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The Dbar formalism for certain linear non-homogeneous elliptic PDEs in two dimensions

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We discuss two different approaches for the analysis of the Poisson and of the non-homogeneous biharmonic equations in two dimensions. The first approach yields the solution as an integral in the complex z-plane (the physical plane), involving explicitly the given boundary conditions. The second approach yields an integral in the complex k-plane (the Fourier plane), involving the Fourier transforms of the given boundary conditions. For simple boundary value problems, such as certain problems formulated in the half complex plane, the first approach is easier. However, for more complicated problems, such as those formulated in the interior of an equilateral triangle, it appears that only the second approach can be used. Furthermore, the second approach also seems more efficient for numerical computations.

1 Introduction

The classical Green's function approach provides a powerful method for obtaining integral representations for the solution of the basic linear elliptic PDEs, such as the Poisson and the Helmholtz equations. However, these representations involve *both* the Dirichlet and the Neumann data. Thus, to solve either the Dirichlet or the Neumann boundary value problem, one must first eliminate either the Neumann or the Dirichlet boundary values respectively. For very simple domains this can be achieved using the *method of images*. However, for more complicated domains or more complicated boundary conditions, such as mixed boundary conditions, it is *not* possible to eliminate the uknown boundary values using the method of images.

We also note that for the Poisson equation in two dimensions, it is possible to determine the Neumann boundary value in terms of the given Dirichlet boundary condition – the so-called *Dirichlet to Neumann map* – by formulating a Hilbert problem[1] in the complex z-plane. This formulation can also be used for other more complicated problems [16, 18].

In what follows we discuss two different approaches for the solution of boundary value problems for a certain class of physically significant non-homogeneous linear elliptic

[1] This problem involves determining an analytic function in a given domain by prescribing a relation between its real and imaginary parts on the boundary of the domain.

equations. These PDEs include the Poisson equation as well as the non-homogeneous biharmonic equation. A crucial role in both our approaches is played by a certain equation which couples all boundary values and which has been called by one of the authors the *global relation.*

The first approach, just like the classical Green's function method, yields the solution in the physical plane. However, in comparison with the Green's function method, it involves the following novel steps:

(a) For the derivation of the integral representation, it uses the Dbar formula.

(b) For the solution of simple boundary value problems, it uses the analysis of the global relation instead of the method of images.

(c) For the solution of more complicated boundary value problems, it also uses the analysis of the global relation instead of the formulation of a Hilbert problem.

Overall, it appears that this approach yields in a straightforward manner the solution of a wide class of boundary value problems.

The second formulation is the extension to non-homogeneous PDEs of the general approach introduced by the first author [4, 5].

The above two approaches are used for the solution of the following concrete boundary value problems:

(1) The Poisson equation in the upper half plane with Dirichlet or Neumann or the more general oblique Neumann boundary conditions (see equation (4.3)).

(2) The analogous problem for the quarter plane (see equations (5.4)–(5.7)).

(3) The Dirichlet-second Neumann boundary value problem for the non homogeneous biharmonic equation in the upper half plane (see equations (7.3)–(7.4)).

(4) The determination of the second Neumann ($\psi_{yy}(x,0)$) and of the third Neumann ($\psi_{yyy}(x,0)$) boundary values for the non homogeneous biharmonic equation in the upper half plane, in terms of the Dirichlet ($\psi(x,0)$) and Neumann ($\psi_y(x,0)$) boundary conditions (see equations (7.18)–(7.20)).

(5) The determination of the Neumann and third Neumann boundary values in terms of Dirichlet and second Neumann boundary conditions (see equations (8.4)–(8.14)) for the non homogeneous biharmonic equation in the interior of an equilateral triangle.

Notation. The usual complex variable will be denoted by z, and its complex conjugate by $\bar{z}$,

$$z = x + iy, \quad \bar{z} = x - iy. \tag{1.1}$$

The exterior product will be denoted by $\wedge$. Since, it is skew symmetric, it follows that

$$dz \wedge dz = 0, \quad d\bar{z} \wedge d\bar{z} = 0, \quad dz \wedge d\bar{z} = -d\bar{z} \wedge dz = -2i\, dx\, dy. \tag{1.2}$$

Subscripts $z, \bar{z}, x, y$, etc, will denote partial derivatives, for example $\Phi_z = \frac{\partial \Phi}{\partial z}$, etc.

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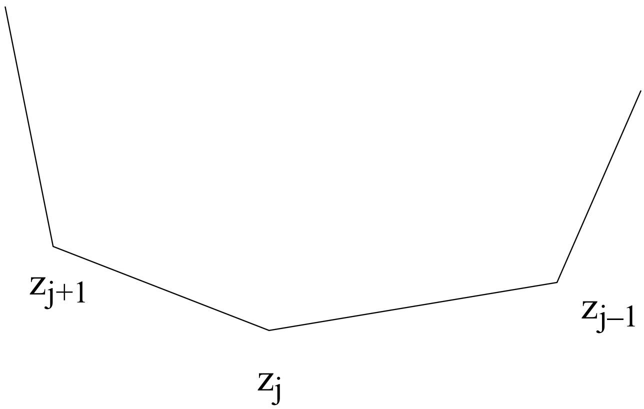

FIGURE 1. Part of the convex polygon Ω.

2 The basic mathematical formalism

Proposition 2.1 *Let the complex-valued function $\Phi(z,\bar z)$ satisfy the equation*

$$\Phi_{\bar z} = G, \quad z \in D \subset \mathbb{R}^2, \tag{2.1}$$

where z is the usual complex variable, $G(z,\bar z)$ is a given complex-valued function with appropriate smoothness, and D is a simply connected, bounded domain of the complex $z-$plane. Then: (a) Φ admits the integral representation

$$\Phi(z,\bar z) = \frac{1}{2\pi i}\int_{\partial D}\frac{\Phi(\zeta,\bar\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2\pi i}\iint_D \frac{G(\zeta,\bar\zeta)}{\zeta - z} d\zeta \wedge d\bar\zeta, \quad z \in D, \tag{2.2}$$

where $\zeta = \zeta_R + i\zeta_I$, and ∂D denotes the boundary of D.

Furthermore, the boundary values of Φ satisfy the global relation

$$\int_{\partial D}\frac{\Phi(\zeta,\bar\zeta)\,d\zeta}{\zeta - z} = -\iint_D \frac{G(\zeta,\bar\zeta)}{\zeta - z} d\zeta \wedge d\bar\zeta, \quad z \notin D. \tag{2.3}$$

(b) Suppose that D is the interior of the convex polygon Ω specified by the corners $z_1,\ldots,z_n,z_{n+1}=z_1$, see Figure 1. Define the function $F(z,\bar z)$ by the equations

$$F_{\bar z} = G, \quad F|_{\partial D} = 0. \tag{2.4}$$

Then, Φ also admits the integral representation

$$\Phi(z,\bar z) = F(z,\bar z) + \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}\widehat{\Phi}_j(k)\,dk - \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}\widehat{F}_j(k)\,dk, \quad z \in \Omega, \tag{2.5}$$

where l_j are the rays in the complex k-plane

$$l_j = \{k \in \mathbb{C} : \arg(k) = -\arg(z_j - z_{j+1})\}, \quad j = 1,\ldots,n, \tag{2.6}$$

oriented from zero to infinity, and the functions $\widehat{\Phi}_j(k), \widehat{F}(k)$ are defined by the following

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integrals along the boundary of the polygon:

$$\widehat{\Phi}_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}\Phi(z,\bar{z})\,dz, \tag{2.7}$$

$$\widehat{F}_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}F(z,\bar{z})\,dz, \quad j=1,\ldots,n, \quad k\in\mathbb{C}. \tag{2.8}$$

Furthermore, the following global relation is valid for all complex k,

$$\sum_{j=1}^{n}\widehat{\Phi}_j(k) = \sum_{j=1}^{n}\widehat{F}_j(k), \quad k\in\mathbb{C}. \tag{2.9}$$

Proof (a) Define the differential form W by

$$W(\zeta,\bar{\zeta};z) = \frac{\Phi(\zeta,\bar{\zeta})}{\zeta - z}d\zeta. \tag{2.10}$$

Then, the differential dW with respect to ζ is given by

$$dW = \frac{\partial}{\partial\bar{\zeta}}\left(\frac{\Phi(\zeta,\bar{\zeta})}{\zeta-z}\right)d\bar{\zeta}\wedge d\zeta = \left[\frac{\Phi_{\bar{\zeta}}(\zeta,\bar{\zeta})}{\zeta-z} + \Phi(\zeta,\bar{\zeta})\frac{\partial}{\partial\bar{\zeta}}\left(\frac{1}{\zeta-z}\right)\right]d\bar{\zeta}\wedge d\zeta. \tag{2.11}$$

Replacing $\Phi_{\bar{\zeta}}$ by $G(\zeta,\bar{\zeta})$, and using the identity

$$\frac{\partial}{\partial\bar{\zeta}}\left(\frac{1}{\zeta-z}\right) = -2\pi i\delta(\zeta - z), \tag{2.12}$$

we obtain

$$dW = \frac{G(\zeta,\bar{\zeta})}{\zeta-z}d\bar{\zeta}\wedge d\zeta + \begin{cases}-2\pi i\delta(\zeta-z)\Phi(\zeta,\bar{\zeta})\,d\bar{\zeta}\wedge d\zeta, & z\in D,\\ 0, & z\notin D.\end{cases} \tag{2.13}$$

Poincaré's lemma (or the complex form of Green's theorem [1]), yields

$$\int_{\partial D} W = \iint_D dW. \tag{2.14}$$

Substituting equations (2.10) and (2.13) in equation (2.14), we obtain equations (2.2) and (2.3).

(b) It is shown in Fokas *et al.* [5, 6] that if $H(z)$ is a holomorphic function, i.e. if $H(z)$ satisfies the equation

$$\partial_{\bar{z}}H = 0, \quad z\in\Omega, \tag{2.15}$$

in a convex polygon Ω specified by the corners $z_1,\ldots,z_n$, then $H(z)$ admits the following integral representation:

$$H(z) = \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}\widehat{H}_j(k)\,dk, \quad z\in\Omega, \tag{2.16}$$

where the rays l_j are defined by equation (2.6) and the functions $\widehat{H}_j(k)$ are defined by

$$\widehat{H}_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}H(z)\,dz, \quad j=1,\ldots,n, \quad k\in\mathbb{C}. \tag{2.17}$$

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Furthermore, the functions $\widehat{H_j}(k)$ satisfy the global relation

$$\sum_{j=1}^{n} \widehat{H_j}(k) = 0, \quad k \in \mathbb{C}. \tag{2.18}$$

If F is defined in terms of G by equations (2.4), then equation (2.1) can be rewritten in the form

$$\partial_{\bar{z}}(\Phi - F) = 0. \tag{2.19}$$

Replacing in equations (2.16)–(2.18), H by $\Phi - F$, we find equations (2.5)–(2.9). □

It is straightforward to show that the global relation (2.3) is equivalent with the following global relation in the complex k-plane

$$\int_{\partial D} e^{-ikz} \Phi(z, \bar{z})\, dz = -\iint_D e^{-ikz} G(z, \bar{z})\, dz \wedge d\bar{z}, \quad k \in \mathbb{C}. \tag{2.20}$$

Indeed, using the expansion

$$\frac{1}{\zeta - z} = -\sum_{j=0}^{\infty} \frac{\zeta^j}{z^{j+1}}, \quad |\zeta| < |z|,$$

in equation (2.3), and equating the coefficients of $\frac{1}{z^{j+1}}$, we find

$$\int_{\partial D} \zeta^j \Phi(\zeta, \bar{\zeta})\, d\zeta = -\iint_D \zeta^j G(\zeta, \bar{\zeta})\, d\zeta \wedge d\bar{\zeta},$$

where j is a non-negative integer. This equation is precisely the equation obtained by inserting the expansion

$$e^{-ikz} = \sum_{j=0}^{\infty} \frac{1}{j!} (-ik)^j z^j,$$

in equation (2.20) and equating the coefficients of k^j.

The global relation (2.9) can be written in the alternative form

$$\sum_{j=1}^{n} \widehat{\Phi_j}(k) = -\iint_\Omega e^{-ikz} G(z, \bar{z})\, dz \wedge d\bar{z}, \quad k \in \mathbb{C}. \tag{2.21}$$

This equation is a direct consequence of equation (2.20). Replacing in this equation G by $F_{\bar{z}}$, and using Poincaré's lemma (equation (2.14)), equation (2.21) becomes equation (2.9).

It is shown in Fokas & Zyskin [12] that equation (2.16) is a consequence of Cauchy's theorem

$$H(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{H(\zeta)\, d\zeta}{\zeta - z}. \tag{2.22}$$

Indeed, if D is the convex polygon Ω, then it can be shown (see Appendix A) that equation (2.22) can be transformed to equation (2.16).

3 The Poisson equation

There exists a large class of elliptic PDEs which can be written in the form of the basic equation (2.1). For such equations, the results of Proposition 2.1 can be used immediately. The first example of an equation that can be written in the form (2.1) is the Poisson equation.

Proposition 3.1 *(Integral representations and global relations). Let the complex-valued function $\Psi(z,\bar{z})$ satisfy the Poisson equation*

$$\Psi_{z\bar{z}} = G, \quad z \in D \subset \mathbb{R}^2, \tag{3.1}$$

where z denotes the usual complex variable, $G(z,\bar{z})$ is a given complex-valued function with appropriate smoothness and D is a simply connected, bounded domain of the complex z-plane. Then: (a) Ψ admits the integral representation

$$\Psi_z(z,\bar{z}) = \frac{1}{2\pi i}\int_{\partial D}\frac{\Psi_\zeta(\zeta,\bar{\zeta})\,d\zeta}{\zeta - z} + \frac{1}{2\pi i}\iint_D \frac{G(\zeta,\bar{\zeta})\,d\zeta\wedge d\bar{\zeta}}{\zeta - z}, \quad z \in D. \tag{3.2}$$

Furthermore, the boundary values of $\Psi(z,\bar{z})$ satisfy the global relation

$$\int_{\partial D}\frac{\Psi_\zeta(\zeta,\bar{\zeta})\,d\zeta}{\zeta - z} = -\iint_D \frac{G(\zeta,\bar{\zeta})}{\zeta - z}d\zeta\wedge d\bar{\zeta}, \quad z \notin D. \tag{3.3}$$

(b) Suppose that D is the interior of the convex polygon Ω specified by the corners $z_1,\dots,z_{n+1} = z_1$ (see Figure 1). Then, Ψ also admits the integral representation

$$\Psi_z(z,\bar{z}) = F(z,\bar{z}) + \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}\widehat{\Psi_j}(k)\,dk - \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}\widehat{F}_j(k)\,dk, \quad z \in \Omega, \tag{3.4}$$

where l_j are the rays in the complex k-plane defined by (2.6), *the functions F and $\widehat{F}_j$ are defined by equations* (2.4) *and* (2.8), *and the function $\widehat{\Psi_j}(k)$ is defined by the integral*

$$\widehat{\Psi_j}(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}\Psi_z(z,\bar{z})\,dz, \quad j = 1,\dots,n, \quad k \in \mathbb{C}. \tag{3.5}$$

Furthermore, the following global relation is valid for all complex k,

$$\sum_{j=1}^{n}\widehat{\Psi_j}(k) = \sum_{j=1}^{n}\widehat{F}_j(k), \quad k \in \mathbb{C}. \tag{3.6}$$

Proof Proposition 3.1 follows immediately from the result of the previous section by replacing Φ with Ψ_z. □

The first term of the right-hand side of (3.2), as well as the functions $\widehat{\Psi_j}(k)$, are defined in terms of Ψ_z evaluated on the boundary ∂D. The function Ψ_z involves a combination of the Dirichlet and Neumann boundary values. However, for a well-posed problem, only one of these values (or a combination) is prescribed as a boundary condition. Thus, in order to solve a concrete boundary value problem, we must first eliminate the unknown boundary values. Depending on the given boundary value problem, this can be done

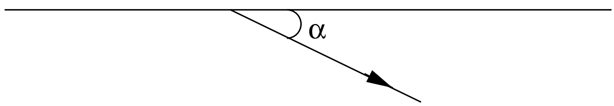

FIGURE 2. The angle α.

either using equations (3.2) and (3.3) (the z-plane approach), or using equations (3.4)–(3.6) (the k-plane approach). In what follows we will illustrate both these approaches.

4 The oblique Neumann problem for the Poisson equation in the upper half complex plane

Proposition 4.1 *(The Oblique Neumann Problem)*

Let the real-valued function $\psi(x, y)$ satisfy the Poisson equation in the upper half complex z-plane,

$$\psi_{z\bar{z}} = g, \quad Im z \geqslant 0, \tag{4.1}$$

where $g(x, y)$ is a given real-valued function with appropriate smoothness and decay. Assume that the derivative of the function ψ is prescribed along the direction making an angle α with the x-axis (see Figure 2), i.e.

$$\psi_y(x,0)\sin\alpha + \psi_x(x,0)\cos\alpha = h(x), \quad -\infty < x < \infty, \tag{4.2}$$

where $h(x)$ is a given function with appropriate smoothness and decay. Then, for $-\infty < x < \infty$, $y \geqslant 0$, $\psi_z(x, y)$ is given by

$$\psi_z = \frac{e^{i\alpha}}{2i\pi}\int_{-\infty}^{\infty}\frac{h(\xi)}{\xi-(x+iy)}d\xi - \frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\left(\frac{1}{\xi-x+i(\eta-y)} - \frac{e^{2i\alpha}}{\xi-x+i(\eta+y)}\right)g(\xi,\eta). \tag{4.3}$$

For the particular cases of the Dirichlet ($\psi_x(x,0) = h(x)$) and of the Neumann ($\psi_y(x,0) = -h(x)$) boundary value problems, the value of α in equations (4.2) and (4.3) is $\alpha = 0$ and $\alpha = \pi/2$, respectively.

Proof If D is the upper half complex plane, equations (3.2) and (3.3) become

$$\psi_z = \frac{1}{4i\pi}\int_{-\infty}^{\infty}\frac{[\psi_\xi(\xi,0) - i\psi_y(\xi,0)]}{\xi-(x+iy)}d\xi - \frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\frac{g(\xi,\eta)}{\xi-x+i(\eta-y)}, \quad -\infty < x < \infty, \quad y \geqslant 0, \tag{4.4}$$

$$0 = \frac{1}{4i\pi}\int_{-\infty}^{\infty}\frac{[\psi_\xi(\xi,0) - i\psi_y(\xi,0)]}{\xi-(x+iy)}d\xi - \frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\frac{g(\xi,\eta)}{\xi-x+i(\eta-y)}, \quad -\infty < x < \infty, \quad y \leqslant 0. \tag{4.5}$$

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Taking the complex conjugate of equation (4.5) and then replacing y by $-y$ in the resulting equation, equation (4.5) becomes

$$0 = \frac{1}{4i\pi}\int_{-\infty}^{\infty}\frac{[-\psi_\xi(\xi,0) - i\psi_y(\xi,0)]}{\xi-(x+iy)}d\xi - \frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\frac{g(\xi,\eta)}{\xi - x + i(-\eta - y)}, \quad (4.6)$$
$$-\infty < x < \infty, \quad y \geqslant 0.$$

By manipulating equations (4.4) and (4.6) it is possible to eliminate the unknown boundary values. Indeed, multiplying equations (4.4) and (4.6) by $e^{-i\alpha/2}$ and $-e^{i\alpha/2}$, respectively, and adding the resulting equations we find equation (4.3). □

We note that since ψ is real, it is straightforward to compute ψ from ψ_z.

For the Dirichlet boundary value problem, we can determine the Neumann boundary value, i.e. $\psi_y(x,0)$, by evaluating equation (4.3) with $\alpha = 0$ at $y = 0$. Alternatively, it is possible to compute *directly* the Neumann boundary value *without* first solving the problem in the interior of the domain. This can be achieved by analysing the global relation as z approaches the boundary of the domain. Actually, the following Dirichlet to Neumann correspondence is valid.

Proposition 4.2 *(The Dirichlet to Neumann correspondence for the half plane).*
Let the real-valued function $\psi(x,y)$ satisfy the Poisson equation (4.1) *in the upper-half complex z-plane.Then, the Neumann boundary value $\psi_y(x,0)$ can be expressed in terms of the Dirichlet boundary value $\psi_x(x,0)$ by the equation*

$$\psi_y(x,0) = \frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{\psi_\xi(\xi,0)}{\xi - x}d\xi - \frac{4}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\frac{\eta g(\xi,\eta)}{(\xi-x)^2+\eta^2}, \quad -\infty < x < \infty, \quad (4.7)$$

where $⨍_{-\infty}^{\infty}$ denotes the principal value integral. Equivalently, the Dirichlet boundary value can be expressed in terms of the Neumann boundary value by the equation

$$\psi_x(x,0) = -\frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{\psi_y(\xi,0)}{\xi - x}d\xi - \frac{4}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\frac{(\xi - x)g(\xi,\eta)}{(\xi-x)^2+\eta^2}, \quad -\infty < x < \infty. \quad (4.8)$$

Proof Taking the limit in equation (4.5) as y approaches zero from negative values, we find

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(\psi_\xi(\xi,0) - i\psi_y(\xi,0)\right)\frac{d\xi}{\xi-(x-i0)} = 2i\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\frac{g(\xi,\eta)}{\xi - x + i\eta}. \quad (4.9)$$

Using the Plemelj formulae [1], the left-hand side of this equation becomes

$$-\frac{i\pi}{2}[\psi_x(x,0) - i\psi_y(x,0)] + \frac{1}{2}⨍_{-\infty}^{\infty}\left(\psi_\xi(\xi,0) - i\psi_y(\xi,0)\right)\frac{d\xi}{\xi - x}. \quad (4.10)$$

Replacing the left-hand side of equation (4.9) by the above expression, and considering the real and imaginary parts of the resulting equation, we find equations (4.7) and (4.8). □

We emphasise that the global relation yields the above maps *without* the need to solve the Poisson equation in the upper half domain. This remarkable feature is true in

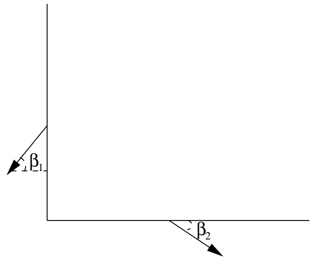

FIGURE 3. The angles β_1 and β_2.

general: The global relation evaluated on the boundary yields the (generalised) Dirichlet to Neumann map via an analysis restricted only on the boundary of the given domain.

The usefulness of equations (4.7) and (4.8) is at least twofold: (a) For some physical problems, one is interested *only* in the unknown boundary value, and *not* in the full solution. (b) For some problems it seems that it is not possible to *eliminate* the unknown boundary values, whereas it is still possible to *determine* the unknown boundary values by analysing the global relation (such a problem is discussed in section 7). After determining the unknown boundary value, equation (3.2) combined with the fact that ψ is real, yield $\psi(x, y)$ for z in D.

5 The oblique Neumann problem for the Poisson equation in the quarter plane

The Laplace equation in the quarter plane, in the semistrip, and in the interior of the isosceles triangle was analysed in [6] using equations (3.4)–(3.6). It is straightforward to extend these results to the Poisson equation. For the sake of economy we only consider the oblique Neumann problem for the quarter plane.

Proposition 5.1 *Let the real-valued function $\psi(x, y)$ satisfy the Poisson equation in the quarter plane*

$$\psi_{z\bar{z}} = g, \quad 0 \leqslant x \leqslant \infty,\ 0 \leqslant y \leqslant \infty, \tag{5.1}$$

where $g(x, y)$ is a given real-valued function with appropriate smoothness and decay. Suppose that the derivative of the function ψ is prescribed along the direction making an angle β_1 with the y-axis as well as along the direction making an angle β_2 with the x-axis (see Figure 3), i.e.

$$-\psi_x(0, y)\sin\beta_1 + \psi_y(0, y)\cos\beta_1 = h_1(y), \quad 0 < y < \infty,$$
$$-\psi_y(x, 0)\sin\beta_2 + \psi_x(x, 0)\cos\beta_2 = h_2(x), \quad 0 < x < \infty, \tag{5.2}$$

where $h_1(y)$ and $h_2(x)$ have appropriate smoothness and decay and $h_1(0) = h_2(0)$.

Assume that $\beta_1 + \beta_2 = n\pi/2, \quad n = 0,$ or $1,$ or $2.$ Define the function f in terms of the function $g(x, y)$ by

$$f(z, \bar{z}) = \int_0^\infty d\xi \int_0^\infty d\eta \frac{g(\xi, \eta)}{\xi + i\eta - z}. \tag{5.3}$$

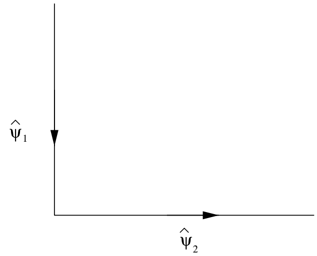

FIGURE 4. The quarter plane and the relevant spectral functions.

Then ψ_z *is given by*

$$\psi_z = f(z,\bar{z}) + \frac{1}{2\pi}\int_0^{i\infty} e^{ikz}\left[2e^{-i\beta_1}H_1(k) + 2e^{-i(2\beta_1+\beta_2)}H_2(ik) - e^{-2i\beta_1}\overline{A(\bar{k})}\right]dk + C(z)$$
$$+ \frac{1}{2\pi}\int_0^{\infty} e^{ikz}\left[2e^{i(\beta_1+2\beta_2)}H_1(-k) + 2e^{i\beta_2}H_2(-ik) - e^{2i(\beta_1+\beta_2)}B(k)\right]dk, \tag{5.4}$$

where

$$H_1(k) = -\frac{1}{2}\int_0^{\infty} e^{ky}h_1(y)\,dy, \quad H_2(k) = \frac{1}{2}\int_0^{\infty} e^{kx}h_2(x)\,dx, \tag{5.5}$$

$$A(k) = \int_0^{\infty} e^{-ikx}f\,dx - i\int_0^{\infty} e^{ky}f\,dy, \quad B(k) = A(-k) + e^{-2i\beta_1}\overline{A(-\bar{k})}, \tag{5.6}$$

$$C(z) = -\frac{1}{2\pi}\int_0^{\infty} e^{ikz}\left(\int_0^{\infty} e^{-ikx}f\,dx\right)dk + \frac{i}{2\pi}\int_0^{i\infty} e^{ikz}\left(\int_0^{\infty} e^{ky}f\,dy\right)dk. \tag{5.7}$$

Proof Let $j_1(y)$ and $j_2(x)$ denote the unknown derivatives in directions normal to the directions of the given derivatives, i.e.

$$-\psi_y(0,y)\sin\beta_1 - \psi_x(0,y)\cos\beta_1 = j_1(y), \quad 0 < y < \infty,$$
$$\psi_x(x,0)\sin\beta_2 + \psi_y(x,0)\cos\beta_2 = j_2(x), \quad 0 < x < \infty. \tag{5.8}$$

Using equations (5.2a) and (5.8a) to express $\{\psi_x(0,y), \psi_y(0,y)\}$ in terms of $\{h_1(y), j_1(y)\}$, as well as equations (5.2b) and (5.8b) to express $\{\psi_x(x,0), \psi_y(x,0)\}$ in terms of $\{h_2(x), j_2(x)\}$, we find

$$\begin{aligned}
\psi_y(0,y) &= h_1(y)\cos\beta_1 - j_1(y)\sin\beta_1,\\
\psi_x(0,y) &= -h_1(y)\sin\beta_1 - j_1(y)\cos\beta_1,\\
\psi_y(x,0) &= j_2(x)\cos\beta_2 - h_2(x)\sin\beta_2,\\
\psi_x(x,0) &= h_2(x)\cos\beta_2 + j_2(x)\sin\beta_2.
\end{aligned} \tag{5.9}$$

Equation (3.5) implies

$$\hat{\psi}_1(k) = e^{-i\beta_1}\left[H_1(k) + iJ_1(k)\right], \quad \hat{\psi}_2(k) = e^{i\beta_2}\left[H_2(-ik) + iJ_2(-ik)\right], \tag{5.10}$$

where the functions H_1, H_2 are defined by equations (5.5) and

$$J_1(k) = \frac{1}{2}\int_0^{\infty} e^{ky}j_1(y)\,dy, \quad J_2(k) = -\frac{1}{2}\int_0^{\infty} e^{kx}j_2(x)\,dx. \tag{5.11}$$

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The global relation (3.6) and its Schwartz conjugate are

$$e^{-i\beta_1}H_1(k) + ie^{-i\beta_1}J_1(k) + e^{i\beta_2}H_2(-ik) + ie^{i\beta_2}J_2(-ik) = A(k), \quad \pi \leqslant argk \leqslant \frac{3\pi}{2}, \tag{5.12}$$

$$e^{i\beta_1}H_1(k) - ie^{i\beta_1}J_1(k) + e^{-i\beta_2}H_2(ik) - ie^{-i\beta_2}J_2(ik) = \overline{A(\bar{k})}, \quad \frac{\pi}{2} \leqslant argk \leqslant \pi, \tag{5.13}$$

where the known function $A(k)$ is defined by equation (5.6).

We supplement the above two equations with the equations obtained from them by substituting $k \to -k$; we will refer to these equations as (5.12)' and (5.13)'. Equations (5.12), (5.13), (5.12)' and (5.13)', are four equations relating the four unknown functions $J_1(k), J_1(-k), J_2(ik), J_2(-ik)$. Using equations (5.12)' and (5.13)' we find

$$J_2(-ik) = e^{2i(\beta_1+\beta_2)}J_2(ik) + N_1(k), \quad k = \mathbb{R}^+, \tag{5.14}$$

where $N_1(k)$ is a known function. Also, equation (5.13) implies

$$J_1(k) = -e^{-i(\beta_1+\beta_2)}J_2(ik) + N_2(k), \quad k = i\mathbb{R}^+, \tag{5.15}$$

where $N_2(k)$ is a known function. Although the function $J_2(ik)$ is an unknown function, it does *not* contribute to the solution. Indeed, substituting the above expressions into the integral representation (3.4) we find

$$\psi_z = \frac{i}{2\pi}\left(e^{i(2\beta_1+3\beta_2)}\int_0^\infty e^{ikz}J_2(ik)\,dk - e^{-i(2\beta_1+\beta_2)}\int_0^{i\infty} e^{ikz}J_2(ik)\,dk\right) + N(z,\bar{z}), \tag{5.16}$$

where $N(z,\bar{z})$ is the right-hand side of equation (5.4). The functions e^{ikz} and $J_2(ik)$ are analytic and bounded in the first quadrant of the complex k-plane. Thus if

$$e^{i(2\beta_1+3\beta_2)} = e^{-i(2\beta_1+\beta_2)}, \quad \text{i.e} \quad e^{4i(\beta_1+\beta_2)} = 1,$$

the application of Cauchy's theorem in the first quadrant of the complex k-plane implies that the unknown function $J_2(ik)$ *does not contribute* to ψ_z. The integral representation (5.4) follows by using the analyticity of $H_2(ik)$. □

5.1 Comparison of the representations in the k- and z-planes

Having constructed an integral representation in the complex k-plane, it is straightforward to obtain an integral representation in the complex z-plane. For simplicity we consider the Laplace equation, i.e.

$$\psi_z = \frac{1}{\pi}\int_0^{i\infty} e^{ikz}\left[e^{-i\beta_1}H_1(k) + e^{-i(2\beta_1+\beta_2)}H_2(ik)\right]dk$$
$$+\frac{1}{\pi}\int_0^{\infty} e^{ikz}\left[e^{i(\beta_1+2\beta_2)}H_1(-k) + e^{i\beta_2}H_2(-ik)\right]dk, \tag{5.17}$$

where $H_1(k)$ and $H_2(k)$ are defined by equations (5.5).

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By computing explicitly the k-integral, equation (5.17) yields

$$\psi_z(z) = \frac{1}{2\pi}\int_0^\infty \left(\frac{e^{i(\beta_1+2\beta_2)}}{iz-\xi} + \frac{e^{-i\beta_1}}{iz+\xi}\right) h_1(\xi)\,d\xi + \frac{i}{2\pi}\int_0^\infty \left(\frac{e^{i\beta_2}}{z-\xi} + \frac{e^{-i(2\beta_1+\beta_2)}}{z+\xi}\right) h_2(\xi)\,d\xi. \tag{5.18}$$

This equation has the advantage that it involves *directly* the boundary conditions h_1 and h_2, whereas equation (5.17) involves the *Fourier transform* of the boundary conditions. However, it appears that the representations in the k-plane are in general is more efficient for numerical computations. Indeed, by appropriately deforming the contours in the complex k-plane, it is possible to obtain contours involving integrands with strong decay.

6 The non-homogeneous Biharmonic equation

Proposition 6.1 *Let the complex-valued function $\Psi(z,\bar{z})$ satisfy the non-homogeneous biharmonic equation*

$$\Psi_{z\bar{z}z\bar{z}} = G, \quad z \in D \subset \mathbb{R}^2, \tag{6.1}$$

where z denotes the usual complex variable, $G(z,\bar{z})$ is a given complex-valued function with appropriate smoothness, and D is a simply connected, bounded domain of the complex z-plane. Then: (a) Ψ admits the integral representation

$$\Psi_{zz} = \frac{1}{2\pi i}\int_{\partial D}\left[(\bar{z}-\bar{\zeta})\Psi_{\zeta\zeta\bar{\zeta}} + \Psi_{\zeta\zeta}\right]\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\iint_D (\bar{z}-\bar{\zeta})G\frac{d\zeta\wedge d\bar{\zeta}}{\zeta - z}, \; z \in D. \tag{6.2}$$

Furthermore, the boundary values of Ψ satisfy the two global relations

$$\int_{\partial D}\frac{\Psi_{\zeta\zeta\bar{\zeta}}}{\zeta - z}d\zeta = -\iint_D \frac{G}{\zeta - z}d\zeta\wedge d\bar{\zeta}, \tag{6.3}$$

and

$$\int_{\partial D}\frac{\Psi_{\zeta\zeta} - \bar{\zeta}\Psi_{\zeta\zeta\bar{\zeta}}}{\zeta - z}d\zeta = \iint_D \frac{\bar{\zeta}G}{\zeta - z}d\zeta\wedge d\bar{\zeta}, \quad z \notin D. \tag{6.4}$$

(b) Suppose that D is the interior of the convex polygon Ω specified by the corners $z_1,\ldots,z_{n+1} = z_1$ (see Figure 1). Then, Ψ also admits the integral representation

$$\Psi_{zz} = F + \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}[\bar{z}\widehat{\Psi}_j^{(1)}(k) + \widehat{\Psi}_j^{(2)}(k)]\,dk - \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}[\bar{z}\widehat{F}_j^{(1)}(k) + \widehat{F}_j^{(2)}(k)]dk, \; z \in \Omega, \tag{6.5}$$

where the rays l_j are defined by equation (2.6), the function F is defined in terms of G by the equation

$$F_{\bar{z}\bar{z}} = G, \quad F|_{\partial D} = F_{\bar{z}}|_{\partial D} = 0,$$

and the functions $\widehat{\Psi}_j^{(m)}, \widehat{F}_j^{(m)}, m = 1, 2$, *are defined by the following equations:*

$$\widehat{\Psi}_j^{(1)}(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}\Psi_{zz\bar{z}}\,dz,\ \ \widehat{\Psi}_j^{(2)}(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}(\Psi_{zz} - \bar{z}\Psi_{zz\bar{z}})\,dz, \tag{6.6}$$

$$\widehat{F}_j^{(1)}(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}F_{\bar{z}}\,dz,\ \ \widehat{F}_j^{(2)}(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}(F - \bar{z}F_{\bar{z}})\,dz,\ \ j = 1, \dots, n,\ \ k \in \mathbb{C}. \tag{6.7}$$

Furthermore, the following global relations are valid for all complex k,

$$\sum_{j=1}^{n} \widehat{\Psi}_j^{(m)}(k) = \sum_{j=1}^{n} \widehat{F}_j^{(m)}(k), \quad m = 1, 2, \quad k \in \mathbb{C}. \tag{6.8}$$

Proof The functions $\Psi_{zz\bar{z}}$ and $\Psi_{zz} - \bar{z}\Psi_{zz\bar{z}}$ satisfy the basic equation (2.1):

$$\partial_{\bar{z}}(\Psi_{zz\bar{z}}) = G, \quad \partial_{\bar{z}}(\Psi_{zz} - \bar{z}\Psi_{zz\bar{z}}) = -\bar{z}G. \tag{6.9}$$

Thus, replacing in equations (2.2) and (2.3) Φ by $\Psi_{zz\bar{z}}$, as well as replacing in equations (2.2) and (2.3) Φ by $\Psi_{zz} - \bar{z}\Psi_{zz\bar{z}}$ and G by $-\bar{z}G$, we find the global relations (6.3) and (6.4) as well as the following equations

$$\Psi_{zz\bar{z}} = \frac{1}{2\pi i}\left(\int_{\partial D} \frac{\Psi_{\zeta\zeta\bar{\zeta}}}{\zeta - z}d\zeta + \iint_D \frac{G}{\zeta - z}d\zeta \wedge d\bar{\zeta}\right), \tag{6.10}$$

$$\Psi_{zz} - \bar{z}\Psi_{zz\bar{z}} = \frac{1}{2\pi i}\left(\int_{\partial D} \frac{\Psi_{\zeta\zeta} - \bar{\zeta}\Psi_{\zeta\zeta\bar{\zeta}}}{\zeta - z}d\zeta - \iint_D \frac{\bar{\zeta}G}{\zeta - z}d\zeta \wedge d\bar{\zeta}\right). \tag{6.11}$$

Multiplying equation (6.10) by $\bar{z}$ and adding the resulting equation to equation (6.11), we obtain equation (6.2).

(b) If F is defined in terms of G by equation (2.4), then

$$-\bar{z}G = \partial_{\bar{z}}(-\bar{z}F + \partial_{\bar{z}}^{-1}F). \tag{6.12}$$

This suggests replacing F by $F_{\bar{z}}$. Thus equations (2.5)–(2.9) yield the global relations (6.8) as well as the following equations:

$$\Psi_{zz\bar{z}} = F_{\bar{z}} + \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}\widehat{\Psi_j^{(1)}}(k)\,dk - \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}\widehat{F_j^{(1)}}(k)\,dk, \tag{6.13}$$

$$\Psi_{zz} - \bar{z}\Psi_{zz\bar{z}} = F - \bar{z}F_{\bar{z}} + \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}\widehat{\Psi_j^{(2)}}(k)\,dk - \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{ikz}\widehat{F_j^{(2)}}(k)\,dk, \tag{6.14}$$

where $\widehat{\Psi}_j^{(m)}, \widehat{F}_j^{(m)}$, $m = 1, 2$, are defined by equations (6.6) and (6.7). Multiplying equation (6.13) by $\bar{z}$ and adding the resulting equation to equation (6.14) we find equation (6.5). □

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7 The non-homogeneous Biharmonic equation in the upper half complex plane

Proposition 7.1 *(The Dirichlet-second Neumann boundary value problem).*

Let the real-valued function $\psi(x,y)$ satisfy the non-homogeneous biharmonic equation in the upper half of the complex z-plane,

$$\psi_{z\bar{z}z\bar{z}} = g, \quad Imz \geqslant 0, \tag{7.1}$$

where $g(x,y)$ is a given real-valued function with appropriate smoothness and decay. Assume that the Dirichlet as well as the second Neumann boundary conditions are prescribed

$$\psi(x,0) = h_1(x), \qquad \psi_{yy}(x,0) = h_2(x), \quad -\infty < x < \infty, \tag{7.2}$$

where the real-valued functions $h_1(x)$ and $h_2(x)$ have appropriate smoothness and decay. Then ψ_{zz} is given by

$$\begin{aligned}\psi_{zz} = &\frac{\bar{z}}{\pi i}\frac{1}{8}\int_{-\infty}^{\infty}(h_1'''(\xi)+h_2'(\xi))\frac{d\xi}{\xi - z}\\ &+\frac{1}{\pi i}\frac{1}{8}\int_{-\infty}^{\infty}[2(h_1''(\xi)-h_2(\xi))-\xi(h_1'''(\xi)+h_2'(\xi))]\frac{d\xi}{\xi - z} + G(z,\bar{z}),\end{aligned} \tag{7.3}$$

where

$$\begin{aligned}G(z,\bar{z}) = &\frac{\bar{z}}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\frac{g(\xi,\eta)}{\xi - i\eta - z} - \frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\frac{(\xi+i\eta)g(\xi,\eta)}{\xi - i\eta - z}\\ &-\frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\frac{[\bar{z}-(\xi - i\eta)]g(\xi,\eta)}{\xi + i\eta - z}.\end{aligned} \tag{7.4}$$

Proof The definitions

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y),$$

imply the identities

$$\psi_{zz\bar{z}} = \frac{1}{8}\left(\psi_{xxx}+\psi_{xyy}-i\psi_{yyy}-i\psi_{xxy}\right), \tag{7.5}$$

$$\psi_{zz} = \frac{1}{4}\left(\psi_{xx}-\psi_{yy}-2i\psi_{xy}\right). \tag{7.6}$$

Using these identities in equation (6.2) as well as in equations (6.3) and (6.4) and then taking the complex conjugate and letting $y \to -y$ in the latter two equations, we find

$$\begin{aligned}\psi_{zz} = &\frac{\bar{z}}{2\pi i}\frac{1}{8}\int_{-\infty}^{\infty}(\psi_{\xi\xi\xi}+\psi_{\xi yy}-i\psi_{yyy}-i\psi_{\xi\xi y})(\xi,0)\frac{d\xi}{\xi - z}\\ &+\frac{1}{2\pi i}\frac{1}{8}\int_{-\infty}^{\infty}[2(\psi_{\xi\xi}-\psi_{yy}-2i\psi_{\xi y})-\xi(\psi_{\xi\xi\xi}+\psi_{\xi yy}-i\psi_{yyy}-i\psi_{\xi\xi y})](\xi,0)\frac{d\xi}{\xi - z}\\ &-\frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_0^{\infty}d\eta\frac{[x-iy-(\xi - i\eta)]g(\xi,\eta)}{\xi + i\eta - z},\end{aligned} \tag{7.7}$$

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as well as

$$
-\frac{\bar{z}}{2\pi i}\frac{1}{8}\int_{-\infty}^{\infty}(\psi_{\xi\xi\xi}+\psi_{\xi yy}+i\psi_{yyy}+i\psi_{\xi\xi y})(\xi,0)\frac{d\xi}{\xi-z}
$$

$$
=\frac{\bar{z}}{\pi}\int_{-\infty}^{\infty}d\xi\int_{0}^{\infty}d\eta\frac{g(\xi,\eta)}{\xi-i\eta-z}, \tag{7.8}
$$

$$
\frac{1}{2\pi i}\frac{1}{8}\int_{-\infty}^{\infty}[2(\psi_{\xi\xi}-\psi_{yy}+2i\psi_{\xi y})-\xi(\psi_{\xi\xi\xi}+\psi_{\xi yy}+i\psi_{yyy}+i\psi_{\xi\xi y})](\xi,0)\frac{d\xi}{\xi-z}
$$

$$
=\frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\int_{0}^{\infty}d\eta\frac{(\xi+i\eta)g(\xi,\eta)}{\xi-i\eta-z}. \tag{7.9}
$$

Subtracting equations (7.8) and (7.9) from equation (7.7), the unknown boundary values are eliminated and we find equation (7.3). □

Having obtained ψ_{zz}, it is straightforward to find ψ.

Suppose that instead of $\psi_{yy}(x,0)$, we prescribe the Neumann boundary condition $\psi_y(x,0)$. In this case it is *not* clear how to manipulate equations (7.7)–(7.9) in order to eliminate the unknown boundary values. However, in this case we can *determine* the unknown boundary value by analysing the generalised Dirichlet to Neumann correspondence.

Proposition 7.2 *(A generalised Dirichlet to Neumann correspondence for the half plane).*

Let the real-valued function $\psi(x,y)$ satisfy the non-homogeneous biharmonic equation (7.1) *in the upper half of the complex z-plane. Then, for $-\infty \leqslant x \leqslant \infty$, the following identities are valid:*

$$
(\psi_{xxx}+\psi_{xyy})(x,0)=-\frac{1}{\pi}⨍_{-\infty}^{\infty}(\psi_{yyy}+\psi_{\xi\xi y})(\xi,0)\frac{d\xi}{\xi-x}
$$

$$
-\frac{16}{\pi}\int_{-\infty}^{\infty}d\xi\int_{0}^{\infty}d\eta\frac{(\xi-x)g(\xi,\eta)}{(\xi-x)^2+\eta^2}, \tag{7.10}
$$

$$
[2(\psi_{xx}-\psi_{yy})-x(\psi_{xxx}+\psi_{xyy})](x,0)=-\frac{1}{\pi}⨍_{-\infty}^{\infty}[4\psi_{\xi y}-\xi(\psi_{yyy}+\psi_{\xi\xi y})](\xi,0)\frac{d\xi}{\xi-x}
$$

$$
-\frac{16}{\pi}\int_{-\infty}^{\infty}d\xi\int_{0}^{\infty}d\eta\frac{[\xi(\xi-x)-\eta^2]g(\xi,\eta)}{(\xi-x)^2+\eta^2}. \tag{7.11}
$$

Equivalently, equations (7.10) *and* (7.11) *can be expressed in the following form*

$$
(\psi_{yyy}+\psi_{xxy})(x,0)=\frac{1}{\pi}⨍_{-\infty}^{\infty}(\psi_{\xi\xi\xi}+\psi_{\xi yy})(\xi,0)\frac{d\xi}{\xi-x}
$$

$$
-\frac{16}{\pi}\int_{-\infty}^{\infty}d\xi\int_{0}^{\infty}d\eta\frac{\eta g(\xi,\eta)}{(\xi-x)^2+\eta^2}, \tag{7.12}
$$

$$
[-4\psi_{xy}+x(\psi_{yyy}+\psi_{xxy})](x,0)=-\frac{1}{\pi}⨍_{-\infty}^{\infty}[2(\psi_{\xi\xi}-\psi_{yy})-\xi(\psi_{\xi\xi\xi}+\psi_{\xi yy})](\xi,0)\frac{d\xi}{\xi-x}
$$

$$
+\frac{16}{\pi}\int_{-\infty}^{\infty}d\xi\int_{0}^{\infty}d\eta\frac{(2\xi\eta-\eta x)g(\xi,\eta)}{(\xi-x)^2+\eta^2}. \tag{7.13}
$$

Proof Letting $\zeta = \xi + i\eta$ in equations (6.3) and (6.4), as well as taking the limit as z approaches the real axis from below,and using Plemelj formulae, we find the following equations:

$$[\psi_{xxx} + \psi_{xyy} - i(\psi_{yyy} + \psi_{xxy})](x,0) + \frac{1}{\pi}\fint_{-\infty}^{\infty}[\psi_{yyy} + \psi_{\xi\xi y} + i(\psi_{\xi\xi\xi} + \psi_{\xi yy})](\xi,0)\frac{d\xi}{\xi - x}$$
$$= -\frac{16}{\pi}\int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta \frac{g(\xi,\eta)}{\xi - x + i\eta}, \tag{7.14}$$

$$\{[2(\psi_{xx} - \psi_{yy}) - x(\psi_{xxx} + \psi_{xyy})] + i[-4\psi_{xy} + x(\psi_{yyy} + \psi_{xxy})]\}(x,0)$$
$$+\frac{1}{\pi}\fint_{-\infty}^{\infty}\{[4\psi_{\xi y} - \xi(\psi_{yyy} + \psi_{\xi\xi y})] + i[2(\psi_{\xi\xi} - \psi_{yy}) - \xi(\psi_{\xi\xi\xi} + \psi_{\xi yy})]\}(\xi,0)\frac{d\xi}{\xi - x}$$
$$= -\frac{16}{\pi}\int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta \frac{(\xi - i\eta)g(\xi,\eta)}{\xi - x + i\eta}. \tag{7.15}$$

The real and imaginary parts of equations (7.14) and (7.15) yield equations (7.10)–(7.13). □

The analysis of equations (7.10) and (7.11) (or equivalently of equations (7.12) and (7.13)) yields the generalised Dirichlet to Neumann map for a variety of boundary conditions. As an illustrative example we consider the Dirichlet-Neumann boundary value problem.

Proposition 7.3 *(The Dirichlet-Neumann boundary value problem).*

Let the real-valued function $\psi(x,y)$ satisfy the non-homogeneous biharmonic equation (7.1) *in the upper half of the complex z-plane with given Dirichlet and Neumann boundary conditions*

$$\psi(x,0) = h_1(x), \quad \psi_y(x,0) = -h_2(x), \quad -\infty < x < \infty. \tag{7.16}$$

Then the second and third Neumann boundary values,

$$\psi_{yy}(x,0) = f_1(x), \quad \psi_{yyy}(x,0) = f_2(x), \tag{7.17}$$

are given by the following expressions:

$$f_1(x) = -\frac{xG_1(x) + G_2(x)}{2} + h_1''(x) - \frac{2}{\pi}\fint_{-\infty}^{\infty}\frac{h_2'(\xi)\,d\xi}{\xi - x}, \tag{7.18}$$

$$f_2(x) = h_2''(x) + \frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{\left[h_1'''(\xi) + f_1'(\xi) - G_1(\xi)\right]d\xi}{\xi - x}, \tag{7.19}$$

where

$$G_1(x) = -\frac{16}{\pi}\int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta \frac{(\xi - x)g(\xi,\eta)}{(\xi - x)^2 + \eta^2},$$
$$G_2(x) = -\frac{16}{\pi}\int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta \frac{[\xi(\xi - x) - \eta^2]g(\xi,\eta)}{(\xi - x)^2 + \eta^2}. \tag{7.20}$$

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Proof Denoting by $G_1(x)$ and $G_2(x)$ the double integrals appearing in the right-hand side of equations (7.10) and (7.11) and denoting $\psi_{yy}(x,0)$ and $\psi_{yyy}(x,0)$ by $f_1(x)$ and $f_2(x)$ respectively, equations (7.10) and (7.11) yield

$$h_1''' + f_1' = -\frac{1}{\pi}\fint_{-\infty}^{\infty}(f_2 - h_2'')\frac{d\xi}{\xi - x} + G_1, \tag{7.21}$$

$$2(h_1'' - f_1) - x(h_1''' + f_1') = \frac{1}{\pi}\fint_{-\infty}^{\infty}[4h_2' + \xi(f_2 - h_2'')]\frac{d\xi}{\xi - x} + G_2. \tag{7.22}$$

Using the identity

$$\fint_{-\infty}^{\infty}\frac{\xi f(\xi)\,d\xi}{\xi - x} = \int_{-\infty}^{\infty} f(\xi)\,d\xi + x\fint_{-\infty}^{\infty}\frac{f(\xi)\,d\xi}{\xi - x}, \tag{7.23}$$

and replacing $h_1''' + f_1'$ by the right-hand side of equation (7.21), equation (7.22) simplifies to the following equation

$$f_1 = -\frac{xG_1 + G_2}{2} + h_1'' - \frac{2}{\pi}\fint_{-\infty}^{\infty}\frac{h_2'\,d\xi}{\xi - x} - \frac{1}{2\pi}\int_{-\infty}^{\infty}(f_2 - h_2'')\,d\xi. \tag{7.24}$$

Taking the limit as $x \to \infty$, it follows that the last term in equation (7.24) vanishes and hence we find equation (7.18).

Also, equation (7.21) implies

$$\frac{1}{\pi}\fint_{-\infty}^{\infty} f_2\frac{d\xi}{\xi - x} = \frac{1}{\pi}\fint_{-\infty}^{\infty} h_2''\frac{d\xi}{\xi - x} - h_1''' - f_1' + G_1. \tag{7.25}$$

Taking the inverse Hilbert transform of equation (7.25) we find equation (7.19). □

8 The non-homogeneous Biharmonic equation in an equilateral triangle

Proposition 8.1 *(A generalised Dirichlet to Neumann map for an equilateral triangle). Let the real-valued function $\psi(x,y)$ satisfy the non homogeneous biharmonic equation in an equilateral triangle*

$$\psi_{zz\bar{z}\bar{z}} = g, \quad z \in D, \tag{8.1}$$

where g is a given real-valued function with appropriate smoothness, and D denotes the interior of the equilateral triangle with the following corners (see figure 5)

$$z_1 = \frac{l}{\sqrt{3}}e^{\frac{i\pi}{3}}, \quad z_2 = \overline{z_1}, \quad z_3 = -\frac{l}{\sqrt{3}}. \tag{8.2}$$

Let ψ satisfy symmetric Dirichlet and second Neumann boundary conditions,

$$\psi^{(j)}(s) = f_0(s), \quad \psi_{nn}^{(j)}(s) = f_1(s), \quad j = 1,2,3, \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right], \tag{8.3}$$

where $\psi^{(j)}(s)$ denotes the value of ψ on the side (j), $\psi_{nn}^{(j)}(s)$ denotes the second normal derivative on the side (j), and $f_0(s), f_1(s)$ have appropriate smoothness and are continuous at the corners of the triangle. Then, the unknown Neumann and third Neumann boundary values,

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i.e. the functions $\psi_n^{(j)}(s) = \psi_n(s)$ and $\psi_{nnn}^{(j)}(s) = \psi_{nnn}(s)$, can be determined as follows. The function $\psi_n(s)$ is obtained by integrating the equation

$$\partial_s \psi_n(s) = \frac{2i}{l}\sum_{-\infty}^{\infty} e^{-\frac{2in\pi s}{l}} \frac{Q^{(1)}\left(\frac{2in\pi}{l}\right)}{\sinh\left(\bar{\alpha}(in\pi)\right)}, \quad n \in \mathbb{Z}, \tag{8.4}$$

where $Q^{(1)}(k)$ is given in terms of the known boundary conditions by the following expressions:

$$Q^{(1)}(k) = iE(-i\bar{\alpha}k)\overline{W^{(1)}(\bar{k})} + iE(i\alpha k)W^{(1)}(k), \tag{8.5}$$

$$E(k) = e^{k\frac{l}{2\sqrt{3}}}, \tag{8.6}$$

$$W^{(1)}(k) = A^{(1)}(k) - [E(-ik)\widetilde{R}(k) + E(-i\bar{\alpha}k)\widetilde{R}(\bar{\alpha}k) + E(-i\alpha k)\widetilde{R}(\alpha k)], \tag{8.7}$$

$$\widetilde{R}(k) = \frac{1}{4}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks}\Bigg\{(f_1(s) - f_0''(s)) - \frac{1}{8}\left(\frac{l}{2\sqrt{3}} - is\right)\left[\frac{8}{l}\sum_{-\infty}^{\infty} e^{-\frac{2in\pi s}{l}} \frac{Q^{(2)}\left(\frac{2in\pi}{l}\right)}{\sinh\left(\bar{\alpha}(in\pi)\right)} - i(f_0'''(s) + f_1'(s))\right]\Bigg\}\, ds, \quad n \in \mathbb{Z}, \tag{8.8}$$

$$A^{(1)}(k) = \iint_D e^{-ikz}\bar{z}g(z,\bar{z})\, dz \wedge d\bar{z}, \qquad k \in \mathbb{C}, \tag{8.9}$$

$$Q^{(2)}(k) = iE(-i\bar{\alpha}k)\overline{W^{(2)}(\bar{k})} + iE(i\alpha k)W^{(2)}(k), \tag{8.10}$$

$$W^{(2)}(k) = A^{(2)}(k) - [E(-ik)R(k) + E(-i\bar{\alpha}k)R(\bar{\alpha}k) + E(-i\alpha k)R(\alpha k)], \tag{8.11}$$

$$R(k) = \frac{1}{8}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks}(f_0''''(s) + f_1'(s))\, ds, \tag{8.12}$$

$$A^{(2)}(k) = -\iint_D e^{-ikz}g(z,\bar{z})\, dz \wedge d\bar{z}, \qquad k \in \mathbb{C}. \tag{8.13}$$

The function $\psi_{nnn}(s)$ is given in terms of the known boundary conditions by the expression

$$\psi_{nnn}(s) = \frac{8}{l}\sum_{-\infty}^{\infty} e^{-\frac{2in\pi s}{l}} \frac{Q^{(2)}\left(\frac{2in\pi}{l}\right)}{\sinh\left(\bar{\alpha}(in\pi)\right)} - \partial_s\left(\frac{2i}{l}\sum_{-\infty}^{\infty} e^{-\frac{2in\pi s}{l}} \frac{Q^{(1)}\left(\frac{2in\pi}{l}\right)}{\sinh\left(\bar{\alpha}(in\pi)\right)}\right), \quad n \in \mathbb{Z}. \tag{8.14}$$

Proof The sides $(z_2, z_1), (z_3, z_2), (z_1, z_3)$, each of length l, will be referred to as sides (1),(2),(3) respectively. On each side we identify the positive direction $\widehat{T}$ and the outward normal $\widehat{N}$, see Figure 5. Let α denote the following complex cube root of unity

$$\alpha = e^{\frac{2i\pi}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}.$$

For the equilateral triangle, the global relations (6.8) become

$$\sum_{j=1}^{3} \widehat{\varphi}_j^{(1)}(k) = A^{(2)}(k), \quad \sum_{j=1}^{3} \widehat{\varphi}_j^{(2)}(k) = A^{(1)}(k), \tag{8.15}$$

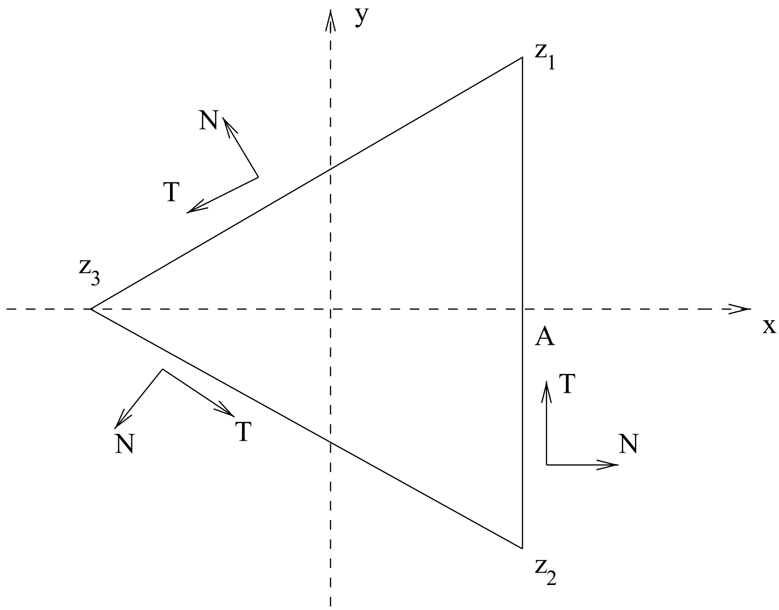

FIGURE 5. The domain D for an equilateral triangle.

where $A^{(1)}(k)$ and $A^{(2)}(k)$ are defined by equations (8.9) and (8.13) respectively, and $\widehat{\psi}_j^{(1)}, \widehat{\psi}_j^{(2)}$ are defined by the following equations

$$\widehat{\psi}_j^{(1)}(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}\psi_{zz\bar{z}}dz, \quad \widehat{\psi}_j^{(2)}(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}\left(\psi_{zz} - \bar{z}\psi_{zz\bar{z}}\right)dz, \quad j = 1, 2, 3, \ k \in \mathbb{C}. \tag{8.16}$$

We next compute the above functions:

Side 1: The variable z can be parametrized as

$$z(s) = \frac{l}{2\sqrt{3}} + is, \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right].$$

Since the normal and the tangential derivatives are parallel to the x and y axes respectively, equations (7.5) and (7.6) give

$$\begin{aligned}\psi_{zz\bar{z}} &= \frac{1}{8}(\psi_{nnn} + \psi_{nss} - i\psi_{sss} - i\psi_{nns}),\\ \psi_{zz} &= \frac{1}{4}(\psi_{nn} - \psi_{ss} - 2i\psi_{ns}).\end{aligned} \tag{8.17}$$

Side 2: If z varies along side (2), ζ varies along side (1), i.e. $z = \zeta\exp(-\frac{2i\pi}{3})$. Thus,

$$z(s) = \left(\frac{l}{2\sqrt{3}} + is\right)e^{-\frac{2i\pi}{3}}, \quad s \in \left[-\frac{l}{2}, \frac{l}{2}\right].$$

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The equations $\partial_{zz} = \exp(\frac{4i\pi}{3})\partial_{\zeta\zeta}$ and $\partial_{zz\bar{z}} = \exp(\frac{2i\pi}{3})\partial_{\zeta\zeta\bar{\zeta}}$, imply

$$\psi_{zz\bar{z}} = \frac{1}{8}\alpha(\psi_{nnn} + \psi_{nss} - i\psi_{sss} - i\psi_{nns}),$$
$$\psi_{zz} = \frac{1}{4}\bar{\alpha}(\psi_{nn} - \psi_{ss} - 2i\psi_{ns}). \tag{8.18}$$

Side 3: In a similar way we find

$$\psi_{zz\bar{z}} = \frac{1}{8}\bar{\alpha}(\psi_{nnn} + \psi_{nss} - i\psi_{sss} - i\psi_{nns}),$$
$$\psi_{zz} = \frac{1}{4}\alpha(\psi_{nn} - \psi_{ss} - 2i\psi_{ns}). \tag{8.19}$$

Substituting the above expressions in the global relations (8.15) we find

$$E(-ik)[R(k) + iI(k)] + E(-i\bar{\alpha}k)[R(\bar{\alpha}k) + iI(\bar{\alpha}k)] + E(-i\alpha k)[R(\alpha k) + iI(\alpha k)] = 0, \tag{8.20}$$
$$E(-ik)[\widetilde{R}(k) + i\widetilde{I}(k)] + E(-i\bar{\alpha}k)\alpha[\widetilde{R}(\bar{\alpha}k) + i\widetilde{I}(\bar{\alpha}k)] + E(-iak)\bar{\alpha}[\widetilde{R}(\alpha k) + i\widetilde{I}(\alpha k)] = 0, \tag{8.21}$$

where the functions $R(k), \widetilde{R}(k)$ are defined by equations (8.12) and (8.8) respectively, and

$$I(k) = \frac{1}{8}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks}(\psi_{nnn}(s) + \psi_{nss}(s))\,ds, \tag{8.22}$$

$$\widetilde{I}(k) = -\frac{i}{2}\int_{-\frac{l}{2}}^{\frac{l}{2}} e^{ks}\psi_{ns}(s)\,ds, \quad k \in \mathbb{C}. \tag{8.23}$$

Hence, equation (8.20) can be written as

$$E(-ik)I(k) + E(-i\bar{\alpha}k)I(\bar{\alpha}k) + E(-i\alpha k)I(\alpha k) = (-i)W^{(2)}(k), \tag{8.24}$$

where $W^{(2)}(k)$ is the known function defined by equation (8.11).

Multiplying equation (8.24) by $E(i\alpha k)$ and multiplying its Schwarz conjugate by $E(-i\alpha k)$ we obtain

$$e(\bar{\alpha}k)I(k) + e(-k)I(\bar{\alpha}k) + I(\alpha k) = (-i)E(i\alpha k)W^{(2)}(k), \tag{8.25}$$
$$e(-\bar{\alpha}k)I(k) + I(\alpha k) + e(k)I(\bar{\alpha}k) = iE(-i\bar{\alpha}k)\overline{W^{(2)}(\bar{k})}, \tag{8.26}$$

where

$$e(k) = e^{k\frac{l}{2}}, \tag{8.27}$$

and we have used the identities

$$E(i\bar{\alpha}k)E(-i\alpha k) = e(k), \quad E(i\alpha k)E(-ik) = e(\bar{\alpha}k). \tag{8.28}$$

Subtracting equation (8.25) from equation (8.26), we find

$$(e(k) - e(-k))\,I(\bar{\alpha}k) = (e(\bar{\alpha}k) - e(-\bar{\alpha}k))\,I(k) + Q^{(2)}(k), \tag{8.29}$$

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where $Q^{(2)}(k)$ is the known function defined by (8.10). By evaluating the last equation at those values of k for which the coefficient of $\psi(\bar{\alpha}k)$ vanishes, i.e. at $e^2(k) = 1$, or

$$k_n = \frac{2in\pi}{l}, \quad n \in \mathbb{Z},$$

it follows that $I(k)$ can be determined. Recalling the definition of $I(k)$ and evaluating equation (8.29) at $k_n = \frac{2in\pi}{l}$, we find

$$\frac{1}{8} \sinh(\bar{\alpha}(in\pi)) \int_{-\frac{l}{2}}^{\frac{l}{2}} e^{2i\pi n \frac{s}{l}} (\psi_{nnn}(s) + \psi_{nss}(s))\, ds = Q^{(2)}\left(\frac{2in\pi}{l}\right), \quad n \in \mathbb{Z}. \tag{8.30}$$

Thus,

$$\psi_{nnn}(s) + \psi_{nss}(s) = \frac{8}{l} \sum_{-\infty}^{\infty} e^{-\frac{2in\pi s}{l}} \frac{Q^{(2)}\left(\frac{2in\pi}{l}\right)}{\sinh(\bar{\alpha}(in\pi))}. \tag{8.31}$$

Similarly, the second global relation (8.21) yields equation (8.4). Using the latter equation as well as equation (8.31), we obtain equation (8.14). □

9 Conclusions

During the twentieth century, the Riemann-Hilbert formalism became a very useful technique for the integration of several types of boundary-value problems in the theory of elasticity. The solution of such problems can be expressed through integral equations formulated in the complex z-plane [16], [18].

There exists an extension of the Riemman-Hilbert formalism, called the Dbar formalism [1]. In this case, the function to be found fails to be analytic in a two-dimensional domain of the complex plane, as opposed to the case of a Riemann-Hilbert problem where the function loses its analyticity only on a curve. The solution of the basic problem appearing in the Dbar formalism, the so-called Dbar problem, was obtained in 1912 by the Romanian mathematician D. Pompeiu.

The Dbar formalism has motivated various elegant generalisations in one and several complex variables [13]–[15]. Also, it has found applications to the solution of integrable nonlinear PDEs [2, 10], as well as in the inversion of important integral transforms such as the Radon and the attenuated Radon transforms [11].

A new method for solving boundary value problems for both linear and integrable nonlinear PDEs was introduced in [4]. The implementation of this method involves the formulation of a Riemann-Hilbert or a Dbar problem in the complex Fourier space (denoted by k), instead of the formulation in the physical space used earlier. In this new framework, the Dbar formalism has found new applications, for example, in the solution of boundary value problems in time-dependent domains [7], as well as, to the spectral analysis of differential operators [8].

In this paper, we have presented a methodology for the analysis of boundary value problems for certain linear non-homogeneous elliptic PDEs in two dimensions. This methodology involves two main features: (a) It yields an explicit integral representation

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for the solution. (b) It characterizes the generalised Dirichlet to Neumann map through the solution of the so called *global relation*. This is an equation which couples both the specified as well as the unknown values of the function and its derivatives on the boundary.

We have introduced two different approaches for the construction of the above integral representations. The first approach involves formulating the solution in the physical space (denoted by z). Although this formulation is well-known in the literature, it seems that results for *non-homogeneous* equations are rather sporadic, as opposed to the systematic approach introduced here. Furthermore, in the classical approach, the integral representation is obtained using Green's functions, and the associated generalised Dirichlet to Neumann map is determined either through the method of images or through the formulation of a Hilbert problem. This is to be contrasted with our approach, where the integral representation is obtained by solving a Dbar problem (see also Pinotsis [17]), and the associated generalised Dirichlet to Neumann map is obtained by analysing the global relation. A novel extension of this approach to multidimensions is presented in Fokas & Pinotsis [9]. The second formulation expresses the solution in the spectral space (denoted by k) and constitutes an extension to *non-homogeneous* elliptic PDEs of the method introduced in Fokas [4]. In particular, the analogous formulations for the Laplace and the biharmonic equations have been given in Fokas & Kapaev [6] and Crowdy & Fokas [3] respectively.

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Appendix

Following Fokas & Zyskin [12] we will show that if the domain D is a convex polygon, then equation (2.22) can be mapped to equation (2.16).

The Fourier representation of Dirac's delta function is given by

$$\delta(z) = \frac{1}{4\pi^2} \iint_{R^2} e^{ik_1 x + ik_2 y}\, dk_1\, dk_2,$$

where k_1, k_2, x, y are real. Using in this equation

$$\delta(z) = -\frac{1}{2\pi i} \partial_{\bar{z}} \left(\frac{1}{z} \right), \quad x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i},$$

and integrating the resulting equation with respect to $\bar{z}$, we find

$$\frac{1}{z} = \frac{1}{i\pi} \iint_{R^2} e^{\frac{1}{2}(ik_1+k_2)z + \frac{1}{2}(ik_1-k_2)\bar{z}} \frac{dk_1 dk_2}{ik_1 - k_2}. \tag{A1}$$

Using this identity in equation (2.22) (with $z - \zeta$ instead of z), and letting $\zeta = \xi + i\eta$, we find

$$H(z) = -\frac{1}{2\pi^2} \iint_{R^2} \left(\sum_{j=1}^{n} \int_{z_{j+1}}^{z_j} e^{ik_1(x-\xi)+ik_2(y-\eta)} H(\xi + i\eta)(d\xi + i d\eta) \right) \frac{dk_1 dk_2}{ik_1 - k_2}. \tag{A2}$$

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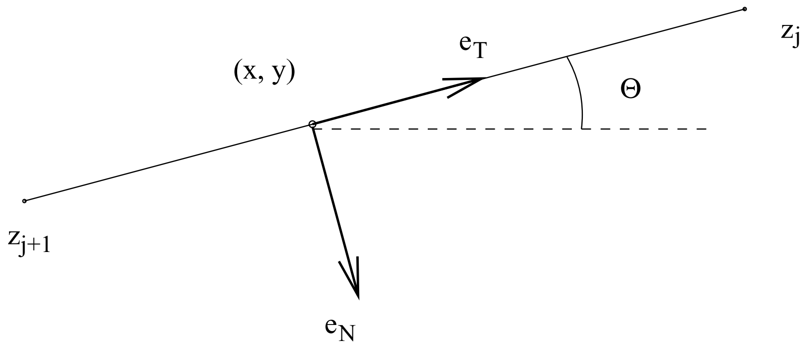

FIGURE 6. The orthonormal vectors e_T and e_N associated with the side (z_{j+1}, z_j).

Let us concentrate on the term involving the side (z_{j+1}, z_j). Suppose that this side makes an angle θ with the x-axis (see Figure 6). Let e_T be the unit vector along this side, and e_N the unit vector perpendicular to this side pointing outwards.

Expanding the vector $\overrightarrow{k} = (k_1, k_2)$ along e_T, e_N, and denoting the corresponding components by k_T, k_N, we find,

$$\begin{pmatrix} k_T \\ k_N \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

Hence,

$$ik_1 - k_2 = e^{i\theta}(k_N + ik_T). \tag{A3}$$

Similarly, expanding the vector $\overrightarrow{X} = (x - \xi, y - \eta)$ along e_T and e_N, and denoting the corresponding components by (X_T, X_N), we find,

$$(X_N + iX_T) = ie^{i\theta}\left[(x - \xi) + i(y - \eta)\right]. \tag{A4}$$

Since the inner product $\overrightarrow{k} \cdot \overrightarrow{X}$ is invariant under rotation, it follows that

$$k_1(\xi - x) + k_2(\eta - y) = k_T X_T + k_N X_N. \tag{A5}$$

The point (x, y) lies inside the convex polygon Ω, and e_N points outwards, therefore $X_N < 0$. Hence, we can compute the relevant integral by integrating in the lower-half complex k_N-plane. Thus the relevant term of the right-hand side of equation (A2) becomes

$$\frac{1}{2\pi}\int_0^\infty dk_T \int_{z_{j+1}}^{z_j} e^{k_T(iX_T + X_N) - i\theta} H(dx + i\,dy). \tag{A6}$$

Using (A5) and reparametrizing k_T by $ke^{i\theta}$, the above equation becomes

$$\frac{1}{2\pi}\int_{l_j} e^{ikz}\widehat{H}_j(k)\,dk. \tag{A7}$$

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