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Codimension one compact center foliations are uniformly compact

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Abstract. Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism whose center foliation has all its leaves compact. We prove that if the unstable bundle of f is one-dimensional, then the volume of center leaves must be bounded in M.

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1. *Introduction*

1.1. *Context.* A diffeomorphism $f : M \to M$ in a closed manifold M is said to be *partially hyperbolic* if the tangent bundle TM decomposes as a direct sum of continuous and Df-invariant subbundles,

$$TM = E^s \oplus E^c \oplus E^u,$$

such that vectors in E^s are uniformly contracted by Df, vectors in E^u are uniformly contracted by Df^{-1}, and vectors in E^c have an intermediate behavior.

The bundles E^s and E^u uniquely integrate to f-invariant foliations $\mathcal{W}^u$ and $\mathcal{W}^s$, respectively (see, for example, [**HPS77**]). The bundles $E^s \oplus E^c$ and $E^c \oplus E^u$ may or may not integrate to foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$. If they do integrate to f-invariant foliations the diffeomorphism is said to be *dynamically coherent*.

If f is dynamically coherent, the bundle E^c also integrates to an invariant foliation $\mathcal{W}^c$ whose leaves are the connected components of the intersections of leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$.

This work fits in the context of studying partially hyperbolic diffeomorphisms where $\mathcal{W}^c$ is a compact foliation (namely, all leaves of $\mathcal{W}^c$ are compact).

Since Sullivan presented his example in [**S76**] of a foliation by circles with unbounded length of leaves, compact foliations have been categorized according to whether the volume of leaves is uniformly bounded or not (see §2.2.3 for a more detailed discussion). In particular, in the uniformly bounded case the leaf space is Hausdorff and has a nice orbifold structure, while in the non-uniformly bounded scenario the leaf space is not Hausdorff and may have a complicate structure (see, for example, [**E76**, **V77**]).

Pugh posed the following questions (see [**RHRHU07**, **G12**]).

Questions. Let $f : M \to M$ be a partially hyperbolic diffeomorphism with a compact center foliation $\mathcal{W}^c$. Is it true that the volume of center leaves is uniformly bounded? Is it true that f can be finitely covered by a partially hyperbolic diffeomorphism $\tilde{f} : \widetilde{M} \to \widetilde{M}$ so that there exist a fibration $p : \widetilde{M} \to N$ whose fibers are the center leaves and an Anosov diffeomorphism $\bar{f} : N \to N$ such that p is a semiconjugacy between $\tilde{f}$ and $\bar{f}$?

Progress on these questions has been made by Bonatti and Wilkinson [**BW05**], Bohnet [**B13**], Bohnet and Bonatti [**BB16**], Carrasco [**C15**] and Gogolev [**G12**].

1.2. *Main result.*

The main result of this work is the following theorem.

THEOREM 1.1. *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism with compact center foliation $\mathcal{W}^c$. If $\dim(E^u) = 1$ then the volume of the center leaves is uniformly bounded.*

In [**B13**] Bohnet has studied the case where the volume of the center leaves is uniformly bounded and $\dim(E^u) = 1$. Combining her results with our main theorem yields the following corollary.

COROLLARY 1.2. *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism with compact center foliation $\mathcal{W}^c$. If $\dim(E^u) = 1$ then, modulo taking a double cover, the leaf space $M/\mathcal{W}^c$ is a torus $\mathbb{T}^d$ and the dynamics $F : M/\mathcal{W}^c \to M/\mathcal{W}^c$ induced by f is topologically conjugate to an Anosov automorphism on $\mathbb{T}^d$, where $d = \operatorname{codim}(\mathcal{W}^c)$.*

Observe that Theorem 1.1 and Corollary 1.2 are valid as well if $\dim(E^s) = 1$ by working with f^{-1} instead of f.

In [**G12**] Gogolev proved that compact center foliations are uniformly compact under the assumptions $\dim(E^c) = 1$, $\dim(E^s) \leq 2$ and $\dim(E^u) \leq 2$. Combining this result with our main theorem then yields the following corollary.

COROLLARY 1.3. *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism with compact center foliation. If $\dim(M) \leq 5$ then the volume of the center leaves is uniformly bounded.*

In particular, with Corollary 1.3 we can completely rule out as a center foliation Sullivan's example [**S76**] of a foliation by circles in a 5-manifold with unbounded length of leaves. Note that for this specific example the result is, as far as we are aware, new. In Remark 3.6 we give a direct proof for this example without the need of the results from §4.

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1.3. *Organization of the paper and structure of the proof.* In §2 we give some preliminaries from partially hyperbolic dynamics and foliation theory. In particular, we address the topic of compact foliations and review some definitions and results that will be useful in the proof of the main theorem.

In §§3 and 4 we give the proof of the main theorem, and the structure is as follows.

The proof will be by contradiction. Assume that the center foliation $\mathcal{W}^c$ does not have uniformly bounded volume of leaves. This is equivalent to the *bad set*

$$\mathcal{B} = \{x : \text{center leaf volume function is not locally bounded at } x\}$$

being non-empty (see §2.2.3).

The main result of §3 is that $\mathcal{B}$ is saturated by the center-unstable foliation: as the unstable holonomy of center leaves is trivial (see Lemma 3.2), the stable holonomy groups of points in the same unstable leave are conjugated (see Lemma 3.3). We deduce that $\mathcal{B}$ is an attractor. In particular, this implies that f cannot be transitive.

Section 4 is dedicated to ruling out the non-transitive case.

The attractor $\mathcal{B}$ induces an associated repeller $\mathcal{R}$. We first show that $\mathcal{R}$ is saturated by the center foliation, implying that the center leaf volume function is bounded in $\mathcal{R}$ (see §4.1).

We then give a sort of topological description of center-stable leaves in $\mathcal{R}$, namely, all of them are bundles over a center leaf with stable manifolds as fibers, thus having trivial transverse holonomy (see §4.2).

Finally, this allow us to adapt Hiraide arguments [**H01**] (see also Bohnet [**B13**]) in order to disprove the existence of the codimension one transversally unstable repeller $\partial\mathcal{R}$.

2. *Preliminaries*

2.1. *Preliminaries from partially hyperbolic dynamics.* Let $f : M \to M$ be a C^1 diffeomorphism on M a closed Riemannian manifold. We say that f is *partially hyperbolic* if there exist a continuous and Df-invariant decomposition

$$TM = E^s \oplus E^c \oplus E^u$$

and some $\ell > 0$ such that, for every $x \in M$ and unit vectors $v^\sigma \in E^\sigma$ for $\sigma \in \{s, c, u\}$, one has that

$$\|D_x f^\ell(v^s)\| < 1, \quad \|D_x f^{-\ell}(v^u)\| < 1,$$

and

$$\|D_x f^\ell(v^s)\| < \|D_x f^\ell(v^c)\| < \|D_x f^\ell(v^u)\|.$$

We call E^s, $E^s \oplus E^c$, E^c, $E^c \oplus E^u$, and E^u *stable*, *center-stable*, *center*, *center-unstable* and *unstable bundles*, respectively.

The stable and unstable bundles are known to be uniquely integrable (see, for example, [**HPS77**]) to foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, respectively. However, the center, center-stable, and center-unstable bundles may not integrate.

We say that a partially hyperbolic diffeomorphism is *dynamically coherent* if the center-stable and center-unstable bundles integrate to f-invariant foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$,

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respectively. In particular, this implies that the center bundle is also integrable, the center leaf through a point $x \in M$ being the connected component of $W^{cs}(x) \cap W^{cu}(x)$ that contains x. The resulting foliation, $\mathcal{W}^c$, is then also f-invariant and tangent to the center bundle. For more information and context on this topic see, for example, [**RHRHU07**].

Given a point $x \in M$, we denote by $W^\sigma(x)$ the leaf of $\mathcal{W}^\sigma$ through x for $\sigma \in \{s, cs, cu, u\}$. We denote by $\mathcal{C}_x$ the leaf of $\mathcal{W}^c$ through x.

Given a center leaf $\mathcal{C}$, we denote by $W^\sigma(\mathcal{C})$ the leaf of $\mathcal{W}^\sigma$ through $\mathcal{C}$ for $\sigma \in \{cs, cu\}$.

For $x \in M$ and $r > 0$, we denote by $\mathrm{B}^\sigma_r(x) \subset W^\sigma(x)$ the intrinsic ball of center x and radius $r > 0$ in $W^\sigma(x)$ for $\sigma \in \{s, cs, c, cu, u\}$. If $\dim(E^\sigma) = 1$ we will simply denote it by $(x - r, x + r)^\sigma$. If $\dim(E^\sigma) = 1$ and $\mathcal{W}^\sigma$ is oriented, we write $(x, x + \delta)^u = \{y \in \mathrm{B}^u_\delta(x) : x < y\}$ and $W^u_+(x) = \{y \in W^u(x) : x < y\}$, and, if $y \in \mathcal{W}^u(x)$, we denote by $[x, y]^u$ and $(x, y)^u$ the oriented closed and open segments in $\mathcal{W}^u(x)$, respectively, from x to y.

Leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$ can be obtained as an increasing union of balls, thus, are homeomorphic to $\mathbb{R}^{\dim E^s}$ and $\mathbb{R}^{\dim E^u}$, respectively.

2.2. *Preliminaries from foliation theory.*

We will consider continuous foliations with C^1-leaves tangent to a continuous distribution. A general reference for this section is [**CC00**].

2.2.1. *Holonomy of a leaf.*

Let us briefly recall the definition of the holonomy group of a leaf.

Consider W a leaf in a foliation $\mathcal{W}$ of codimension q. Fix x_0 a point in W and let D be a disk of dimension q transversal to $\mathcal{W}$ through x_0.

For every loop $\gamma : [0, 1] \to W$ based on x_0 one can consider $h_\gamma : D' \to D$ the *holonomy return map* to D of the leaves of $\mathcal{W}$ through points from a smaller transversal disk $D' \subset D$. Consider adequate small transversal disks $D_{\gamma(t)}$ through each point $\gamma(t)$ and, given y in $D' = D_{\gamma(0)}$, define $h_\gamma(y)$ as the end point of the continuous curve γ_y defined by $\gamma_y(t) \in D_{\gamma(t)} \cap W(y)$ and $\gamma_y(0) = y$.

For a local homeomorphism $h : D' \subset D \to D$ fixing x_0 one defines the *germ of h* as the class of all local homeomorphisms that coincide with h in a neighborhood of x_0. With the operation given by composition, these classes form $G(x_0, D)$, the *group of germs of local homeomorphisms at* x_0.

One can see that for basepoint fixed homotopic curves in W based at x_0 there corresponds the same holonomy germ. Since concatenation of curves corresponds to composition of holonomy maps (whenever well defined) one has a well-defined homomorphism

$$\phi : \pi_1(W, x_0) \to G(x_0, D)$$

where $\pi_1(W, x_0)$ is the fundamental group of W based in x_0.

Define the *holonomy group of W at* x_0 as $\phi(\pi_1(W, x_0))$. The isomorphism class of $\phi(\pi_1(W, x_0))$ does not depend of x_0 or D, so we call it *holonomy group of W* and denote it by $\mathrm{Hol}(W)$.

2.2.2. *Reeb stability.* This next theorem is classical in foliation theory. See, for example, [**CC00**, Theorem 2.4.3] and [**CLN85**, Theorem 3].

THEOREM 2.1. (Generalized Reeb stability theorem) *Let W be a compact leaf in a foliation $\mathcal{W}$ such that W has a finite holonomy group* $\mathrm{Hol}(W)$. *Then there exists $\mathcal{U}(W)$ a neighborhood of W, saturated by leaves of $\mathcal{W}$, in which all leaves of $\mathcal{W}$ are compact with finite holonomy group. Moreover, $\mathcal{U}(W)$ has an associated projection $\pi : \mathcal{U}(W) \to W$ such that for every $W' \subset \mathcal{U}(W)$ the map $\pi|_{W'} : W' \to W$ is a finite covering with k sheets,* $k \leq |\mathrm{Hol}(W)|$, *and for each $y \in W$ the set $\pi^{-1}(\{y\})$ is a disk transversal to $\mathcal{W}$. The neighborhood $\mathcal{U}(W)$ can be taken to be arbitrarily small.*

2.2.3. *Compact foliations.* We say that a foliation $\mathcal{W}$ is *compact* if every leaf W of $\mathcal{W}$ is compact.

Given a compact foliation $\mathcal{W}$ and a Riemannian metric in M we can consider the *volume function*

$$\mathrm{vol} : M \to [0, +\infty)$$

that assigns to each point $x \in M$ the volume of the leaf $W(x)$ with respect to the metric in $W(x)$ induced by the metric of M.

It may be the case that a compact foliation does not have uniformly bounded volume of leaves, meaning that vol is an unbounded function (see, for example, [**S76**] or [**EV78**]).

Given a compact foliation $\mathcal{W}$, define the *bad set of $\mathcal{W}$* as

$$\mathcal{B} := \{x \in M : \mathrm{vol} \text{ is not locally bounded at } x\}.$$

The set $\mathcal{B}$ does not depend on the choice of the metric in M.

Remark 2.2. The fact that $\mathcal{W}$ has uniformly bounded volume of leaves is equivalent to $\mathcal{B}$ being the empty set.

Remark 2.3. Observe that if $\mathcal{W}$ is invariant by a C^1 diffeomorphism f then $\mathcal{B}$ is invariant by f (namely, $f(\mathcal{B}) = \mathcal{B}$). This is going to be used in our context where $\mathcal{W}$ will be the center foliation of a partially hyperbolic diffeomorphism.

Let us show some of the properties of $\mathcal{B}$. For a proof of the following result see, for example, Epstein [**E76**] or Lessa [**L15**].

PROPOSITION 2.4. *Let $\mathcal{W}$ be a compact foliation. Then the volume function is lower semicontinuous. That is,* $\liminf_{x_n \to x} \mathrm{vol}(W(x_n)) \geq \mathrm{vol}(W(x))$.

Using the previous proposition we can prove our next result (see, for example, [**EMS77**]).

PROPOSITION 2.5. *The bad set $\mathcal{B}$ is closed with empty interior and is saturated by leaves of $\mathcal{W}$.*

Proof. The set $\mathcal{B}$ is clearly closed.

Semicontinuous functions are continuous in a residual set and $\mathcal{B} \subset \{x \in M :$ vol is not continuous in $x\}$. Thus, from vol being lower semicontinuous we deduce that $\mathcal{B}$ has empty interior.

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The fact that the volume function is constant along leaves implies that $\mathcal{B}$ is saturated by leaves of $\mathcal{W}$: if $x \in \mathcal{B}$ then x has arbitrarily large leaves arbitrarily close to it, so if $y \in W(x)$ then these arbitrarily long leaves pass close to y as well (due to the continuity of the foliation $\mathcal{W}$), meaning that y also belongs to $\mathcal{B}$. □

Leaves in $\mathcal{B}$ can be completely characterized in terms of holonomy (see, for example, Epstein [**E76**, Theorem 4.2]).

PROPOSITION 2.6. *Let $\mathcal{W}$ be a compact foliation and W a leaf of $\mathcal{W}$. Then $W \subset \mathcal{B}$ if and only if* $|\mathrm{Hol}(W)| = \infty$.

2.3. *Stable and unstable holonomies and product neighborhoods.* In this section we give particular definitions and results we will use later. Throughout this section $f : M \to M$ will be a dynamically coherent partially hyperbolic diffeomorphism with compact center foliation.

2.3.1. *Stable and unstable holonomies.* A consequence of dynamical coherence is that each leaf W of $\mathcal{W}^{cs}$ is foliated by center leaves. We denote the restricted foliation by $\mathcal{W}^c|_W$.

Given a center leaf $\mathcal{C}$, denote by $\mathrm{Hol}(\mathcal{C})$ the holonomy group of $\mathcal{C}$ as a leaf of $\mathcal{W}^c$. Denote by $\mathrm{Hol}^s(\mathcal{C})$ the holonomy group of $\mathcal{C}$ as a leaf of $\mathcal{W}^c|_{W^{cs}(\mathcal{C})}$.

If W is a leaf of $\mathcal{W}^{cs}$, denote by $\mathrm{Hol}_{cs}(W)$ the holonomy group of W as a leaf of $\mathcal{W}^{cs}$. More generally, if $V \subset W$, we denote $\mathrm{Hol}_{cs}(V)$ the subgroup of $\mathrm{Hol}_{cs}(W)$ that corresponds to holonomy return maps along closed curves inside V.

We define Hol^u and Hol_{cu} analogously.

Remark 2.7. We have that $\mathrm{Hol}_{cs}(\mathcal{C}) \simeq \mathrm{Hol}^u(\mathcal{C})$ and $\mathrm{Hol}_{cu}(\mathcal{C}) \simeq \mathrm{Hol}^s(\mathcal{C})$ for every center leaf $\mathcal{C}$.

Proof. We prove the first equality; the second one is analogous. Let γ be a loop in a center leaf $\mathcal{C}$ based in some point $x_0 \in \mathcal{C}$. Consider $D^u(x_0)$ a small unstable disk through x_0. One can consider, on the one hand, the holonomy return map h_γ associated to γ of the leaf $W^{cs}(\mathcal{C})$ in the foliation $\mathcal{W}^{cs}$ and, on the other hand, the holonomy return map h'_γ of the leaf $\mathcal{C}$ in the foliation $\mathcal{W}^c|_{W^u(\mathcal{C})}$. Dynamical coherence gives us that h_γ coincides with h'_γ. The result follows. □

Remark 2.8. For a set V inside a center-stable leaf W that deformation retracts inside W to some center leaf $\mathcal{C}$, we have that $\mathrm{Hol}_{cs}(V) \simeq \mathrm{Hol}_{cs}(\mathcal{C}) \simeq \mathrm{Hol}^u(\mathcal{C})$. For such sets we write $\mathrm{Hol}^u(V)$ instead of $\mathrm{Hol}_{cs}(V)$, and analogously for $\mathrm{Hol}_{cu}(V)$ and $\mathrm{Hol}^s(V)$.

Remark 2.9. Each one of the previous holonomies is invariant by f.

Dynamical coherence gives a relationship between the holonomy $\mathrm{Hol}(\mathcal{C})$ and the holonomies $\mathrm{Hol}^s(\mathcal{C})$ and $\mathrm{Hol}^u(\mathcal{C})$. In particular, we will need the following proposition (see, for example, Carrasco [**C15**, Proposition 2.5]).

PROPOSITION 2.10. *For every center leaf $\mathcal{C}$ we have that*

$$\max\{|\mathrm{Hol}^s(\mathcal{C})|, |\mathrm{Hol}^u(\mathcal{C})|\} \leq |\mathrm{Hol}(\mathcal{C})| \leq |\mathrm{Hol}^s(\mathcal{C})||\mathrm{Hol}^u(\mathcal{C})|.$$

In particular, $\mathrm{Hol}(\mathcal{C})$ *is finite if and only if* $\mathrm{Hol}^s(\mathcal{C})$ *and* $\mathrm{Hol}^u(\mathcal{C})$ *are finite.*

Proof. Let $\mathcal{C}$ be a center leaf and x_0 a point in $\mathcal{C}$.

Consider D a small disk transversal to $\mathcal{W}^c$ and tangent to $E^s \oplus E^u$ in x_0. Denote by D^s and D^u the connected components of $D \cap W^{cs}(x_0)$ and $D \cap W^{cu}(x_0)$ that contain x_0, respectively. More generally, for every x in D denote by $D^s(x)$ and $D^u(x)$ the connected components of $D \cap W^{cs}(x)$ and $D \cap W^{cu}(x)$ that contain x, respectively. We have, in a neighborhood $D' \subset D$ of x_0, local stable/unstable coordinates. That is, for every x in D' there exist unique $x_s \in D^s$ and $x_u \in D^u$ such that $x = D^u(x_s) \cap D^s(x_u)$. Denote the coordinates of x by (x_s, x_u).

Given a loop γ based at x_0 we have well-defined holonomy return maps, $h^s_\gamma : D^{s\prime} \to D^s$ and $h^u_\gamma : D^{u\prime} \to D^u$, for center leaves inside $W^{cs}(x_0)$ and $W^{cu}(x_0)$, respectively. Then dynamical coherence says that the holonomy return map associated to γ in D can be written in local coordinates as

$$h_\gamma = (h^s_\gamma, h^u_\gamma),$$

where h_γ is given by $h_\gamma(x) = (h^s_\gamma(x_s), h^u_\gamma(x_u))$, with $x = (x_s, x_u)$ the stable/unstable coordinates defined before.

Elements of $\mathrm{Hol}^s(\mathcal{C})$ and $\mathrm{Hol}^u(\mathcal{C})$ are germs of local homeomorphisms fixing x_0 in D^s and D^u, respectively. The previous discussion shows that the map

$$\psi : \mathrm{Hol}^s(\mathcal{C}) \times \mathrm{Hol}^u(\mathcal{C}) \to \mathrm{Hol}(\mathcal{C})$$

given by $\psi(h^s_\gamma, h^u_\gamma) = h_\gamma$ is surjective. This implies that $|\mathrm{Hol}(\mathcal{C})| \leq |\mathrm{Hol}^s(\mathcal{C})||\mathrm{Hol}^u(\mathcal{C})|$. Moreover, since $h_\gamma(x_s, x_0) = (h^s(x_s), x_0)$ and $h_\gamma(x_0, x_u) = (x_0, h^u(x_u))$, we obtain that $\max\{|\mathrm{Hol}^s(\mathcal{C})|, |\mathrm{Hol}^u(\mathcal{C})|\} \leq |\mathrm{Hol}(\mathcal{C})|$. □

2.3.2. *Product neighborhoods.* Assume throughout the rest of this section that $\dim(E^u) = 1$ and $\mathrm{Hol}^u(\mathcal{C}) = \mathrm{Id}$ for every center leaf. This hypothesis will be verified later during the proof of the main theorem.

The following proposition will be of use many times (for similar results see, for example, [**CC00**] or [**HH87**]).

PROPOSITION 2.11. (Product neighborhoods) *Let $\mathcal{C}$ be a center leaf and consider $E = \bigcup_{x\in\mathcal{C}} \mathrm{B}^s(x)$, where $\mathrm{B}^s(x) \subset W^s(x)$ is a disk such that $\mathrm{B}^s(x) \cap \mathcal{C} = \{x\}$ for every $x \in \mathcal{C}$ and $\mathrm{B}^s(x)$ varies continuously with x. Then there exists an homeomorphism over its image $\varphi : E \times [-1, 1] \to M$ such that:*

(1) $\varphi(E \times \{0\}) = E$;

(2) $\varphi(E \times \{y\})$ *lies inside a $\mathcal{W}^{cs}$ leaf for every $y \in [-1, 1]$;*

(3) $\varphi(\{x\} \times [-1, 1])$ *lies inside a $\mathcal{W}^u$ leaf for every $x \in E$.*

In this case, $\varphi(E \times [-1, 1])$ will be called a $(\mathcal{W}^{cs}, \mathcal{W}^u)$*-product neighborhood* of E and, in an abuse of notation, we will simply refer to it as $E \times [-1, 1]$.

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Proof. We can consider $\epsilon > 0$ such that for every distinct $x, x' \in E$ we get $(x - \epsilon, x + \epsilon)^u \cap (x' - \epsilon, x' + \epsilon)^u = \emptyset$.

Let us fix $x_0 \in \mathcal{C}$. By the continuity of $\mathcal{W}^{cs}$ we can consider $\delta > 0$ such that for every curve $\gamma : [0, 1] \to E$ with $\gamma(0) = x_0$ and $\text{length}(\gamma) \leq 2\,\text{diam}(E)$ we have a well-defined holonomy map

$$h_\gamma : (x_0 - \delta, x_0 + \delta)^u \to (\gamma(1) - \epsilon, \gamma(1) + \epsilon)^u.$$

Observe that if such a γ is closed, since $E = \bigcup_{x \in \mathcal{C}} \mathrm{B}^s(x)$ with each $\mathrm{B}^s(x)$ a stable disk, then γ would be homotopic to a loop in $\mathcal{C}$. Since $\mathrm{Hol}^u(\mathcal{C}) = \mathrm{Id}$ we can take $\delta > 0$ small enough to ensure that h_γ coincides with the identity from $(x_0 - \delta, x_0 + \delta)^u$ to itself for every γ such that $\text{length}(\gamma) \leq 2\,\text{diam}(E)$.

For every $x \in E$ consider $\gamma_x : [0, 1] \to E$ a curve from x_0 to x such that $\text{length}(\gamma_x) \leq \text{diam}(E)$. Now, for every $y \in (x_0 - \delta, x_0 + \delta)^u$ define

$$\varphi(x, y) = h_{\gamma_x}(y).$$

This definition is independent of the choice of the curve γ_x. Indeed, if $\gamma'_x : [0, 1] \to E$ is another such a curve, the concatenation $\gamma_x * \gamma'_x$ has length at most $2\,\text{diam}(E)$ and then $\mathrm{id} = h_{\gamma_x * \gamma'_x} = h_{\gamma_x} \circ h^{-1}_{\gamma'_x}$ implies $h_{\gamma_x}(y) = h_{\gamma'_x}(y)$ for every $y \in (x_0 - \delta, x_0 + \delta)^u$.

The properties of the map φ follow directly from its definition. □

We will often need a particular instance of the previous proposition.

Remark 2.12. For every center leaf $\mathcal{C}$ such that $|\mathrm{Hol}(\mathcal{C})| < \infty$ we can consider $\mathcal{U}^s(\mathcal{C})$ a neighborhood of $\mathcal{C}$ in $W^{cs}(\mathcal{C})$ given by the generalized Reeb stability theorem (see Theorem 2.1) such that:

- the associated projection $\pi : \mathcal{U}^s(\mathcal{C}) \to \mathcal{C}$ is such that $\pi^{-1}(x)$ is a disk in $W^s(x)$ for every $x \in \mathcal{C}$;
- there exists a $(\mathcal{W}^{cs}, \mathcal{W}^u)$-product neighborhood $\mathcal{U}^s(\mathcal{C}) \times [-1, 1]$ of $\mathcal{U}^s(\mathcal{C})$.

Proof. By the transversality of the foliations $\mathcal{W}^s$ and $\mathcal{W}^c$ inside $W^{cs}(\mathcal{C})$ and the fact that $\mathcal{U}^s(\mathcal{C})$ can be taken arbitrarily small, we obtain that the projection π can be taken along leaves of $\mathcal{W}^s$.

The neighborhood $\mathcal{U}^s(\mathcal{C})$ verifies the hypothesis of Proposition 2.11, which implies the existence of the product neighborhood. □

3. *The bad set is saturated by the center-unstable foliation*

From now on, let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism such that $\dim(E^u) = 1$.

As before, we assume that $\mathcal{W}^c$ is a compact foliation. We will see that $\mathcal{W}^c$ is in fact uniformly compact (meaning that the leaf volume function is bounded in M).

Remark 3.1. We can assume from now on that all bundles E^s, E^{cs}, E^c, E^{cu} and E^u are orientable.

Proof. By taking a finite cover of M we can lift all bundles E^s, E^{cs}, E^c, E^{cu} and E^u to orientable bundles. Then f lifts to a dynamically coherent partially hyperbolic diffeomorphism $\tilde{f}$ whose center foliation $\tilde{\mathcal{W}}^c$ is the lift of $\mathcal{W}^c$. The lifted center foliation

$\tilde{\mathcal{W}}^c$ remains compact, and each one of its leaves is a finite cover of some leaf of $\mathcal{W}^c$. Then if $\tilde{\mathcal{W}}^c$ is uniformly compact, so is $\mathcal{W}^c$. □

Let us first show a simple but yet crucial consequence of the codimension one hypothesis.

LEMMA 3.2. *The group* $\mathrm{Hol}^u(\mathcal{C})$ *is trivial for every center leaf* $\mathcal{C}$.

Proof. Let $\mathcal{C}$ be a center leaf. Recall that $\mathrm{Hol}^u(\mathcal{C})$ consists of the holonomy maps associated to $\mathcal{C}$ inside the center-unstable leaf $W^{cu}(\mathcal{C})$.

Let x be a point in $\mathcal{C}$ and $\gamma : [0, 1] \to \mathcal{C}$ a closed curve based in x. Consider $\lambda' \subset \lambda \subset \mathcal{W}^u(x)$ small enough one-dimensional transversals through x such that the holonomy return map associated to γ is a well-defined map $h_\gamma : \lambda' \to \lambda$. As we are assuming that the unstable foliation is orientable, the map h_γ preserves the orientation of λ.

Assume that h_γ is not the identity. Then there exists $y \in \lambda'$ such that $\{h_\gamma^n(y)\}_{n\geq 0}$ or $\{h_\gamma^{-n}(y)\}_{n\geq 0}$ constitutes an infinite set of points lying in λ. Assume without loss of generality it is the former.

Fix U a small foliated neighborhood of $\mathcal{W}^c$ containing λ. Each point of $h_\gamma^n(y)$ lies on a different plaque of U. Since each one of these plaques belong to $\mathcal{C}_y$, this contradicts the fact that $\mathcal{C}_y$ is compact. □

From the previous lemma, we deduce that each center leaf $\mathcal{C}$ has in $W^{cu}(\mathcal{C})$ a product neighborhood of the form $\mathcal{C} \times (-\delta, \delta)^u$, where each $\mathcal{C} \times \{y\}$ corresponds to a center leaf and each $\{x\} \times (-\delta, \delta)^u$ corresponds to an unstable arc (see Figure 1). This will allow us to use Remark 2.12.

This kind of 'stacking' of center leaves along the unstable direction implies that stable holonomy is constant along unstable leaves (see Figure 2).

LEMMA 3.3. *For every* $x \in M$ *and* $y \in W^u(x)$ *the groups* $\mathrm{Hol}^s(\mathcal{C}_x)$ *and* $\mathrm{Hol}^s(\mathcal{C}_y)$ *are isomorphic.*

Proof. Observe that it is enough to give a local argument. Suppose that for every $x \in M$ there is an unstable arc $(x - \delta, x + \delta)^u$ such that $\mathrm{Hol}(\mathcal{C}_y) \simeq \mathrm{Hol}(\mathcal{C}_x)$ for every $y \in (x - \delta, x + \delta)^u$. This implies that the set $\{y \in W^u(x) : \mathrm{Hol}(\mathcal{C}_y)$ is isomorphic to $\mathrm{Hol}(\mathcal{C}_x)\}$ is an open subset of $W^u(x)$ as well as its complement, and thus the result follows.

Let x be a point in M and denote by $\mathcal{C}$ the center leaf through x. Since $\mathcal{C}$ is compact there exists $\delta > 0$ such that $\mathrm{B}^s_\delta(z) \cap \mathrm{B}^s_\delta(z') = \emptyset$ for every distinct $z, z' \in \mathcal{C}$. Thus $\bigcup_{z\in\mathcal{C}} \mathrm{B}^s_\delta(z)$ (denote it by $\mathrm{B}^s_\delta(\mathcal{C})$) is in the hypothesis of the Proposition 2.11 and we can consider a $(\mathcal{W}^{cs}, \mathcal{W}^u)$-product neighborhood $\mathrm{B}^s_\delta(\mathcal{C}) \times (-1, 1)$.

Given $t \in (-1, 1)$, we can define a projection $p_t : \mathrm{B}^s_\delta(\mathcal{C}) \times \{0\} \to \mathrm{B}^s_\delta(\mathcal{C}) \times \{t\}$ along unstable leaves inside $\mathrm{B}^s_\delta(\mathcal{C}) \times (-1, 1)$, namely,

$$p_t(z, 0) = (z, t).$$

The projection p_t then identifies $\mathcal{W}^c|_{\mathrm{B}^s_\delta(\mathcal{C})\times\{0\}}$ homeomorphically with $\mathcal{W}^c|_{\mathrm{B}^s_\delta(\mathcal{C})\times\{t\}}$ since the leaves of $\mathcal{W}^c$ are the connected components of the intersection of leaves of $\mathcal{W}^{cs}$

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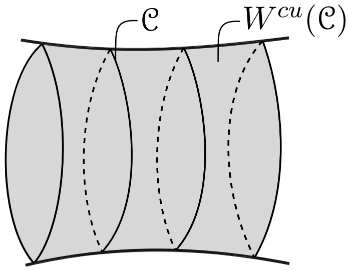

FIGURE 1. Local picture of a center-unstable leaf.

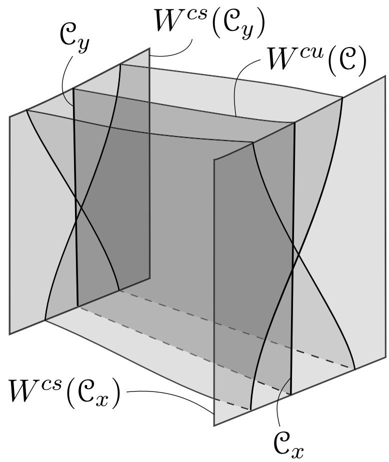

FIGURE 2. Stable holonomy is preserved along the unstable direction.

with leaves of $\mathcal{W}^{cu}$. This implies that

$$\mathrm{Hol}^s(\mathcal{C}_y) \simeq \mathrm{Hol}^s(\mathcal{C}_x)$$

for every $y \in \{x\} \times (-1, 1)$. □

Recall from the preliminaries that we denote the bad set $\mathcal{B}$ of $\mathcal{W}^c$ as the points of M in which the leaf volume function is not locally bounded.

COROLLARY 3.4. *The bad set $\mathcal{B}$ of $\mathcal{W}^c$ is saturated by the center-unstable foliation.*

Proof. From Proposition 2.6 we have that a center leaf belongs to $\mathcal{B}$ if and only if $|\mathrm{Hol}(\mathcal{C})| = \infty$. By Proposition 2.10 and Lemma 3.2, we have that $|\mathrm{Hol}(\mathcal{C})| < \infty$ if and only if $|\mathrm{Hol}^s(\mathcal{C})| < \infty$.

Lemma 3.3 then implies that $\mathcal{B}$ is saturated by the unstable foliation. As it is also saturated by the center foliation (see Remark 2.5) the result follows. □

We obtain the following corollary.

COROLLARY 3.5. *The bad set $\mathcal{B}$ of $\mathcal{W}^c$ is a proper attractor. In particular, there are no transitive, codimension one, dynamically coherent partially hyperbolic diffeomorphisms with compact center foliation and unbounded volume of leaves.*

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Proof. The set $\mathcal{B}$ is compact, f-invariant, has empty interior and is saturated by the center-unstable foliation (see Corollary 3.4). In particular, $\mathcal{B}$ is transversally stable and not all M, and thus a proper attractor. This implies that f cannot be transitive. □

The aim of the rest of the work is to show that same result follows in the non-transitive scenario.

We finish this section by noting that, by what we have done up to this point and the work of Gogolev in [**G12**], one is already able to disregard the Sullivan foliation [**S76**] as the center foliation of a dynamically coherent partially hyperbolic system.

Remark 3.6. The Sullivan foliation [**S76**] cannot be the center foliation of a dynamically coherent partially hyperbolic diffeomorphism.

Proof. The example given by Sullivan is a foliation by circles in a five-dimensional compact space M with unbounded length of leaves.

Assume that Sullivan's foliation is the center foliation of a dynamically coherent partially hyperbolic diffeomorphism. As the center foliation is one-dimensional (in a five-dimensional manifold), Gogolev's work implies that it has uniformly bounded volume of leaves if $\dim(E^s) = 2$ and $\dim(E^s) = 2$ (see the main theorem in [**G12**]).

It remains to rule out the codimension one case. Assume without loss of generality that $\dim(E^u) = 1$.

Let us denote the Sullivan foliation by $\mathcal{F}$. In this particular case, the bad set $\mathcal{B}$ of $\mathcal{F}$ has the structure of the unitary tangent bundle T^1S^2 of a 2-sphere. Moreover, the leaves of the foliation in $\mathcal{B}$ are exactly the fibers of this unitary tangent bundle.

By Corollary 3.4, the set $\mathcal{B}$ is saturated by the center-unstable foliation, and so center-unstable leaves foliate $\mathcal{B}$.

Given a center leaf $\mathcal{C}$, we have that $\mathrm{Hol}^u(\mathcal{C}) = \mathrm{Id}$ and then $\mathcal{C}$ has a neighborhood $\mathcal{C} \times (-\delta, \delta)^u$ in $W^{cu}(\mathcal{C})$ such that each $\mathcal{C} \times \{y\}$ is a center leaf (see Figure 1).

This implies that the center-unstable foliation in $\mathcal{B} \simeq T^1S^2$ projects to a (topological) foliation without singularities in the base S^2. This is impossible. □

4. *Proof of the non-transitive case*

We saw in the previous section that f cannot be transitive if the bad set $\mathcal{B}$ of $\mathcal{W}^c$ is not empty. Lose the transitivity hypothesis and assume that $\mathcal{B}$ is non-empty. We will see that this yields a contradiction.

4.1. *Construction of the repeller $\mathcal{R}$.*

Let us consider $\mathcal{R}$ the repelling set induced by the attractor set $\mathcal{B}$ (see Figure 3):

$$\mathcal{R} = M \Big\backslash \bigcup_{x \in \mathcal{B}} W^s(x).$$

We will next closely study $\mathcal{R}$. In particular, we will see that is saturated by center-stable leaves.

For every $r > 0$, let us write $\mathrm{B}^s_r(\mathcal{B}) = \bigcup_{x\in\mathcal{B}} \mathrm{B}^s_r(x)$.

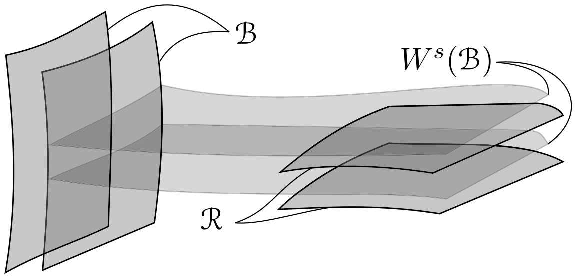

FIGURE 3. Repeller $\mathcal{R}$.

LEMMA 4.1. *The set $\mathcal{R}$ is non-empty, f-invariant, compact and saturated by the stable foliation.*

Proof. Since $\mathcal{B}$ is saturated by leaves of $\mathcal{W}^{cu}$, it follows that $\bigcup_{x\in\mathcal{B}} W^s(x)$ is open. This implies that $\mathcal{R}$ is compact.

Let us show that $\mathcal{R}$ is not empty. The set $\bigcup_{x\in\mathcal{B}} W^s(x)$ is the increasing union of the open sets $\{\bigcup_{x\in\mathcal{B}} \mathrm{B}^s_n(x)\}_n$. If it were the case that $M = \bigcup_{x\in\mathcal{B}} W^s(x)$, then M would coincide with $\mathrm{B}^s_{n_0}(\mathcal{B})$ for some n_0. Then $f^{-n}(M)$ would be contained in a small neighborhood of the proper compact subset $\mathcal{B}$ for some big enough n, which of course is impossible.

Since $\bigcup_{x\in\mathcal{B}} W^s(x)$ is f-invariant and saturated by $\mathcal{W}^s$, the same holds for $\mathcal{R}$. □

We are on the way to proving that $\mathcal{R}$ is also saturated by the center foliation. The proof will rely on the following three lemmas.

The next lemma is the main observation that will allow us to continue to work in a neighborhood of $\mathcal{R}$ as if the center foliation were uniformly compact.

LEMMA 4.2. *For every $\epsilon > 0$ there exists $K > 0$ such that if $\mathcal{C}$ is a center leaf with* $\mathrm{vol}(\mathcal{C}) > K$ *then* $\mathcal{C} \subset \mathrm{B}^s_\epsilon(\mathcal{B})$.

Proof. Fix $\epsilon > 0$ and suppose that there is no such K. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $M\setminus \mathrm{B}^s_\epsilon(\mathcal{B})$ such that $\mathrm{vol}(\mathcal{C}_{x_n}) \xrightarrow{n} \infty$. By taking a convergent subsequence $x_{n_k} \xrightarrow{k} x \in M\setminus \mathrm{B}^s_\epsilon(\mathcal{B})$ we obtain that vol is not locally bounded in x. So x should be a point of $\mathcal{B}$, but that is impossible since $x \in M\setminus \mathrm{B}^s_\epsilon(\mathcal{B})$. □

The following lemma relates the diameter and volume of leaves.

LEMMA 4.3. *For every $K > 0$ there exists $D > 0$ such that if $\mathcal{C}$ is a center leaf with* $\mathrm{vol}(\mathcal{C}) < K$ *then* $\mathrm{diam}(\mathcal{C}) < D$.

Proof. Take $\{U_i\}_{i=1,\dots,l}$ a finite covering of M by foliated boxes of the center foliation. For each $i \in \{1, \dots, l\}$ denote by d_i the supremum of the diameter of center plaques in U_i, and by v_i the infimum of the volume of center plaques in U_i. Write $d = \max_i\{d_i\}$ and $v = \min_i\{v_i\}$.

Now let $\mathcal{C}$ be a center leaf with $\mathrm{vol}(\mathcal{C}) < K$. We have that $\mathcal{C}$ has less than $K/v + 1$ plaques in each U_i and then $\mathrm{diam}(\mathcal{C}) < ld(K/v + 1) = D$. □

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From the continuity of $\mathcal{W}^c$ we have our next lemma.

LEMMA 4.4. *Given $D > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that for any x and y with* $d(x, y) < \delta$ *we have that* $B^c_D(y) \subset B_\epsilon(\mathcal{C}_x)$.

We can now prove the following proposition.

PROPOSITION 4.5. *The set $\mathcal{R}$ is saturated by the center foliation.*

Proof. Suppose that there exist $x \in \mathcal{R}$ and $y \in \mathcal{C}_x$ such that $y \in M \backslash \mathcal{R}$. Since $y \in M \backslash \mathcal{R}$ there exists $w \in \mathcal{B}$ such that $y \in W^s(w)$.

Denote $d = d(\mathcal{B}, \mathcal{R}) > 0$. Note by Lemma 4.2 that vol is bounded in $\mathcal{R}$, say by some constant $K > 0$.

By Lemma 4.3 there exists $D > 0$ such that for every center leaf $\mathcal{C}$ with $\text{vol}(\mathcal{C}) < K$ the diameter of $\mathcal{C}$ is less than D. So for every $z \in \mathcal{R}$ we have that $B^c_D(z) = \mathcal{C}_z$.

We can now consider N large enough so that, by Lemma 4.4, the points $f^N(w)$ and $f^N(y)$ are close enough to ensure that $B^c_D(f^N(y)) \subset B_{d/2}(\mathcal{B})$. This yields a contradiction since $f^N(y) \in \mathcal{C}_{f^N(x)}$ and $f^N(x) \in \mathcal{R}$ since $\mathcal{R}$ is f-invariant. This shows that for every $x \in \mathcal{R}$ the leaf $\mathcal{C}_x \subset \mathcal{R}$. □

4.2. *Completeness and trivial holonomy for center-stable leaves in $\mathcal{R}$.* In this subsection we prove some properties of $\mathcal{R}$ in order to implement the proof of Hiraide in §4.3.

Let us first show in the following proposition that center-stable leaves in $\mathcal{R}$ are *complete* (this terminology is used in [**BW05**, **C15**, **BB16**]).

PROPOSITION 4.6. *For every center leaf $\mathcal{C}$ in $\mathcal{R}$ we have that* $W^{cs}(\mathcal{C}) = \bigcup_{x\in\mathcal{C}} W^s(x)$.

Proof. Let $\mathcal{C}$ be a center leaf in $\mathcal{R}$. If we prove that $\bigcup_{x\in\mathcal{C}} W^s(x)$ is saturated by center leaves then $\bigcup_{x\in\mathcal{C}} W^s(x)$ will be a non-empty open and closed subset of $W^{cs}(\mathcal{C})$ and then will coincide with $W^{cs}(\mathcal{C})$.

Let us consider $y_0 \in \bigcup_{x\in\mathcal{C}} W^s(x)$. We want to show that $\mathcal{C}_{y_0} \subset \bigcup_{x\in\mathcal{C}} W^s(x)$. Let x_0 be any point in $\mathcal{C}$ and let $\gamma : [0, 1] \to W^{cs}(\mathcal{C})$ be a continuous path from x_0 to y_0.

For every $t \in [0, 1]$ denote by $\mathcal{C}_t$ the center leaf through $\gamma(t)$.

Recall that by Proposition 2.6 the center leaves in $\mathcal{B}$ coincide with those with infinite holonomy group. Thus, since $\mathcal{R} \cap \mathcal{B} = \emptyset$, for every $t \in [0, 1]$ we have that $\mathcal{C}_t \subset W^{cs}(\mathcal{C})$ satisfies $|\text{Hol}^s(\mathcal{C}_t)| < \infty$.

We can then take for each $\mathcal{C}_t$ a neighborhood $\mathcal{U}^s(\mathcal{C}_t)$ of $\mathcal{C}_t$ in $W^{cs}(\mathcal{C}_t)$ given by the generalized Reeb stability theorem (see Theorem 2.1). The associated projection $\pi_t : \mathcal{U}^s(\mathcal{C}_t) \to \mathcal{C}_t$ can be taken such that $\pi_t^{-1}(x)$ is a disk in $W^s(x)$ for every $x \in \mathcal{C}_t$ (see Remark 2.12).

Then $\{\mathcal{U}^s(\mathcal{C}_t)\}_{t\in[0,1]}$ is an open cover of $\gamma([0, 1])$. Let us take a finite subcover $\{\mathcal{U}^s(\mathcal{C}_{t_0}), \ldots, \mathcal{U}^s(\mathcal{C}_{t_k})\}$ such that $\mathcal{C}_{x_0} = \mathcal{C}_{t_0}$, $\mathcal{C}_{y_0} = \mathcal{C}_{t_k}$ and $\mathcal{U}^s(\mathcal{C}_{t_i}) \cap \mathcal{U}^s(\mathcal{C}_{t_{i+1}}) \neq \emptyset$ for every $0 \le i \le k - 1$.

Observe that, if $\mathcal{C}'$ is a center leaf in some $\mathcal{U}^s(\mathcal{C}_{t_i})$, then each stable disk of a point of $\mathcal{C}_{t_i}$ intersects $\mathcal{C}'$.

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Observe also that since each $\mathcal{U}^s(\mathcal{C}_{t_i})$ is saturated by center leaves, so it is each $\mathcal{U}^s(\mathcal{C}_{t_i}) \cap \mathcal{U}^s(\mathcal{C}_{t_{i+1}})$.

Then, by taking $\mathcal{C}_i' \subset \mathcal{U}^s(\mathcal{C}_{\gamma(t_i)}) \cap \mathcal{U}^s(\mathcal{C}_{\gamma(t_{i+1})})$ we deduce that each stable leaf of $\mathcal{C}_i$ intersects $\mathcal{C}_{i+1}$.

This implies that $\mathcal{C}_{y_0} \subset \bigcup_{x \in \mathcal{C}} W^s(x)$ as we wanted. □

The following result is a mild extension of Lemma 4.2 that will come in handy later.

LEMMA 4.7. *There exists a constant $C > 0$ such that for every x in $\mathcal{R}$ we have $\#\{\mathcal{C}_x \cap W^s(x)\} < C$.*

Proof. Cover $\mathcal{R}$ by a finite number $\{U_i\}_{1 \leq i \leq k}$ of foliated boxes for the $\mathcal{W}^c$ foliation. Let $d > 0$ be such that each plaque of each U_i has volume larger than d.

By Lemma 4.2 the volume function is bounded in $\mathcal{R}$, say by some constant K.

Let us show that $\#\{\mathcal{C}_x \cap W^s(x)\} < K/d + 1$ for every $x \in \mathcal{R}$.

Suppose there exist $x \in \mathcal{R}$ and distinct points $\{x = x_0, \ldots, x_l\} \subset \{\mathcal{C}_x \cap W^s(x)\}$ with $l \geq K/d + 1$.

Let $\gamma > 0$ be a Lebesgue number for the covering $\{U_i\}_{1 \leq i \leq k}$. Then we can consider N large enough to ensure that $\operatorname{diam}(\{f^N(x_0), \ldots, f^N(x_l)\}) < \gamma$ in $W^s(f^N(x_0))$ with the intrinsic topology. So $\{f^N(x_0), \ldots, f^N(x_l)\}$ is contained in some member U_{i_0} of the covering.

Since the points $\{f^N(x_0), \ldots, f^N(x_l)\}$ are close in $W^s(f^N(x_0))$ in the intrinsic topology, then each one of them lies on a different plaque of U_{i_0}. On the other hand, $\{f^N(x_0), \ldots, f^N(x_l)\} \subset f^N(\mathcal{C}_x)$. This contradicts the fact that $\operatorname{vol}(f^N(\mathcal{C}_x)) \leq K$. □

We can now give some kind of a description of center-stable leaves in $\mathcal{R}$ (see Bohnet [**B13**, Corollary 4.10] for a similar result).

PROPOSITION 4.8. *Let W be a center-stable leaf in $\mathcal{R}$. Then there exists a center leaf $\mathcal{C}$ in W such that for every $x \in \mathcal{C}$ we have that $\mathcal{C} \cap W^s(x) = \{x\}$. Therefore, W is a bundle with base $\mathcal{C}$ and fibers $\{W^s(x)\}_{x \in \mathcal{C}}$.*

Proof. Observe that it is enough to prove what we want for some $f^N(W)$.

Cover $\mathcal{R}$ by a finite $(\mathcal{W}^{cs}, \mathcal{W}^u)$-product neighborhood $\{\mathcal{U}^s(\mathcal{C}_i) \times [-1, 1]\}_{1 \leq i \leq k}$ with each $\mathcal{U}^s(\mathcal{C}_i)$ being a Generalized Reeb stability neighborhood of the center leaf $\mathcal{C}_i$ (see Remark 2.12).

Let $\gamma > 0$ be a Lebesgue number for the covering.

Let W be a center-stable leaf in $\mathcal{R}$. By the previous lemma, we have that $\#\{\mathcal{C}_x \cap W^s(x)\}$ is bounded by a constant $C > 0$ for every $x \in W$. So, let us consider $x_0 \in W$ such that $l = \#\{\mathcal{C}_{x_0} \cap W^s(x_0)\}$ is maximal in W. Let $\{x_0, \ldots, x_{l-1}\} = \mathcal{C}_{x_0} \cap W^s(x_0)$.

Let $N > 0$ be large enough so that $\operatorname{diam}(\{f^N(x_0), \ldots, f^N(x_{l-1})\}) < \gamma$ in $W^s(f^N(x_0))$ with the intrinsic topology. Then there exist $i \in \{1, \ldots, k\}$ and $t \in [-1, 1]$ such that $\{f^N(x_0), \ldots, f^N(x_{l-1})\} \subset \mathcal{U}^s(\mathcal{C}_i) \times \{t\}$. Moreover, $\{f^N(x_0), \ldots, f^N(x_{l-1})\}$ belongs to the same s-disk in the generalized Reeb stability neighborhood $\mathcal{U}^s(\mathcal{C}_i) \times \{t\}$. So each s-disk of $\mathcal{U}^s(\mathcal{C}_i) \times \{t\}$ intersects $f^N(\mathcal{C}_{x_0})$ in at least l distinct points.

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This implies that $(\mathcal{C}_i \times \{t\}) \cap W^s(x) = \{x\}$ for every $x \in \mathcal{C}_i \times \{t\}$. Otherwise, $f^N(\mathcal{C}_{x_0})$ would intersect some stable leaf in at least $2l$ distinct points and this is impossible since $l = \#\{\mathcal{C}_{x_0} \cap W^s(x_0)\}$ is maximal in W and, therefore, also in $f^N(W)$. □

Remark 4.9. For W as in Proposition 4.8 we have that the group $\mathrm{Hol}(W)$ is trivial. Indeed, any closed curve in W is freely homotopic to a closed curve in $\mathcal{C}$, which has trivial unstable holonomy (see Lemma 3.2).

4.3. *Adapted Hiraide arguments to rule out the existence of $\mathcal{R}$.* This last subsection is dedicated to prove that the set $\mathcal{R}$ as described before cannot exist. The proof we give is an adaptation of the work by Hiraide in [**H01**] and by Bohnet in [**B13**]. However, the proof itself is self-contained.

The key advantage of Hiraide's proof over Newhouse's (Anosov case; see [**N70**]) is that the former takes place in a neighborhood of the repeller while the latter makes a more global argument. As we want to avoid dealing with the bad set, we find it more convenient to follow Hiraide's proof. It is worth mentioning that for the reasons just mentioned the authors could not directly adapt Newhouse's proof.

From now on we will work with both $\mathcal{R}$ and its boundary $\partial\mathcal{R}$ in M. Note that, as well as $\mathcal{R}$, the set $\partial\mathcal{R}$ is non-empty, closed, saturated by the center-stable foliation, has trivial transversal holonomy, and the volume of its center leaves is uniformly bounded. The set $\partial\mathcal{R}$ has empty interior.

Let us fix an orientation of $\mathcal{W}^u$.

For every $x \in \mathcal{R}$ we can consider $\mathcal{U}^s(x) \times [-1, 1]$ a $(\mathcal{W}^{cs}, \mathcal{W}^u)$-product neighborhood of x with $\mathcal{U}^s(x)$ a small center-saturated generalized Reeb stability neighborhood of $\mathcal{C}_x$ (see Remark 2.12).

Let $\{V_i = \mathcal{U}^s(x_i) \times (-1, 1)\}_{0 \leq i \leq k}$ be a finite cover of $\mathcal{R}$. Define $\mathcal{V} = \bigcup_{0 \leq i \leq k} V_i$.

We will show that for certain points near $\partial\mathcal{R}$ the center-stable leaf through this point must remain in $\mathcal{V}$ (see Lemma 4.13) while it must also intersect $\mathcal{B}$, thus yielding a contradiction.

Remark 4.10. We can assume that:

- $\mathcal{V} \cap \mathcal{B} = \emptyset$ (this is obtained by taking each $\mathcal{U}^s(x) \times [-1, 1]$ disjoint from $\mathcal{B}$);
- $\mathcal{U}^s(x_i) \times \{1\} \cap \partial\mathcal{R} = \emptyset$ (we can assume this since $\partial\mathcal{R}$ has empty interior).

For every $i \in \{0, \ldots, k\}$ let $0 < t_i < 1$ be such that $\mathcal{U}(x_i) \times [t_i, 1] \cap \partial\mathcal{R} = \mathcal{U}(x_i) \times \{t_i\}$. Denote each $\mathcal{U}(x_i) \times \{t_i\}$ by P_i^+. Informally speaking, P_i^+ is the last (according to the orientation of $\mathcal{W}^u$) center-stable plaque of V_i that is contained in $\partial\mathcal{R}$.

LEMMA 4.11. *There exists a pair $(x_0, \delta) \in \partial\mathcal{R} \times \mathbb{R}^+$ such that either $(x_0, x_0 + \delta)^u \cap \mathcal{R} = \emptyset$ or $(x_0 - \delta, x_0)^u \cap \mathcal{R} = \emptyset$.*

Proof. Observe first that the set $\mathcal{R}$ cannot be saturated by the unstable foliation (because in that case it would be all M), so there must exist a point $x \in \mathcal{R}$ such that $W^u(x) \cap M \backslash \mathcal{R} \neq \emptyset$.

As $\mathcal{R} \cap W^u(x)$ is closed in $W^u(x)$ with the intrinsic topology then there must be at least one connected component I of $W^u(x) \backslash \mathcal{R}$. Choose x_0 an endpoint of I. □

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Fix (x_0, δ) given by the previous lemma. Assume that $(x_0, x_0 + \delta)^u \cap \mathcal{R} = \emptyset$ (otherwise, simply change the orientation of $\mathcal{W}^u$).

By Propositions 4.6 and 4.8 we have $W^{cs}(x_0) = \bigcup_{x \in \mathcal{C}} W^s(x)$ for some center leaf $\mathcal{C} \subset W^{cs}(x_0)$ such that $\mathcal{C} \cap W^s(x) = \{x\}$ for every $x \in \mathcal{C}$. Assume, without loss of generality, that $\mathcal{C}_{x_0}$ is such a center leaf.

Some of the plaques $\{P_1^+, \ldots, P_k^+\}$ could possibly be contained in $W^{cs}(x_0)$. Denote them all by $\{P_{i_1}^+, \ldots, P_{i_m}^+\}$. We can now consider $N > 0$ large enough to ensure that

$$(P_{i_1}^+ \cup \cdots \cup P_{i_m}^+) \subset \bigcup_{x \in \mathcal{C}_{x_0}} \mathrm{B}_N^s(x).$$

If $W^{cs}(x_0)$ contains none of the plaques $\{P_1^+, \ldots, P_k^+\}$ then take $N > 0$ to be any positive number.

For simplicity, let us denote $\bigcup_{x \in \mathcal{C}_{x_0}} \mathrm{B}_N^s(x)$ as E. The subset E of $W^{cs}(x_0)$ is then in the hypothesis of the Proposition 2.11 (in particular, $\mathrm{Hol}^u(E) = \mathrm{Id}$) so we can consider a $(\mathcal{W}^{cs}, \mathcal{W}^u)$-product neighborhood $E \times [-1, 1]$ of it. Recall that this means that each $E \times \{y\}$ lies in a center-stable leaf, each $\{x\} \times [-1, 1]$ lies in an unstable leaf and $E \times \{0\} = E$.

By eventually shrinking it in the unstable direction, we can assume that $E \times [-1, 1]$ is contained in $\mathcal{V}$.

LEMMA 4.12. *Through every $x \in W^{cs}(x_0) \backslash E$ there exists $L_x = [x, x + \delta_x]^u$ a closed, non-trivial unstable segment that intersects $\mathcal{R}$ just in its endpoints. Moreover, L_x varies continuously with x in $W^{cs}(x_0) \backslash E$ and is contained in any element V_i of the covering that contains x.*

Proof. Let $x \in W^{cs}(x_0) \backslash E$. The point x lies in some V_i. Then it must be that the connected component of $(W_+^u(x) \cup \{x\}) \cap V_i$ that contains x intersects $\mathcal{R}$ in at least some other point distinct from x (since $x \notin \bigcup_{0 \leq i \leq k} P_i^+$).

Observe that the fact that x lies inside $W^{cs}(x_0)$ and $(x_0, x_0 + \delta)^u$ is disjoint from $\mathcal{R}$ implies that x cannot be accumulated in $W_+^u(x) \cup \{x\}$ by points of $W_+^u(x) \cap \mathcal{R}$. The existence of the stated $\delta_x > 0$ follows.

The definition of L_x does clearly not depend on the choice of the V_i containing x. The continuous dependence on x follows. □

By shrinking $E \times [-1, 1]$ even more (if necessary) in the unstable direction, we can assume that $\{x\} \times [0, 1] \subset L_x$ for every $x \in \partial E$.

Now, let $y \in (x_0, x_0 + \delta)^u$ be close enough to x_0 such that $y \in E \times [0, 1]$. Since $y \in M \backslash \mathcal{R}$ we have that $W^{cs}(y) \cap \mathcal{B} \neq \emptyset$. On the other hand, we will see that $W^{cs}(y) \subset \mathcal{V}$, and this will yield a contradiction since $\mathcal{V} \cap \mathcal{B} = \emptyset$.

Then, the proof of the main theorem will be finished with the following.

LEMMA 4.13. (Sandwich lemma) *For y as above we have $W^{cs}(y) \subset \mathcal{V}$.*

Proof. First, note that $\bigcup_{x \in W^{cs}(x_0) \backslash E} L_x$ is a foliated interval bundle with base $W^{cs}(x_0) \backslash E$ and fibers the L_xs that are transversal to $\mathcal{W}^{cs}$ for every x.

We then have a well-defined projection along fibers $\pi : \bigcup_{x \in W^{cs}(x_0) \backslash E} L_x \to W^{cs}(x_0) \backslash E$ given by $\pi([x, x + \delta_x)) = x$ for $L_x = [x, x + \delta_x]$.

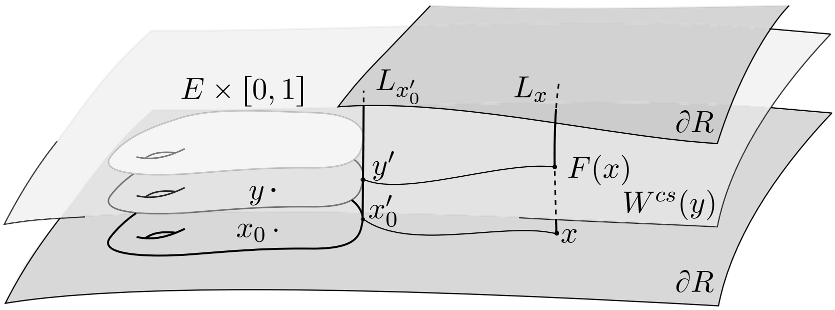

FIGURE 4. The leaf $W^{cs}(y)$ gets enclosed between the leaves of $\partial\mathcal{R}$.

For every $\gamma : [0, 1] \to W^{cs}(x_0)\backslash E$ and $z \in L_{\gamma(0)}$ we can lift γ to $\gamma_z : [0, 1] \to W^{cs}(z)$ such that $\pi \circ \gamma_z = \gamma$. This lift defines a projection $p_\gamma : L_{\gamma(0)} \to L_{\gamma(1)}$ given by $p_\gamma(z) = \gamma_z(1)$.

Fix $x_0' \in \partial E$.

CLAIM. *For every curve $\gamma : [0, 1] \to W^{cs}(x_0)\backslash E$ such that $\gamma(0) = \gamma(1) = x_0'$ there exists a homotopy $\gamma_s : [0, 1] \to W^{cs}(x_0)\backslash E$ with endpoints fixed such that $\gamma_0 = \gamma$ and $\gamma_1([0, 1]) \subset \partial E$.*

Proof of claim. For every $s \in (0, 1]$ we can consider a retraction $r_s : W^{cs}(x_0) = \bigcup_{x\in\mathcal{C}_{x_0}} W^s(x) \to \bigcup_{x\in\mathcal{C}_{x_0}} \mathrm{B}^s_{N/s}(x)$ such that r_s varies continuously with s and is the identity in $E = \bigcup_{x\in\mathcal{C}_{x_0}} \mathrm{B}^s_N(x)$. For every $s \in (0, 1]$ compose γ with r_s to get a curve γ_s and set γ_0 as γ. Then the homotopy γ_s is as desired, and this proves the claim. □

Now, for every $x \in W^{cs}(x_0)\backslash E$ consider $\gamma_x : [0, 1] \to W^{cs}(x_0)\backslash E$ such that $\gamma_x(0) = x_0'$ and $\gamma_x(1) = x$.

Remark. The existence of such a γ_x is guaranteed if $\dim(E^s) \geq 2$ for this implies that $W^{cs}(x_0)\backslash E$ is path connected. Otherwise, if $\dim(E^s) = 1$ and $W^{cs}(x_0)\backslash E$ has two connected components simply add $x_0'' \in \partial E$ not in the same connected component as x_0' and reproduce the arguments that follow separately in each connected component, with x_0' and x_0'' playing the same role.

Denote $y = (x_0, t)$ in $E \times (0, 1)$. Observe that we can extend $E \times (0, 1)$ to a $(\mathcal{W}^{cs}, \mathcal{W}^u)$-product neighborhood $(E \cup \partial E) \times (0, 1)$. Denote by y' the point (x_0', t) in these coordinates (see Figure 4).

Define $F : W^{cs}(x_0)\backslash E \to W^{cs}(y)$ by

$$F(x) = p_{\gamma_x}(y').$$

CLAIM. *The definition of F does not depend on the choice of γ_x.*

Proof of claim. If $\gamma_x' : [0, 1] \to W^{cs}(x_0)\backslash E$ is another path such that $\gamma_x'(0) = x_0'$ and $\gamma_x'(1) = x$, we have to prove that $\gamma_x'^{-1} * \gamma_x$ lifts to a closed path $(\gamma_x * \gamma_x'^{-1})_{y'}$ from y' to itself.

By the first claim, $\gamma_x'^{-1} * \gamma_x$ is homotopic to some closed path $\alpha : [0, 1] \to W^{cs}(x_0)\backslash E$ such that $\alpha([0, 1]) \subset \partial E$. This homotopy lifts to a homotopy contained in $W^{cs}(y')$ from $(\gamma_x'^{-1} * \gamma_x)_{y'}$ to $\alpha_{y'}$, with $\alpha_{y'}$ being the lift of α from y'.

Recall that we have $y' \in (E \cup \partial E) \times \{t\}$. Then the lift $\alpha_{y'}$ has to be closed because $\alpha_{y'}([0, 1])$ is contained in $W^{cs}(y')$ and this implies $\alpha_{y'}([0, 1]) \subset (\partial E \times \{t\})$ and then $\alpha_{y'}(1) = (\partial E \times \{t\}) \cap L_{x_0'} = y'$. This proves the claim. □

So we have a well-defined map $F : W^{cs}(x_0)\backslash E \to W^{cs}(y)$. We can extend F to E in the natural way: $F(x) = (x, t)$ for every $x \in E$. Then we have $F : W^{cs}(x_0) \to W^{cs}(y)$.

The map F is clearly an injective local homeomorphism. Given a Lebesgue number $\eta > 0$ for the covering $\mathcal{V} = \bigcup_{0 \le i \le k} V_i$, there exists $\delta > 0$ such that for every $x \in W^{cs}(x_0)$ we have that $\mathrm{B}^{cs}_\delta(F(x)) \subset F(\mathrm{B}^{cs}_{\eta/2}(x))$. We deduce from this that F is a proper map. Then F is also surjective, and this implies that $W^{cs}(y) \subset \mathcal{V}$ since $E \times \{t\}$ and every L_x are contained in $\mathcal{V}$. □

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