ and $\bar{\Gamma}$ are the sheaves of germs of differentiable functions on M with values in G, $\mathfrak{g}$ and Γ. Since Γ is discrete, the Čech cohomology $H^*(M; \bar{\Gamma})$ coincides with the singular cohomology $H^*(M; \Gamma)$. (7) induces a homomorphism $j_\# : H^2(M; \Gamma) \rightarrow H^2(M; \mathfrak{g})$ (cf. Proposition 2), and (8) induces a long exact sequence of Čech cohomology, part of which is

$$H^1(M; \bar{\mathfrak{g}}) \rightarrow H^1(M; \bar{G}) \xrightarrow{\partial} H^2(M; \bar{\Gamma}) \rightarrow H^2(M; \bar{\mathfrak{g}}). \tag{9'}$$

$\bar{\mathfrak{g}}$ is a fine sheaf. Therefore the connecting homomorphism is an isomorphism:

$$\partial : H^1(M; \bar{G}) \cong H^2(M; \Gamma). \tag{9}$$

Now, a G-bundle ξ over M can be regarded as an element of $H^1(M; \bar{G})$, which we denote again by ξ (cf. [**4**]).

THEOREM 2. *If H is the curvature form of the G-bundle ξ over M with respect to some connection, then*

$$[H] = j_\# \partial \xi.$$

Proof. We use essentially the same method by which de Rham's theorem is proved. (See [**3**].) Let $\mathfrak{U}$ be a 1-simple open cover of M, i.e. the U_i and the intersections $U_{ij} = U_i \cap U_j$ are simply connected. We write $C^p(\bar{G})$ instead of $C^p(\mathfrak{U}; \bar{G})$, and similarly for $\bar{\mathfrak{g}}$ and $\bar{\Gamma}$. $C^p(\mathfrak{g})$ denotes the p-cochains with respect to $\mathfrak{U}$ of locally constant $\mathfrak{g}$-valued functions, similarly $C^p(\Gamma) = C^p(\bar{\Gamma})$. ($C^p(\mathfrak{g})$ may also be described as the set of p-cochains on the nerve of $\mathfrak{U}$ with coefficients in $\mathfrak{g}$.) Furthermore, $A_q(\mathfrak{g})$ denotes the $\mathfrak{g}$-valued q-forms on M, while $C_q^p(\mathfrak{g})$ denotes the p-cochains with respect to $\mathfrak{U}$ of $\mathfrak{g}$-valued q-forms. In particular, $C_0^p(\mathfrak{g}) = C^p(\bar{\mathfrak{g}})$, and there is a canonical injection $A_q(\mathfrak{g}) \to C_q^0(\mathfrak{g})$.

If d denotes the exterior differentiation of differential forms and δ the coboundary operator for cochains with respect to $\mathfrak{U}$, we have the following two commutative diagrams:

(10)
$$\begin{tikzcd}
0 \arrow[r] & C^1(\bar{\Gamma}) \arrow[r, "j"] \arrow[d, "\delta"'] & C^1(\bar{\mathfrak{g}}) \arrow[r, "\exp"] \arrow[d, "\delta"'] & C^1(\bar{G}) \arrow[r] \arrow[d, "\delta"'] & 0 \\
0 \arrow[r] & C^2(\bar{\Gamma}) \arrow[r, "j"] & C^2(\bar{\mathfrak{g}}) \arrow[r, "\exp"] & C^2(\bar{G}) \arrow[r] & 0
\end{tikzcd}$$

(11)
$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{g} \arrow[r] \arrow[d] & A_0(\mathfrak{g}) \arrow[r, "d"] \arrow[d] & A_1(\mathfrak{g}) \arrow[r, "d"] \arrow[d] & A_2(\mathfrak{g}) \arrow[d] \\
0 \arrow[r] & C^0(\mathfrak{g}) \arrow[r] \arrow[d, "\delta"'] & C_0^0(\mathfrak{g}) \arrow[r, "d"] \arrow[d, "\delta"'] & C_1^0(\mathfrak{g}) \arrow[r, "d"] \arrow[d, "\delta"'] & C_2^0(\mathfrak{g}) \\
0 \arrow[r] & C^1(\mathfrak{g}) \arrow[r] \arrow[d, "\delta"'] & C_0^1(\mathfrak{g}) \arrow[r, "d"] \arrow[d, "\delta"'] & C_1^1(\mathfrak{g}) & \\
0 \arrow[r] & C^2(\mathfrak{g}) \arrow[r] & C_0^2(\mathfrak{g}) & &
\end{tikzcd}$$
.

Since $\mathfrak{U}$ is 1-simple, ξ is representable by a 1-cocycle $g = (g_{ij}) \in C^1(\bar{G})$, which is of the form $g = \exp h$ for a 1-cochain $h = (h_{ij}) \in C^1(\mathfrak{g})$. By definition of the connecting homomorphism, $j_\# \partial \xi$ is represented by δh, which is in $C^2(\mathfrak{g})$, since $j(C^2(\bar{\Gamma})) \subset C^2(\mathfrak{g}) \subset C^2(\bar{\mathfrak{g}})$.

In Diagram (11), the homology of the first column is the singular cohomology of M with coefficients in $\mathfrak{g}$, while the homology of the first row is the de Rham cohomology of M with coefficients in $\mathfrak{g}$. If both cohomologies are identified under the usual isomorphism, the differential form $H \in A_2(\mathfrak{g})$ and the 2-cochain δh represent the same cohomology-class, if there is a correspondence along the arrows of (11):

$$\begin{tikzcd}
 & & & H \arrow[d] \\
 & & * \arrow[r, "d"] \arrow[d, "\delta"'] & H \\
 & h \arrow[r, "d"] \arrow[d, "\delta"'] & dh & \\
\delta h \arrow[r] & \delta h & &
\end{tikzcd}$$
.

If we take $\theta = (\theta_i)$ for $*$, where the θ_i are the connection forms for the bundle ξ, we have $d\theta = H$ by the definition of H, while $\delta\theta_{ij} = \theta_j - \theta_i = dh_{ij}$ by (5).

Now let $G = U(1) \times \ldots \times U(1)$ (m factors). We may identify $\mathfrak{g}$ with $\mathbf{C}^m$ and Γ with $\mathbf{Z}^m$, if we take the exponential map $\exp(z_1, \ldots, z_m) := (e^{2\pi i z_1}, \ldots, e^{2\pi i z_m})$. Then $H^2(M; \Gamma) \cong (H^2(M; \mathbf{Z}))^m$. This isomorphism depends on the representation of G as a product. But if $\sigma_i : H^2(M; \mathbf{Z})^m \to H^{2i}(M; \mathbf{Z})$ denotes the ith elementary-symmetric function, the maps $\hat{\sigma}_i : H^2(M; \Gamma) \to H^{2i}(M; \mathbf{Z})$ are well-defined for $i = 0, 1, \ldots, m$. The composition

$$c_i = \hat{\sigma}_i \cdot \partial : H^1(M; \bar{G}) \to H^{2i}(M; \mathbf{Z})$$

is the ith Chern-class, i.e. for a G-bundle ξ over M,

$$c_i(\xi) = \hat{\sigma}_i(\partial\xi).$$

By Theorem 2, the cohomology-classes $j_\#(c_i(\xi)) \in H^{2i}(M; \mathbf{C})$ may be expressed in terms of the curvature tensor H, and one obtains the known formula

$$j_\# c_i(\xi) = \left[i\text{th coefficient of } \det\left(1 + \frac{1}{2\pi i} H\right)\right].$$

Here the $1/2\pi i$ appears, because we have identified Γ with $\mathbf{Z}^m$.

By the technique of the splitting principle for vector bundles, the last result can be extended for non-splitting complex vector bundles, i.e. vector bundles with structure group $GL(n; \mathbf{C})$.

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