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Generalized energetics for inertially stable parallel shear flows

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For simple parallel shear flows on the f-plane and the equatorial β-plane we derive an energy norm for zonally invariant perturbations. It is used to derive the linear stability boundary for when these flows are inertially stable in the classical sense but may be destabilized due to unequal rates of diffusion of momentum and heat. The analysis is valid when there are arbitrary, zonally invariant, no-slip boundaries which are perfect thermal conductors.

1. Introduction

For a zonally symmetric flow $U(y, z)$ in the x-direction subjected to zonally invariant perturbations, the classical criterion for inertial (or symmetric) instability is that

$$N^2\Phi < f^2U_z^2, \quad \text{where} \quad \Phi = f(f - U_y), \tag{1.1}$$

is the Rayleigh discriminant, f the Coriolis parameter, $U_y = \partial_y U$ the horizontal shear, $U_z = \partial_z U$ the vertical shear and $N^2 = -(g/\rho_0)\partial\rho/\partial z$ the square of the buoyancy frequency with g the gravitational constant; ρ_0 is the constant reference density that arises in the context of the Boussinesq approximation while ρ is the density. When $N^2\Phi > f^2U_z^2$ the flow is stable in the classical sense. However, for stability one cannot have both $\Phi < 0$ and $N^2 < 0$: such a flow is unstable (Ooyama 1966; Charney 1973). In what follows we will assume that there is statically stable stratification $N^2 > 0$.

By ‘classically’ stable or unstable we mean that stability/instability for inviscid and adiabatic fluids can be established in a variety of ‘classical’ ways. One is to modify Rayleigh’s (1916) fluid ring exchange argument (for circular vortices) to include rotation and stratification and replace the conservation of absolute angular momentum by the conservation of absolute linear momentum $m = U - fy$. This (infinitely long) fluid ‘rod’ exchange argument is described in Holton (1992). This is essentially an energy argument where the flow is considered stable if the exchange leads to a hypothetical increase in total (kinetic and potential) energy. A Lagrangian displacement argument, first introduced by Solberg (1936), leads to the same stability/instability conditions. A fluid element is imagined to be displaced, again conserving angular momentum or linear momentum and density, while the ambient pressure field is presumed unchanged. If such an element is found to accelerate away from its original position, the flow is unstable; if ‘pushed’ back, it is stable. Mathematically more rigorous is Ooyama’s (1966) method which utilizes a Lagrangian displacement field, taking continuity and pressure perturbations into account. In the small-displacement limit, conditions for growth/decay of the material displacements are established as well as time-evolution equations for the perturbation energy and total displacement field (volume integrated). Ooyama essentially used Fjørtoft's (1950) energy method but augmented it with a clever construction to prove instability/stability, having realized that Fjørtoft's assumption of the existence of normal-modes solutions may be erroneous.

In (1.1) U_y, U_z and N^2 can be variable but it is only valid if diffusion of heat and momentum are ignored. McIntyre (1970) found through normal-modes analysis on an unbounded domain that when U_y, U_z and N^2 are constant, the instability criterion is

$$N^2\Phi < \frac{(1+P)^2}{4P} f^2 U_z^2, \quad \text{where} \quad P = \nu/\kappa \tag{1.2}$$

is the Prandtl number with ν the kinematic viscosity and κ the diffusivity (of heat). Since $(1+P)^2/4P > 1$ except when $P = 1$, it is seen that for all $P \neq 1$ the flow is 'more unstable' than the classical criterion (1.1) would indicate. That is, flows with $P \neq 1$ and

$$f^2 U_z^2 < N^2\Phi < \frac{(1+P)^2}{4P} f^2 U_z^2$$

are unstable whereas (1.1) would predict stability. For small or large P, this 'extra' instability range can become large. If there is no vertical shear ($U_z = 0$) the criteria (1.1) and (1.2) coincide. Hence, the 'diffusive destabilization' discovered by McIntyre only occurs for vertically sheared flows.

Through a consideration of what one might call the evolution of 'effective' energy or generalized energetics, we show in §2 the converse of (1.2), namely that when $N^2\Phi > (1+P) f^2 U_z^2/4P$ the flow is linearly stable. The derivation is simple and does not depend on the assumption of the existence of normal modes. It works equally well for infinite domains as for flows bounded in the meridional (y, z)-plane by arbitrary no-slip rigid boundaries that are thermally conducting. We construct a positive-definite (Lyapunov) functional which defines a norm $\|\cdot\|$ for the perturbations. With viscosity and diffusion added we determine under which conditions $\mathrm{d}\|\cdot\|/\mathrm{d}t \leqslant 0$ (t is time). If these conditions are met, (asymptotic) stability is established.

In §3 we use the same approach to derive the stability criterion for a simple horizontally and vertically sheared flow on the equatorial β-plane with or without the non-traditional horizontal component of the Earth's rotation included. There is no need to make the customary hydrostatic approximation or a normal-modes assumption, which would involve the introduction of Hermite expansions in latitude (Dunkerton 1981; Hua, Moore & Le Gentil 1997). Such Hermite expansions seem no longer an option if one allows for horizontal diffusion in a normal-modes stability analysis. This is the reason why in for example Griffiths (2003) only vertical diffusion is considered. In §4 we summarize the main results and discuss some open questions.

2. A simple zonal flow on the f-plane

For a steady, zonally symmetric parallel shear flow $U(y, z)$ in the x-direction on the f-plane, the balance equations with the Boussinesq approximation are

$$fU = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 U, \quad \frac{g}{\rho_0}\rho = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}, \quad \kappa\nabla^2\rho = 0 \quad \text{with} \quad \nabla^2 = \partial_y^2 + \partial_z^2;$$

p and ρ are the equilibrium pressure and density, respectively. As in McIntyre's study we consider $U = U_0 + U_y y + U_z z$ with U_y = constant and U_z = constant so that $\nabla^2 U = 0$. The usual thermal-wind balance equation then holds:

$$\frac{g}{\rho_0}\frac{\partial \rho}{\partial y} = fU_z. \tag{2.1}$$

For ρ we take $\rho(y,z) = \rho_0(1 - (N^2/g)z + (fU_z/g)y)$, with the square of the buoyancy frequency $N^2 = -(g/\rho_0)\,\mathrm{d}\rho/\mathrm{d}z$ = constant, so that also $\nabla^2\rho = 0$. Introducing zonally invariant velocity perturbations u', v', $w'(y,z,t)$, a density perturbation $\rho'(y,z,t)$ and pressure perturbation $p'(y,z,t)$, and linearizing about the steady state, one finds

$$\mathrm{D}_t^{\nu} u' + [U_y - f]v' + U_z w' = 0, \tag{2.2}$$

$$\mathrm{D}_t^{\nu} v' + fu' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}, \tag{2.3}$$

$$\mathrm{D}_t^{\nu} w' - b = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}, \tag{2.4}$$

$$\mathrm{D}_t^{\kappa} b - v'\frac{g}{\rho_0}\frac{\partial \rho}{\partial y} - w'\frac{g}{\rho_0}\frac{\partial \rho}{\partial z} = 0, \tag{2.5}$$

$$\frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \tag{2.6}$$

where

$$\mathrm{D}_t^{\nu} = \partial_t - \nu\nabla^2, \quad \mathrm{D}_t^{\kappa} = \partial_t - \kappa\nabla^2.$$

In (2.4) and (2.5), $b = -g\rho'/\rho_0$ is the buoyancy. Equation (2.6) indicates that we assume the fluid to be incompressible. Using the thermal-wind balance equation (2.1) and the definition of N^2, (2.5) becomes

$$\mathrm{D}_t^{\kappa} b - fU_z v' + N^2 w' = 0. \tag{2.7}$$

In view of (2.6), it is customary to introduce a streamfunction ψ for the meridional motions, that is, $v' = \partial_z\psi$, $w' = -\partial_y\psi$. By elimination between (2.2)–(2.4) and (2.7), a single equation for ψ results:

$$\mathrm{D}_t^{\kappa}(\mathrm{D}_t^{\nu})^2\nabla^2\psi + \mathrm{D}_t^{\kappa}\partial_z[\Phi\partial_z\psi + fU_z\partial_y\psi] + \mathrm{D}_t^{\nu}\partial_y[fU_z\partial_z\psi + N^2\partial_y\psi] = 0, \tag{2.8}$$

where Φ is the Rayleigh discriminant defined in (1.1). Equation (2.8) is the viscous/diffusive version of the so-called Eliassen–Sawyer equation (Sawyer 1949; Eliassen 1951). If Φ, N^2 and fU_z are constant, as in McIntyre's (1970) study, (2.8) may allow normal-modes solutions of the form $\psi = \exp(\omega t)\Psi(y,z)$. Considering an unbounded domain, McIntyre chose $\Psi = \exp[\mathrm{i}k(y\cos\phi + z\sin\phi)]$, where the real part is understood. If this is substituted in equation (2.8), the result is a cubic polynomial in ω. If this cubic has roots ω with a positive real part for some k, ϕ, the flow can be considered unstable. McIntyre (1970) found (1.2) through inspection of the properties of the roots in the limit of vanishing viscosity.

Let us note that, for example, when $U_z = 0$ and $\nu = \kappa = 0$, normal modes requiring $\Psi = 0$ at rigid boundaries (no normal flow) exist when, say, these boundaries are at $y = 0$ (vertical sidewall) and $z = 0$ (horizontal bottom). In that case one can take $\Psi = \sin(ly)\sin(mz)$, with l, m the horizontal and vertical wavenumber, respectively. However, it can be shown (see Høiland 1962) that if the sidewall or the bottom is slightly tilted, no normal-modes solutions exist with $\Psi = 0$ at these non-perpendicular walls. Similarly, if additionally $U_z \neq 0$, normal-modes solutions do not exist when the flow is confined to, for example, a rectangular domain (Yanai & Tokiaka 1969). An interesting consequence of the non-existence of normal modes in a container with a slanted sidewall filled with a stratified fluid is discussed by Maas *et al.* (1997).

In this paper we derive a sufficient condition for stability by constructing an energy norm, avoiding the use of normal modes. With (2.2) and (2.7) it follows that

$$v' = \frac{f}{D}\left[N^2 \mathrm{D}_t^{\nu} u' - f U_z \mathrm{D}_t^{\kappa} b\right], \quad w' = \frac{f}{D}\left[f U_z \mathrm{D}_t^{\nu} u' - \frac{\Phi}{f}\mathrm{D}_t^{\kappa} b\right], \tag{2.9}$$

where

$$D = N^2 \Phi - f^2 U_z^2. \tag{2.10}$$

The classical condition for stability is that $D > 0$. Adding $v' \times$ (2.3) + $w' \times$ (2.4) we get

$$\frac{\partial}{\partial t}\frac{1}{2}(v'^2 + w'^2) + f u'v' - w'b = -\boldsymbol{u}' \cdot \nabla p'/\rho_0 + \nu(v'\nabla^2 v' + w'\nabla^2 w'). \tag{2.11}$$

Substituting (2.9) in (2.11), the sum

$$f u'v' - w'b = \frac{f^2}{D}\left[N^2 u' \mathrm{D}_t^{\nu} u' - U_z\left(u'\mathrm{D}_t^{\kappa} b + b\mathrm{D}_t^{\nu} u'\right) + \frac{\Phi}{f^2} b \mathrm{D}_t^{\kappa} b\right]. \tag{2.12}$$

Keeping all time-derivatives on the left-hand side, we obtain

$$\frac{\partial}{\partial t}\frac{1}{2}\left[v'^2 + w'^2 + \frac{f^2}{D}\left(N^2 u'^2 - 2U_z u'b + \frac{\Phi}{f^2}b^2\right)\right] = -\boldsymbol{u}' \cdot \nabla p'/\rho_0$$
$$+ \nu(v'\nabla^2 v' + w'\nabla^2 w') + \frac{f^2}{D}\left[N^2 \nu u'\nabla^2 u' - U_z(\kappa u'\nabla^2 b + \nu b \nabla^2 u') + \frac{\Phi}{f^2}\kappa b \nabla^2 b\right]. \tag{2.13}$$

By completing the square on the left-hand side in the term $(f^2/D)(N^2 u'^2 - \cdots)$, one finds after integration over the meridional plane that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}\left[v'^2 + w'^2 + \frac{f^2 N^2}{D}\left(u' - \frac{U_z}{N^2}b\right)^2 + \frac{b^2}{N^2}\right] \mathrm{d}\mathscr{V} \quad (\mathrm{d}\mathscr{V} = \mathrm{d}y\,\mathrm{d}z)$$
$$= -\iint \left[\nu(|\nabla v'|^2 + |\nabla w'|^2) + \frac{f^2}{D}\left(\nu N^2 |\nabla u'|^2 - (\nu + \kappa) U_z \nabla u' \cdot \nabla b + \kappa \frac{\Phi}{f^2}|\nabla b|^2\right)\right] \mathrm{d}\mathscr{V} \tag{2.14}$$

provided that

$$\iint \nabla \cdot \left[\nu(v'\nabla v' + w'\nabla w') + (f^2/D)\left(\nu N^2 u'\nabla u' - U_z(\kappa u'\nabla b + \nu b \nabla u')\right.\right.$$
$$\left.\left. + \kappa\frac{\Phi}{f^2} b\nabla b\right)\right] \mathrm{d}\mathscr{V} = 0 \quad \text{and} \quad \int \nabla \cdot (\boldsymbol{u}' p/\rho_0)\, \mathrm{d}\mathscr{V} = 0. \tag{2.15}$$

If there are rigid boundaries, we prescribe the no-slip condition at such boundaries, i.e. $u', v', w' = 0$. Also at such boundaries we prescribe $b = 0$. If the flow is unbounded in one or more direction, we assume that for large distances from the origin somewhere located in the domain, the perturbations vanish. With these conditions, (2.15) will hold. To obtain the right-hand side in (2.14), we used the fact that N^2, U_z, U_y (and therefore D and Φ) are constant.

But, if $\nu = \kappa = 0$, the right-hand side in (2.14) vanishes and D, N^2 and U_z can be variable. In that case we have $\mathrm{d}E/\mathrm{d}t = 0$ where

$$E = \iint \frac{1}{2}\left[v'^2 + w'^2 + \frac{f^2 N^2}{D}\left(u' - \frac{U_z}{N^2}b\right)^2 + \frac{b^2}{N^2}\right]\mathrm{d}\mathscr{V} \tag{2.16}$$

(the effective energy) is a positive-definite functional (a Lyapunov functional) for this system if $N^2 > 0$ (statically stable stratification) and $D > 0$ (the classical condition for stability). This is valid for arbitrary $U(y,z)$, $\rho(y,z)$, as long as $N^2 > 0$ and $D > 0$ everywhere in the domain. With our approach we thus find that in that case $\|u'\| \equiv [\iint u'^2 \mathrm{d}\mathscr{V}]^{1/2}$, $\|v'\|$, $\|w'\|$ and $\|b\|$ can be kept arbitrarily small by choosing the initial disturbance amplitudes small enough. With this norm as a measure for the magnitude of the perturbations, the flow is stable 'in the mean' (Drazin 2002).

Returning to the case $\nu \neq 0$, $\kappa \neq 0$ and U_y, U_z and N^2 constant, we find with (2.14) and $U_z \nabla u' \cdot \nabla b \leqslant |U_z||\nabla u'||\nabla b|$ that if $D > 0$

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &\leqslant -\iint \frac{f^2}{D}\left(\nu N^2|\nabla u'|^2 - (\nu+\kappa)|U_z||\nabla u'||\nabla b| + \kappa\frac{\Phi}{f^2}|\nabla b|^2\right)\mathrm{d}\mathscr{V} \\
&\quad - \iint \nu(|\nabla v'|^2 + |\nabla w'|^2)\mathrm{d}\mathscr{V} \\
&= -\iint \frac{f^2}{D}\left[\nu N^2\left(|\nabla u'| - \frac{1+P}{2P}\frac{|U_z|}{N^2}|\nabla b|\right)^2 + \frac{\kappa}{N^2}\left(\frac{N^2\Phi}{f^2} - \frac{(1+P)^2}{4P}U_z^2\right)|\nabla b|^2\right]\mathrm{d}\mathscr{V} \\
&\quad - \iint \nu(|\nabla v'|^2 + |\nabla w'|^2)\,\mathrm{d}\mathscr{V},
\end{aligned} \tag{2.17}$$

where the final form of the right-hand side is obtained by completing the square. If

$$N^2\Phi > \frac{(1+P)^2}{4P} f^2 U_z^2, \tag{2.18}$$

we also have $D = N^2\Phi - f^2U_z^2 > 0$ for all positive P (because $(1+P)^2/4P \geqslant 1$) and E (2.16) is positive-definite. Thus (2.17) shows that when (2.18) is satisfied, we have $\mathrm{d}E/\mathrm{d}t \leqslant 0$, where equality only occurs when $\nabla u'$, $\nabla v'$, $\nabla w'$ and $\nabla b = 0$ everywhere. In a closed domain, with our boundary conditions, these gradients can only vanish when $u' = v' = w' = b = 0$. Thus it follows that if the flow is entirely enclosed, it is asymptotically stable, that is, $\lim_{t\to\infty} E = 0$, and the flow is forced toward the state of rest with constant N.

3. Zonal flow on the equatorial β-plane

In the same fashion as in §2 we now derive a stability criterion for a simple flow on the equatorial β-plane. For a steady, zonally symmetric parallel shear flow $U(y,z)$ in the x-direction (west–east) on the equatorial β-plane, the (Boussinesq) balance equations would be

$$\beta y U = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 U, \quad -\gamma U + \frac{g}{\rho_0}\rho = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}, \quad \kappa\nabla^2\rho = 0;$$

βy replaces f as the 'vertical' Coriolis parameter while we retained the non-traditional horizontal component $\gamma = 2\Omega$, with Ω Earth's angular velocity. We will assume that ν and κ are very small and that the unperturbed flow and corresponding density field evolve very slowly. In the analysis we thus assume that essentially the basic flow can be considered steady. With this assumption we can set $\nabla^2 U = 0$ so that the thermal-wind balance is

$$\frac{g}{\rho_0}\frac{\partial \rho}{\partial y} = \beta y U_z + \gamma U_y. \tag{3.1}$$

The linear perturbation equations for zonally invariant perturbations are

$$\mathrm{D}_t^\nu u' + (U_y - \beta y)v' + (U_z + \gamma)w' = 0, \tag{3.2}$$

$$\mathrm{D}_t^\nu v' + \beta y u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}, \tag{3.3}$$

$$\mathrm{D}_t^\nu w' - \gamma u' - b = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}, \tag{3.4}$$

$$\mathrm{D}_t^\kappa b - (\beta y U_z + \gamma U_y)v' + N^2 w' = 0, \tag{3.5}$$

plus the incompressibility condition (2.6) (see also Hua *et al.* 1997). The thermal-wind balance equation (3.1) was used to cast the buoyancy equation in the form shown in (3.5). Proceeding as in § 2, we first establish that

$$\left.\begin{aligned} v' &= \frac{\beta y}{D}\left[N^2 \mathrm{D}_t^\nu u' - (U_z + \gamma)\mathrm{D}_t^\kappa b\right] \\ w' &= \frac{\beta y}{D}\left[(\beta y U_z + \gamma U_y)\mathrm{D}_t^\nu u' - (\beta y - U_y)\mathrm{D}_t^\kappa b\right], \end{aligned}\right\} \tag{3.6}$$

where

$$D = \beta y(\beta y - U_y)[N^2 + \gamma(U_z + \gamma)] - (\beta y)^2(\gamma + U_z)^2. \tag{3.7}$$

When $\nu = \kappa = 0$, Hua *et al.* (1997) give the criterion necessary for instability

$$\beta y(\beta y - U_y)[N^2 + \gamma(U_z + \gamma)] < (\beta y)^2(\gamma + U_z)^2. \tag{3.8}$$

To get this, the normal-modes assumption was made by Hua *et al.* (1997) in conjunction with the inviscid, non-diffusive Eliassen–Sawyer equation, similar to (2.8). Changing '$<$' to '$>$' in (3.8) would imply that the condition for stability is that $D > 0$ everywhere. Note that if we identify $f = \beta y$ then in the traditional dynamics ($\gamma = 0$), $D > 0$ is the same as $N^2\Phi > f^2U_z^2$ in the f-plane dynamics, the classical condition for stability.

Adding $v' \times$ (3.3) + $w' \times$ (3.4) we get

$$\frac{\partial}{\partial t}\frac{1}{2}(v'^2 + w'^2) + \beta y u'v' - \gamma u'w' - w'b = -\boldsymbol{u}' \cdot \nabla p'/\rho_0 + \nu(v'\nabla^2 v' + w'\nabla^2 w'). \tag{3.9}$$

Substituting (3.6) in (3.9), we find that

$$\beta y u'v' - \gamma u'w' - w'b = \frac{1}{D}\left[A u'\mathrm{D}_t^\nu u' - B(u'\mathrm{D}_t^\kappa b + b\mathrm{D}_t^\nu u') + \Phi b \mathrm{D}_t^\kappa b\right], \tag{3.10}$$

with

$$A = (\beta y)^2(N^2 - \gamma U_z) - \beta y \gamma^2 U_y, \quad B = \beta y(\beta y U_z + \gamma U_y), \quad \Phi = \beta y(\beta y - U_y). \tag{3.11}$$

Now consider the case $\nu=\kappa=0$ so that $\mathrm{D}_t^\nu=\mathrm{D}_t^\kappa=\partial/\partial t$. Integration over the meridional plane results in $\mathrm{d}E/\mathrm{d}t=0$ with

$$
\begin{aligned}
E &= \iint \frac{1}{2}\left(v'^2+w'^2+\frac{1}{D}[Au'^2-2Bu'b+\Phi b^2]\right)\mathrm{d}\mathscr{V} \\
&= \iint \frac{1}{2}\left(v'^2+w'^2+\frac{1}{D}\left[\Phi\left(b-\frac{B}{\Phi}u'\right)^2+\frac{(A\Phi-B^2)}{\Phi}u'^2\right]\right)\mathrm{d}\mathscr{V} \\
&= \iint \frac{1}{2}\left(v'^2+w'^2+\frac{\Phi}{D}\left(b-\frac{B}{\Phi}u'\right)^2+\frac{(\beta y)^2}{\Phi}u'^2\right)\mathrm{d}\mathscr{V}.
\end{aligned} \tag{3.12}
$$

To obtain the last form we used the fact that $A\Phi-B^2=(\beta y)^2D$. Equation (3.12) holds when the same (boundary) conditions as in §2 are prescribed. The effective energy E will again be positive-definite if $\Phi/D>0$ and $\Phi>0$, or, when $D>0$ and $\Phi>0$ everywhere. The point $y=0$ is special in that there $D=0$. However, assuming that for small positive y, $U_y\approx U_y^0\times y^{1+\varepsilon}$ $(\varepsilon\geqslant 0)$, we find that in the limit $y\downarrow 0$, the factor $(\beta y)^2/\beta y(\beta y-U_y)$ remains finite except when both $\varepsilon=0$ and $U_y^0=\beta$. In that case B/Φ also becomes infinite. The term Φ/D remains finite unless $N^2-U_z(U_z+\gamma)=0$ (when $\varepsilon>0$) or $(\beta-U_y^0)[N^2+\gamma(U_z+\gamma)]-\beta(\gamma+U_z)^2=0$ (when $\varepsilon=0$). For small negative y we arrive at the same conclusion, assuming that $U_y\approx U_0\times -|y|^{1+\varepsilon}$. Excluding these special cases, we have a stability proof for arbitrary $U(y,z)$ and $\rho(y,z)$ (and corresponding $N^2(y,z)$) when $\Phi>0$ and $D>0$ for $y\neq 0$.

If $\gamma=0$ (traditional β-plane dynamics), D will be negative when there is a negative Rayleigh discriminant $\Phi<0$. In that case Φ/D in (3.12) is positive, but the factor $(\beta y)^2/\Phi$ which multiplies u'^2 is negative. Hence perturbations can grow without violating the conservation law $\mathrm{d}E/\mathrm{d}t=0$. Hence we expect the flow to be unstable. This is well-known from normal-modes analysis. See for example Dunkerton (1981) and Griffiths (2003) who show that any flow with $U_y\neq 0$ at and around $y=0$ is unstable: there is then always a range of y-values for which the Rayleigh discriminant $\Phi<0$. Hence any flow with finite horizontal shear at the equator is inertially unstable (in the y-range where $\Phi<0$). On the other hand a flow for example with $U(y)=U_0\exp(-\frac{1}{2}y^2/L_h^2)$ is inertially stable for any $U_0>0$ (an eastward flowing Gaussian jet, similar to an equatorial Kelvin wave). When $U_0<0$ however, it is stable only when $|U_0|/L_h^2<\beta$ (L_h is some arbitrary horizontal length scale).

Knowing that such a flow can be classically stable, we will study the stability of $U=U_0(1-\frac{1}{2}y^2/L_h^2)+U_zz$ while $N^2=$constant and $\nu>0$ and $\kappa>0$. This flow can be considered a local (small y) approximation to the Gaussian jet, with additional constant vertical shear U_z. This is a convenient choice since the coefficients A/D, B/D and Φ/D in

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{1}{2}(v'^2+w'^2)+\frac{1}{D}\left[Au'\mathrm{D}_t^\nu u'-B(u'\mathrm{D}_t^\kappa b+b\mathrm{D}_t^\nu u')+\Phi b\mathrm{D}_t^\kappa b\right] \\
=-\boldsymbol{u}'\cdot\nabla p'/\rho_0+\nu(v'\nabla^2v'+w'\nabla^2w')
\end{aligned} \tag{3.13}
$$

do not depend on the spatial variables. By partial integration we find an equation essentially the same as (2.14):

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} = &-\iint \frac{1}{D'}[\nu A'|\nabla u'|^2-(\nu+\kappa)B'\nabla u'\cdot\nabla b+\kappa C'|\nabla b|^2]/\mathrm{d}\mathscr{V} \\
&-\iint \nu\left(|\nabla v'|^2+|\nabla w'|^2\right)\mathrm{d}\mathscr{V},
\end{aligned} \tag{3.14}
$$

with

$$E = \iint \frac{1}{2}\left(v'^2 + w'^2 + \frac{C'}{D'}\left(b - \frac{B'}{C'}u'\right)^2 + \frac{\beta}{C'}u'^2\right)\mathrm{d}\mathscr{V}, \tag{3.15}$$

$$D' = \left(\beta + U_0/L_h^2\right)[N^2 + \gamma(U_z + \gamma)] - \beta(\gamma + U_z)^2, \tag{3.16}$$

and

$$A' = \beta(N^2 - \gamma U_z) + \gamma^2 U_0/L_h^2, \quad B' = \beta U_z - \gamma U_0/L_h^2, \quad C' = \beta + U_0/L_h^2. \tag{3.17}$$

Thus, E is positive definite if $D' > 0$ and $C' > 0$. With $B'\nabla u' \cdot \nabla b \leqslant |B'||\nabla u'||\nabla b|$, it follows that if $D' > 0$

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &\leqslant -\iint \frac{1}{D'}[\nu A'|\nabla u'|^2 - (\nu+\kappa)|B'||\nabla u'||\nabla b| + \kappa C'|\nabla b|^2]\,\mathrm{d}\mathscr{V} \\ &\quad - \iint \nu(|\nabla v'|^2 + |\nabla w'|^2)\,\mathrm{d}\mathscr{V} \\ &= -\iint \frac{1}{D'}\left[\kappa C'\left(|\nabla b| - \frac{1+P}{2}\frac{|B'|}{C'}|\nabla u'|\right)^2 + \frac{\nu}{C'}\left(A'C' - \frac{(1+P)^2}{4P}B'^2\right)|\nabla u'|^2\right]\mathrm{d}\mathscr{V} \\ &\quad - \iint \nu(|\nabla v'|^2 + |\nabla w'|^2)\,\mathrm{d}\mathscr{V}.\end{aligned} \tag{3.18}$$

This is of the same form as equation (2.17) for the parallel shear flow on the f-plane. With (3.11) we find that $A'C' - B'^2 = \beta D'$. Hence if $A'C' > (1+P)^2 B'^2/4P$ then automatically $D' > 0$ because $(1+P)^2/4P \geqslant 1$ for positive P. If also $C' > 0$, E is positive definite and at all times $\mathrm{d}E/\mathrm{d}t \leqslant 0$. In summary if $A'C' > (1+P)^2 B'^2/4P$, or

$$\left(\beta + U_0/L_h^2\right)\left[\beta(N^2 - \gamma U_z) + \gamma^2 U_0/L_h^2\right] > \frac{(1+P)^2}{4P}\left(\beta U_z - \gamma U_0/L_h^2\right)^2 \tag{3.19}$$

and $\beta + U_0/L_h^2 > 0$, the flow is stable.

If $U_0 = 0$ (no horizontal shear), (3.19) reduces to

$$N^2 - \gamma U_z > \frac{(1+P)^2}{4P}U_z^2.$$

If $\gamma = 0$ (traditional β-plane dynamics), this coincides with (2.18) if we set $U_y = 0$ there since then $\Phi = f^2$ so that f^2 cancels out. With $\gamma \neq 0$, the sign of the vertical shear affects the criterion, unlike in the parallel shear flow case on the f-plane. But, in view of the smallness of $\gamma \approx 1.5 \times 10^{-4}\,\mathrm{s}^{-1}$, the correction is negligible. If we retain γ and let the vertical shear vanish ($U_z = 0$), the condition for stability (3.19) can be rewritten as

$$\beta\left[\beta N^2 + (\gamma^2 + N^2)U_0/L_h^2\right] > \frac{(1-P)^2}{4P}\gamma^2\left(U_0/L_h^2\right)^2. \tag{3.20}$$

This suggests that when $\gamma \neq 0$, the horizontally sheared flow $U = U_0(1 - \frac{1}{2}y^2/L_h^2)$ may be unstable if the left-hand side is smaller than the right-hand side, even if $U_0 > 0$, which would always be stable when $\nu = \kappa = 0$. But, it is a mere curiosity in view of the smallness of $\beta \approx 2.3 \times 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}$. If it is accepted that changing '$>$' to '$<$' in (3.20) would imply instability, then in view of the magnitudes of γ and β, the dominant term on the left-hand side is the term $\beta N^2 U_0/L_h^2$ since generally one has $\gamma^2 \approx 10^{-8}\,\mathrm{s}^{-2} \ll N^2$ in the oceans. With a Prandtl number of 7, say, the term $(1-P)^2/4P \approx 1.3$ and instability would only be possible if (approximately) $\beta N^2 < \gamma^2 U_0/L_h^2$. With a typical large oceanic value, $N^2 = 10^{-4}\,\mathrm{s}^{-2}$, this would be accomplished with $U_0 > 10\,\mathrm{cm\,s^{-1}}$ and $L_h = 1\,\mathrm{km}$, which corresponds to extremely high shear. On the other hand if $N \approx \gamma$ (very weak stratification), then $U_0 > 1\,\mathrm{cm\,s^{-1}}$ and $L_h \approx 30\,\mathrm{km}$ may be sufficient. But still, since the profile is an approximation for small y of $U = U_0 \exp(-\frac{1}{2}y^2/L_h^2)$, this corresponds to a very narrow Gaussian flow at the equator. These extreme values are not needed if the Prandtl number is very large or very small, but this does not apply to terrestrial oceanic or atmospheric flows.

4. Summary

We have derived sufficient conditions for linear stability of parallel shear flows subjected to zonally invariant perturbations. With elementary manipulations of the linear perturbation equations we derived what we call the effective energy E, which defines a norm for the perturbations when the classical conditions for stability are met. The essential step in both § 2 and § 3 was to utilize the presumed invariance of the perturbations in the along-flow (x) direction in conjunction with the buoyancy equation. This allowed us to express the meridional perturbation velocities v', w' in terms of u' and the buoyancy b (equations (2.9) and (3.6), respectively). When substituted in the equation for the meridional kinetic energy $\frac{1}{2}(v'^2 + w'^2)$, in each case this leads to an expression for the effective energy E, (2.16) and (3.12), which is positive definite when the classical condition for stability is satisfied. That is, if D defined in (2.10) and (3.7) is positive, plus either $N^2 > 0$ or $\Phi > 0$. In either case, linear stability follows for arbitrary flows that satisfy these conditions in the absence of viscous and diffusive effects. We find that this method also works for, for example, baroclinic circular vortices on the f-plane (for perturbations that are also circularly symmetric) or parallel flows on the mid-latitude β-plane (subjected to zonally invariant perturbations), but this will not be shown here.

The conditions for stability incorporating viscosity and diffusion were obtained for flows with no-slip boundary conditions (if boundaries are present), which were assumed thermally conducting but can be of arbitrary shape in the meridional plane. We essentially re-derived McIntyre's (1970) stability boundary (2.18), but it is more general in that it is sufficient also when normal-modes cannot possibly exist. The extension to the equatorial β-plane (3.19) is novel we believe. Both criteria were derived for the simplest possible flows that allowed us to make the crucial step from (2.13) to (2.14) in § 2 and from (3.13) to (3.14) in § 3. That is, since all coefficients multiplying the perturbative quantities were constant, we could convert terms such as $(A/D)\nu u'\nabla^2 u'$ by integration over the domain into $-(A/D)\nu|\nabla u'|^2$ using the boundary conditions (2.15) because A and D were constant. In the two examples discussed in this paper, the positive-definite effective energy E generally diminishes with time if (2.18) and (3.19) are satisfied and $N^2 > 0$ or $\Phi > 0$.

An open question is whether for example (2.18) is more generally valid. That is, could it be a sharp stability boundary for when U_z, U_y and N^2 are variable? The same question can be asked with regard to arbitrary parallel flows on the equatorial β-plane. Equation (3.19) suggests that in that case the equivalent condition for stability might be

$$(\beta y - U_y)[\beta y(N^2 - \gamma U_z) - \gamma^2 U_y] > \frac{(1+P)^2}{4P}(\beta y U_z + \gamma U_y)^2, \tag{4.1}$$

plus the additional classical condition that $\Phi = \beta y(\beta y - U_y) > 0$. If we follow the same strategy as used in this paper, in order to obtain a time-evolution equation for the positive definite E, such as (2.17) and (3.18), with a negative-definite right-hand side, we will obtain terms such as $-\nu\nabla(Au'/D)\cdot\nabla u'$ there. For variable A/D this would have to be judiciously estimated in conjunction with the additional terms involving the buoyancy b. If this can be accomplished, the stability boundaries (if they exist) most likely will become somewhat 'fuzzy', involving, say, maxima or minima attained by $|\nabla(A/D)|$ within the domain considered and other such estimates. This is a rather formidable task we have not ventured to undertake.

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