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LINEAR TIME IN HYPERSEQUENT FRAMEWORK

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Abstract. Hypersequent calculus (HC), developed by A. Avron, is one of the most interesting proof systems suitable for nonclassical logics. Although HC has rather simple form, it increases significantly the expressive power of standard sequent calculi (SC). In particular, HC proved to be very useful in the field of proof theory of various nonclassical logics. It may seem surprising that it was not applied to temporal logics so far. In what follows, we discuss different approaches to formalization of logics of linear frames and provide a cut-free HC formalization of **Kt4.3**, the minimal temporal logic of linear frames, and some of its extensions. The novelty of our approach is that hypersequents are defined not as finite (multi)sets but as finite lists of ordinary sequents. Such a solution allows both linearity of time flow, and symmetry of past and future, to be incorporated by means of six temporal rules (three for future-necessity and three dual rules for past-necessity). Extensions of the basic calculus with simple structural rules cover logics of serial and dense frames. Completeness is proved by Schütte/Hintikka-style argument using models built from saturated hypersequents.

§1. Introduction. G. Pottinger [41], and independently, A. Avron [1] introduced a generalised form of Gentzen's sequent calculus (SC) called hypersequent calculus (HC). The main feature of HC is the application of hypersequents which are (multi)sets of ordinary sequents. It may be of interest that at the same time Došen [15] proposed even more general framework where one is dealing with a hierarchy of sequents of order $n + 1$ with arguments being finite sets of sequents of order n. In particular, sequents of order 2 consist of finite sets of ordinary sequents (of order 1) on both sides, where elements of the antecedent are treated conjunctively, and elements of the succedent disjunctively. Thus one can treat hypersequents as sequents of order 2 with empty antecedents. Similar ideas based on embedding of sequents inside other sequents were later exploited by Kashima [28], Stouppa [46], Brünnler [10], Poggiolesi [40], and one may notice that HC may be interpreted as a restricted version of such approaches.

HC may be seen also as a special kind of display calculus (DC) introduced by Belnap [9] (see also [14]). While in DC a family of structural connectives of fixed arity is introduced, in HC a separator of sequents may be treated as the only added structural connective of nonfixed arity. One may find results concerning embedding of HC in DC in Wansing [50] and Ramanayake [42].

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It is also easy to establish similar relationship of HC to much stronger framework of labelled sequent calculi (see Negri [36]).

The above considerations seem to show that HC is rather a weak generalization of standard sequent calculus. But even this simple and straightforward modification significantly increases the expressive power of ordinary Gentzen apparatus by allowing an additional transfer of information between different sequents. It proved to be very useful for construction of cut-free formalizations of many nonclassical logics including many-valued, relevant, paraconsistent, and fuzzy logics (see for example Avron [2, 3] Baaz, Ciabattoni, and Fermüller [6], Metcalfe, Olivetti, and Gabbay [35]).

Although HC was originally introduced for modal logics (Pottinger [41]), after more than 30 years one can find rather limited applications of hypersequent calculi in this field. In fact, there are surprisingly many different cut-free systems for **S5** (Pottinger [41], Avron [2], Restall [44], Poggiolesi [39], Lahav [32], Kurokawa [31], Bednarska, and Indrzejczak [8]) but for other modal logics the situation is worse. One can find case studies of some logics of linear frames; there are HC for **S4.3** (Indrzejczak [25], Kurokawa [31]), later generalised to **K4.3**, and **KD4.3** (Indrzejczak [27]). Kurokawa [31] provided also HC for **K4.2**. Recently some more general approaches were provided. Lahav [32] proposed a uniform treatment of various normal modal logics based on translation of semantic conditions. Some general approach of a different character is developed by Lellmann [33]. However, temporal logics with Priorean operators were not formalised in HC setting so far.

In what follows, we introduce a variant of HC for the minimal linear time logic **Kt4.3** and some of its extensions. It is based on a different idea than HC formalizations of monomodal linear logics from Indrzejczak [25, 27], Kurokawa [31], or Lahav [32]. The solution applied in these systems was essentialy based on Avron's [2,4] HC for Gödel–Dummett's logic, the logic of intuitionistic relational frames with linear accessibility relation. It works well for these logics but it is not flexible enough to cover the symmetry of future and past operators. For instance, Lahav's HC systems for symmetric modal logics require some kind of analytic cut. To provide a cut-free HC system for logics of linear time we decided to treat hypersequents not as (multi)sets but as finite lists of sequents[1]. A similar idea of using noncommutative hypersequents was already exploited by Ciabattoni and Ferrari [12] in the framework of intermediate logics. In the context of temporal logics such a solution naturally corresponds to linear order of time points. As a result we get a syntactic modelling of both forward and backward transmission of data on the time axis by means of suitably defined rules for temporal operators.

The basic cut-free calculus **HCKt4.3**, and its simple extensions with additional structural rules are investigated rather semantically in this paper. Thus, we show the completeness by means of Schütte/Hintikka's style

[1]The first version of this system was presented in Indrzejczak [26].

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argument showing how to construct a countermodel for any unprovable hypersequent. However, the rules for temporal operators are defined with care for having good syntactic properties. An open question is the existence of syntactical proof of cut admissibility for it. The problem is that strategies of proof developed so far for other forms of HC are not suitable for dealing with hypersequents defined as lists of sequents.

§**2. Temporal logic.** This section is mainly for reference and to establish notation; good introductions to the field are provided by Rescher and Urquhart [43], Goldblatt [18], or van Benthem [49]. Let us first recall a few basic facts concerning the minimal temporal logic of linear frames in standard characterization, i.e., as an axiomatic system adequate with respect to suitable class of relational (Kripke) frames. We will use standard bimodal language with countable set of propositional variables VAR, ordinary boolean constants and two Priorean unary temporal operators: $\Box^F$ (always in the future) and $\Box^P$ (always in the past). Dual operators $\Diamond^F$ and $\Diamond^P$ are treated as definitional shortcuts. One can axiomatize **Kt4.3** by adding to any Hilbert system for classical propositional logic the following schemata:

KF $\Box^F(\varphi \rightarrow \psi) \rightarrow (\Box^F \varphi \rightarrow \Box^F \psi)$
KP $\Box^P(\varphi \rightarrow \psi) \rightarrow (\Box^P \varphi \rightarrow \Box^P \psi)$
PF $\varphi \rightarrow \Box^P \Diamond^F \varphi$
FP $\varphi \rightarrow \Box^F \Diamond^P \varphi$
4 $\Box^F \varphi \rightarrow \Box^F \Box^F \varphi$
LF $\Box^P \varphi \wedge \Box^F \varphi \wedge \varphi \rightarrow \Box^F \Box^P \varphi$
LP $\Box^P \varphi \wedge \Box^F \varphi \wedge \varphi \rightarrow \Box^P \Box^F \varphi$

The system is closed under MP (modus ponens) and two rules of necessitation:

$\vdash \varphi \;/\; \vdash \Box^F \varphi$
$\vdash \varphi \;/\; \vdash \Box^P \varphi$

Kt4.3 is adequate with respect to the class of relational frames $\langle T, R\rangle$ where acessibility relation R on the nonempty set of time points T is transitive and satisfies the conditions of future and past linearity (or connectedness):

$$\forall t, t', t''(Rtt' \wedge Rtt'' \rightarrow Rt't'' \vee Rt''t' \vee t' = t''),$$
$$\forall t, t', t''(Rt't \wedge Rt''t \rightarrow Rt't'' \vee Rt''t' \vee t' = t'').$$

It may be shown that **Kt4.3** is also characterised by the narrower class of linear structures, where instead of these two conditions it holds the condition of trichotomy[2]:

$$\forall t, t'(Rtt' \vee Rt't \vee t = t').$$

[2]In fact, no axioms in standard temporal language correspond to conditions of trichotomy and dichotomy; see Goldblatt [18] for details. A possible remedy for this problem is provided by the application of stronger languages, for instance those of hybrid temporal logics; see e.g. [24]

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We will prove directly the adequacy of our hypersequent calculi with respect to these narrower class of frames.

Models $\mathfrak{M}$ are built on frames by the addition of valuation function $v : VAR \longrightarrow \mathcal{P}(T)$. Formulae are evaluated at the points of the model in the standard way by means of recursively defined satisfaction relation $\models$. In particular, the clauses for temporal operators are the following:

$\mathfrak{M}, t \models \Box^F \varphi$ iff $\mathfrak{M}, t' \models \varphi$ for every t' such that Rtt'
$\mathfrak{M}, t \models \Box^P \varphi$ iff $\mathfrak{M}, t' \models \varphi$ for every t' such that $Rt't$
$\mathfrak{M}, t \models \Diamond^F \varphi$ iff $\mathfrak{M}, t' \models \varphi$ for some t' such that Rtt'
$\mathfrak{M}, t \models \Diamond^P \varphi$ iff $\mathfrak{M}, t' \models \varphi$ for some t' such that $Rt't$

A formula φ is **Kt4.3**-valid ($\models \varphi$) iff it is satisfied at every point of every **Kt4.3**-Model. A formula is falsifiable if it is nonvalid, i.e., $\mathfrak{M}, t \not\models \varphi$ for some $\mathfrak{M}$ and t.

We consider also some extensions of **Kt4.3** which are obtained by the addition of any of the following axioms:

FS $\Box^F \varphi \rightarrow \Diamond^F \varphi$
PS $\Box^P \varphi \rightarrow \Diamond^P \varphi$
D $\Box^F \Box^F \varphi \rightarrow \Box^F \varphi$

which correspond to conditions of future seriality (no ending point), past seriality (no starting point), and density of R, i.e.,:

FS $\forall t, \exists t' Rtt'$
PS $\forall t, \exists t' Rt't$
D $\forall t, t'(Rtt' \rightarrow \exists t''(Rtt'' \wedge Rt''t'))$

Since the conditions are independent we obtain 7 different extensions of **Kt4.3** determined by classes of frames with R satisfying at least one of the above conditions.

We provide a formalization of all these logics in HC setting. In this paper, we define hypersequents as finite lists of Gentzen's sequents and apply the following notation:

- $\Gamma \Rightarrow \Delta$ or s (usually with subscripts) for sequents; note that Γ, Δ are finite, possibly empty, multisets of formulae.
- $\wedge\Gamma$ ($\vee\Gamma$) for the conjunction (disjunction) of all elements of Γ.
- $s_1 \mid \cdots \mid s_n$, G, H, G_i, H_i stand for hypersequents; in particular G, G_i will always denote nonempty hypersequent, whereas H, H_i will be used if we admit that it is empty.
- $H_1 \mid s \mid H_2$ (or $H_1 \mid \Gamma \Rightarrow \Delta \mid H_2$) stand for hypersequents with displayed sequent s (or $\Gamma \Rightarrow \Delta$).
- $s_i < s_j$ means that the occurrence of a sequent s_i is on the left of the occurrence of s_j in some hypersequent, i.e., a hypersequent is of the form $H_1 \mid s_i \mid \cdots \mid s_j \mid H_2$.

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- $\Gamma \Rightarrow \Delta \subseteq \Gamma' \Rightarrow \Delta'$ (or $s \subseteq s'$) iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.
- $G \subseteq G'$ iff for every occurrence of s in G there is a corresponding occurrence of s' in G' such that $s \subseteq s'$ and whenever $s_i < s_j$ (or $s_j < s_i$) in G, then also $s'_i < s'_j$ (or $s'_j < s'_i$) in G'.

For simplicity we have used in all cases the set-theoretic symbol $\subseteq$ despite the differences in meaning (multisets-, sequent-, and hypersequent-inclusion). In two last cases we say that s' (resp. G') is an extension of s (resp. G) and that s (resp. G) is a reduction of s' (resp. G'). Note that in case $G \subseteq G'$, G' may contain additional occurrences of sequents which do not extend any sequent occurring in G.

§3. **How to deal with linearity.** So far modal and temporal logics of linear frames were formalised in the framework of various calculi like natural deduction, tableaux e.t.c.; also sequent calculi of different sorts were used for this aim. Very often the proposed solutions are based on the use of several forms of labelling which encode selected elements of relational semantics[3]. Before we present our solution, a brief survey of proposed approaches may appear helpful. Note however that most of the proposals are suitable only for monomodal logics like **S4.3** and do not admit resources necessary for dealing with symmetry inherent in temporal logics. Note also that in case of reflexive R (like in **S4.3**) conditions of connectedness and trichotomy may be replaced by their stronger versions with deleted identity of states in disjunction (hence trichotomy is changed into dichotomy). All the systems for linear modal logics may be divided roughly according to two criteria: (a) the shape of the rules and (b) the implicit strategy of linearization (of attempted falsifying model).

The rules devised to express suitable conditions of connectedness, dichotomy or trichotomy may be divided into nonbranching and branching. The former solution is rather rare, one can mention here a labelled tableau system of Marx, Mikulas, and Reynolds [34] and of Baldoni [7], as well as a labelled natural deduction system of Indrzejczak [23]. Since construction of nonbranching rules for (strong) connectedness is based on the following form of this condition:

$$\forall xyz(Rxy \wedge Rxz \wedge \neg Ryz \rightarrow Rzy),$$

one must have in the system sufficient resources not only for expressing that some points are in the relation R but also that some are not, i.e. either the apparatus of labels must be sufficiently strong (like in Baldoni's system) or some forms of (analytic) cut must be involved (in the remaining proposals). Surprisingly enough, in case of richer language of bimodal temporal logics of linear frames one can define suitable nonbranching rules for natural deduction system even without using labels (see Indrzejczak [22]).

[3]See the 9th chapter of Indrzejczak [24] for a comprehensive survey of different approaches; Goré [19] provides a description of standard nonlabelled tableau formalizations of respective logics.

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Most solutions are based on the application of branching rules. In some systems the number of branches is fixed—two in case of strong connectedness or dichotomy, and three in case of weak connectedness (trichotomy). This is obvious since suitable rules are directly modelled on semantic conditions involving disjunction. One may mention here, e.g., a display calculus of Wansing [50], labelled sequent calculus of Negri [36], or nested tableau system of Kashima [28]. This approach is in the most straightforward way realised in Negri's calculus since there we have a direct representation of frame conditions in the syntax by means of relational formulae. Consider, for example a rule for strong connectedness. This condition is a special case of a universal implication of the form $\forall x_1 \dots x_i(\varphi_1 \wedge \dots \wedge \varphi_k \rightarrow \psi_1 \vee \dots \vee \psi_n)$, with all φ_i and ψ_j being atoms. Since every universal implication is characterised by means of the following rule-schema:

$$\frac{\psi_1, \varphi_1, \dots, \varphi_k, \Gamma \Rightarrow \Delta, \dots, \psi_n, \varphi_1, \dots, \varphi_k, \Gamma \Rightarrow \Delta}{\varphi_1, \dots, \varphi_k, , \Gamma \Rightarrow \Delta},$$

then for strong connectedness we obtain:

$$\frac{yRz, xRy, xRz, \Gamma \Rightarrow \Delta \quad zRy, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}.$$

There is no place for demonstrating here in what ways this strategy is realised in the framework of display calculus or nested system of Kashima since this would require the presentation of some technicalities of these approaches first. The strategy of generating such rules in an uniform fashion in display calculus is discussed at length in Kracht [30] and Ciabattoni and Ramanayake [13]. Kashima's solution is closest to that applied for HC which will be discussed below. Incidentally, it is worth noting that display calculus is a very natural framework for the development of bimodal temporal logics because it directly generates the dualities between future and past operators.

Both nonbranching and fixed-branching approaches realize the local strategy of linearization of attempted falsifying model. It means that we compare only two states at a time and put them in some order, either disjunctively (by means of branching rules) or by choosing one possibility if the other ones are excluded (nonbranching rules). However, there are formal systems for linear logics which realize global strategy of linearization, i.e., their rules are defined in such a way that all points which are generated at the same moment are immediately put in the sequence. This group consists of systems which use rules with the number of branches not fixed in advance. It depends on the number of modal formulae which are responsible for creation of new points in the attempted model. One may distinguish here between solutions which make it in the decreasing way (Zeman's sequent calculus [51], Goré tableau systems [19]) or increasing way (Rescher and Urquhart's tableau system for linear temporal logics [43]). In the former approach one is activating all modal formulae which create new states in one step (see below the schema of Zeman's rule). In the latter, one is activating only one modal formula but generates all possible ways of locating the new state in an attempted model. In both approaches, we eventually create all possible sequences of points (but in a different order) hence the number of branches is exponentially dependent on the number of state's creating modal formulae.

Zeman's approach to linearity, developed by Góre, is of particular interest from the standpoint of proof-search due to the the property of confluency. In case of sequent calculus, it means that if a sequent s is provable, then any proof-tree with this sequent as the root may be extended in such a way that we obtain a proof of s. Of course, we talk about the application of rules in the root-first (or backwards) manner characteristic for the process of proof-search. In practice it means that no matter which choices we have made during proof-search, we finally obtain a proof, if the input is provable. On the other hand, if some branch ends with atomic but not axiomatic sequent, we are done negatively, we know for sure that there is no proof. That is why confluent systems are very convenient from the point of view of automated theorem proving. We are not forced at some stage of proof-search for backtracking to earlier stages, if we made 'wrong' choices. Consequently, confluent systems are less expensive for the program memory. From the standpoint of the result, in confluent systems there are no wrong choices (although our choices may have strong influence on the length of obtained proof or the time needed for performance).

Unfortunately, it is a well known fact that even cut-free standard sequent calculi for modal logics usually are not confluent. It is connected with the fact that before we apply (backwards) suitable rule for the introduction of $\Box$ in the succedent we must usually make a choice which formula with $\Box$ from the succedent is being dealt with; the remaining ones are deleted from the succedent of the premiss. Wrong choices may lead to a failure even if the input-sequent is provable, so we must try with other choices. On the other hand, to make a falsifying model, we must use all failed proof-trees constructed during the proof-search. Several techniques were provided for dealing with such an inconveniency[4]; in particular, Zeman's solution is very natural. In his rule for introduction of $\Box$ in the succedent, all $\Box$-formulae are activated at the same time, each one leading to one premiss and with other $\Box$-formulae still present in the succedent. Here is the scheme for **S4.3**:

$$(\Rightarrow\Box)\ \frac{\Gamma^{\Box} \Rightarrow \varphi_1, \Box\varphi_2, \ldots, \Box\varphi_n \qquad \ldots \qquad \Gamma^{\Box} \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_{n-1}, \varphi_n}{\Gamma \Rightarrow \Delta, \Box\varphi_1, \ldots, \Box\varphi_n},$$

where $\Gamma^{\Box} = \{\Box\varphi : \Box\varphi \in \Gamma\}$ and Δ has no $\Box$-formulae. Clearly in case $n = 1$ we get a standard right introduction rule for $\Box$ in **S4**.

One should notice here the unquestioned advantage of HC over ordinary SC in this respect. In the context of HC, the problem of confluency finds a very natural solution also for other modal logics needing a special treatment in the setting of SC. The apparatus of hypersequents allows all $\Box$-formulae

[4] See, e.g. chapter 7 of Indrzejczak [24] for more details.

to be activated at the same time in a parallel fashion for any modal logic. For example, one can provide for the basic normal modal logic **K** the following rule for introduction of $\Box$ in the succedent:

$$(\Rightarrow\Box)\ \ \frac{\Gamma^{\Box} \Rightarrow \varphi_1 \mid \ \dots \ \mid \Gamma^{\Box} \Rightarrow \varphi_n \mid H}{\Gamma \Rightarrow \Delta, \Box\varphi_1, \dots, \Box\varphi_n \mid H},$$

where $\Gamma^{\Box} = \{\varphi : \Box\varphi \in \Gamma\}$ and Δ has no $\Box$-formulae.

This nice feature of HC was unnoticed before Indrzejczak [25] and usually more standard-looking rules of $\Box$ introduction into succedent were provided. In fact, due to external structural rule of contraction, one generally does not loose confluency in HC, but our format of $\Box$-introduction rule makes it explicit and directly introduces this property into the system. However, the main point of [25] is that we provided a characterization of monomodal linear logic **S4.3** (later in [27] extended to weaker monomodal linear logics **K4.3** and **KD4.3**) in terms of HC which closely follows the lines of Avron's [4] solution for Gödel–Dummett's logic. In terms of described solutions these calculi should be characterised based on fixed-branching rules realizing local strategy of searching for linear model. In fact, this solution for linearization of states is similar to that of Kashima [28] but realised by means of hypersequents instead of nested tableaux or some kind of labels or structural constants. The characteristic rule for dichotomy in HC for **S4.3** was:

$$(\texttt{Dich})\ \ \frac{\Gamma, \Pi^{\Box} \Rightarrow \Delta \mid H \qquad \Pi, \Gamma^{\Box} \Rightarrow \Sigma \mid H}{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H},$$

where $\Gamma^{\Box} = \{\Box\varphi : \Box\varphi \in \Gamma\}$

whereas for trichotomy (in **K4.3** and **KD4.3**) it was:

$$(\texttt{Trich})\ \ \frac{\Gamma, \Pi^{\boxplus} \Rightarrow \Delta \mid H \qquad \Pi, \Gamma^{\boxplus} \Rightarrow \Sigma \mid H \qquad \Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H}{\Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma \mid H},$$

where $\Gamma^{\boxplus} = \{\Box\varphi : \Box\varphi \in \Gamma\} \cup \{\varphi : \Box\varphi \in \Gamma\}$.

Unfortunately this solution does not work for temporal logics of linear frames where additionally the symmetry of past and future must be dealt with. In order to provide a solution in the framework of hypersequents, it is useful to introduce hypersequents being neither sets nor multisets of sequents but rather their finite lists. This approach will be developed below.

§4. **Hypersequent calculus.** The calculus for **Kt4.3** consists of the following rules:

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(AX) $H_1 \mid \varphi, \Gamma \Rightarrow \Delta, \varphi \mid H_2$

$$(C\Rightarrow)\ \frac{H_1 \mid \varphi, \varphi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid H_2} \qquad (\Rightarrow C)\ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi, \varphi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2}$$

$$(\neg\Rightarrow)\ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2}{H_1 \mid \neg\varphi, \Gamma \Rightarrow \Delta \mid H_2} \qquad (\Rightarrow\neg)\ \frac{H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \neg\varphi \mid H_2}$$

$$(\wedge\Rightarrow)\ \frac{H_1 \mid \varphi, \psi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \varphi \wedge \psi, \Gamma \Rightarrow \Delta \mid H_2} \qquad (\Rightarrow\wedge)\ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2 \qquad H_1 \mid \Gamma \Rightarrow \Delta, \psi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \wedge \psi \mid H_2}$$

$$(\Rightarrow\vee)\ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi, \psi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \vee \psi \mid H_2} \qquad (\vee\Rightarrow)\ \frac{H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid H_2 \qquad H_1 \mid \psi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \varphi \vee \psi, \Gamma \Rightarrow \Delta \mid H_2}$$

$$(\Rightarrow\rightarrow)\ \frac{H_1 \mid \varphi, \Gamma \Rightarrow \Delta, \psi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \rightarrow \psi \mid H_2} \qquad (\rightarrow\Rightarrow)\ \frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2 \qquad H_1 \mid \psi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta \mid H_2}$$

$$(\Rightarrow\Box^F)\ \frac{H \mid \Gamma \Rightarrow \Delta \mid\Rightarrow \varphi}{H \mid \Gamma \Rightarrow \Delta, \Box^F\varphi} \qquad (\Box^F\Rightarrow)\ \frac{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \Pi \Rightarrow \Sigma \mid H_2}$$

$$(\Rightarrow\Box^P)\ \frac{\Rightarrow \varphi \mid \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \Box^P\varphi \mid H} \qquad (\Box^P\Rightarrow)\ \frac{H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \Box^P\varphi, \Pi \Rightarrow \Sigma \mid H_2}$$

$$(\Rightarrow\Box^{F\prime})\ \frac{H_1 \mid \Gamma \Rightarrow \Delta \mid\Rightarrow \varphi \mid \Pi \Rightarrow \Sigma \mid H_2 \qquad H_1 \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \varphi \mid H_2 \qquad H_1 \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \Box^F\varphi \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta, \Box^F\varphi \mid \Pi \Rightarrow \Sigma \mid H_2}$$

$$(\Rightarrow\Box^{P\prime})\ \frac{H_1 \mid \Gamma \Rightarrow \Delta \mid\Rightarrow \varphi \mid \Pi \Rightarrow \Sigma \mid H_2 \qquad H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid \Pi \Rightarrow \Sigma \mid H_2 \qquad H_1 \mid \Gamma \Rightarrow \Delta, \Box^P\varphi \mid \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \Box^P\varphi \mid H_2}$$

Note that:

1. All rules satisfy the subformula property.
2. All rules, except $(\Box^F\Rightarrow)$ and $(\Box^P\Rightarrow)$, are invertible in the semantic sense, i.e., if the conclusion of the rule is valid, then all premisses are also valid (in the sense specified in Definition 5.1 below).
3. Defining hypersequents as finite lists of sequents allows linear time to be syntactically representable in a direct way. Informally, every sequent corresponds to some point on the time axis and if $H_1 \mid s_i \mid \cdots \mid s_j \mid H_2$, then the time point represented by s_i is earlier than that represented by s_j.
4. One needs two rules of right introduction for each temporal operator. $(\Rightarrow\Box^F)$ works when $\Box^F\varphi$ is in the succedent of the rightmost sequent, whereas $(\Rightarrow\Box^{F\prime})$ is applied if there are some other sequents on the right (for $\Box^P$ we have dual rules). The former rule corresponds to the situation where $\Box^F\varphi$ is present in the last time point t in the list and all that we need is to introduce the next time point t' in which this formula is fulfilled. The latter corresponds to the situation where the time point t has some successors, at least one t'. Due to the linearity, $\Box^F\varphi$ must be fulfilled either in some point t'' which is between t and t' or it is fulfilled in t' or in some later points. All these eventualities are represented by three premisses. In fact, these two rules (i.e., $(\Rightarrow\Box^{F\prime})$ and $(\Rightarrow\Box^{P\prime})$) have recursive character due to the shape of the third premise. $\Box^F\varphi$ is copied to the next (if any) sequent and the application of suitable rule

to it is postponed. In terms of described solutions this calculus is also based on fixed-branching rules realizing the local strategy of searching for falsifying linear model. Similar solution for modal logics of linear frames was applied in labelled tableau system of Catach [11].

5. Although the idea underlying the rules of the present system is different from that of [25] it also provides a confluent system.

The proof of a hypersequent G is defined in the usual way as a tree of hypersequents with G as the root, axioms as leaves and all other nodes regulated by the application of rules. Here is the proof of the instance of LF as an example.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box^P p, p \Rightarrow \mid p \Rightarrow p \mid \Rightarrow}{\Box^P p, \Box^F p, p \Rightarrow \mid \Rightarrow p \mid \Rightarrow}(\Box^F \Rightarrow) \qquad \Box^P p, \Box^F p, p \Rightarrow p \mid \Rightarrow \qquad \Box^P p, \Box^F p, p \Rightarrow \Box^P p \mid \Rightarrow}{\Box^P p, \Box^F p, p \Rightarrow \mid \Rightarrow \Box^P p}(\Rightarrow \Box^{P\prime})}{\Box^P p, \Box^F p, p \Rightarrow \Box^F \Box^P p}(\Rightarrow \Box^F)}{\Box^P p \wedge \Box^F p \wedge p \Rightarrow \Box^F \Box^P p}(\wedge \Rightarrow)\text{ twice}}{\Rightarrow \Box^P p \wedge \Box^F p \wedge p \rightarrow \Box^F \Box^P p}(\Rightarrow \rightarrow)$$

.

In what follows we will denote with **HCKt4.3** $\vdash G$, or with $\vdash G$ simply, the provability of G in **HCKt4.3**.

§5. Soundness.

Satisfaction of a sequent at a state in a model is defined as usual by:

$\mathfrak{M}, t \vDash \Gamma \Rightarrow \Delta$ iff $\mathfrak{M}, t \vDash \wedge\Gamma \rightarrow \vee\Delta$,

and

$\mathfrak{M}, t \nvDash \Gamma \Rightarrow \Delta$ iff $\mathfrak{M}, t \nvDash \wedge\Gamma \rightarrow \vee\Delta$.

We extend semantical notions to noncommutative hypersequents in the following way:

Definition 5.1. For any **Kt4.3**-model $\mathfrak{M}$ and hypersequent $G = s_1 \mid \cdots \mid s_n$:

- $\mathfrak{M} \models G$ iff for all states $t_1, \ldots t_n$ of $\mathfrak{M}$:
  if $t_1 R t_2 R \ldots R t_n$, then for some $i \leq n, \mathfrak{M}, t_i \vDash s_i$;
- $\models G$ (G is **Kt4.3**-valid) iff $\mathfrak{M} \models G$ for every $\mathfrak{M}$.

Note that in consequence: $\not\models G$ iff there is $\mathfrak{M}$ such that $\mathfrak{M} \not\models G$ and this means that there are $t_1, \ldots t_n$ such that $t_1 R t_2 R \ldots R t_n$ and $t_1 \nvDash s_1, \ldots t_n \nvDash s_n$.

One can easily prove:

Lemma 5.2 (Validity-preservation). *All rules of* **HCKt4.3** *are validity-preserving.*

Proof. We will show by contraposition two cases: $(\Box^F \Rightarrow)$ and $(\Rightarrow \Box^{F\prime})$.

$(\Box^F \Rightarrow)$. Assume that $\not\models H_1 \mid \Box^F \varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \Pi \Rightarrow \Sigma \mid H_2$. Hence there is a model $\mathfrak{M}$ such that $\mathfrak{M}, t \nvDash \Box^F \varphi, \Gamma \Rightarrow \Delta$ and $\mathfrak{M}, t' \nvDash \Pi \Rightarrow \Sigma$, for

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some t, t' such that Rtt'. In particular, $\mathfrak{M}, t \vDash \Box^F\varphi$, hence by transitivity $\mathfrak{M}, t' \vDash \varphi$ which means that this model falsifies also $H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \varphi, \Pi \Rightarrow \Sigma \mid H_2$.

$(\Rightarrow\Box^{F'})$. Assume that $\nvDash H_1 \mid \Gamma \Rightarrow \Delta, \Box^F\varphi \mid \Pi \Rightarrow \Sigma \mid H_2$. Hence there is a model $\mathfrak{M}$ such that $\mathfrak{M}, t \nvDash \Gamma \Rightarrow \Delta, \Box^F\varphi$ and $\mathfrak{M}, t' \nvDash \Pi \Rightarrow \Sigma$, for some t, t' such that Rtt'. In particular, $\mathfrak{M}, t \nvDash \Box^F\varphi$ which means that there is some t'' such that Rtt'' and $\mathfrak{M}, t'' \nvDash \varphi$. Future linearity implies $Rt't'' \vee Rt''t' \vee t' = t''$. If we take the first case, then $\mathfrak{M}, t' \nvDash \Box^F\varphi$ and the third premiss, i.e. $H_1 \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \Box^F\varphi \mid H_2$ is falsified by this model. If we take the second choice, then we can assign t'' to $\Rightarrow \varphi$. This way we obtain an augmented model with t'' inserted between t and t' which falsifies the first premiss, i.e. $H_1 \mid \Gamma \Rightarrow \Delta \mid\Rightarrow \varphi \mid \Pi \Rightarrow \Sigma \mid H_2$. The last choice falsifies the second premiss. i.e. $H_1 \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \varphi \mid H_2$. Hence if the conclusion is falsifiable, then at least one of the premisses must be falsifiable. ⊣

As a consequence we obtain

THEOREM 5.3 (Soundness). *If* **HCKt4.3** $\vdash G$, *then* $\vDash G$.

Now we can also substantiate the claim on invertibility of rules made in Section 4.

LEMMA 5.4 (Invertibility). *All logical rules of* **HCKt4.3** *except* $(\Box^F\Rightarrow)$ *and* $(\Box^P\Rightarrow)$ *are semantically invertible.*

PROOF. The proof is obvious for rules characterising boolean connectives. So we provide a proof for $(\Rightarrow\Box^{F'})$ as an illustration.

Assume that the conclusion is valid and that the leftmost premise is not. Hence there is a model $\mathfrak{M}$ such that $\mathfrak{M}, t \nvDash \Gamma \Rightarrow \Delta$, $\mathfrak{M}, t' \nvDash \varphi$, and $\mathfrak{M}, t'' \nvDash \Pi \Rightarrow \Sigma$, for some t, t', t'' such that Rtt' and $Rt't''$. But then the conclusion must be falsifiable as well, because the only way of keeping it valid in this $\mathfrak{M}$ is to have $\mathfrak{M}, t \vDash \Box^F\varphi$. This however contradicts $\mathfrak{M}, t' \nvDash \varphi$ since Rtt'. Similarly we demonstrate the result for the remaining premisses with additional reference to transitivity of R in case of the rightmost premiss. ⊣

§6. Variations and extensions. Let us point out some possible refinements of this calculus and introduce some extensions for stronger logics.

The last section ended with a result on invertibility of rules. Note that one may easily obtain invertibility of all logical rules by changing a bit rules $(\Box^F\Rightarrow)$ and $(\Box^P\Rightarrow)$ into:

$$(\Box^F\Rightarrow') \quad \frac{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Box^F\varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \Pi \Rightarrow \Sigma \mid H_2},$$

$$(\Box^P\Rightarrow') \quad \frac{H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid \cdots \mid \Box^P\varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta \mid \cdots \mid \Box^P\varphi, \Pi \Rightarrow \Sigma \mid H_2}.$$

Both rules are easily derivable by original rules and internal contraction. Since invertibility of rules plays no role in completeness proof from Section 7,

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we prefer the simpler rules with no repetition of principal formulae in the premiss.

One may consider the question of structural rules like contraction or weakening in this framework. In the present calculus, there are two primitive internal contraction rules but no external contraction, and there are no rules of weakening, neither internal nor external. Concerning contraction, the internal rules are necessary for completeness. As for the external contraction it is not even clear what form should have such a rule in this framework. Note however that in principle we could get rid of internal contraction, and provide a fully logical calculus, i.e., with no primitive structural rules. One could instead prove both internal contraction rules as admissible in the calculus where temporal rules are slightly changed. Namely, both $(\Box^F\Rightarrow)$ and $(\Box^P\Rightarrow)$ are replaced with $(\Box^F\Rightarrow')$ and $(\Box^P\Rightarrow')$ introduced above. Similarly, for the remaining temporal rules, we must add a principal formula to the succedents of suitable sequents in all premisses. It is routine to check validity-preservation of these new rules. For a calculus with such rules, one can provide a proof of height-preserving admissibility of internal contraction following the strategy of Dragalin (see e.g. the exposition in Negri and von Plato [38]). However such a proof presupposes a proof of syntactical invertibility of all rules[5], which in turn presupposes admissibility of weakening rules and axioms restricted to atomic formula on both sides (see e.g. Poggiolesi [39] for such a solution in hypersequent calculus for **S5**). Since the lack of contraction as primitive rule may be important for syntactical proofs of cut admissibility—an issue which is not dealt with in this paper—but is not significant for the completeness proof from Section 7, we will omit the details.

On the other hand, the lack of weakening rules, especially external ones, may seem defective. For example, it is not possible to show derivability of both rules of necessitation without introducing an empty sequent by means of (some special form of?) external weakening. Strictly speaking it is not necessary since we show only weak completeness of our calculus, however it is possible to prove that both kinds of weakening are admissible in it. The proof of height-preserving admissibility of internal weakening is trivial. As for the external weakening the proof is more involved and we can prove only its sheer admissibility.

LEMMA 6.1 (Admissibility of external weakening). *If* $\vdash H_1 \mid H_2$, *then* $\vdash H_1 \mid s \mid H_2$, *for any* s *and at least one of* H_1, H_2 *nonempty.*

PROOF. The proof is by induction on the height of the proof of $H_1 \mid H_2$. The problem is only with four temporal rules in the succedent. We consider two of them, for $\Box^F\varphi$; the proof for the remaining two rules is similar.

Let our H_2 be empty and $H_1 = H_1' \mid \Gamma \Rightarrow \Delta, \Box^F\varphi$ obtained by $(\Rightarrow\Box^F)$ from $H_1' \mid \Gamma \Rightarrow \Delta \mid\Rightarrow \varphi$. We must show that the addition of some s to the right is also provable. This may be provable only via $(\Rightarrow\Box^{F\prime})$, so we must

[5]Note that semantical invertibility is not enough, in particular, the original four rules for temporal connectives in succedents are (semantically) invertible, but the syntactical proof of admissibility of contraction does not hold for them.

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provide proofs of three premisses. (1) $H_1' \mid \Gamma \Rightarrow \Delta \mid\Rightarrow \varphi \mid s$ is obtained from the premiss by the induction hypothesis. (2) $H_1' \mid \Gamma \Rightarrow \Delta, \mid s, \varphi$, where s, φ is s with φ added to the succedent, is provable from the premiss by (admissible) internal weakening on $\Rightarrow \varphi$. Finally, from the premiss we get $H_1' \mid \Gamma \Rightarrow \Delta \mid s \mid\Rightarrow \varphi$ by the induction hypothesis, and then by $(\Rightarrow\Box^F)$ we get (3) $H_1' \mid \Gamma \Rightarrow \Delta \mid s, \Box^F\varphi$, where $s, \Box^F\varphi$ is s with $\Box^F\varphi$ added to the succedent. From (1), (2), (3), $H_1' \mid \Gamma \Rightarrow \Delta, \Box^F\varphi \mid s$ follows by $(\Rightarrow\Box^{F\prime})$.

The case of $(\Rightarrow\Box^{F\prime})$ is also problematic when we want to insert an additional sequent immediately after active sequent, i.e., if $H_1 = H_1' \mid \Gamma \Rightarrow \Delta, \Box^F\varphi$ and $H_2 = \Pi \Rightarrow \Sigma \mid H_2'$. Now the premisses are: (1) $H_1' \mid \Gamma \Rightarrow \Delta \mid\Rightarrow \varphi \mid \Pi \Rightarrow \Sigma \mid H_2'$, (2) $H_1' \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \varphi \mid H_2'$ and (3) $H_1' \mid \Gamma \Rightarrow \Delta \mid \Pi \Rightarrow \Sigma, \Box^F\varphi \mid H_2'$. By the induction hypothesis we get (1') $H_1' \mid \Gamma \Rightarrow \Delta \mid\Rightarrow \varphi \mid s \mid \Pi \Rightarrow \Sigma \mid H_2'$ and by internal weakening (2') $H_1' \mid \Gamma \Rightarrow \Delta \mid s, \varphi \mid \Pi \Rightarrow \Sigma \mid H_2'$; both from the first premiss. It remains to prove (3') $H_1' \mid \Gamma \Rightarrow \Delta \mid s, \Box^F\varphi \mid \Pi \Rightarrow \Sigma \mid H_2'$. We again need three premisses for deducing (3') by $(\Rightarrow\Box^{F\prime})$: (1") $H_1' \mid \Gamma \Rightarrow \Delta \mid s \mid\Rightarrow \varphi \mid \Pi \Rightarrow \Sigma \mid H_2'$ is from (1). (2") $H_1' \mid \Gamma \Rightarrow \Delta \mid s \mid \Pi \Rightarrow \Sigma, \varphi \mid H_2'$ is from (2). (3") $H_1' \mid \Gamma \Rightarrow \Delta \mid s \mid \Pi \Rightarrow \Sigma, \Box^F\varphi \mid H_2'$ is from (3); all by the induction hypothesis. ⊣

One can also avoid two different rules for introduction of each temporal connective in the succedent. We could use instead a pair of rules $(\Rightarrow\Box^{F\prime\prime})$ and $(\Rightarrow\Box^{P\prime\prime})$ of the form:

$$\frac{H \mid \Gamma_1 \Rightarrow \Delta_1 \mid\Rightarrow \varphi \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \quad \ldots \quad H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n, \varphi \quad H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n, \mid\Rightarrow \varphi}{H \mid \Gamma_1 \Rightarrow \Delta_1, \Box^F\varphi \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n}$$

$$\frac{\Rightarrow \varphi \mid \Gamma_n \Rightarrow \Delta_n \mid \cdots \mid \Gamma_1 \Rightarrow \Delta_1 \mid H \quad \Gamma_n \Rightarrow \Delta_n, \varphi \mid \cdots \mid \Gamma_1 \Rightarrow \Delta_1 \mid H \quad \ldots \quad \Gamma_n \Rightarrow \Delta_n \mid \cdots \mid\Rightarrow \varphi \mid \Gamma_1 \Rightarrow \Delta_1 \mid H}{\Gamma_n \Rightarrow \Delta_n \mid \cdots \mid \Gamma_1 \Rightarrow \Delta_1, \Box^P\varphi \mid H}.$$

Both rules are validity-preserving and invertible. However, from the standpoint of proof theory the new rules are not so elegant as primitive rules of **HCKt4.3**. In particular, in $(\Rightarrow\Box^{F\prime\prime})$ and $(\Rightarrow\Box^{P\prime\prime})$ the number of premisses depends on the number of sequents to the right, or to the left of the active sequent in the conclusion. Specifically, we generate $2n - 1$ premisses if there is $n - 1$ sequents to the right (or left) of the active sequent. One can easily notice that both rules realize in HC framework, the global strategy of searching for falsifying model due to Rescher and Urquhart (see Section 3).

However, thanks to their global character, the new rules have some advantages. In fact, $(\Rightarrow\Box^{F\prime\prime})$ and $(\Rightarrow\Box^{P\prime\prime})$ are more general in the sense that they cover all respective rules of **HCKt4.3** (in the case of $n = 1$ they are just $(\Rightarrow\Box^F)$ and $(\Rightarrow\Box^P)$).

Instead of proving validity-preservation of these new rules we will show their derivability:

Lemma 6.2 (Derivability). *Both* $(\Rightarrow\Box^{F\prime\prime})$ *and* $(\Rightarrow\Box^{P\prime\prime})$ *are derivable in* **HCKt4.3**.

Proof. The derivability of $(\Rightarrow\Box^{F''})$ is schematically demonstrated by the following figure:

$$\dfrac{s_{21} \qquad s_{22} \qquad \dfrac{\dfrac{\vdots}{\dfrac{s_{n1} \qquad s_{n2} \qquad \dfrac{s_{n3}}{s_{n3'}}(\Rightarrow\Box^{F})}{s_{n-1}}(\Rightarrow\Box^{F'})}}{s_{23}}}{s_1}(\Rightarrow\Box^{F'})$$

where from $s_{n3} = H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \mid\Rightarrow \varphi$ we derive $s_{n3'} = H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n, \Box^F\varphi$ by $(\Rightarrow\Box^F)$. $s_{n3'}$ together with $s_{n1} = H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid\Rightarrow \varphi \mid \Gamma_n \Rightarrow \Delta_n$ and $s_{n2} = H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n, \varphi$ yields $s_{n-1} = H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_{n-1} \Rightarrow \Delta_{n-1}, \Box^F\varphi \mid \Gamma_n \Rightarrow \Delta_n$ by $(\Rightarrow\Box^{F'})$. We continue systematically applications of $(\Rightarrow\Box^{F'})$ to remaining premisses till the end round with $s_{21} = H \mid \Gamma_1 \Rightarrow \Delta_1 \mid\Rightarrow \varphi \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$, $s_{22} = H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2, \varphi \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$, $s_{23} = H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2, \Box^F\varphi \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ as premisses, and $s_1 = H \mid \Gamma_1 \Rightarrow \Delta_1, \Box^F\varphi \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ as the conclusion.

Similarly we prove the derivability of $(\Rightarrow\Box^{P''})$. ⊣

One can easily extend our calculus to cover serial logics (i.e., with no starting point or no ending point). It is sufficient to add the following structural rules for past seriality and future seriality:

$$(\mathtt{PS})\ \frac{\Rightarrow\mid G}{G} \quad \text{or} \quad (\mathtt{FS})\ \frac{G \mid\Rightarrow}{G}.$$

Also the condition of density may be easily expressed in terms of suitable structural rule. The corresponding rule is:

$$(\mathtt{D})\ \frac{G_1 \mid\Rightarrow\mid G_2}{G_1 \mid G_2}.$$

These rules are certainly sufficient for cut-free deductions of suitable axioms, as the reader may easily check. The formalization of linear logics of reflexive frames is easy but requires additional (obvious) rules for introduction of $\Box^F$ ($\Box^P$) in the antecedent; it is not clear how to obtain the same result by means of structural rules. On the other hand, dropping all rules for $\Box^P$ gives us formalizations of respective monomodal linear logics.

One may easily check that the additional structural rules are validity-preserving in the corresponding classes of frames. The key observation needed for all of them is that the sequent with empty antecedent and succedent is read (in a standard way) as $\top \Rightarrow \bot$ and so it is falsified at any point. Thus our soundness theorem may be strengthened:

Theorem 6.3 (Soundness 2). *If* **HCL** $\vdash G$, *then* $\models G$ *in the class of frames characterising* **L**, *where* **HCL** *is one of the extensions of* **HCKt4.3** *with any of* (FS), (PS), (D).

§7. Completeness. Completeness proofs for cut-free versions of tableau or sequent calculi are usually based on the process of (downward) saturation,

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in the sense of Hintikka, of branches of a proof tree. Two main strategies are applied for this aim, called by Hodges [20] 'direct' and 'tree' arguments. The latter is based on the application of some procedure for construction of a reduction tree $T(s)$ for any sequent s. Such a tree is built up starting with s and systematically applying the rules of the calculus in the root-first manner in all possible ways. This way we are building a, possibly infinite, inductively defined chain $T_0(s), T_1(s), T_2(s), \ldots$ of finite trees with $T(s)$ being its limit. In case s is provable, we will finish at some stage of construction with finite $T(s)$ having all the leaves labelled with instances of axioms. Otherwise, by König's lemma, there is an infinite branch which allows for the construction of a countermodel for s. Such method of proof was used, e.g., by Schütte [45] and Takeuti [48] (see also Negri [37] for modal logics) for SC, it works well also in the framework of hypersequents[6]. Note that 'tree arguments' (in Hodges' terminology) are particularly useful when we want to obtain decidability results as by-products of completeness proofs.

In completeness proofs based on direct method we devise a procedure for unprovable sequent and, instead of trees, we construct an infinite chain of sequents constantly extending unprovable input. The most important thing is to show that each step of the procedure yields unprovable sequent on the assumption that the preceding step resulted in unprovable sequent either. Completeness proofs of this kind were offered for tableau systems by Fitting [16] and Goré [19], for example. In what follows we will provide a completeness proof also based on this strategy since we do not attempt to prove decidability and, in general, a description of such procedure is simpler. In brief, we will show how to construct, for every unprovable hypersequent G, an infinite chain of hypersequents which evantually delivers a special (possibly infinite) hypersequent called a linear saturation of G. The resulting hypersequent is also unprovable and allows for construction of a falsifying model for G. We will show that the procedure is fair, i.e., all formulae must be eventually used, and that each extension step yields an unprovable hypersequent.

For better readability, we separate the proof into three different procedures of downward saturation, coherency saturation, and fulfillment.

Let us start with the most basic notion:

Definition 7.1. $\Gamma \Rightarrow \Delta$ is downward saturated iff the following holds:

(i) $\neg\varphi \in \Gamma$ implies $\varphi \in \Delta$,
(ii) $\neg\varphi \in \Delta$ implies $\varphi \in \Gamma$,
(iii) $\varphi \wedge \psi \in \Gamma$ implies $\varphi \in \Gamma$ and $\psi \in \Gamma$,
(iv) $\varphi \wedge \psi \in \Delta$ implies $\varphi \in \Delta$ or $\psi \in \Delta$,
(v) $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$,
(vi) $\varphi \vee \psi \in \Delta$ implies $\varphi \in \Delta$ and $\psi \in \Delta$,
(vii) $\varphi \rightarrow \psi \in \Gamma$ implies $\varphi \in \Delta$ or $\psi \in \Gamma$,
(viii) $\varphi \rightarrow \psi \in \Delta$ implies $\varphi \in \Gamma$ and $\psi \in \Delta$.

[6] For example, it was successfully applied in Indrzejczak [25] for proving the completeness of **HCS4.3**.

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We will say that: G is saturated iff all its elements are downward saturated; s' is a saturated extension of s iff s' is downward saturated and $s \subseteq s'$, and G' is a saturated extension of G iff it is saturated and $G \subseteq G'$.

LEMMA 7.2 (Saturation lemma). *For every hypersequent G, if $\nvdash G$, then it has an unprovable saturated extension G'.*

PROOF. Let $\nvdash G = s_1 \mid \cdots \mid s_n$, we start with the leftmost sequent and successively proceed to the right, until we get s_n. In each case we check if s_i is saturated, if it is, then we move to the next sequent. Otherwise, we perform the following operations. Let $s_i = \Gamma \Rightarrow \Delta$ be not saturated, then some of the conditions (i)–(viii) do not hold. As an example, we consider the cases of $\wedge$—we proceed as follows:

Assume that $\varphi \wedge \psi \in \Gamma$, but either $\varphi \notin \Gamma$ or $\psi \notin \Gamma$. Then add lacking formula to Γ. The resulting G' containing enriched sequent $\varphi, \psi, \Gamma \Rightarrow \Delta$ cannot be provable, otherwise by $(\wedge\Rightarrow)$ and $(C \Rightarrow)$, G with $\Gamma \Rightarrow \Delta$ would be provable as well.

Assume that $\varphi \wedge \psi \in \Delta$ but neither $\varphi \in \Delta$ nor $\psi \in \Delta$. Then add to Δ, one of the lacking formula, namely this one which yields unprovable hypersequent G'. At least one of them must be unprovable because if both $\vdash G'$ with $\Gamma \Rightarrow \Delta, \varphi$ and $\vdash G''$ with $\Gamma \Rightarrow \Delta, \psi$, then by $(\Rightarrow\wedge)$ and $(\Rightarrow C)$, also G with $\Gamma \Rightarrow \Delta$ would be provable.

Note that each step of the procedure corresponds to root-first application of some extensional rule in a contraction-absorbing manner, i.e. with a duplication of the main formula in premisses. Thus we may treat it as a (part of a) failed proof search procedure building a proof tree. This may be defined precisely but we leave the details (see e.g. Negri [37]; the difference is that here the sequents are saturated, not the branches). By subformula property, we eventually obtain a finite unprovable hypersequent G' which is by construction a saturated extension of G having the same number of sequents. ⊣

DEFINITION 7.3. G is coherent iff for any $\Gamma_i \Rightarrow \Delta_i \in G$ and $\Gamma_j \Rightarrow \Delta_j \in G$ such that $\Gamma_i \Rightarrow \Delta_i < \Gamma_j \Rightarrow \Delta_j$:

- *If $\Box^F \varphi \in \Gamma_i$, then $\varphi \in \Gamma_j$,*
- *If $\Box^P \varphi \in \Gamma_j$, then $\varphi \in \Gamma_i$.*

LEMMA 7.4 (Coherency lemma). *For every hypersequent G, if $\nvdash G$, then it has an unprovable coherent extension G'.*

PROOF. If G is not coherent, then for some $\Box^F \varphi \in \Gamma_i$ (or $\Box^P \varphi$), we have a sequent $\Gamma_j \Rightarrow \Delta_j$ such that $\Gamma_i \Rightarrow \Delta_i < \Gamma_j \Rightarrow \Delta_j$ and $\varphi \notin \Gamma_j$. Add φ to Γ_j. The resulting hypersequent must be unprovable; otherwise, by $(\Box^F \Rightarrow)$ and $(C \Rightarrow)$ also G would be provable. We repeat this procedure for every case violating the conditions of coherency. Since there is only a finite number of such cases, then processing of this procedure must terminate. In order to keep it evident it may be performed in some fixed order, e.g. start with the leftmost sequent and move to the rightmost one, adding φ for each $\Box^F \varphi$ in each Γ_i. Then start with the rightmost sequent and move to the left doing the same for every $\Box^P\varphi$ in each Γ_i. Of course, new temporal formulae may appear in some antecedents not satisfying the conditions of coherency, so if necessary we repeat both rounds until coherency conditions will be satisfied for all temporal formulae. By subformula property the procedure is terminating. ⊣

Clearly, a saturated hypersequent after the application of the above procedure most probably will become unsaturated, and a coherent hypersequent after saturation will become incoherent. But, due to subformula property, by iterating both procedures sufficiently often, we eventually get for every unprovable hypersequent its finite extension which is both saturated and coherent.

Now consider some unprovable G and let $SF(G)$ denote the set of all subformulae of formulae in G. We can make a finite list $LTF = \psi_1, \dots, \psi_k$ of all temporal formulae in $SF(G)$. Additionaly, for the sake of completeness proof, we admit the existence of infinite hypersequents. By infinite hypersequent we mean here a countable linear order with elements labelled by finite sequents[7]. We will say that an infinite hypersequent is provable iff it is an extension of some finite provable hypersequent, otherwise it is unprovable (i.e., all its finite reductions are unprovable). Let us define a special, possibly infinite, hypersequent $LS(G)$ made of $SF(G)$ which is a linear saturation of G, namely:

Definition 7.5. $LS(G)$ is a linear saturation of G iff:

(i) $LS(G)$ is unprovable, saturated, and coherent extension of G;
(ii) for any $\Gamma_i \Rightarrow \Delta_i \in LS(G)$:
- if $\Box^F\varphi \in \Delta_i$, then there is $\Gamma_j \Rightarrow \Delta_j \in LS(G)$ such that $\Gamma_i \Rightarrow \Delta_i < \Gamma_j \Rightarrow \Delta_j$ and $\varphi \in \Delta_j$;
- if $\Box^P\varphi \in \Delta_i$, then there is $\Gamma_j \Rightarrow \Delta_j \in LS(G)$ such that $\Gamma_i \Rightarrow \Delta_i > \Gamma_j \Rightarrow \Delta_j$ and $\varphi \in \Delta_j$.

We say that temporal formula occurring in the succedent of at least one sequent in G is fulfilled in a hypersequent iff all its succedent occurrences in this hypersequent satisfy the condition (ii).

Lemma 7.6 (LS lemma:). *For every hypersequent G, if $\nvdash G$, then it has an unprovable linear saturation.*

Proof. Assume that $\nvdash G$. We build $LS(G)$ in stages each time checking whether the result is unprovable. Essentially we are building a (possibly infinite) inductively defined chain of unprovable hypersequents, each one being an extension of its predecessor. We will do it in stages
$S_0(G), S_1(G), S_2(G), \dots, S_k(G), S_{k+1}(G), \dots, S_{2k}(G), \dots$.
The k-series are dictated by the number of formulae in LTF, we will call each such series $(m1 - mk,\ m \geq 0)$ a round. In stage 0, we do not consider any formula from LTF but for each $1 \leq l \leq k$, we start with taking

[7] In this respect, our notion is different from the notion of infinite sequent used by Kleene [29] or Gallier [17], or from that of Tait [47] where infinite formulae are also admitted. Similar notion of infinte hypersequent is also used by Lahav [32].

the ψ_l from the list LTF until all of them will be examined in a round, then we repeat the procedure and make the next round. Repeated consideration of all formulae from LTF is necessary to guarantee fairness of the procedure.

In stage 0, we transform G into its saturated and coherent extension G_0 by repeated application of Lemma 7.2 and 7.4. By construction the output of $S_0(G)$ must be a finite and unprovable hypersequent. In particular, the application of Lemma 7.4 is required to put all sequents in G_0 into linear order corresponding to their placement from left to right. Clearly, it may be the case that G is already (trivially) saturated and coherent, e.g., in case of G being of the form $\Rightarrow \Box^F \varphi$, then the output of $S_0(G)$ is just G and we go to the next stage.

Suppose we have finished stage n (where $n = mk + l$ for some $m \geq 0$ and $0 \leq l \leq k$) with finite unprovable saturated and coherent hypersequent G_n, we start construction of stage $n + 1$ by examining ψ_{l+1} from the list LTF. First, we check if it is applicable. It is nonapplicable if either it has no occurrence in any succedent of any element of G_n or if all such occurrences are fulfilled (i.e., satisfy the condition (ii) from the definition of $LS(G)$). In case our ψ_{l+1} is nonapplicable in G_n we finish stage $n + 1$ by declaring $S_{n+1}(G) = S_n(G)$.

ψ_{l+1} is applicable if for some saturated $\Gamma_i \Rightarrow \Delta_i$ and $\psi_{l+1} = \Box^F \varphi \in \Delta_i$ (or $\psi_{l+1} = \Box^P \varphi \in \Delta_i$) either there is no j such that $\Gamma_i \Rightarrow \Delta_i < \Gamma_j \Rightarrow \Delta_j$ or for all such j, $\varphi \notin \Delta_j$. We take the rightmost of such an occurrence of $\Box^F \varphi$ in G_n (or the leftmost of $\Box^P \varphi$) and show that there is a finite unprovable extension of G_n which fulfills this formula. It is handy to consider this as a separate claim which we prove by induction on the number of sequents to the right of $\Gamma_i \Rightarrow \Delta_i$ (to the left of it in case of $\Box^P \varphi$). In the basis $\Gamma_i \Rightarrow \Delta_i$ is the rightmost sequent of G_n and we consider $G_n \mid \Rightarrow \varphi$. It must be unprovable, otherwise by $(\Rightarrow \Box^F)$ and $(\Rightarrow C)$ we would obtain a proof of G_n. As the induction hypothesis, we assume that the claim holds for the case where there are m sequents to the right of $\Gamma_i \Rightarrow \Delta_i$, and prove it for the case of $m + 1$ sequents. We must consider 3 options: (a) G_n with $\Rightarrow \varphi$ inserted between $\Gamma_i \Rightarrow \Delta_i$ and $\Gamma_{i+1} \Rightarrow \Delta_{i+1}$ (b) G_n with φ added to Δ_{i+1} and (c) G_n with $\Box^F \varphi$ added to Δ_{i+1}. If all three hypersequents were provable, then G_n would be provable by $(\Rightarrow \Box^{F\prime})$ and $(\Rightarrow C)$, hence at least one of (a), (b), (c) must be unprovable. In cases (a) and (b), our ψ_{l+1} is fulfilled but not in (c), however if (c) is unprovable, then the claim holds by the induction hypothesis since there are m sequents to the right of $\Gamma_{i+1} \Rightarrow \Delta_{i+1}, \Box^F \varphi$. In practice we just continue with this new occurrence of ψ_{l+1} and repeat the procedure until ψ_{l+1} is fulfilled[8]. In any case, we obtain some G'_n which is a finite unprovable extension of G_n fulfilling ψ_{l+1}, but not necessarily saturated and coherent. We continue $S_{n+1}(G)$ with it. By saturation and coherency lemma, any unprovable finite hypersequent is extendible to a finite unprovable saturated and coherent hypersequent.

[8]One may observe that in the calculus with $(\Rightarrow \Box^{F\prime\prime})$, discussed in section 6, we obtain this result directly in one step.

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Thus repeated application of both lemmata yields some finite hypersequent G_{n+1} of this kind, being the output of $S_{n+1}(G)$. Note that in case of unprovable extension obtained by the addition of a new sequent $\Rightarrow \varphi$, the process of saturation and making it coherent assures that if all sequents in G_n were linearly ordered, then this order is preserved with respect to a new inserted sequent.

Let us summarize the most important features of our procedure. It produces a , possibly infinite, chain $G \subseteq G_0 \subseteq G_1 \subseteq \cdots$ of finite unprovable hypersequents, where each G_n is the output of the corresponding stage $S_n(G)$. Each stage, except $S_0(G)$, starts with fulfillment phase for some ψ_i from LTF, and continues with repeated phases of downward and coherency saturation. As a result each G_n is a finite unprovable, saturated, and coherent extension of every preceding hypersequent in the chain. It implies that if some s was added in stage $S_n(G)$ for the first time, then it obtains its downward saturated extension s_n in G_n and that we have an ascending chain $s \subseteq s_n \subseteq s_{n+1} \subseteq \cdots$ of corresponding saturated extensions of s for each G_m such that $G_n \subseteq G_m$. Conversely, for any G_n and any s_n belonging to it we have a finite descending chain of reductions $s_n \supseteq s_{n-1} \supseteq \cdots \supseteq s_k$, for some $k \geq 0$ where s_k was introduced for the first time in stage $S_k(G)$.

The procedure is fair, since every compound formula is decomposed and every temporal formula is dealt with infinitely often. This assures that the conditions of saturation for occurrences of boolean formulae, as well as that of coherency and fulfillment for occurrences of temporal formulae must be satisfied at some stage. Suppose that at some stage some boolean formula is not saturated yet, then, by construction, it must be saturated by the end of this stage. Similarly for any temporal formulae in the antecedents, since the next stage may start only after we got a saturated and coherent hypersequent. Suppose that some temporal formula ψ_i in some succedent is not yet fulfilled at some stage, then even if it remains unfulfilled until the end of this round, it must be activated at the beginning of the i-th stage of the next round. Note that it is crucial for fairness that fulfillment phase for some ψ_i is performed only once in a suitable stage. After repeated application of downward and coherency saturation, we do not consider ψ_i again (it may be again unfulfilled!) but move to the next stage and fulfill ψ_{i+1}. ψ_i will be dealt with again after k stages in the next round.

Although a performance of each stage is terminating and each round is also terminating, the process of building $LS(G)$ may be infinite. Generally after finishing every round (mk stage for some $m \geq 0$) we start the next one by examination of ψ_1, and then the next formulae from LTF. Now, it may be the case that after finishing some stage $S_n(G)$ we obtain a finite unprovable, saturated, and coherent extension of G where all elements of LTF are fulfilled. In this case, the chain would extend forever by repeating the last hypersequent and we can choose G_n as $LS(G)$. By construction it is saturated, coherent, and fulfilled for all temporal formulae occurring in the succedents of all sequents.

Otherwise, in every round we are finding at least one unfulfilled formula and our chain is extended in a nontrivial way. In this case $LS(G)$ is reached in the omega-step as an infinite hypersequent G_∞ being the limit of all hypersequents occurring in the infinite chain. Thus every element of G_∞ is a sequent s being an ordered pair of unions of formulae occuring in all sequents belonging to the corresponding chain of reductions of s. One may check that G_∞ is indeed $LS(G)$. G_∞ is unprovable, otherwise some finite G_n must be provable which is impossible. G_∞ is saturated, for suppose not, then there is some s in G_∞ with a boolean formula φ which is not decomposed. But φ occurred for the first time in some stage $S_n(G)$ in the reduction of s and by fairness it must have been dealt with in this stage. G_∞ is coherent, for suppose not. Then there is some temporal formula $\Box^F\psi$ or $\Box^P\psi$ in the antecedent of some s_i, and some s_j to the right of s_i (if it is of the form $\Box^F\psi$) or to the left (if it is of the form $\Box^P\psi$) where ψ is not in the antecedent. There must have been some stages $S_n(G)$ and $S_m(G)$ where this formula was introduced for the first time to some reduction of s_i, and where a descending chain of reductions of s_j started. But then, by construction, in the maximum of n, m, ψ was added to the antecedent of s_j, contrary to our assumption. Finally, G_∞ must satisfy condition (ii) of the definition of $LS(G)$. For assume that some temporal formula is not fulfilled in G_∞. But it is impossible since by construction it is dealt with infinitely often during the construction of the chain so it must be fulfilled in the omega step.

This is the basic construction of $LS(G)$ for **HCKt4.3**. In case of stronger logics satisfying at least one of the conditions of seriality, or density we must make some adjustments to our procedure.

For future seriality (and similarly for past seriality) after termination of every round we just add $\Rightarrow$ as the rightmost sequent to every nonaxiomatic leaf G'. It must be unprovable since otherwise, by (FS) also G' would be provable. Then we start the next round. This way we secure future seriality of our infinite $LS(G)$.

For density we must secure that in $LS(G)$ for all s and s' such that $s < s'$ we have s'' such that $s < s''$ and $s'' < s$. To get this result after termination of every round we take two leftmost sequents of every nonaxiomatic leaf $G' = s_1 \mid s_2 \mid \cdots \mid s_n$ and insert $\Rightarrow$ between them. $s_1 \mid\Rightarrow\mid s_2 \mid \cdots \mid s_n$ is unprovable, otherwise, by (D) G' would be provable. We repeat the procedure with the next pair of sequents until we get s_n, then we start the next round. This way in the limit we get an infinite $LS(G)$ where for every pair of sequents an intermediate one was inserted at some stage. ⊣

Now we define a model $\mathfrak{M}_G$ for unprovable G as follows:

- T is the set of all occurrences of sequents in $LS(G)$,
- $R(\Gamma_i \Rightarrow \Delta_i, \Gamma_j \Rightarrow \Delta_j)$ iff $\Gamma_i \Rightarrow \Delta_i < \Gamma_j \Rightarrow \Delta_j$,
- $V(p) = \{\Gamma \Rightarrow \Delta \in T : p \in \Gamma\}$.

Linearity and transitivity of R follows directly from the definition of $LS(G)$. Seriality and density are also straightforward to demonstrate for the corresponding $LS(G)$. We need to prove:

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LEMMA 7.7 (Truth lemma). *For each* $\varphi \in SF(G)$ *and each* $\Gamma \Rightarrow \Delta \in T$ *it holds true*:

- $\varphi \in \Gamma$ *implies* $\Gamma \Rightarrow \Delta \vDash \varphi$,
- $\varphi \in \Delta$ *implies* $\Gamma \Rightarrow \Delta \nvDash \varphi$.

PROOF. By induction on the complexity of formulae. The basis is obvious from the definition of V. For complex extensional formulae the result follows directly from the definition of saturated sequents and the induction hypothesis. We will only show the case of $\Box^F$.

Assume that $\Box^F \varphi \in \Gamma$. In order to show that $\Gamma \Rightarrow \Delta \vDash \Box^F \varphi$, we must show that if $R(\Gamma \Rightarrow \Delta, \Pi \Rightarrow \Sigma)$, then $\Pi \Rightarrow \Sigma \vDash \varphi$. From the definition of R and coherency of $LS(G)$ it follows that $\varphi \in \Pi$ and by the induction hypothesis $\Pi \Rightarrow \Sigma \vDash \varphi$.

Assume that $\Box^F \varphi \in \Delta$. In order to show that $\Gamma \Rightarrow \Delta \nvDash \Box^F \varphi$ we must show that there is some $\Pi \Rightarrow \Sigma$ such that $R(\Gamma \Rightarrow \Delta, \Pi \Rightarrow \Sigma)$ and $\Pi \Rightarrow \Sigma \nvDash \varphi$. By condition (ii) of the definition of $LS(G)$ there is some $\Pi \Rightarrow \Sigma$ on the right of $\Gamma \Rightarrow \Delta$ with $\varphi \in \Sigma$. Hence by the induction hypothesis $\Pi \Rightarrow \Sigma \nvDash \varphi$. By the definition of R it holds that $R(\Gamma \Rightarrow \Delta, \Pi \Rightarrow \Sigma)$, hence we are done. ⊣

Consequently, we obtain:

THEOREM 7.8 (Completeness). *If* $\models G$, *then* $\vdash G$ *for all considered logics.*

PROOF. Assume that $\nvdash G$, then by LS Lemma, there exists some $LS(G)$. On the basis of $LS(G)$, we obtain a model $\mathfrak{M}_G$ of suitable character which falsifies every element of $LS(G)$. But then, by the condition (i) of the definition of $LS(G)$, every element of G is falsified, as well. Therefore, $\not\models G$. ⊣

§8. Concluding remarks.

We finish the paper with some prospects for future work. First of all there is an open question of how to prove syntactical cut-elimination theorem for **HCKt4.3** and its extensions. One must note that suitable cut rule must be either of the form:

$$\frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_2 \qquad H_1 \mid \varphi, \Gamma \Rightarrow \Delta \mid H_2}{H_1 \mid \Gamma \Rightarrow \Delta \mid H_2}$$

or

$$\frac{H_1 \mid \Gamma \Rightarrow \Delta, \varphi \mid H_1 \qquad H_1 \mid \varphi, \Pi \Rightarrow \Sigma \mid H_2}{H_1 \mid \Gamma, \Pi \Rightarrow \Delta, \Sigma \mid H_2}.$$

One can easily check that both rules are validity-preserving in our system. Both rules are globally additive in the sense of having the same external contexts (H_1 and H_2). Moreover the former (locally additive) seems to be better since internal contraction rule is derivable in this case. For the time being the problem of the syntactical proof of cut elimination for this system is open. No existing strategy[9] seems to work for the present system because

[9]See, e.g. the "history technique" of Avron [1], the "decoration technique" of Baaz and Ciabattoni [5], Poggiolesi's [40] adaptation of Dragalin's proof for HC, or the general and elegant method of Metcalfe, Olivetti and Gabbay [35].

hypersequents are lists of sequents and this fact strongly restricts admissible syntactic manipulations. Of course one can obtain a simple indirect proof of the admissibility of both rules as a by-product of our completeness proof and their validity-preservation. Assume that both premisses are schemata of derivable sequents, then by Theorem 5.3 they are both valid and the conclusion is also valid due to validity-preservation of the rule. Then, by Theorem 7.8 the conclusion is also derivable, and the rule is admissible in the system.

The next problem worth investigating is to provide a decidability proof for respective logics on the basis of exhaustive proof-search procedure devised for completeness proof. The problem is not only with careful detection of loops generated by all modal logics of transitive frames; one can find a discussion of such strategies for monomodal logics in Goré [19]. The symmetry of past and future introduces additional complications because it may appear that during a proof search some points previously identified may eventually change their content. To deal with such problems we need for instance an adaptation of some kind of dynamic blocking introduced by Horrocks, Sattler, and Tobies [21] in the framework of description logics.

Finally—another related question is to find rules for other temporal operators like e.g. "since" and "until" or "next". We leave these problems for future work.

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