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Harmonic analysis operators associated with multidimensional Bessel operators

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We establish that the maximal operator and the Littlewood–Paley g-function associated with the heat semigroup defined by multidimensional Bessel operators are of weak type $(1, 1)$. We also prove that Riesz transforms in the multidimensional Bessel setting are of strong type (p, p), for every $1 < p < \infty$, and of weak type $(1, 1)$.

1. Introduction

Muckenhoupt and Stein [14] made an in-depth study about harmonic analysis in the one-dimensional ultraspherical and Bessel settings. In this paper we are interested in the L^p-boundedness properties for the maximal operator and Littlewood–Paley g-function associated with the heat semigroup for the multidimensional Bessel operators and the Riesz transforms in the multidimensional Bessel context. As far as we know, harmonic analysis operators in the Bessel settings have always been studied in the one-dimensional case; only recently have the authors considered multipliers of Laplace transform type in this n-dimensional Bessel context [9]. Following the publication of [14], both Andersen and Kerman [1, 2, 10, 11] established L^p-weighted inequalities for the Bessel–Riesz transforms. Stempak [18] studied Littlewood–Paley g-functions and Mihlin–Hörmander multipliers in the Bessel setting, and, together with Nowak [16], established transplantation theorems for Hankel transforms. Recently, in [3–8], the one-dimensional harmonic analysis in the Bessel context has been completed by investigating properties of g-functions and Riesz transforms of every order and maximal operators associated with Poisson and heat semigroups.

We now present some notational conventions that will allow us to simplify the presentation of our results. We denote by $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ elements of $(0, \infty)^n$ or $\mathbb{R}^n$ and we represent by u and v positive or real numbers.

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By $\lambda = (\lambda_1, \dots, \lambda_n)$ we denote an element of $(-\frac{1}{2}, \infty)^n$ (n-dimensional index) and we consider always $\alpha \in (-\frac{1}{2}, \infty)$ (one-dimensional index). Also, we write $\bar{x} = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ and $\bar{\lambda} = (\lambda_2, \dots, \lambda_n) \in (-\frac{1}{2}, \infty)^{n-1}$ when $x \in \mathbb{R}^n$ or $\lambda \in (-\frac{1}{2}, \infty)^n$. If $x \in \mathbb{R}^n$ is a vector,

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}$$

represents its Euclidean norm, while $|\lambda| = \lambda_1 + \cdots + \lambda_n$ is the length of the multi-index $\lambda \in (-\frac{1}{2}, \infty)^n$. Suppose that we have a function $g(u, v, \alpha, t)$, $t, u, v \in (0, \infty)$ and $\alpha \in (-\frac{1}{2}, \infty)$. We define

$$g(x, y, \lambda, t) = \prod_{j=1}^{n} g(x_j, y_j, \lambda_j, t), \quad t \in (0, \infty),\ x, y \in (0, \infty)^n,\ \lambda \in (-\tfrac{1}{2}, \infty)^n.$$

For instance, we shall use the function

$$\left(\frac{y}{x}\right)^{\lambda} = \prod_{j=1}^{n} \left(\frac{y_j}{x_j}\right)^{\lambda_j}, \quad x, y \in (0, \infty)^n,\ \lambda \in (-\tfrac{1}{2}, \infty)^n.$$

We think this notation does not produce any confusion.

By $\mathrm{d}\mu_\lambda(x)$ we represent the product measure

$$\prod_{j=1}^{n} x^{2\lambda_j}\, \mathrm{d}x_j \quad \text{on } (0, \infty)^n.$$

We consider the n-dimensional Bessel operator Δ_λ defined by

$$\Delta_\lambda = \sum_{j=1}^{n} \Delta_{\lambda_j, x_j},$$

where

$$\Delta_{\lambda_j, x_j} = -x_j^{-2\lambda_j} \frac{\partial}{\partial x_j}\left(x_j^{2\lambda_j} \frac{\partial}{\partial x_j}\right) \quad \text{for every } j = 1, \dots, n,\ n \geqslant 2.$$

The heat semigroup generated by $-\Delta_\lambda$ is represented by $\{W_t^\lambda\}_{t>0}$. This semigroup is a symmetric diffusion semigroup in the sense of [17] with respect to the measure $\mathrm{d}\mu_\lambda$. Then, according to [17, p. 73] the maximal operator

$$W_*^\lambda(f) = \sup_{t>0} |W_t^\lambda(f)|$$

is bounded from $L^p((0, \infty)^n, \mathrm{d}\mu_\lambda)$ into itself, for every $1 < p \leqslant \infty$. Inspired by the ideas developed by Nowak and Sjögren in [15], we establish that W_*^λ is of weak type $(1, 1)$ with respect to the measure $\mathrm{d}\mu_\lambda$.

Theorem 1.1. *Let $\lambda \in (-\frac{1}{2}, \infty)^n$. The maximal operator W_*^λ is bounded from $L^1((0, \infty)^n, \mathrm{d}\mu_\lambda)$ into $L^{1,\infty}((0, \infty)^n, \mathrm{d}\mu_\lambda)$.*

Since, for every $f \in C_c^\infty((0,\infty)^n)$ (the space of C^∞-functions on $(0,\infty)^n$ that have compact support),

$$\lim_{t\to 0^+} W_t^\lambda(f)(x) = f(x), \quad x \in (0,\infty)^n,$$

and $C_c^\infty((0,\infty)^n)$ is a dense subspace of $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ for every $1 \leqslant p < \infty$, standard arguments give the following result.

Corollary 1.2. *Let*

$$\lambda \in (-\tfrac{1}{2},\infty)^n \quad \textit{and} \quad 1 \leqslant p < \infty.$$

For every $f \in L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$,

$$\lim_{t\to 0^+} W_t^\lambda(f)(x) = f(x) \quad \textit{a.e. } x \in (0,\infty)^n.$$

According to [17, p. 67], the semigroup $\{W_t^\lambda\}_{t>0}$ admits an analytic extension to $\Omega = \{t \in \mathbb{C}\colon |\arg(t)| < \frac{1}{2}\pi(1 - |2/p - 1|)\}$ in $L^p((0,\infty)^n, d\mu_\lambda)$, $1 < p < \infty$. The Littlewood–Paley g-function of the first order associated with $\{W_t^\lambda\}_{t>0}$ is defined by

$$g^\lambda(f)(x) = \left\{ \int_0^\infty \left| t \frac{\partial}{\partial t} W_t^\lambda(f)(x) \right|^2 \frac{\mathrm{d}t}{t} \right\}^{1/2}, \quad x \in (0,\infty)^n.$$

By [17, p. 111], g^λ defines a bounded operator from $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ into itself for every $1 < p < \infty$. We complete this result by analysing the behaviour of g^λ on $L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)$.

Theorem 1.3. *Let* $\lambda \in (-\frac{1}{2},\infty)^n$. *Then* g^λ *is bounded from* $L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)$ *into* $L^{1,\infty}((0,\infty)^n, \mathrm{d}\mu_\lambda)$.

The Bessel operator Δ_α can be factorized as follows:

$$\Delta_\alpha = -x^{-2\alpha} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{2\alpha} \frac{\mathrm{d}}{\mathrm{d}x} \right) = D^* D,$$

where $D = \mathrm{d}/\mathrm{d}x$ and D^* denotes the formal adjoint of D in $L^2((0,\infty), \mathrm{d}\mu_\alpha)$. According to [14, § 16] and [17], we define, for every $i = 1,\dots,n$, the ith Riesz transform R_i^λ associated with Δ_λ by

$$R_i^\lambda f = \frac{\partial}{\partial x_i} \Delta_\lambda^{-1/2} f, \quad f \in C_c^\infty((0,\infty)^n).$$

Here $\Delta_\lambda^{-1/2}$ represents the negative square root of Δ_λ, whose definition will be specified later (see § 4). L^p-boundedness properties of the Riesz transforms R_i^λ, $i = 1,\dots,n$, are established in the following.

Theorem 1.4. *Let* $\lambda \in (-\frac{1}{2},\infty)^n$ *and* $i = 1,\dots,n$. *For every* $f \in C_c^\infty((0,\infty)^n)$, $\Delta_\lambda^{-1/2} f$ *admits a derivative with respect to* x_i *on almost all* $(0,\infty)^n$ *and*

$$\frac{\partial}{\partial x_i} \Delta_\lambda^{-1/2} f(x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} R_i^\lambda(x,y) f(y) \, \mathrm{d}\mu_\lambda(y) \quad \textit{a.e. } x \in (0,\infty)^n,$$

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where

$$R_i^\lambda(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\partial}{\partial x_i} W_t^\lambda(x,y) \frac{\mathrm{d}t}{\sqrt{t}}, \quad x,y \in (0,\infty)^n, \tag{1.1}$$

and $W_t^\lambda(x,y)$ *represents the kernel of the operator* W_t^λ *for every* $t > 0$ *(see (1.3) for definitions).*

Moreover, the maximal operator $R_{i,*}^\lambda$ *defined by*

$$R_{i,*}^\lambda(f)(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} R_i^\lambda(x,y) f(y) \, \mathrm{d}\mu_\lambda(y) \right|, \quad x \in (0,\infty)^n,$$

is bounded from $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ *into itself, for every* $1 < p < \infty$*, and from* $L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)$ *into* $L^{1,\infty}((0,\infty)^n, \mathrm{d}\mu_\lambda)$*. The Riesz transform* R_i^λ *can be extended to* $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ *by*

$$R_i^\lambda(f)(x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} R_i^\lambda(x,y) f(y) \, \mathrm{d}\mu_\lambda(y) \quad \text{a.e. } x \in (0,\infty)^n$$

for every $f \in L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$, $1 \leqslant p < \infty$*, that is a bounded operator from* $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ *into itself, for* $1 < p < \infty$*, and from* $L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)$ *into* $L^{1,\infty}((0,\infty)^n, \mathrm{d}\mu_\lambda)$*.*

We now recall some definitions and properties that will be useful in the following. If J_ν denotes the Bessel function of the first kind and order $\nu > -1$, we have that, when $\alpha > -\frac{1}{2}$,

$$\Delta_{\alpha,u}((uv)^{-\alpha+1/2} J_{\alpha-1/2}(uv)) = v^2 (uv)^{-\alpha+1/2} J_{\alpha-1/2}(uv), \quad u,v \in (0,\infty).$$

Then, the heat semigroup $\{W_t^\alpha\}_{t>0}$ generated by $-\Delta_\alpha$ is given by

$$W_t^\alpha(f)(u) = \int_0^\infty W_t^\alpha(u,v) f(v) v^{2\alpha} \, \mathrm{d}v,$$

where

$$W_t^\alpha(u,v) = \int_0^\infty \mathrm{e}^{-tz^2} (uz)^{-\alpha+1/2} J_{\alpha-1/2}(uz) (vz)^{-\alpha+1/2} J_{\alpha-1/2}(vz) z^{2\alpha} \, \mathrm{d}z, \qquad t,u,v \in (0,\infty).$$

Moreover, according to [20, p. 395], we can write

$$W_t^\alpha(u,v) = \frac{(uv)^{-\alpha+1/2}}{2t} I_{\alpha-1/2}\left(\frac{uv}{2t}\right) \mathrm{e}^{-(u^2+v^2)/4t}, \quad t,u,v \in (0,\infty), \tag{1.2}$$

where I_ν represents the modified Bessel function of the first kind and order $\nu > -1$.

The heat semigroup $\{W_t^\lambda\}_{t>0}$ generated by the multidimensional Bessel operator $-\Delta_\lambda$ is defined by

$$W_t^\lambda(f)(x) = \int_{(0,\infty)^n} W_t^\lambda(x,y) f(y) \, \mathrm{d}\mu_\lambda(y),$$

where, according to our notational convention,

$$W_t^\lambda(x,y) = \prod_{j=1}^{n} W_t^{\lambda_j}(x_j,y_j), \quad x=(x_1,\dots,x_n),\ y=(y_1,\dots,y_n)\in(0,\infty)^n. \quad (1.3)$$

Asymptotic expansion of I_ν (see (3.3)) allows us to connect the operators (maximal operators, g-functions and Riesz transforms) associated with Δ_λ with the corresponding operators in the classical Euclidean setting in a local region defined by

$$L = \{(x,y)\in(0,\infty)^n\times(0,\infty)^n\colon \tfrac{1}{2}x_j < y_j < 2x_j,\ j=1,\dots,n\}.$$

Note that L contains the diagonal of $(0,\infty)^n\times(0,\infty)^n$. This is a crucial point in the proof of our theorems because in that diagonal are the singularities of the kernels of the operators under consideration. We describe in § 2 in a quite general way the procedure that we use to prove the L^p-boundedness properties of the treated operators. In § 3 we present some auxiliary results that will be very useful in the proof of theorem 1.4. The proofs of theorems 1.3 and 1.4 are more involved than that of theorem 1.1 because the maximal operator W_*^λ is positive, while the Littlewood–Paley g-function and Riesz transforms are not positive operators. For brevity, we present only a complete proof of theorem 1.4 (see § 4). By using the ideas developed in the proof of theorem 1.4 and applying our general procedure, the interested reader will thus be able to prove theorems 1.1 and 1.3.

It is remarkable that, as can be observed in § 4, the proof of the results in the n-dimensional setting cannot be made by iterating one-dimensional results, and much more work is needed.

Throughout this paper we shall use repeatedly without further mention the fact that, for every $k\geqslant 0$, there exists $C_k>0$ such that $z^k\mathrm{e}^{-z}\leqslant C_k$, $z>0$. By C we denote a suitable positive constant that can change from one line to another.

2. A general procedure

In this section we describe a general procedure that can be used to show theorems 1.1, 1.3 and 1.4.

Suppose now that P is a monomial in $\mathbb{R}^n$ and $\mathcal{A}$ is a linear space of continuous functions on $(0,\infty)$ such that $P(\partial_{x_1},\dots,\partial_{x_n})W_t^\lambda(x,y)\in\mathcal{A}$, $x,y\in(0,\infty)^n$, $x\neq y$, and $P(\partial_{x_1},\dots,\partial_{x_n})\mathbb{W}_t(x,y)\in\mathcal{A}$, $x,y\in\mathbb{R}^n$, $x\neq y$, where $\mathbb{W}_t(x,y)$ represents the classical heat kernel defined by

$$\mathbb{W}_t(x,y) = \frac{\mathrm{e}^{-|x-y|^2/4t}}{(4\pi t)^{n/2}}, \quad x,y\in\mathbb{R}^n,\ t>0.$$

Assume that $\mathbb{B}$ is a Banach space and $\mathcal{L}$ is a linear mapping from $\mathcal{A}$ into $\mathbb{B}$. We define

$$K^\lambda(x,y) = \mathcal{L}\{P(\partial_{x_1},\dots,\partial_{x_n})W_t^\lambda(x,y)\}, \quad x,y\in(0,\infty)^n,\ x\neq y,$$

and

$$\mathbb{K}(x,y) = \mathcal{L}\{P(\partial_{x_1},\dots,\partial_{x_n})\mathbb{W}_t(x,y)\}, \quad x,y\in\mathbb{R}^n,\ x\neq y.$$

Assume also that the functions

$$K^\lambda(x,\cdot)\colon (0,\infty)^n \setminus \{x\} \to \mathbb{B}, \quad x \in (0,\infty)^n,$$

and

$$\mathbb{K}(x,\cdot)\colon \mathbb{R}^n \times \setminus \{x\} \to \mathbb{B}, \quad x \in \mathbb{R}^n,$$

are strongly measurable, and that, for every $f \in C_c^\infty((0,\infty)^n)$ (respectively, $f \in C_c^\infty(\mathbb{R}^n)$), the limit

$$\lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} \mathcal{K}(x,y) f(y)\,\mathrm{d}y$$

exists, for almost all $x \in (0,\infty)^n$, with respect to the Lebesgue measure on $(0,\infty)^n$ (respectively, $\mathbb{R}^n$), where $\mathcal{K} = K^\lambda$ (respectively, $\mathcal{K} = \mathbb{K}$). Here, the integral and the limit are understood in the $\mathbb{B}$-Bochner sense and in $\mathbb{B}$, respectively.

We consider the operators T^λ and $\mathbb{T}$ defined by

$$T^\lambda(f)(x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} K^\lambda(x,y) f(y)\,\mathrm{d}\mu_\lambda(y) \qquad \text{a.e. } x \in (0,\infty)^n,\ f \in C_c^\infty((0,\infty)^n),$$

and

$$\mathbb{T}(f)(x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} \mathbb{K}(x,y) f(y)\,\mathrm{d}y \quad \text{a.e. } x \in \mathbb{R}^n,\ f \in C_c^\infty(\mathbb{R}^n), \tag{2.1}$$

and the corresponding local operators given by

$$T^\lambda_{\mathrm{loc}}(f)(x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon, y\in L(x)} K^\lambda(x,y) f(y)\,\mathrm{d}\mu_\lambda(y) \qquad \text{a.e. } x \in (0,\infty)^n,\ f \in C_c^\infty((0,\infty)^n),$$

and

$$\mathbb{T}^\lambda_{\mathrm{loc}}(f)(x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon, y\in L(x)} \mathbb{K}(x,y) \left(\frac{y}{x}\right)^\lambda f(y)\,\mathrm{d}y \qquad \text{a.e. } x \in (0,\infty)^n,\ f \in C_c^\infty((0,\infty)^n), \tag{2.2}$$

where, for every $x \in (0,\infty)^n$, the local region $L(x)$ is defined by

$$L(x) = \{y \in (0,\infty)^n \colon \tfrac{1}{2}x_j < y_j < 2x_j,\ j = 1,\dots,n\}.$$

We apply the general procedure described in this section to study L^p-boundedness properties for maximal operators and Littlewood–Paley g-functions associated with the heat semigroup and the Riesz transforms in the Bessel setting. In each case it is not hard to identify the Banach space $\mathbb{B}$, the linear space $\mathcal{A}$, the linear mapping $\mathcal{L}$ and the monomial P. Although some of these operators can be only defined as a pointwise principal value limit and not as a principal value limit in the corresponding Banach spaces, the procedure still works. We describe the procedure in this general setting looking for a unified presentation.

We denote by $L^p_{\mathbb{B}}(\Omega, \mathrm{d}\omega)$, $1 \leqslant p < \infty$, and $L^{1,\infty}_{\mathbb{B}}(\Omega, \mathrm{d}\omega)$ the $\mathbb{B}$-valued Bochner–Lebesgue L^p-space, $1 \leqslant p < \infty$, and the $\mathbb{B}$-valued weak L^1-space, respectively.

Our objective is to obtain L^p-boundedness properties for the operator T^λ from the corresponding L^p-boundedness properties for $\mathbb{T}$. Hence, we need to know these properties for the operator $\mathbb{T}$. In many cases (Riesz transforms, Littlewood–Paley g-functions and the maximal operator for the heat semigroup, in the classical setting) the boundedness properties that we need are known.

In the following result we prove that boundedness of the operator $\mathbb{T}$ implies the boundedness of the local operator $\mathbb{T}^\lambda_{\mathrm{loc}}$ provided that

$$\|\mathbb{K}(x, y)\|_{\mathbb{B}} \leqslant \frac{C}{|x-y|^n}, \quad x, y \in \mathbb{R}^n,\ x \neq y. \tag{2.3}$$

Proposition 2.1. *Let $\mathbb{B}$ be a Banach space. We consider the operators $\mathbb{T}$ and $\mathbb{T}^\lambda_{\mathrm{loc}}$ defined by (2.1) and (2.2), respectively, where $\mathbb{K}$ satisfies (2.3). If $1 < p < \infty$ and $\mathbb{T}$ can be extended to $L^p(\mathbb{R}^n, \mathrm{d}x)$ as a bounded operator from $L^p(\mathbb{R}^n, \mathrm{d}x)$ into $L^p_{\mathbb{B}}(\mathbb{R}^n, \mathrm{d}x)$, then $\mathbb{T}^\lambda_{\mathrm{loc}}$ can be extended to $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ as a bounded operator from $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ into $L^p_{\mathbb{B}}((0,\infty)^n, \mathrm{d}\mu_\lambda)$. Also, if $\mathbb{T}$ can be extended to $L^1(\mathbb{R}^n, \mathrm{d}x)$ as a bounded operator from $L^1(\mathbb{R}^n, \mathrm{d}x)$ into $L^{1,\infty}_{\mathbb{B}}(\mathbb{R}^n, \mathrm{d}x)$, then $\mathbb{T}^\lambda_{\mathrm{loc}}$ can be extended to $L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)$ as a bounded operator from $L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)$ into $L^{1,\infty}_{\mathbb{B}}((0,\infty)^n, \mathrm{d}\mu_\lambda)$.*

Proof. Let $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$. We consider

$$Q_m = \{y \in (0,\infty)^n : 2^{m_j} \leqslant y_j \leqslant 2^{m_j+1},\ j = 1, \dots, n\}$$

and

$$\tilde{Q}_m = \{y \in (0,\infty)^n : 2^{m_j-1} \leqslant y_j \leqslant 2^{m_j+2},\ j = 1, \dots, n\}.$$

Note that if $x \in Q_m$ and $y \in L(x)$, then $y \in \tilde{Q}_m$.

We define the operator

$$\mathcal{Y}^\lambda_m(f)(x) = \chi_{Q_m}(x)\mathbb{T}(\chi_{\tilde{Q}_m}(y)f(y)(y/x)^\lambda)(x), \quad f \in C^\infty_{\mathrm{c}}((0,\infty)^n).$$

We then assume that $\mathbb{T}$ can be extended to $L^p(\mathbb{R}^n, \mathrm{d}x)$ as a bounded operator from $L^p(\mathbb{R}^n, \mathrm{d}x)$ into $L^p_{\mathbb{B}}(\mathbb{R}^n, \mathrm{d}x)$, where $1 < p < \infty$. Let $f \in C^\infty_{\mathrm{c}}((0,\infty)^n)$. We can write

$$\begin{aligned}
\Big\| \sum_{m \in \mathbb{Z}^n} \mathcal{Y}^\lambda_m(f) \Big\|^p_{L^p_{\mathbb{B}}((0,\infty)^n, \mathrm{d}\mu_\lambda)} &= \sum_{m \in \mathbb{Z}^n} \int_{Q_m} \|\mathbb{T}(\chi_{\tilde{Q}_m}(y)f(y)(y/x)^\lambda)(x)\|^p_{\mathbb{B}} \,\mathrm{d}\mu_\lambda(x) \\
&\leqslant C \sum_{m \in \mathbb{Z}^n} 2^{(2-p)m\lambda} \int_{\mathbb{R}^n} \|\mathbb{T}(\chi_{\tilde{Q}_m}(y)f(y)y^\lambda)(x)\|^p_{\mathbb{B}} \,\mathrm{d}x \\
&\leqslant C \sum_{m \in \mathbb{Z}^n} 2^{(2-p)m\lambda} \int_{\tilde{Q}_m} |f(y)y^\lambda|^p \,\mathrm{d}y \\
&\leqslant C \sum_{m \in \mathbb{Z}^n} \int_{\tilde{Q}_m} |f(y)|^p \,\mathrm{d}\mu_\lambda(y) \\
&\leqslant C\|f\|^p_{L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)},
\end{aligned}$$

where $m\lambda = \sum_{j=1}^n m_j\lambda_j$.

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We now analyse the difference operator

$$\tau^\lambda = \sum_{m\in\mathbb{Z}^n} \mathcal{Y}_m^\lambda - \mathbb{T}_{\text{loc}}^\lambda.$$

Fix $m = (m_1,\dots,m_n) \in \mathbb{Z}^n$. We have that

$$\tilde{Q}_m \setminus L(x) = \bigcup_{\ell=1}^n (B_\ell(x) \cup C_\ell(x)),$$

where, for every $\ell = 1,\dots,n$,

$$B_\ell(x) = \{y \in (0,\infty)^n \colon 2^{m_j-1} \leqslant y_j \leqslant 2^{m_j+2},\\ j=1,\dots,n,\ j\neq\ell;\ 2^{m_\ell-1} \leqslant y_\ell < \tfrac{1}{2}x_\ell\}$$

and

$$C_\ell(x) = \{y \in (0,\infty)^n \colon 2^{m_j-1} \leqslant y_j \leqslant 2^{m_j+2},\\ j=1,\dots,n,\ j\neq\ell;\ 2x_\ell < y_\ell \leqslant 2^{m_\ell+2}\}.$$

We can write, for each $x \in Q_m$,

$$\begin{aligned} \|(\mathcal{Y}_m^\lambda - \mathbb{T}_{\text{loc}}^\lambda)(f)(x)\|_{\mathbb{B}} &= \|\mathbb{T}(\chi_{\tilde{Q}_m\setminus L(x)}(y) f(y)(y/x)^\lambda)(x)\|_{\mathbb{B}} \\ &\leqslant C\sum_{\ell=1}^n \biggl(\int_{B_\ell(x)} |f(y)| \Bigl(\frac{y}{x}\Bigr)^\lambda \frac{1}{|x-y|^n}\,\mathrm{d}y + \int_{C_\ell(x)} |f(y)| \Bigl(\frac{y}{x}\Bigr)^\lambda \frac{1}{|x-y|^n}\,\mathrm{d}y\biggr). \end{aligned} \tag{2.4}$$

We estimate only the integral on $B_1(x)$. The other integrals can be analysed similarly. Let $x \in Q_m$. Since $|x_1-y_1| = x_1-y_1 > \frac{1}{2}x_1 > 2^{m_1-1}$ when $2^{m_1-1} \leqslant y_1 < \frac{1}{2}x_1$, we get

$$\begin{aligned} \int_{B_1(x)} |f(y)| \Bigl(\frac{y}{x}\Bigr)^\lambda \frac{1}{|x-y|^n}\,\mathrm{d}y &\leqslant C\int_{B_1(x)} \frac{|f(y)|}{(2^{2m_1-2} + \sum_{j=2}^n (x_j-y_j)^2)^{n/2}} \Bigl(\frac{y}{x}\Bigr)^\lambda \mathrm{d}y \\ &\leqslant C\int_{\tilde{Q}_m} \frac{|f(y)|}{(2^{2m_1-2} + \sum_{j=2}^n (x_j-y_j)^2)^{n/2}}\,\mathrm{d}y \\ &\leqslant C\int_{\mathbb{R}^{n-1}} \frac{\tilde{f}_m(\bar{y})}{(2^{2m_1-2} + |\bar{x}-\bar{y}|^2)^{n/2}}\,\mathrm{d}\bar{y}, \end{aligned} \tag{2.5}$$

where

$$\tilde{f}_m(\bar{y}) = \chi_{\prod_{j=2}^n [2^{m_j-1},2^{m_j+2}]}(\bar{y}) \int_{2^{m_1-1}}^{2^{m_1+2}} |f(y_1,\bar{y})|\,\mathrm{d}y_1, \quad \bar{y} \in \mathbb{R}^{n-1}.$$

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Then, by standard arguments, we get, for every $\bar{x} \in \mathbb{R}^{n-1}$,

$$\int_{B_1(x)} |f(y)| \left(\frac{y}{x}\right)^{\lambda} \frac{1}{|x-y|^n} \, \mathrm{d}y \leqslant C \sum_{k=0}^{\infty} \frac{1}{2^{(m_1+k)n}} \int_{|\bar{x}-\bar{y}|<2^{k+m_1}} \tilde{f}_m(\bar{y}) \, \mathrm{d}\bar{y}$$
$$\leqslant C 2^{-m_1} \mathcal{M}_{n-1}(\tilde{f}_m)(\bar{x}),$$

where $\mathcal{M}_{n-1}$ denotes the Hardy–Littlewood maximal function on $\mathbb{R}^{n-1}$.

Since the maximal function $\mathcal{M}_{n-1}$ is bounded from $L^p(\mathbb{R}^{n-1}, \mathrm{d}\bar{x})$ into itself, it follows, by using Jensen's inequality, that

$$\int_{Q_m} \left| \int_{B_1(x)} |f(y)| \left(\frac{y}{x}\right)^{\lambda} \frac{1}{|x-y|^n} \, \mathrm{d}y \right|^p \mathrm{d}\mu_\lambda(x)$$
$$\leqslant C 2^{2m\lambda} \int_{2^{m_1}}^{2^{m_1+1}} 2^{-m_1 p} \int_{\mathbb{R}^{n-1}} |\mathcal{M}_{n-1}(\tilde{f}_m)(\bar{x})|^p \, \mathrm{d}\bar{x} \, \mathrm{d}x_1$$
$$\leqslant C 2^{2m\lambda + m_1(1-p)} \int_{\prod_{j=2}^{n}[2^{m_j-1}, 2^{m_j+2}]} \left(\int_{2^{m_1-1}}^{2^{m_1+2}} |f(y_1, \bar{y})| \, \mathrm{d}y_1 \right)^p \mathrm{d}\bar{y}$$
$$\leqslant C 2^{2m\lambda} \int_{\tilde{Q}_m} |f(y)|^p \, \mathrm{d}y$$
$$\leqslant C \int_{\tilde{Q}_m} |f(y)|^p \, \mathrm{d}\mu_\lambda(y).$$

By combining the above estimates we obtain

$$\int_{(0,\infty)^n} \|\tau^\lambda(f)(x)\|_{\mathbb{B}}^p \, \mathrm{d}\mu_\lambda(x) = \sum_{m \in \mathbb{Z}^n} \int_{Q_m} \|(\mathcal{Y}_m^\lambda - \mathbb{T}_{\mathrm{loc}}^\lambda)(f)(x)\|_{\mathbb{B}}^p \, \mathrm{d}\mu_\lambda(x)$$
$$\leqslant C \sum_{m \in \mathbb{Z}^n} \int_{\tilde{Q}_m} |f(y)|^p \, \mathrm{d}\mu_\lambda(y)$$
$$\leqslant C \|f\|_{L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)}^p.$$

Hence, we conclude that $\mathbb{T}_{\mathrm{loc}}^\lambda$ can be extended to $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ as a bounded operator from $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ into $L_{\mathbb{B}}^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$.

Suppose now that $\mathbb{T}$ can be extended to $L^1(\mathbb{R}^n, \mathrm{d}x)$ as a bounded operator from $L^1(\mathbb{R}^n, \mathrm{d}x)$ into $L_{\mathbb{B}}^{1,\infty}(\mathbb{R}^n, \mathrm{d}x)$. Let $f \in C_{\mathrm{c}}^\infty((0,\infty)^n)$ and $\gamma > 0$. We have that

$$\mu_\lambda(\{x \in (0,\infty)^n \colon \|\mathbb{T}_{\mathrm{loc}}^\lambda(f)(x)\|_{\mathbb{B}} > \gamma\})$$
$$\leqslant \mu_\lambda\left(\left\{x \in (0,\infty)^n \colon \left\|\mathbb{T}_{\mathrm{loc}}^\lambda(f)(x) - \sum_{m \in \mathbb{Z}^n} \mathcal{Y}_m^\lambda(f)(x)\right\|_{\mathbb{B}} > \tfrac{1}{2}\gamma\right\}\right)$$
$$+ \mu_\lambda\left(\left\{x \in (0,\infty)^n \colon \left\|\sum_{m \in \mathbb{Z}^n} \mathcal{Y}_m^\lambda(f)(x)\right\|_{\mathbb{B}} > \tfrac{1}{2}\gamma\right\}\right)$$
$$\leqslant \mu_\lambda(\{x \in (0,\infty)^n \colon \|\tau^\lambda(x)\|_{\mathbb{B}} > \tfrac{1}{2}\gamma\})$$
$$+ \sum_{m \in \mathbb{Z}^n} \mu_\lambda(\{x \in Q_m \colon \|\mathcal{Y}_m^\lambda(f)(x)\|_{\mathbb{B}} > \tfrac{1}{2}\gamma\}).$$

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Moreover, for every $m \in \mathbb{Z}^n$, we get

$$\begin{aligned}\mu_\lambda(\{x \in Q_m\colon \|\mathcal{Y}_m^\lambda(f)(x)\|_{\mathbb{B}} > \tfrac{1}{2}\gamma\}) &\leqslant C 2^{2m\lambda} \mu_0^{(n)}(\{x \in \mathbb{R}^n\colon \|\mathbb{T}(\chi_{\tilde{Q}_m}(y) f(y) y^\lambda)(x)\|_{\mathbb{B}} > 2^{m\lambda}\tfrac{1}{2}\gamma\}) \\ &\leqslant C \frac{2^{m\lambda}}{\gamma} \int_{\tilde{Q}_m} |f(y)| y^\lambda \, \mathrm{d}y \leqslant \frac{C}{\gamma} \int_{\tilde{Q}_m} |f(y)| \, \mathrm{d}\mu_\lambda(y).\end{aligned}$$

Henceforth, $\mu_0^{(\ell)}$ represents the Lebesgue measure on $\mathbb{R}^\ell$, $1 \leqslant \ell \leqslant n$.

Hence,

$$\sum_{m \in \mathbb{Z}^n} \mu_\lambda(\{x \in Q_m\colon \|\mathcal{Y}_m^\lambda(f)(x)\|_{\mathbb{B}} > \tfrac{1}{2}\gamma\}) \leqslant \frac{C}{\gamma} \|f\|_{L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)}.$$

Also, since $\mathcal{M}_{n-1}$ is bounded from $L^1(\mathbb{R}^{n-1}, \mathrm{d}\bar{x})$ into $L^{1,\infty}(\mathbb{R}^{n-1}, \mathrm{d}\bar{x})$, we obtain, for every $m \in \mathbb{Z}^n$,

$$\begin{aligned}\mu_\lambda\bigg(\bigg\{x \in Q_m\colon \int_{B_1(x)} |f(y)| \bigg(\frac{y}{x}\bigg)^\lambda \frac{1}{|x-y|^n} \, \mathrm{d}y > \gamma\bigg\}\bigg) &\leqslant \mu_\lambda(\{x \in Q_m\colon \mathcal{M}_{n-1}(\tilde{f}_m)(\bar{x}) > \gamma 2^{m_1} M\}) \\ &\leqslant 2^{2m\lambda} \mu_0^{(n)}(\{x \in Q_m\colon \mathcal{M}_{n-1}(\tilde{f}_m)(\bar{x}) > \gamma 2^{m_1} M\}) \\ &\leqslant 2^{2m\lambda + m_1} \mu_0^{(n-1)}(\{\bar{x} \in \mathbb{R}^{n-1}\colon \mathcal{M}_{n-1}(\tilde{f}_m)(\bar{x}) > \gamma 2^{m_1} M\}) \\ &\leqslant \frac{C}{\gamma} 2^{2m\lambda} \|\tilde{f}_m\|_{L^1(\mathbb{R}^{n-1}, \mathrm{d}\bar{x})} \\ &\leqslant \frac{C}{\gamma} 2^{2m\lambda} \int_{\tilde{Q}_m} |f(y)| \, \mathrm{d}y \\ &\leqslant \frac{C}{\gamma} \int_{\tilde{Q}_m} |f(y)| \, \mathrm{d}\mu_\lambda(y).\end{aligned}$$

Here $C, M > 0$ are constants that do not depend on $m \in \mathbb{Z}^n$.

Hence, we can deduce from (2.4) that

$$\begin{aligned}\mu_\lambda(\{x \in (0,\infty)\colon \|\tau^\lambda(f)(x)\|_{\mathbb{B}} > \gamma\}) &\leqslant \frac{C}{\gamma} \sum_{m \in \mathbb{Z}^n} \int_{\tilde{Q}_m} |f(y)| \, \mathrm{d}\mu_\lambda(y) \\ &\leqslant \frac{C}{\gamma} \|f\|_{L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)}.\end{aligned}$$

Combining the above estimates, it is possible to extend $\mathbb{T}_{\mathrm{loc}}^\lambda$ to $L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)$ as a bounded operator from $L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)$ into $L_{\mathbb{B}}^{1,\infty}((0,\infty)^n, \mathrm{d}\mu_\lambda)$. □

The key point of our procedure is to show the L^p-boundedness properties for the operator S^λ defined by

$$S^\lambda(f) = T^\lambda(f) - \mathbb{T}_{\mathrm{loc}}^\lambda(f), \quad f \in C_c^\infty((0,\infty)^n).$$

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We write $P(x_1, \dots, x_n) = x_1^{k_1} \cdots x_n^{k_n}$, $x \in \mathbb{R}^n$ and $k = (k_1, \dots, k_n) \in \mathbb{N}^n$. Let $f \in C_c^\infty((0,\infty)^n)$. We have that

$$S^\lambda(f) = \sum_{\ell=1}^n S_\ell^\lambda(f),$$

where

$$S_\ell^\lambda(f)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \mathcal{S}_\ell^\lambda(x,y) f(y) \, \mathrm{d}\mu_\lambda(y) \quad \text{a.e. } x \in (0,\infty)^n, \ \ell = 1, \dots, n,$$

and

$$\begin{aligned}\mathcal{S}_\ell^\lambda(x,y) &= \mathcal{L}\biggl\{ \prod_{j=1}^{\ell-1} \chi_{L_j(x)}(y_j)(x_j y_j)^{-\lambda_j} \frac{\partial^{k_j}}{\partial x_j^{k_j}} \mathbb{W}_t(x_j, y_j) \\ &\quad \times \biggl(\frac{\partial^{k_\ell}}{\partial x_\ell^{k_\ell}} W_t^{\lambda_\ell}(x_\ell, y_\ell) - \chi_{L_\ell(x)}(y_\ell)(x_\ell y_\ell)^{-\lambda_\ell} \frac{\partial^{k_\ell}}{\partial x_\ell^{k_\ell}} \mathbb{W}_t(x_\ell, y_\ell) \biggr) \\ &\quad \times \prod_{j=\ell+1}^{n} \frac{\partial^{k_j}}{\partial x_j^{k_j}} W_t^{\lambda_j}(x_j, y_j) \biggr\}, \\ &\qquad x, y \in (0,\infty)^n, \ x \neq y, \ \ell = 1, \dots, n.\end{aligned}$$

Here, $L_j(x) = \{v \in (0,\infty) \colon \frac{1}{2}x_j < v < 2x_j\}$ denotes the one-dimensional local region with respect to the jth variable, $j = 1, \dots, n$.

Let $\ell \in \{1, \dots, n\}$. We consider the positive operator H_ℓ^λ given by

$$H_\ell^\lambda(f)(x) = \int_{(0,\infty)^n} \|\mathcal{S}_\ell^\lambda(x,y)\|_{\mathbb{B}} f(y) \, \mathrm{d}\mu_\lambda(y) \quad \text{a.e. } x \in (0,\infty)^n.$$

According to the properties of the Bessel function I_ν established in § 3, in order to analyse the operator H_ℓ^λ we split the integral over $(0,\infty)^n$ into a sum of integrals as follows. We define, for every $u > 0$, the sets

$$\begin{aligned} A_1(u) &= \{v \in (0,\infty) \colon 0 < v \leqslant \tfrac{1}{2}u\}, \\ A_2(u) &= \{v \in (0,\infty) \colon \tfrac{1}{2}u < v < 2u\}, \\ A_3(u) &= \{v \in (0,\infty) \colon 2u \leqslant v < \infty\}, \end{aligned}$$

and, for every $x \in (0,\infty)^n$ and $s \in \{1,2,3\}^n$,

$$A_s(x) = \{y \in (0,\infty)^n \colon y_i \in A_{s_i}(x_i), \ i = 1, \dots, n\}.$$

We can write

$$H_\ell^\lambda(f) = \sum_{s \in \{1,2,3\}^n} H_{\ell,s}^\lambda(f),$$

where

$$H_{\ell,s}^\lambda(f)(x) = \int_{A_s(x)} \|\mathcal{S}_\ell^\lambda(x,y)\|_{\mathbb{B}} f(y) \, \mathrm{d}\mu_\lambda(y) \quad \text{a.e. } x \in (0,\infty)^n, \ s \in \{1,2,3\}^n.$$

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We must now establish the L^p-boundedness properties of the operators $H^{\lambda}_{\ell,s}$, $s \in \{1,2,3\}^n$. To make this we use estimations of the Bessel function I_ν (see § 3) adapted to the operator T^λ considered, that is, to the linear mapping $\mathcal{L}$ and the monomial P. Note that if $1 \leqslant p < \infty$, the composition operator $\mathcal{H}_1 \circ \mathcal{H}_2$ is not always of weak type (p,p) when $\mathcal{H}_1$ is of strong type (p,p) and $\mathcal{H}_2$ is of weak type (p,p). This fact means that the L^p-boundedness properties of the operator $H^{\lambda}_{\ell,s}$, $s \in \{1,2,3\}^n$, cannot be obtained by iteration of one-dimensional type operators. More involved manipulations are needed to control our operators by others whose L^p-boundedness properties can be established. In § 3 we present some of the auxiliary operators that will be used in the proof of theorem 1.4.

3. Some auxiliary results

In this section we present L^p-boundedness properties for some operators that will be very useful in the following. We also establish some properties of Bessel functions that will be needed in the proof of our results.

3.1. Auxiliary operators

The Hardy-type operator H_∞ defined by

$$H_\infty(g)(u) = \int_u^\infty \frac{g(v)}{v}\,\mathrm{d}v, \quad u \in (0,\infty),$$

is bounded from $L^1((0,\infty), u^{2\alpha}\,\mathrm{d}u)$ into itself when $\alpha > -\frac{1}{2}$ (see [13]). Then, for every $k \in \mathbb{N}$, the operator

$$H^k_\infty(g)(x) = \int_{x_1}^\infty \cdots \int_{x_k}^\infty \frac{g(y_1,\dots,y_k)}{y_1\cdots y_k}\,\mathrm{d}y_1\cdots\mathrm{d}y_k, \quad x \in (0,\infty)^k,$$

is bounded from $L^1((0,\infty)^k, \mathrm{d}\mu_\beta)$ into itself, provided that $\beta \in (-\frac{1}{2},\infty)^k$.

If $\beta \in \mathbb{R}^k$, with $k \in \mathbb{N}$, we define the local Hardy-type operator H^β_{loc} by

$$H^\beta_{\mathrm{loc}}(g)(x) = \frac{1}{x^{2\beta+1}} \int_{x_1/2}^{2x_1} \cdots \int_{x_k/2}^{2x_k} g(y)\,\mathrm{d}\mu_\beta(y), \quad x \in (0,\infty)^k.$$

It is not hard to see that H^β_{loc} is bounded from $L^1((0,\infty)^k, \mathrm{d}\mu_\beta)$ into itself.

Also, we consider, for every $1 \leqslant \ell \leqslant k$, $\ell, k \in \mathbb{N}$ and $\beta \in (-\frac{1}{2},\infty)^k$, the operator

$$\mathcal{H}^\beta_{\ell,k}(g)(x) = \int_0^{x_1/2} \cdots \int_0^{x_\ell/2} \int_{x_{\ell+1}/2}^{2x_{\ell+1}} \cdots \int_{x_k/2}^{2x_k} \prod_{j=\ell+1}^{k} (x_j y_j)^{-\beta_j} \frac{g(y)}{|x-y|^{2\varepsilon}}\,\mathrm{d}\mu_\beta(y), \quad x \in (0,\infty)^k,$$

where

$$\varepsilon = \sum_{j=1}^{\ell} \tfrac{1}{2}(\beta_j + \tfrac{1}{2}) + (k-\ell).$$

We can prove that $\mathcal{H}^\beta_{\ell,k}$ is bounded from $L^1((0,\infty)^k, \mathrm{d}\mu_\beta)$ into $L^{1,\infty}((0,\infty)^k, \mathrm{d}\mu_\beta)$ by proceeding as in [15, case 3].

Lemma 3.1. *Let $k \in \mathbb{N}$ and $\beta \in (-\frac{1}{2}, \infty)^k$. The operator L_β defined by*

$$L_\beta(g)(x) = \left(\sum_{j=1}^k x_j\right)^{-2|\beta|-k} \int_0^{x_1} \cdots \int_0^{x_k} g(y)\,\mathrm{d}\mu_\beta(y), \quad x \in (0,\infty)^k,$$

is bounded from $L^1((0,\infty)^k, \mathrm{d}\mu_\beta)$ into $L^{1,\infty}((0,\infty)^k, \mathrm{d}\mu_\beta)$.

Proof. Let $g \in L^1((0,\infty)^k, \mathrm{d}\mu_\beta)$ and $\gamma > 0$. We have that

$$\begin{aligned}
&\mu_\beta(\{x \in (0,\infty)^k : |L_\beta(g)(x)| > \gamma\}) \\
&\qquad \leqslant \mu_\beta\left(\left\{x \in (0,\infty)^k : \left(\sum_{j=1}^k x_j\right)^{-2|\beta|-k} \|g\|_{L^1((0,\infty)^k,\mathrm{d}\mu_\beta)} > \gamma\right\}\right) \\
&\qquad = \mu_\beta\left(\left\{x \in (0,\infty)^k : \sum_{j=1}^k x_j < \left(\frac{1}{\gamma}\|g\|_{L^1((0,\infty)^k,\mathrm{d}\mu_\beta)}\right)^{1/(2|\beta|+k)}\right\}\right) \\
&\qquad \leqslant \mu_\beta(Q),
\end{aligned}$$

where

$$Q = \left[0, \left(\frac{1}{\gamma}\|g\|_{L^1((0,\infty)^k,\mathrm{d}\mu_\beta)}\right)^{1/(2|\beta|+k)}\right]^k.$$

Since,

$$\mu_\beta(Q) \leqslant \frac{C}{\gamma}\|g\|_{L^1((0,\infty)^k,\mathrm{d}\mu_\beta)},$$

it follows that

$$\mu_\beta(\{x \in (0,\infty)^k : |L_\beta(g)(x)| > \gamma\}) \leqslant \frac{C}{\gamma}\|g\|_{L^1((0,\infty)^k,\mathrm{d}\mu_\beta)}.$$

Thus, the proof is finished. □

3.2. Properties of the Bessel function I_ν

The modified Bessel function I_ν, $\nu > -1$, admits the following series representation:

$$I_\nu(z) = \sum_{k=0}^\infty \frac{(\frac{1}{2}z)^{2k+\nu}}{\Gamma(k+1)\Gamma(\nu+k+1)}, \quad z \in (0,\infty). \tag{3.1}$$

Then,

$$I_\nu(z) \sim \frac{1}{2^\nu \Gamma(\nu+1)} z^\nu \quad \text{as } z \to 0^+. \tag{3.2}$$

Also, according to [12, p. 123] we have that

$$I_\nu(z) = \frac{\mathrm{e}^z}{\sqrt{2\pi z}}\left(\sum_{k=0}^m (-1)^k [\nu,k](2z)^{-k} + \mathcal{O}\left(\frac{1}{z^{m+1}}\right)\right), \quad m \in \mathbb{N},\ z \in (0,\infty), \tag{3.3}$$

where $[\nu, 0] = 1$ and

$$[\nu,k] = \frac{(4\nu^2-1)(4\nu^2-3^2)\cdots(4\nu^2-(2k-1)^2)}{2^{2k}\Gamma(k+1)}, \quad k = 1,2,\ldots.$$

From (3.1) it is easy to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^{-\nu}I_\nu(z)) = z^{-\nu}I_{\nu+1}(z), \quad z \in (0,\infty). \tag{3.4}$$

In the following we establish some properties of the heat kernel for the Bessel operator that will be useful in the proof of our results.

LEMMA 3.2. *Let* $\lambda \in (-\frac{1}{2},\infty)^n$. *Then*

$$W_t^\lambda(x,y) = (xy)^{-\lambda}\mathbb{W}_t(x,y)\biggl(1 + \sum_{\substack{k\in\{0,1,\dots,n\}^n\\ k\neq(0,\dots,0)}} c_k t^{|k|}(xy)^{-k} + g_n(x,y,t)\biggr), \tag{3.5}$$

where $c_k \in \mathbb{R}$ *and* $g_n \in C^\infty((0,\infty)^n \times (0,\infty)^n \times (0,\infty))$ *satisfy the condition that, for every compact set* $K \subset (0,\infty)^n$ *and* $a > 0$, *there exists* $C > 0$ *for which*

$$|g_n(x,y,t)| \leqslant Ct^{n+1}, \quad t \in (0,a) \text{ and } x,y \in K,$$

and, for every $j = 1,\dots,n$,

$$\left|\frac{\partial}{\partial x_j} g_n(x,y,t)\right| \leqslant Ct^{n+1}, \quad t \in (0,a) \text{ and } x,y \in K. \tag{3.6}$$

Proof. According to (1.2), (1.3) and (3.3), we have, for every $x,y \in (0,\infty)^n$ and $t > 0$,

$$\begin{aligned} W_t^\lambda(x,y) &= \frac{(xy)^{-\lambda}}{(2t)^{n/2}}\left(\frac{xy}{2t}\right)^{1/2} I_{\lambda-1/2}\left(\frac{xy}{2t}\right)\mathrm{e}^{-(|x|^2+|y|^2)/4t} \\ &= (xy)^{-\lambda}\mathbb{W}_t(x,y)\Biggl(\sum_{k_j=0}^{n}(-1)^{k_j}[\lambda_j - \tfrac{1}{2}, k_j]\left(\frac{t}{x_j y_j}\right)^{k_j} + f_n^{\lambda_j-1/2}\left(\frac{x_j y_j}{2t}\right)\Biggr), \end{aligned}$$

where, for every $\nu > -1$, f_n^ν is a $C^\infty(0,\infty)$-function and $f_n^\nu(z) = O(z^{-(n+1)})$, as $z \to \infty$. Then, (3.5) holds, where $g_n \in C^\infty((0,\infty)^n \times (0,\infty)^n \times (0,\infty))$ and, for every $a > 0$ and every compact set $K \subset (0,\infty)^n$ there exists $C > 0$ such that

$$|g_n(x,y,t)| \leqslant Ct^{n+1}, \quad t \in (0,a) \text{ and } x,y \in K.$$

We now verify that g_n satisfies (3.6). Let $\nu > -1$. By (3.3) and (3.4) we obtain, for each $z \in (0,\infty)$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^{-z}z^{-\nu}I_\nu(z)) \\ &= \mathrm{e}^{-z}z^{-\nu-1/2}(\sqrt{z}I_{\nu+1} - \sqrt{z}I_\nu(z)) \\ &= \frac{z^{-\nu-1/2}}{\sqrt{2\pi}}\Biggl(\sum_{k=0}^{n+1}(-1)^k([\nu+1,k]-[\nu,k])(2z)^{-k} + f_{n+1}^{\nu+1}(z) - f_{n+1}^{\nu}(z)\Biggr) \\ &= -\frac{z^{-\nu-1/2}}{\sqrt{2\pi}}\Biggl(\sum_{k=0}^{n}(-1)^k([\nu+1,k+1]-[\nu,k+1])(2z)^{-k-1} + f_{n+1}^{\nu+1}(z) - f_{n+1}^{\nu}(z)\Biggr) \end{aligned} \tag{3.7}$$

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and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^{-z}z^{-\nu}I_\nu(z)) \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\biggl(\frac{z^{-\nu-1/2}}{\sqrt{2\pi}}\biggl(\sum_{k=0}^{n}(-1)^k[\nu,k](2z)^{-k} + f_n^\nu(z)\biggr)\biggr) \\
&= -\frac{z^{-\nu-1/2}}{\sqrt{2\pi}}\biggl(\sum_{k=0}^{n}(-1)^k[\nu,k](2\nu+2k+1)(2z)^{-k-1} + \frac{\nu+\frac{1}{2}}{z}f_n^\nu(z) - \frac{\mathrm{d}}{\mathrm{d}z}f_n^\nu(z)\biggr).
\end{aligned} \tag{3.8}$$

Moreover,

$$[\nu+1,k+1] - [\nu,k+1] = [\nu,k](2\nu+2k+1) \quad \text{for } k \in \mathbb{N}.$$

Indeed, let $k \geqslant 1$. We have that

$$\begin{aligned}
&[\nu+1,k+1] - [\nu,k+1] \\
&\qquad = \frac{1}{2^{2k+2}\Gamma(k+2)}\biggl\{\prod_{j=0}^{k}(4(\nu+1)^2 - (2j+1)^2) - \prod_{j=0}^{k}(4\nu^2 - (2j+1)^2)\biggr\} \\
&\qquad = \frac{1}{2^{2k+2}\Gamma(k+2)}\biggl\{\prod_{j=0}^{k}(2\nu+2(j+1)+1)(2\nu-2(j-1)-1) \\
&\qquad\qquad\qquad\qquad - \prod_{j=0}^{k}(2\nu+2j+1)(2\nu-2j-1)\biggr\} \\
&\qquad = \frac{(2\nu+2k+3)-(2\nu-2k-1)}{4(k+1)2^{2k}\Gamma(k+1)} \\
&\qquad\qquad \times\biggl\{\prod_{j=0}^{k-1}(2\nu+2j+1)(2\nu-2j-1)\biggr\}(2\nu+2k+1) \\
&\qquad = [\nu,k](2\nu+2k+1).
\end{aligned}$$

Then, from (3.7) and (3.8) we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}z}f_n^\nu = \frac{\nu+\frac{1}{2}}{z}f_n^\nu - f_{n+1}^{\nu+1} + f_{n+1}^\nu.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}z}f_n^\nu(z) = \mathcal{O}\biggl(\frac{1}{z^{n+2}}\biggr) \quad \text{as } z \to \infty,$$

and we conclude that if $a > 0$ and K is a compact subset of $(0,\infty)^n$, there exists $C > 0$ such that, for every $j = 1,\dots,n$,

$$\biggl|\frac{\partial}{\partial x_j}g_n(x,y,t)\biggr| \leqslant Ct^{n+1}, \quad t \in (0,a) \text{ and } x,y \in K.$$

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□

Let $\alpha > -\frac{1}{2}$. According to (3.2) and (3.3) we obtain

$$0 \leqslant W_t^\alpha(u,v) \leqslant C \begin{cases} \dfrac{e^{-(u^2+v^2)/4t}}{t^{\alpha+1/2}}, & uv \leqslant t, \\ \dfrac{(uv)^{-\alpha}}{\sqrt{t}} e^{-(u-v)^2/4t}, & uv \geqslant t. \end{cases} \tag{3.9}$$

From (3.9) it follows that

$$0 \leqslant W_t^\alpha(u,v) \leqslant C\biggl(\frac{(uv)^{-\alpha}}{\sqrt{t}} e^{-(u-v)^2/4t} + \frac{1}{u^{2\alpha+1}}\biggr), \quad t,u,v \in (0,\infty). \tag{3.10}$$

Also, (3.9) implies that

$$0 \leqslant W_t^\alpha(u,v) \leqslant C\frac{(uv)^{-\alpha-1/2}}{\sqrt{t}} u e^{-u^2/16t} \leqslant C\frac{e^{-u^2/32t}}{t^{\alpha+1/2}},$$

provided that $0 < v < \frac{1}{2}u < \infty$, $0 < t \leqslant uv$ and

$$0 \leqslant W_t^\alpha(u,v) \leqslant C\frac{e^{-u^2/4t}}{t^{\alpha+1/2}},$$

when $u, v \in (0,\infty)$, $t \geqslant uv$. Then, we conclude that

$$0 \leqslant W_t^\alpha(u,v) \leqslant C\frac{e^{-u^2/32t}}{t^{\alpha+1/2}}, \quad 0 < v < \tfrac{1}{2}u \text{ and } t > 0. \tag{3.11}$$

On the other hand, according to (3.2), (3.3) again, and (3.9), we obtain

$$|W_t^\alpha(u,v) - (uv)^{-\alpha}\mathbb{W}_t(u,v)| \leqslant C \begin{cases} \biggl(\dfrac{1}{t^{\alpha+1/2}} + \dfrac{(uv)^{-\alpha}}{\sqrt{t}}\biggr) e^{-(u-v)^2/4t}, & uv \leqslant t, \\ (uv)^{-\alpha-1}\sqrt{t} e^{-(u-v)^2/4t}, & uv \geqslant t. \end{cases} \tag{3.12}$$

By using (3.2)–(3.4) we can prove that

$$\biggl|W_t^\alpha(u,v) - \frac{t^{-\alpha-1/2}}{2^{2\alpha}\Gamma(\alpha+\frac{1}{2})}\biggr| \leqslant \frac{u^2+v^2}{t^{\alpha+3/2}}\biggl(\frac{uv}{t} + 1\biggr), \quad t,u,v \in (0,\infty). \tag{3.13}$$

Indeed, we have that

$$\begin{aligned} &\biggl|W_t^\alpha(u,v) - \frac{t^{-\alpha-1/2}}{2^{2\alpha}\Gamma(\alpha+\frac{1}{2})}\biggr| \\ &\quad = \biggl|\frac{1}{(2t)^{\alpha+1/2}}\biggl(\frac{uv}{2t}\biggr)^{-\alpha+1/2} I_{\alpha-1/2}\biggl(\frac{uv}{2t}\biggr) e^{-(u^2+v^2)/4t} - \frac{t^{-\alpha-1/2}}{2^{2\alpha}\Gamma(\alpha+\frac{1}{2})}\biggr| \\ &\quad \leqslant \frac{1}{(2t)^{\alpha+1/2}} e^{-(u^2+v^2)/4t}\biggl|\biggl(\frac{uv}{2t}\biggr)^{-\alpha+1/2} I_{\alpha-1/2}\biggl(\frac{uv}{2t}\biggr) - \frac{1}{2^{\alpha-1/2}\Gamma(\alpha+\frac{1}{2})}\biggr| \\ &\qquad + \frac{t^{-\alpha-1/2}}{2^{2\alpha}\Gamma(\alpha+\frac{1}{2})}|e^{-(u^2+v^2)/4t} - 1| \end{aligned}$$

$$\leqslant \frac{uv}{(2t)^{\alpha+3/2}} \mathrm{e}^{-(u^2+v^2)/4t} \sup_{z\in(0,uv/2t)} \left| \frac{\mathrm{d}}{\mathrm{d}z}(z^{-\alpha+1/2} I_{\alpha-1/2}(z)) \right| + C \frac{u^2+v^2}{t^{\alpha+3/2}}$$

$$\leqslant C\left(\frac{(uv)^2}{t^{\alpha+5/2}} + \frac{u^2+v^2}{t^{\alpha+3/2}} \right) \leqslant C \frac{u^2+v^2}{t^{\alpha+3/2}} \left(\frac{uv}{t} + 1 \right), \quad t,u,v \in (0,\infty).$$

By (3.9) and (3.13) it follows that, for every compact set $K \subset (0,\infty)^n$, $b > 0$ and $\lambda \in (-\frac{1}{2},\infty)^n$,

$$\begin{aligned} &\frac{1}{\sqrt{t}} \left| W_t^\lambda(x,y) - \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})} - (xy)^{-\lambda}\mathbb{W}_t(x,y) \right| \\ &\qquad \leqslant \sum_{i=1}^n \frac{1}{\sqrt{t}} \prod_{j=1}^{i-1} W_t^{\lambda_j}(x_j,y_j) \left| W_t^{\lambda_i}(x_i,y_i) - \frac{t^{-\lambda_i-1/2}}{2^{2\lambda_i}\Gamma(\lambda_i+\frac{1}{2})} \right| \\ &\qquad\qquad \times \prod_{j=i+1}^n \frac{t^{-\lambda_j-1/2}}{2^{2\lambda_j}\Gamma(\lambda_j+\frac{1}{2})} + \frac{C}{t^{(n+1)/2}} \\ &\qquad \leqslant C(t^{-|\lambda|-n/2-3/2} + t^{-(n+1)/2}), \quad b < t < \infty,\ x,y \in K. \end{aligned} \tag{3.14}$$

Now we estimate some derivatives of the heat Bessel kernel. By combining (3.1)–(3.3), we obtain

$$\left| \frac{\partial}{\partial u} W_t^\alpha(u,v) \right| \leqslant C \begin{cases} \dfrac{u+v}{t^{\alpha+3/2}} \mathrm{e}^{-(u^2+v^2)/4t}, & uv \leqslant t, \\ \dfrac{(uv)^{-\alpha}}{t} \mathrm{e}^{-(u-v)^2/8t}, & uv \geqslant t. \end{cases} \tag{3.15}$$

Inequality (3.15) leads to

$$\left| \frac{\partial}{\partial u} W_t^\alpha(u,v) \right| \leqslant C \begin{cases} \dfrac{\mathrm{e}^{-u^2/32t}}{t^{\alpha+1}}, & 0 < v < \frac{1}{2}u, \\ \dfrac{\mathrm{e}^{-v^2/32t}}{t^{\alpha+1}}, & 2u < v < \infty. \end{cases} \tag{3.16}$$

According to (3.4) we have that

$$\begin{aligned} &\frac{\partial}{\partial x_1} \left(W_t^\lambda(x,y) - \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})} - (xy)^{-\lambda}\mathbb{W}_t(x,y) \right) \\ &= \prod_{j=2}^n W_t^{\lambda_j}(x_j,y_j) \frac{\mathrm{e}^{-(x_1^2+y_1^2)/4t}}{(2t)^{\lambda_1+1/2}} \\ &\quad \times \left[\left(\frac{x_1y_1}{2t} \right)^{-\lambda_1+1/2} I_{\lambda_1+1/2}\left(\frac{x_1y_1}{2t} \right) \frac{y_1}{2t} - \frac{x_1}{2t} \left(\frac{x_1y_1}{2t} \right)^{-\lambda_1+1/2} I_{\lambda_1-1/2}\left(\frac{x_1y_1}{2t} \right) \right] \\ &\quad + \frac{\lambda_1}{x_1}(xy)^{-\lambda}\mathbb{W}_t(x,y) + \frac{x_1-y_1}{2t}(xy)^{-\lambda}\mathbb{W}_t(x,y), \quad t > 0,\ x,y \in (0,\infty)^n. \end{aligned}$$

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Then, (3.2) and (3.9) lead to

$$\left|\frac{\partial}{\partial x_1}\left(W_t^\lambda(x,y)-\frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})}-(xy)^{-\lambda}\mathbb{W}_t(x,y)\right)\right|\leqslant C(t^{-|\lambda|-n/2-1}+t^{-n/2}) \tag{3.17}$$

for every $t>b$ and $x,y\in K$; where $b>0$ and K is a compact subset of $(0,\infty)^n$. Here

$$2^{2\lambda}\Gamma(\lambda+\tfrac{1}{2})=\prod_{j=1}^{n}2^{2\lambda_j}\Gamma(\lambda_j+\tfrac{1}{2}).$$

Finally, we recall the following properties established in [8, lemma 4.3]:

$$\int_0^\infty\left|\frac{\partial}{\partial u}W_t^\alpha(u,v)\right|\frac{\mathrm{d}t}{\sqrt{t}}\leqslant C\begin{cases}\dfrac{1}{u^{2\alpha+1}}, & 0<v<\frac{1}{2}u,\\[2mm] \dfrac{u}{v^{2\alpha+2}}, & 2u<v<\infty,\end{cases} \tag{3.18}$$

and also, from [8, proposition 4.2], we have that

$$\left|\frac{\partial}{\partial u}W_t^\alpha(u,v)-(uv)^{-\alpha}\frac{\partial}{\partial u}\mathbb{W}_t(u,v)\right|\leqslant C\frac{(uv)^{-\alpha-1/2}}{\sqrt{t}}\mathrm{e}^{-(u-v)^2/4t},\quad uv\geqslant t>0. \tag{3.19}$$

4. Proof of theorem 1.4

In this section we prove that the Riesz transforms R_i^λ, $i=1,\dots,n$, associated with the Bessel operator Δ_λ are bounded from $L^p((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into itself for every $1<p<\infty$, and from $L^1((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n,\mathrm{d}\mu_\lambda)$.

For every $\lambda\in(-\frac{1}{2},\infty)^n$ and $i=1,\dots,n$, the Riesz transform R_i^λ is formally defined by

$$R_i^\lambda=\frac{\partial}{\partial x_i}\Delta_\lambda^{-1/2}, \tag{4.1}$$

where $\Delta_\lambda^{-1/2}$ denotes the negative square root of the operator Δ_λ. We shall now make precise the definition (4.1).

Assume that $0<\beta<|\lambda|+\frac{1}{2}n+1$. We define the negative power $\Delta_\lambda^{-\beta}$ on $C_\mathrm{c}^\infty((0,\infty)^n)$ as follows:

$$\Delta_\lambda^{-\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_0^\infty\left(W_t^\lambda(f)(x)-\chi_{(1,\infty)}(t)\frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})}\int_{(0,\infty)^n}f(y)\,\mathrm{d}\mu_\lambda(y)\right)t^{\beta-1}\,\mathrm{d}t, \tag{4.2}$$

where $f\in C_\mathrm{c}^\infty((0,\infty)^n)$.

Let $f\in C_\mathrm{c}^\infty((0,\infty)^n)$. We can write

$$\Delta_\lambda^{-\beta}f(x)=\int_{(0,\infty)^n}f(y)K_\beta^\lambda(x,y)\,\mathrm{d}\mu_\lambda(y),\quad x\in(0,\infty)^n, \tag{4.3}$$

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where, for each $x, y \in (0,\infty)^n$, $x \neq y$,

$$K_\beta^\lambda(x,y) = \frac{1}{\Gamma(\beta)} \int_0^\infty \left(W_t^\lambda(x,y) - \chi_{(1,\infty)}(t) \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})} \right) t^{\beta-1}\,\mathrm{d}t.$$

In order to prove the integral representation (4.3) for $\Delta_\lambda^{-\beta}$, it is sufficient to show that, for every $x \in (0,\infty)^n$,

$$\int_{(0,\infty)^n} |f(y)| \int_0^\infty \left| W_t^\lambda(x,y) - \chi_{(1,\infty)}(t) \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})} \right| t^{\beta-1}\,\mathrm{d}t\,\mathrm{d}\mu_\lambda(y) < \infty.$$

Indeed, let $x \in (0,\infty)^n$. According to (3.9) and defining $K = \operatorname{supp}(f)$, we have that

$$\begin{aligned} \int_{(0,\infty)^n} |f(y)| \int_0^1 W_t^\lambda(x,y) t^{\beta-1}\,\mathrm{d}t\,\mathrm{d}\mu_\lambda(y) &\leqslant C \int_K \int_0^1 t^{\beta-1-n/2} \mathrm{e}^{-|x-y|^2/4t}\,\mathrm{d}t\,\mathrm{d}y \\ &\leqslant C \int_K \frac{1}{|x-y|^{n-2\beta}}\,\mathrm{d}y \\ &< \infty, \end{aligned}$$

because $\beta > 0$. In the last inequality we have used [19, lemma 1.1].

On the other hand, by (3.9) and (3.13) we get

$$\begin{aligned} &\int_{(0,\infty)^n} |f(y)| \int_1^\infty \left| W_t^\lambda(x,y) - \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})} \right| t^{\beta-1}\,\mathrm{d}t\,\mathrm{d}\mu_\lambda(y) \\ &\quad\leqslant C \sum_{i=1}^n \int_K \int_1^\infty \prod_{j=1}^{i-1} \frac{t^{-(\lambda_j+1/2)}}{2^{2\lambda_j}\Gamma(\lambda_j+\frac{1}{2})} \left| W_t^{\lambda_i}(x_i,y_i) - \frac{t^{-(\lambda_i+1/2)}}{2^{2\lambda_i}\Gamma(\lambda_i+\frac{1}{2})} \right| \\ &\qquad\times \prod_{j=i+1}^n W_t^{\lambda_j}(x_j,y_j) t^{\beta-1}\,\mathrm{d}t\,\mathrm{d}y \\ &\quad\leqslant C \int_K \int_1^\infty t^{-|\lambda|-n/2-2+\beta}\,\mathrm{d}t\,\mathrm{d}y \\ &\quad< \infty, \end{aligned}$$

because $\beta < |\lambda| + \frac{1}{2}n + 1$.

Remark 4.1. Note that if $0 < \beta < |\lambda| + \frac{1}{2}n$ for every $f \in C_c^\infty((0,\infty)^n)$,

$$\int_{(0,\infty)^n} |f(y)| \int_0^\infty W_t^\lambda(x,y) t^{\beta-1}\,\mathrm{d}t\,\mathrm{d}\mu_\lambda(y) < \infty, \quad x \in (0,\infty)^n,$$

and we can define the function

$$(\Delta_\lambda)^{-\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_{(0,\infty)^n} f(y) \int_0^\infty W_t^\lambda(x,y) t^{\beta-1}\,\mathrm{d}t\,\mathrm{d}\mu_\lambda(y), \quad x \in (0,\infty)^n, \tag{4.4}$$

when $f \in C_c^\infty((0,\infty)^n)$. Then, if $0 < \beta < |\lambda| + \frac{1}{2}n$ and $f \in C_c^\infty((0,\infty)^n)$, for each $x \in (0,\infty)^n$,

$$\Delta_\lambda^{-\beta} f(x) - (\Delta_\lambda)^{-\beta} f(x) = \frac{-1}{\Gamma(\beta)} \int_{(0,\infty)^n} f(y) \int_1^\infty \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})} t^{\beta-1}\,\mathrm{d}t\,\mathrm{d}\mu_\lambda(y)$$

and, for every $i = 1, \ldots, n$,

$$\frac{\partial}{\partial x_i}\Delta_\lambda^{-\beta} f(x) = \frac{\partial}{\partial x_i}(\Delta_\lambda)^{-\beta} f(x), \tag{4.5}$$

provided that these derivatives exist. Equation (4.5) tells us that once we have taken derivatives, it is irrelevant to consider definition (4.2) or (4.4). However, we are interested in the particular case $\beta = \frac{1}{2}$, and when λ is very close to $(-\frac{1}{2}, \ldots, -\frac{1}{2})$ the integrals in (4.4) are not convergent.

For every $0 < \beta < \frac{1}{2}n$, the fractional power $\Delta^{-\beta}$ is defined on $C_c^\infty(\mathbb{R}^n)$ by

$$\Delta^{-\beta} f(x) = \int_{\mathbb{R}^n} f(y)\mathbb{K}_\beta(x,y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

where, for every $x, y \in \mathbb{R}^n$, $x \neq y$,

$$\mathbb{K}_\beta(x,y) = \frac{1}{\Gamma(\beta)}\int_0^\infty \mathbb{W}_t(x,y)t^{\beta-1}\,\mathrm{d}t = \frac{\Gamma(\frac{1}{2}n-\beta)}{\pi^{n/2}4^\beta\Gamma(\beta)}\frac{1}{|x-y|^{n-2\beta}}. \tag{4.6}$$

In particular, since $n > 1$, we have that, for every $f \in C_c^\infty(\mathbb{R}^n)$,

$$\Delta^{-1/2} f(x) = \frac{1}{2^n\pi^{(n+1)/2}}\int_{\mathbb{R}^n} f(y)\int_0^\infty \frac{\mathrm{e}^{-|x-y|^2/4t}}{t^{(n+1)/2}}\,\mathrm{d}t\,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

A crucial result to prove theorem 1.4 is the following.

Proposition 4.2. *Let $f \in C_c^\infty((0,\infty)^n)$. Assume that $\lambda \in (-\frac{1}{2},\infty)^n$. Then, for every $i = 1, \ldots, n$,*

$$\begin{aligned}\frac{\partial}{\partial x_i}&(\Delta_\lambda^{-1/2} f(x) - x^{-\lambda}\Delta^{-1/2}(y^\lambda f)(x))\\ &= \int_{(0,\infty)^n} f(y)\Big(R_i^\lambda(x,y) - \frac{\partial}{\partial x_i}((xy)^{-\lambda}\mathbb{K}_{1/2}(x,y))\Big)\,\mathrm{d}\mu_\lambda(y) \quad \text{a.e. } x \in (0,\infty)^n,\end{aligned} \tag{4.7}$$

where R_i^λ is defined in (1.1). Moreover, the integral in (4.7) is absolutely convergent.

Proof. We shall prove (4.7) for $i = 1$. Suppose that Ω is a compact subset of $(0,\infty)^n$. There exist $0 < a < 1$ and $b > 1$ such that, for every $y \in \operatorname{supp}(f)$ and $x \in \Omega$, $x_jy_j/t \geqslant 1$ when $0 < t < a$, and $x_jy_j/t \leqslant 1$ when $b < t < \infty$ for every $j = 1, \ldots, n$. We can write, for each $x, y \in (0,\infty)^n$,

$$\begin{aligned}K_{1/2}^\lambda(x,y) &- (xy)^{-\lambda}\mathbb{K}_{1/2}(x,y)\\ &= \frac{1}{\sqrt{\pi}}\bigg\{\int_0^a (W_t^\lambda(x,y) - (xy)^{-\lambda}\mathbb{W}_t(x,y))\frac{\mathrm{d}t}{\sqrt{t}}\\ &\qquad + \int_a^1 (W_t^\lambda(x,y) - (xy)^{-\lambda}\mathbb{W}_t(x,y))\frac{\mathrm{d}t}{\sqrt{t}}\\ &\qquad + \int_1^b \bigg(W_t^\lambda(x,y) - \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})} - (xy)^{-\lambda}\mathbb{W}_t(x,y)\bigg)\frac{\mathrm{d}t}{\sqrt{t}}\\ &\qquad + \int_b^\infty \bigg(W_t^\lambda(x,y) - \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})} - (xy)^{-\lambda}\mathbb{W}_t(x,y)\bigg)\frac{\mathrm{d}t}{\sqrt{t}}\bigg\}.\end{aligned} \tag{4.8}$$

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If $r \in \mathbb{N}$, according to [19, lemma 1.1] we get, for every $x, y \in (0,\infty)^n$, $x \neq y$,

$$\int_0^a \mathrm{e}^{-|x-y|^2/4t} t^{r-(n+1)/2} \, \mathrm{d}t \leqslant C \int_0^a \mathrm{e}^{-|x-y|^2/4t} t^{-(n+1)/2} \, \mathrm{d}t \leqslant \frac{C}{|x-y|^{n-1}}. \tag{4.9}$$

Suppose that $k \in \{0, 1, \dots, n\}^n$ and $k \neq (0, \dots, 0)$. From (4.9) we deduce that the functions

$$h_k(z) = \int_0^a \frac{\mathrm{e}^{-|z|^2/4t}}{t^{(n+1)/2-|k|}} \, \mathrm{d}t, \quad H_k(z) = \int_0^a \frac{\mathrm{e}^{-|z|^2/4t}}{t^{n/2+1-|k|}} \, \mathrm{d}t, \quad z \in \mathbb{R}^n \setminus \{0\},$$

are in $L^1(\Lambda)$ for every compact subset $\Lambda \subset \mathbb{R}^n$. Moreover,

$$\frac{\partial}{\partial x_1} h_k(x) = \int_0^a \frac{\partial}{\partial x_1} \mathrm{e}^{-|x|^2/4t} t^{-(n+1)/2+|k|} \, \mathrm{d}t, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Since $f \in C_{\mathrm{c}}^\infty((0,\infty)^n)$, by defining $f(y) = 0$, $y \in \mathbb{R}^n \setminus (0,\infty)^n$, the function

$$G_k(x) = \int_{\mathbb{R}^n} (xy)^{-\lambda-k} f(y) h_k(x-y) \, \mathrm{d}\mu_\lambda(y), \quad x \in (0,\infty)^n,$$

is derivable with respect to x_1 on $(0,\infty)^n$, and

$$\begin{aligned}
&\frac{\partial}{\partial x_1} G_k(x) \\
&= -\frac{\lambda_1 + k_1}{x_1} \int_{\mathbb{R}^n} (xy)^{-\lambda-k} f(y) h_k(x-y) \, \mathrm{d}\mu_\lambda(y) \\
&\quad - \int_{\mathbb{R}^{n-1}} x^{-\lambda-k} (\bar{x} - \bar{y})^{\bar{\lambda}-\bar{k}} \int_{-\infty}^{\infty} \frac{\partial}{\partial y_1} [(x_1 - y_1)^{\lambda_1-k_1} f(x-y)] h_k(y) \, \mathrm{d}y \\
&= -\frac{\lambda_1 + k_1}{x_1} \int_{\mathbb{R}^n} (xy)^{-\lambda-k} f(y) h_k(x-y) \, \mathrm{d}\mu_\lambda(y) \\
&\quad + \int_{\mathbb{R}^{n-1}} \lim_{\varepsilon \to 0} \bigg\{ -[(x_1 - y_1)^{\lambda_1-k_1} f(x-y) h_k(y)]_{x_1+\varepsilon}^{\infty} \\
&\qquad\qquad - [(x_1 - y_1)^{\lambda_1-k_1} f(x-y) h_k(y)]_{-\infty}^{x_1-\varepsilon} \\
&\qquad\qquad + \int_{\mathbb{R} \setminus (x_1-\varepsilon, x_1+\varepsilon)} (x_1 - y_1)^{\lambda_1-k_1} f(x-y) \frac{\partial}{\partial y_1} h_k(y) \, \mathrm{d}y_1 \bigg\} \\
&\qquad\qquad \times x^{-\lambda-k} (\bar{x} - \bar{y})^{\bar{\lambda}-\bar{k}} \, \mathrm{d}\bar{y} \\
&= -\frac{\lambda_1 + k_1}{x_1} \int_{\mathbb{R}^n} (xy)^{-\lambda-k} f(y) h_k(x-y) \, \mathrm{d}\mu_\lambda(y) \\
&\quad + \int_{\mathbb{R}^n} x^{-\lambda-k} (x-y)^{\lambda-k} f(x-y) \frac{\partial}{\partial y_1} h_k(y) \, \mathrm{d}y
\end{aligned}$$

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$$= -\frac{\lambda_1 + k_1}{x_1} \int_{\mathbb{R}^n} (xy)^{-\lambda-k} f(y) h_k(x-y)\,\mathrm{d}\mu_\lambda(y)$$
$$+ \int_{\mathbb{R}^n} (xy)^{-\lambda-k} f(y) \frac{\partial}{\partial x_1} h_k(x-y)\,\mathrm{d}\mu_\lambda(y)$$
$$= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_1}[(xy)^{-\lambda-k} h_k(x-y)] f(y)\,\mathrm{d}\mu_\lambda(y), \quad x \in (0,\infty)^n.$$

Hence, we obtain, for each $x \in (0,\infty)^n$,

$$\frac{\partial}{\partial x_1}\biggl(\int_{(0,\infty)^n} f(y) \int_0^a (xy)^{-\lambda} \mathbb{W}_t(x,y) \sum_{\substack{k\in\{0,1,\dots,n\}^n \\ k\neq(0,\dots,0)}} c_k (xy)^{-k} t^{|k|} \frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}\mu_\lambda(y)\biggr)$$
$$= \int_{(0,\infty)^n} f(y) \int_0^a \frac{\partial}{\partial x_1}\biggl((xy)^{-\lambda} \mathbb{W}_t(x,y) \sum_{\substack{k\in\{0,1,\dots,n\}^n \\ k\neq(0,\dots,0)}} c_k (xy)^{-k} t^{|k|}\biggr) \frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}\mu_\lambda(y). \tag{4.10}$$

Also, by using (3.6) we can see that

$$\frac{\partial}{\partial x_1} \int_{(0,\infty)^n} f(y) \int_0^a (xy)^{-\lambda} \mathbb{W}_t(x,y) g_n(x,y,t) \frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}\mu_\lambda(y)$$
$$= \int_{(0,\infty)^n} f(y) \int_0^a \frac{\partial}{\partial x_1}((xy)^{-\lambda} \mathbb{W}_t(x,y) g_n(x,y,t)) \frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}\mu_\lambda(y), \quad x \in (0,\infty)^n. \tag{4.11}$$

From (3.5), (4.10) and (4.11) we deduce that

$$\frac{\partial}{\partial x_1} \int_{(0,\infty)} f(y) \int_0^a (W_t^\lambda(x,y) - (xy)^{-\lambda} \mathbb{W}_t(x,y)) \frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}\mu_\lambda(y)$$
$$= \int_{(0,\infty)^n} f(y) \int_0^a \frac{\partial}{\partial x_1}(W_t^\lambda(x,y) - (xy)^{-\lambda} \mathbb{W}_t(x,y)) \frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}\mu_\lambda(y), \quad x \in (0,\infty)^n. \tag{4.12}$$

By taking into account (3.14) and (3.17), for each $x \in (0,\infty)^n$, we can differentiate under the integral sign, obtaining

$$\frac{\partial}{\partial x_1} \int_{(0,\infty)^n} f(y) \int_b^\infty \biggl(W_t^\lambda(x,y) - \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})} - (xy)^{-\lambda} \mathbb{W}_t(x,y)\biggr) \frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}y$$
$$= \int_{(0,\infty)^n} f(y) \int_b^\infty \frac{\partial}{\partial x_1}\biggl(W_t^\lambda(x,y) - \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})}$$
$$- (xy)^{-\lambda} \mathbb{W}_t(x,y)\biggr) \frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}\mu_\lambda(y). \tag{4.13}$$

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Finally, it is not hard to see that

$$\frac{\partial}{\partial x_1}\int_{(0,\infty)^n} f(y)\biggl(\int_a^b (W_t^\lambda(x,y)-(xy)^{-\lambda}\mathbb{W}_t(x,y))\frac{\mathrm{d}t}{\sqrt{t}} - \int_1^b \frac{t^{-|\lambda|-n/2}}{2^{2\lambda}\Gamma(\lambda+\frac{1}{2})}\frac{\mathrm{d}t}{\sqrt{t}}\biggr)\,\mathrm{d}\mu_\lambda(y)$$
$$= \int_{(0,\infty)^n} f(y)\int_a^b \frac{\partial}{\partial x_1}(W_t^\lambda(x,y)-(xy)^{-\lambda}\mathbb{W}_t(x,y))\frac{\mathrm{d}t}{\sqrt{t}}\,\mathrm{d}\mu_\lambda(y),\quad x\in(0,\infty)^n. \tag{4.14}$$

By combining (4.8) and (4.12)–(4.14) we prove (4.7). Moreover, the estimations that we have established show the absolute convergence of the integral in (4.7).

Thus, the proof of proposition 4.2 is finished. □

Proposition 4.2 allows us to define the Riesz transforms R_i^λ, $i=1,\dots,n$, on $C_\mathrm{c}^\infty((0,\infty)^n)$.

Proposition 4.3. *Let*

$$f\in C_\mathrm{c}^\infty((0,\infty)^n)\quad \textit{and}\quad \lambda\in(-\tfrac{1}{2},\infty)^n.$$

Then, for every $i=1,\dots,n$, *the function* $\Delta_\lambda^{-1/2}f$ *admits derivative* $\partial\Delta_\lambda^{-1/2}f/\partial x_i$ *with respect to* x_i *on almost all* $(0,\infty)^n$, *and*

$$\frac{\partial}{\partial x_i}\Delta_\lambda^{-1/2}f(x)=\lim_{\varepsilon\to0^+}\int_{|x-y|>\varepsilon} R_i^\lambda(x,y)f(y)\,\mathrm{d}\mu_\lambda(y)\quad \textit{a.e. } x\in(0,\infty)^n.$$

Proof. Let $i=1,\dots,n$. As is well known, for every $g\in C_\mathrm{c}^\infty(\mathbb{R}^n)$, $\Delta^{-1/2}g$ admits derivative $\partial\Delta^{-1/2}g/\partial x_i$ with respect to x_i on almost all $\mathbb{R}^n$ and

$$\frac{\partial}{\partial x_i}\Delta^{-1/2}g(x)=\lim_{\varepsilon\to0^+}\int_{|x-y|>\varepsilon} g(y)\mathbb{R}_i(x,y)\,\mathrm{d}y\quad \text{a.e. } x\in(0,\infty)^n,$$

where $\mathbb{R}_i$ represents the kernel of the classical Riesz transform

$$\mathbb{R}_i(x,y)=\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\partial}{\partial x_i}\mathbb{W}_t(x,y)\frac{\mathrm{d}t}{\sqrt{t}},\quad x,y\in\mathbb{R}^n,\ x\neq y.$$

Moreover, for every $x\in(0,\infty)^n$, it follows that, by (4.6),

$$\int_{(0,\infty)^n}|f(y)|\mathbb{K}_{1/2}(x,y)y^\lambda\,\mathrm{d}y\leqslant C\int_{\mathrm{supp}(f)}\frac{|f(y)|}{|x-y|^{n-1}}y^\lambda\,\mathrm{d}y<\infty.$$

Then,

$$\frac{\partial}{\partial x_i}(x^{-\lambda}\Delta^{-1/2}(y^\lambda f)(x)) = \lim_{\varepsilon\to0^+}\int_{|x-y|>\varepsilon} f(y)\frac{\partial}{\partial x_i}(x^{-\lambda}\mathbb{K}_{1/2}(x,y))y^\lambda\,\mathrm{d}y\quad \text{a.e. } x\in(0,\infty)^n.$$

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Hence, from proposition 4.2 we conclude that $\Delta_\lambda^{-1/2} f$ admits derivative $\partial \Delta_\lambda^{-1/2}/\partial x_i$ with respect to x_i on almost all $(0,\infty)^n$ and

$$\begin{aligned}\frac{\partial}{\partial x_i}\Delta_\lambda^{-1/2} f(x) &= \frac{\partial}{\partial x_i}(\Delta_\lambda^{-1/2} f(x) - x^{-\lambda}\Delta^{-1/2}(y^\lambda f)(x)) + \frac{\partial}{\partial x_i}(x^{-\lambda}\Delta^{-1/2}(y^\lambda f)(x)) \\ &= \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} f(y)\biggl(R_i^\lambda(x,y) - \frac{\partial}{\partial x_i}((xy)^{-\lambda}\mathbb{K}_{1/2}(x,y))\biggr)\,\mathrm{d}\mu_\lambda(y) \\ &\quad + \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} f(y)\frac{\partial}{\partial x_i}((xy)^{-\lambda}\mathbb{K}_{1/2}(x,y))\,\mathrm{d}\mu_\lambda(y) \\ &= \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} f(y) R_i^\lambda(x,y)\,\mathrm{d}\mu_\lambda(y) \quad \text{a.e. } x\in(0,\infty)^n.\end{aligned}$$

□

We now prove that, for every $f \in L^p((0,\infty)^n, \mathrm{d}\mu_\lambda(y))$, $1 \leqslant p < \infty$, there exists the limit

$$\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} R_i^\lambda(x,y) f(y)\,\mathrm{d}\mu_\lambda(y) \quad \text{a.e } x\in(0,\infty)^n,$$

with $i = 1,\dots,n$. In order to show this we consider, for every $i = 1,\dots,n$, the maximal operator $R_{i,*}^\lambda$ defined by

$$R_{i,*}^\lambda(f)(x) = \sup_{\varepsilon>0}\biggl|\int_{|x-y|>\varepsilon} R_i^\lambda(x,y) f(y)\,\mathrm{d}\mu_\lambda(y)\biggr|.$$

Proposition 4.4. *Let $\lambda \in (-\frac{1}{2},\infty)^n$ and $i = 1,\dots,n$. The maximal operator $R_{i,*}^\lambda$ is bounded from $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ into itself, $1 < p < \infty$, and from $L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n, \mathrm{d}\mu_\lambda)$.*

Proof. We consider $i = 1$. For other values of i we can proceed analogously. We can write

$$\begin{aligned}R_1^\lambda(x,y) &= (R_1^\lambda(x,y) - \chi_{L(x)}(y)(xy)^{-\lambda}\mathbb{R}_1(x,y)) + \chi_{L(x)}(y)(xy)^{-\lambda}\mathbb{R}_1(x,y) \\ &= \frac{1}{\sqrt{\pi}}\int_0^\infty \biggl(\frac{\partial}{\partial x_1}W_t^{\lambda_1}(x_1,y_1) - \chi_{L_1(x)}(y_1)(x_1y_1)^{-\lambda_1}\frac{\partial}{\partial x_1}\mathbb{W}_t(x_1,y_1)\biggr) \\ &\qquad \times W_t^{\bar\lambda}(\bar x,\bar y)\frac{\mathrm{d}t}{\sqrt{t}} + \sum_{\ell=2}^n \frac{1}{\sqrt{\pi}}\int_0^\infty \chi_{L_1(x)}(y_1)(x_1y_1)^{-\lambda_1}\frac{\partial}{\partial x_1}\mathbb{W}_t(x_1,y_1) \\ &\qquad \times \prod_{j=2}^{\ell-1}\chi_{L_j(x)}(y_j)(x_jy_j)^{-\lambda_j}\mathbb{W}_t(x_j,y_j) \\ &\qquad \times (W_t^{\lambda_\ell}(x_\ell,y_\ell) - \chi_{L_\ell(x)}(y_\ell)(x_\ell y_\ell)^{-\lambda_\ell}\mathbb{W}_t(x_\ell,y_\ell)) \\ &\qquad \times \prod_{j=\ell+1}^n W_t^{\lambda_j}(x_j,y_j)\frac{\mathrm{d}t}{\sqrt{t}} + \chi_{L(x)}(y)(xy)^{-\lambda}\mathbb{R}_1(x,y) \\ &= \sum_{\ell=1}^{n+1} R_{1,\ell}^\lambda(x,y), \quad x,y\in(0,\infty)^n.\end{aligned}$$

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We now study the maximal operator $R^{\lambda}_{1,\ell,*}$ associated with $R^{\lambda}_{1,\ell}$ in the usual way. First, we consider $R^{\lambda}_{1,1,*}$. In the forthcoming analysis we follow the general procedure described in § 2. We distinguish three situations: $0 < y_1 < \frac{1}{2}x_1$, $\frac{1}{2}x_1 < y_1 < 2x_1$ and $2x_1 < y_1 < \infty$.

Assume first that $0 < y_1 < \frac{1}{2}x_1$ and $1 \leqslant \ell \leqslant m \leqslant n$. According to (3.10), (3.11) and (3.16), we get

$$\begin{aligned} &S^{\lambda,1}_{\ell,m}(f)(x)\\ &\quad= \int_0^{x_1/2}\cdots\int_0^{x_\ell/2}\int_{x_{\ell+1}/2}^{2x_{\ell+1}}\cdots\int_{x_m/2}^{2x_m}\int_{2x_{m+1}}^{\infty}\cdots\int_{2x_n}^{\infty}|R^{\lambda}_{1,1}(x,y)||f(y)|\,\mathrm{d}\mu_\lambda(y)\\ &\quad\leqslant C\int_0^{x_1/2}\cdots\int_0^{x_\ell/2}\int_{x_{\ell+1}/2}^{2x_{\ell+1}}\cdots\int_{x_m/2}^{2x_m}\int_{2x_{m+1}}^{\infty}\cdots\int_{2x_n}^{\infty}\prod_{j=m+1}^{n}\frac{1}{y_j^{2\lambda_j+1}}|f(y)|\\ &\qquad\times\int_0^{\infty}\frac{\exp(-(\sum_{j=1}^{\ell}x_j^2)/32t)}{t^{\sum_{j=1}^{\ell}(\lambda_j+1/2)+1}}\\ &\qquad\times\prod_{j=\ell+1}^{m}\biggl(\frac{(x_jy_j)^{-\lambda_j}}{\sqrt{t}}\mathrm{e}^{-(x_j-y_j)^2/4t}+\frac{1}{x_j^{2\lambda_j+1}}\biggr)\,\mathrm{d}t\,\mathrm{d}\mu_\lambda(y). \end{aligned}$$

Since H^k_∞ and H^β_{loc} are bounded from $L^1((0,\infty)^k,\mathrm{d}\mu_\beta)$ into itself for every $k\in\mathbb{N}$ and $\beta\in(-\frac{1}{2},\infty)^k$ (see § 3.1), the operator $S^{\lambda,1}_{\ell,m}$ is bounded from $L^1((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n,\mathrm{d}\mu_\lambda)$, provided that, for every $r,k\in\mathbb{N}$, $1\leqslant r\leqslant k$ and $\beta\in(-\frac{1}{2},\infty)^k$, the operator

$$\begin{aligned} &T^{\beta,1}_r(g)(x)\\ &\quad=\int_0^{x_1/2}\cdots\int_0^{x_r/2}\int_{x_{r+1}/2}^{2x_{r+1}}\cdots\\ &\qquad\int_{x_k/2}^{2x_k}|g(y)|\int_0^{\infty}\frac{1}{t^{\sum_{j=1}^{r}(\beta_j+1/2)+1+(k-r)/2}}\\ &\qquad\times\exp\biggl(-\frac{1}{32t}\biggl(\sum_{j=1}^{r}x_j^2+\sum_{j=r+1}^{k}(x_j-y_j)^2\biggr)\biggr)\\ &\qquad\times\prod_{j=r+1}^{k}(x_jy_j)^{-\beta_j}\,\mathrm{d}t\,\mathrm{d}\mu_\beta(y)\\ &\quad\leqslant C\int_0^{x_1/2}\cdots\int_0^{x_r/2}\int_{x_{r+1}/2}^{2x_{r+1}}\cdots\\ &\qquad\int_{x_k/2}^{2x_k}|g(y)|\frac{\prod_{j=r+1}^{k}(x_jy_j)^{-\beta_j}}{(\sum_{j=1}^{r}x_j^2+\sum_{j=r+1}^{k}(x_j-y_j)^2)^{\sum_{j=1}^{r}(\beta_j+1/2)+(k-r)/2}}\,\mathrm{d}\mu_\beta(y) \end{aligned}$$

is bounded from $L^1((0,\infty)^k,\mathrm{d}\mu_\beta)$ into $L^{1,\infty}((0,\infty)^k,\mathrm{d}\mu_\beta)$. This last property is true because the operators $\mathcal{H}^\beta_{r,k}$ (see § 3.1) and L_β (see lemma 3.1) are bounded from $L^1((0,\infty)^k,\mathrm{d}\mu_\beta)$ into $L^{1,\infty}((0,\infty)^k,\mathrm{d}\mu_\beta)$, when $\beta\in(-\frac{1}{2},\infty)^k$ and $1\leqslant r\leqslant k$, $r,k\in\mathbb{N}$.

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According to (3.18), we have, for every $x \in (0,\infty)^n$,

$$\begin{aligned}
S^{\lambda,1}_{\ell,m}(f)(x) &\leqslant C \int_0^{x_1/2} \int_0^\infty \left| \frac{\partial}{\partial x_1} W_t^{\lambda_1}(x_1,y_1) \right| \frac{\mathrm{d}t}{\sqrt{t}} \\
&\qquad \times \left(\sup_{t>0} \int_{(0,\infty)^{n-1}} W_t^{\bar\lambda}(\bar x,\bar y)|f(y_1,\bar y)|\,\mathrm{d}\mu_{\bar\lambda}(\bar y) \right) \mathrm{d}\mu_{\lambda_1}(y_1) \\
&\leqslant \frac{C}{x_1^{2\lambda_1+1}} \int_0^{x_1/2} \left(\sup_{t>0} \int_{(0,\infty)^{n-1}} W_t^{\bar\lambda}(\bar x,\bar y)|f(y_1,\bar y)|\,\mathrm{d}\mu_{\bar\lambda}(\bar y) \right) \mathrm{d}\mu_{\lambda_1}(y_1).
\end{aligned}$$

The Hardy-type operator L_{λ_1} is bounded from $L^p((0,\infty),\mathrm{d}\mu_{\lambda_1})$ into itself [13] and $W_*^{\bar\lambda}$ is bounded from $L^p((0,\infty)^{n-1},\mathrm{d}\mu_{\bar\lambda})$ into itself. Hence, $S^{\lambda,1}_{\ell,m}$ is bounded from $L^p((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into itself for every $1<p<\infty$.

Suppose now $2x_1 < y_1 < \infty$ and $1 \leqslant \ell \leqslant m \leqslant n$. By (3.10), (3.11) and (3.16), for each $x \in (0,\infty)^n$, it follows that

$$\begin{aligned}
S^{\lambda,3}_{\ell,m}(f)(x) &= \int_{2x_1}^\infty \cdots \int_{2x_\ell}^\infty \int_{x_{\ell+1}/2}^{2x_{\ell+1}} \cdots \int_{x_m/2}^{2x_m} \int_0^{x_{m+1}/2} \cdots \\
&\qquad \int_0^{x_n/2} |R^\lambda_{1,1}(x,y)||f(y)|\,\mathrm{d}\mu_\lambda(y) \\
&\leqslant C \int_{2x_1}^\infty \cdots \int_{2x_\ell}^\infty \int_{x_{\ell+1}/2}^{2x_{\ell+1}} \cdots \int_{x_m/2}^{2x_m} \int_0^{x_{m+1}/2} \cdots \\
&\qquad \int_0^{x_n/2} |f(y)| \int_0^\infty \frac{1}{t^{\sum_{j=1}^{\ell}(\lambda_j+1/2)+\sum_{j=m+1}^{n}(\lambda_j+1/2)+1}} \\
&\qquad \times \exp\left(-\frac{1}{32t} \left(\sum_{j=1}^{\ell} y_j^2 + \sum_{j=m+1}^{n} x_j^2 \right) \right) \\
&\qquad \times \prod_{j=\ell+1}^{m} \left(\frac{(x_jy_j)^{-\lambda_j}}{\sqrt{t}} \mathrm{e}^{-(x_j-y_j)^2/4t} + \frac{1}{x_j^{2\lambda_j+1}} \right) \mathrm{d}t\,\mathrm{d}\mu_\lambda(y).
\end{aligned}$$

Since the operator H^β_{loc} (see § 3.1) is bounded from $L^1((0,\infty)^k,\mathrm{d}\mu_\beta)$ into itself, provided that $\beta \in (-\frac{1}{2},\infty)^k$, in order to see that the operator $S^{\lambda,3}_{\ell,m}$ is bounded from $L^1((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n,\mathrm{d}\mu_\lambda)$ it is sufficient to show that, for every $\beta \in (-\frac{1}{2},\infty)^k$ and $1 \leqslant s \leqslant r \leqslant k$, $s,r,k \in \mathbb{N}$, the operator

$$\begin{aligned}
&T^{\beta,3}_{s,r}(g)(x) \\
&= \int_{2x_1}^\infty \cdots \int_{2x_s}^\infty \int_{x_{s+1}/2}^{2x_{s+1}} \cdots \int_{x_r/2}^{2x_r} \int_0^{x_{r+1}/2} \cdots \int_0^{x_k/2} \mathrm{d}\mu_\beta(y)\,|g(y)| \prod_{j=s+1}^{r} (x_jy_j)^{-\beta_j} \\
&\quad \times \left(\sum_{j=1}^{s} y_j^2 + \sum_{j=s+1}^{r} (x_j-y_j)^2 + \sum_{j=r+1}^{k} x_j^2 \right)^{-1/(\sum_{j=1}^{s}(\beta_j+1/2)+\sum_{j=r+1}^{k}(\beta_j+1/2)+(r-s)/2)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in (0,\infty)^k
\end{aligned}$$

is bounded from $L^1((0,\infty)^k,\mathrm{d}\mu_\beta)$ into $L^{1,\infty}((0,\infty)^k,\mathrm{d}\mu_\beta)$. Let $\beta \in (-\frac{1}{2},\infty)^k$ and $1 \leqslant s \leqslant r \leqslant k$. By proceeding as in the proof of [15, case 3] we can prove that

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the operator $T^{\beta,3}_{s,r}$ is bounded from $L^1((0,\infty)^k, \mathrm{d}\mu_\beta)$ into $L^{1,\infty}((0,\infty)^k, \mathrm{d}\mu_\beta)$ when $s < r$. Also, the operator $T^{\beta,3}_{s,s}$ is of weak type $(1,1)$ with respect to the measure $\mathrm{d}\mu_\beta$, because

$$T^{\beta,3}_{s,s}(g)(x) \leqslant \frac{C}{(\sum_{j=s+1}^k x_j^2)^{\sum_{j=s+1}^k(\beta_j+1/2)}} \int_0^{x_{s+1}/2} \cdots \int_0^{x_k/2} \int_{2x_1}^\infty \cdots \int_{2x_s}^\infty \frac{|g(y)|}{y_1\cdots y_s} \times \prod_{j=s+1}^k y_j^{2\beta_j}\,\mathrm{d}y, \quad x \in (0,\infty)^k,$$

and H^s_∞ is bounded from $L^1((0,\infty)^s, \prod_{j=1}^s x_j^{2\beta_j}\,\mathrm{d}x)$ into itself and, by lemma 3.1, $L_{\beta_{s+1},\dots,\beta_k}$ is bounded from

$$L^1\biggl((0,\infty)^{k-s}, \prod_{j=s+1}^k x_j^{2\beta_j}\,\mathrm{d}x\biggr) \quad \text{into} \quad L^{1,\infty}\biggl((0,\infty)^{k-s}, \prod_{j=s+1}^k x_j^{2\beta_j}\,\mathrm{d}x\biggr).$$

Then, the operator $S^{\lambda,3}_{\ell,m}$ is bounded from $L^1((0,\infty)^n, \mathrm{d}\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n, \mathrm{d}\mu_\lambda)$.

By using (3.18), for each $x \in (0,\infty)^n$,

$$S^{\lambda,3}_{\ell,m}(f)(x) \leqslant C \int_{2x_1}^\infty \frac{1}{y_1}\biggl(\sup_{t>0} \int_{(0,\infty)^{n-1}} W_t^{\bar\lambda}(\bar x,\bar y)|f(y_1,\bar y)|\,\mathrm{d}\mu_{\bar\lambda}(\bar y)\biggr)\mathrm{d}y_1.$$

Since H^1_∞ is bounded from $L^p((0,\infty), \mathrm{d}\mu_{\lambda_1})$ into itself and $W^{\bar\lambda}_*$ is bounded from $L^p((0,\infty)^{n-1}, \mathrm{d}\mu_{\bar\lambda})$ into itself, $S^{\lambda,3}_{\ell,m}$ is bounded from $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$ into itself for every $1 < p < \infty$.

Finally, we consider the case $x_1/2 < y_1 < 2x_1$. By using (3.15) and (3.19) we obtain that, for every $1 \leqslant \ell \leqslant m \leqslant n$,

$$\begin{aligned}
&S^{\lambda,2}_{\ell,m}(f)(x)\\
&= \int_{x_1/2}^{2x_1} \cdots \int_{x_\ell/2}^{2x_\ell} \int_0^{x_{\ell+1}/2} \cdots \int_0^{x_m/2} \int_{2x_{m+1}}^\infty \cdots \int_{2x_n}^\infty |R^\lambda_{1,1}(x,y)||f(y)|\,\mathrm{d}\mu_\lambda(y)\\
&\leqslant C \int_{x_1/2}^{2x_1} \cdots \int_{x_\ell/2}^{2x_\ell} \int_0^{x_{\ell+1}/2} \cdots \int_0^{x_m/2} \int_{2x_{m+1}}^\infty \cdots\\
&\quad \int_{2x_n}^\infty |f(y)|\biggl[\int_0^{x_1y_1} \frac{(x_1y_1)^{-\lambda_1-1/2}}{t} \mathrm{e}^{-(x_1-y_1)^2/4t} W_t^{\bar\lambda}(\bar x,\bar y)\,\mathrm{d}t\\
&\qquad + \int_{x_1y_1}^\infty \biggl(\frac{\mathrm{e}^{-x_1^2/8t}}{t^{\lambda_1+3/2}} + \frac{(x_1y_1)^{-\lambda_1}}{t^{3/2}}\mathrm{e}^{-(x_1-y_1)^2/8t}\biggr) W_t^{\bar\lambda}(\bar x,\bar y)\,\mathrm{d}t\biggr]\mathrm{d}\mu_\lambda(y)\\
&\leqslant C\biggl[\int_0^{2x_1^2} \frac{\mathrm{d}t}{\sqrt{t}} \sup_{t>0} \int_{x_1/2}^{2x_1} \int_{(0,\infty)^{n-1}} \frac{(x_1y_1)^{-\lambda_1-1/2}}{\sqrt{t}}\\
&\qquad \times \exp\biggl(-\frac{(x_1-y_1)^2}{4t}\biggr) W_t^{\bar\lambda}(\bar x,\bar y)|f(y)|\,\mathrm{d}\mu_\lambda(y)\\
&\qquad + \biggl(x_1^{-2\lambda_1}\int_{x_1^2/2}^\infty \frac{\mathrm{d}t}{t^{3/2}} + \int_{x_1^2/2}^\infty \frac{\mathrm{d}t}{t^{\lambda_1+3/2}}\biggr)\\
&\qquad \times \sup_{t>0} \int_{x_1/2}^{2x_1} \int_{(0,\infty)^{n-1}} W_t^{\bar\lambda}(\bar x,\bar y)|f(y)|\,\mathrm{d}\mu_\lambda(y)\biggr]
\end{aligned}$$

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$$\leqslant C\bigg[\sup_{t>0}\int_{x_1/2}^{2x_1}\int_{(0,\infty)^{n-1}}\frac{(x_1y_1)^{-\lambda_1}}{\sqrt{t}}\exp\bigg(-\frac{(x_1-y_1)^2}{4t}\bigg)W_t^{\bar{\lambda}}(\bar{x},\bar{y})|f(y)|\,\mathrm{d}\mu_\lambda(y)$$
$$+\sup_{t>0}\int_{(0,\infty)^{n-1}}W_t^{\bar{\lambda}}(\bar{x},\bar{y})\frac{1}{x_1^{2\lambda_1+1}}\int_{x_1/2}^{2x_1}|f(y)|\,\mathrm{d}\mu_{\lambda_1}(y_1)\,\mathrm{d}\mu_{\bar{\lambda}}(\bar{y})\bigg].$$

The operator $H_{\mathrm{loc}}^{\lambda_1}$ is bounded from $L^1((0,\infty),\mathrm{d}\mu_{\lambda_1})$ into itself, and the maximal operator $W_*^{\bar{\lambda}}$ is bounded from $L^1((0,\infty)^{n-1},\mathrm{d}\mu_{\bar{\lambda}})$ into $L^{1,\infty}((0,\infty)^{n-1},\mathrm{d}\mu_{\bar{\lambda}})$ (theorem 1.1). By splitting $(0,\infty)^{n-1}$ into global and local regions and using the arguments developed above, we can show that the maximal operator,

$$\mathcal{W}_*^{\lambda}(f)(x)=\sup_{t>0}\int_{x_1/2}^{2x_1}\int_{(0,\infty)^{n-1}}\frac{(x_1y_1)^{-\lambda_1}}{\sqrt{t}}\mathrm{e}^{-(x_1-y_1)^2/4t}W_t^{\bar{\lambda}}(\bar{x},\bar{y})f(y)\,\mathrm{d}\mu_\lambda(y),$$

is bounded from $L^1((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n,\mathrm{d}\mu_\lambda)$ for each $x\in(0,\infty)^n$. We conclude that $S_{\ell,m}^{\lambda,2}$ is bounded from $L^1((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n,\mathrm{d}\mu_\lambda)$.

Moreover, since, for each $p\in(1,\infty)$, $H_{\mathrm{loc}}^{\lambda_1}$, $W_*^{\bar{\lambda}}$ and $\mathcal{W}_*^{\lambda}$ are bounded from $L^p((0,\infty),\mathrm{d}\mu_{\lambda_1})$ into itself, $L^p((0,\infty)^{n-1},\mathrm{d}\mu_{\bar{\lambda}})$ into itself and $L^p((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into itself, respectively, $S_{\ell,m}^{\lambda,2}$ is bounded from $L^p((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into itself for every $1<p<\infty$.

Hence, we have proved that the operator $R_{1,1}^{\lambda}$ defined by

$$R_{1,1}^{\lambda}(f)(x)=\int_{(0,\infty)^n}|R_{1,1}^{\lambda}(x,y)f(y)|\,\mathrm{d}\mu_\lambda(y),\quad x\in(0,\infty)^n,$$

is bounded from $L^1((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into $L^{1,\infty}((0,\infty)^n,\mathrm{d}\mu_\lambda)$ and from $L^p((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into itself for every $1<p<\infty$. Since

$$R_{1,1,*}^{\lambda}(f)(x)=\sup_{\varepsilon>0}\bigg|\int_{|x-y|>\varepsilon}R_{1,1}^{\lambda}(x,y)f(y)\,\mathrm{d}\mu_\lambda(y)\bigg|\leqslant R_{1,1}^{\lambda}(f)(x),\quad x\in(0,\infty)^n,$$

$R_{1,1,*}^{\lambda}$ is bounded from

$$L^1((0,\infty)^n,\mathrm{d}\mu_\lambda)\quad\text{into}\quad L^{1,\infty}((0,\infty)^n,\mathrm{d}\mu_\lambda)$$

and from $L^p((0,\infty)^n,\mathrm{d}\mu_\lambda)$ into itself, for every $1<p<\infty$.

In order to study the maximal operators $R_{1,\ell,*}^{\lambda}$, $\ell=2,\dots,n$, we can proceed as for $R_{1,1,*}^{\lambda}$ by taking into account (3.12).

Finally, the L^p-boundedness properties of the maximal operator defined by

$$R_{1,n+1,*}^{\lambda}(f)(x)=\sup_{\varepsilon>0}\bigg|\int_{|x-y|>\varepsilon,y\in L(x)}\mathbb{R}_1(x,y)\bigg(\frac{y}{x}\bigg)^{\lambda}f(y)\,\mathrm{d}y\bigg|,\quad x\in(0,\infty)^n,$$

are a consequence of proposition 2.1.

Thus, the proof of proposition 4.4 is finished. □

According to propositions 4.3 and 4.4 standard arguments allow us to conclude that, for every $f\in L^p((0,\infty)^n,\mathrm{d}\mu_\lambda)$, $1\leqslant p<\infty$, and $i=1,\dots,n$, there exists the limit

$$\lim_{\varepsilon\to0^+}\int_{|x-y|>\varepsilon}R_i^{\lambda}(x,y)f(y)\,\mathrm{d}\mu_\lambda(y)\quad\text{a.e. }x\in(0,\infty)^n.$$

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We define, for every $f \in L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$, $1 \leqslant p < \infty$ and $i = 1, \ldots, n$, the Riesz transform $R_i^\lambda(f)$ of f by

$$R_i^\lambda(f)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R_i^\lambda(x,y) f(y) \, \mathrm{d}\mu_\lambda(y) \quad \text{a.e. } x \in (0,\infty)^n.$$

Note that, by proposition 4.3, for every $i = 1, \ldots, n$, this definition extends the initial definition of the Riesz transform R_i^λ from $C_c^\infty((0,\infty)^n)$ to $L^p((0,\infty)^n, \mathrm{d}\mu_\lambda)$, $1 \leqslant p < \infty$.

Finally, from proposition 4.4 we infer the desired L^p-boundedness properties for the Riesz transform R_i^λ, $i = 1, \ldots, n$, and the proof of theorem 1.4 is complete.

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