Higher rank rigidity for Berwald spaces

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Abstract. We generalize the higher rank rigidity theorem to a class of Finsler spaces, i.e. Berwald spaces. More precisely, we prove that a complete connected Berwald space of finite volume and bounded non-positive flag curvature with rank at least two whose universal cover is irreducible is a locally symmetric space or a locally Minkowski space.

1. Introduction

In the 1980s, based on the work in Ballmann–Brin–Eberlein [**2**] and Ballmann–Brin–Spatzier [**3**], Ballmann and Burns–Spatzier proved the following higher rank rigidity theorem independently in [**1**, **7**].

THEOREM 1.1. (Cf. [**1**, **7**]) *Let M be a complete connected Riemannian manifold of finite volume and bounded non-positive sectional curvature. Then the universal cover $\widetilde{M}$ of M is a flat Euclidean space, a symmetric space of non-compact type, a space of rank one or a product of the above types.*

If we assume that $\widetilde{M}$ is irreducible, i.e. not a Riemannian product of another two Riemannian manifolds of lower dimensions, we have the following.

THEOREM 1.2. (Cf. [**1**, **7**]) *Let M be a complete connected Riemannian manifold of finite volume and bounded non-positive sectional curvature with rank at least two whose universal cover $\widetilde{M}$ is irreducible. Then M is locally symmetric.*

Our goal is to generalize the above higher rank rigidity theorem from Riemannian manifolds to a broader class, i.e. Finsler manifolds. A Finsler structure is given by a smooth family of Minkowski norms defined on each tangent space. Tensors such as the fundamental tensor are defined on TM instead of M and hence not only depend on the

position x on M, but also depend on the direction y in the tangent space. Let $\pi : TM \to M$ be the natural projection. The pulled-back bundle π^*TM admits a unique linear connection $\triangledown$ which satisfies torsion freeness and almost g-compatibility, called the Chern connection. The curvature 2-form of the Chern connection consists of two parts: hh- and hv-curvature parts. The flag curvature is then defined via the hh-curvature tensors, which coincides with the sectional curvature if the Finsler manifold is Riemannian. See §2 for details. We want to prove the higher rank rigidity theorem for Finsler spaces of non-positive flag curvature following the scheme in [2, 3, 7]. However, there are some technical difficulties we cannot overcome at this time, such as no flat strip lemma and no proper angle notion for general Finsler spaces.

In this paper we restrict to Berwald spaces, a special class of Finsler spaces larger than the class of Riemannian manifolds. In Berwald spaces, the Chern connection coefficients Γ^i_{jk} have no y dependence (see §2.3). It follows that hv-curvature vanishes and the behavior of the geodesics is well controlled by the non-positivity of the flag curvature. More precisely, on the universal cover $\widetilde{M}$ of a Berwald space M, the non-positivity of the flag curvature implies the convexity in t of the distance function $d(\alpha(t), \beta(t)))$ between two geodesics α and β (see Proposition 3.1 below). Based on this convexity, we prove a flat strip lemma (see Lemma 3.10) and thus the higher rank implies a great deal of flats. Then we construct the stable and unstable manifolds for the geodesic flows. Finally, we are able to construct Weyl chambers and Tits buildings in the sphere at infinity for the universal cover of Berwald spaces with non-positive flag curvature. To this end, we need to overcome some difficulties caused by the Berwald metric. We define a notion of angle using the convexity (see Definition 3.2), which in turn defines a metric on $\widetilde{M}(\infty)$, the sphere at infinity. Furthermore, we obtain a type of coarse estimation on distance functions to deal with the issue of reference vector (see Proposition 4.3).

Therefore, we can go through all steps in [2, 3, 7] and prove the following.

MAIN THEOREM 1. *Let (M, F) be a complete connected Berwald space of finite volume and bounded non-positive flag curvature with rank at least two whose universal cover $\widetilde{M}$ is irreducible. Then (M, F) is a locally symmetric space or a locally Minkowski space.*

We note that our main theorem also deals with the irreversible case. A locally symmetric Finsler space is reversible unless it is an irreversible locally Minkowski space. It follows that a complete connected irreversible Berwald space of finite volume and bounded non-positive flag curvature with rank at least two whose universal cover $\widetilde{M}$ is irreducible must be an (irreversible) locally Minkowski space.

Our proof of Main Theorem 1 in this paper is based on the study of the global properties of Berwald spaces with non-positive flag curvature. We give a clear dynamical picture of the geodesic flow and reveal a rigidity phenomenon that dynamical behavior of the geodesic flow remains the same even in a broader geometric context.

The paper is organized as follows. In §2, we present some preliminaries on Finsler spaces and Berwald spaces, particularly on flag curvature and Jacobi fields. The key properties of Berwald spaces of non-positive flag curvature are studied in §3. In §4, we prove Main Theorem 1 by adapting the methods in [2, 3, 7].

2. *Preliminaries on Finsler spaces*

In this section, we give definitions and present some basic results first on Finsler spaces and then on Berwald spaces. We will adapt the notation in [**5**].

2.1. *The Chern connection and flag curvature.*

Definition 2.1. A Minkowski norm on an n-dimensional real vector space is a function $F : V \to [0, \infty)$ with the following properties:

(1) F is smooth on $V - \{0\}$;

(2) $F(\lambda y) = \lambda F(y)$ for all $\lambda > 0$;

(3) the Hessian matrix

$$(g_{ij}) := ([\tfrac{1}{2} F^2]_{y^i y^j})$$

is positive definite at every point of $V - \{0\}$.

The simplest example of a Minkowski norm is a Euclidean norm $F(y) := \sqrt{\langle y, y \rangle}$, $y \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product. There are other examples, such as

$$F(y^1, y^2) = \sqrt{\sqrt{(y^1)^4 + (y^2)^4} + \lambda((y^1)^2 + (y^2)^2)}, \quad y^1, y^2 \in \mathbb{R},$$

where λ can be any positive number.

A Finsler manifold is a manifold with a smoothly varying family of Minkowski norms, one on each tangent space. We give a precise definition of Finsler manifolds.

Definition 2.2. A Finsler manifold is a manifold M with an $F : TM \to [0, \infty)$ (Finsler metric) such that:

(1) F is smooth on $TM \backslash \{0\}$;

(2) $F \mid_{T_x M} : T_x M \to [0, \infty)$ is a Minkowski norm for all $x \in M$.

We recall some notions of tensors for a Finsler manifold M. In general, they are defined on the tangent bundle TM instead of M, as distinct from the case for Riemannian manifolds. There is a natural coordinate system $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ on TM, where $(x^1, \ldots, x^n)$ is a local coordinate system on M, and, for any $y \in T_x M$, $y = y^j (\partial / \partial x^j)$. Here we list some tensors and quantities which will be useful later. We use the Einstein summation convention throughout this paper.

(1) The fundamental tensor $g = g_{ij} \, dx^i \otimes dx^j$, where

$$g_{ij}(x, y) = (\tfrac{1}{2} F^2)_{y^i y^j}.$$

(2) The Cartan tensor $C = C_{ijk} \, dx^i \otimes dx^j \otimes dx^k$, where

$$C_{ijk}(x, y) = (\tfrac{1}{4} F^2)_{y^i y^j y^k}.$$

The Cartan tensor vanishes for Riemannian manifolds.

(3) The formal Christofell symbols of the second type are

$$\gamma^i_{jk}(x, y) = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{g_{ks}}{\partial x^j} \right),$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

(4)   The nonlinear connection is the quantities

$$N_j^i(x, y) := \gamma_{jk}^i y^k - g^{il} C_{ljk} \gamma_{rs}^k y^r y^s,$$

which provides us with a splitting of the second tangent bundle TTM into a horizontal subspace $H(TM)$ spanned by $\{\delta/\delta x^j\}$ and a vertical subspace $V(TM)$ spanned by $\{\partial/\partial y^j\}$, where

$$\frac{\delta}{\delta x^j} := \frac{\partial}{\partial x^j} - N_j^i \frac{\partial}{\partial y^i}.$$

We can also define the corresponding 1-forms $\{dx^i\}$ and $\{\delta y^i\}$, where

$$\delta y^i := dy^i + N_j^i \, dx^j.$$

The pulled-back tangent bundle π^*TM is induced by the projection $\pi : TM \to M$. It is a vector bundle over the slit tangent bundle $TM\backslash\{0\}$, with fiber over a typical point (x, y) a copy of $T_x M$. The fundamental tensor and Cartan tensor are symmetric sections of $\pi^*T^*M \otimes \pi^*T^*M$ and $\otimes^3 \pi^*T^*M$, respectively. Now the Chern connection is defined on π^*TM.

THEOREM 2.3. (Chern, cf. [**5**]) *The pulled-back bundle π^*TM admits a unique linear connection $\bigtriangledown$ which satisfies torsion freeness and almost g-compatibility. In the natural coordinates,*

$$\bigtriangledown_v \frac{\partial}{\partial x^j} := \omega_j^i(v) \frac{\partial}{\partial x^i},$$

$$\omega_j^i = \Gamma_{jk}^i \, dx^k \quad and \quad \Gamma_{jk}^i = \Gamma_{kj}^i \quad (torsion\ freeness)$$

and

$$\Gamma_{jk}^l = \gamma_{jk}^l - g^{li}(C_{ijs} N_k^s - C_{jks} N_i^s + C_{kis} N_j^s) \quad (almost\ g\text{-}compatibility).$$

The curvature 2-forms on $TM\backslash\{0\}$ of the Chern connection are

$$\Omega_j^i := d\omega_j^i - \omega_j^k \wedge w_k^i.$$

As *a priori*, they can be expanded as

$$\Omega_j^i = \frac{1}{2} R_{j\ kl}^i \, dx^k \wedge dx^l + P_{j\ kl}^i \, dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q_{j\ kl}^i \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}.$$

The tensors R, P and Q are respectively the hh-, hv- and vv-curvature tensors of the Chern connection. It turns out that $Q_{j\ kl}^i = 0$. Let $R_{jikl} = g_{is} R_{j\ kl}^s$. Now we can define the notion of flag curvature, which is a generalization of the sectional curvature of Riemannian manifolds. Since the fundamental tensors and curvature tensors are defined on $TM\backslash\{0\}$, we need to specify that $y \in T_x M$ as the flagpole and $V = V^i(\partial/\partial x^i)$ as a transverse edge. The flag curvature of the plane span$\{y, V\}$ is defined as

$$K(y, V) := \frac{V^i(y^j R_{jikl} y^l) V^k}{g(y, y)g(V, V) - (g(y, V))^2}, \tag{1}$$

where

$$g := g_{ij}(x, y) \, dx^i \otimes dx^j = (\tfrac{1}{2} F^2)_{y^i y^j} \, dx^i \otimes dx^j$$

is the lift of the fundamental tensor defined by the Hessian of the Finsler metric. When M is a Riemannian manifold, the flag curvature defined above is exactly the sectional curvature.

2.2. *Geodesics and Jacobi fields.* Let $c : [a, b] \to M$ be a piecewise C^1 curve on a Finsler manifold (M, F). The length of c is defined as

$$L(c) := \int_a^b F(c(t), c'(t)) \, dt.$$

For a pair of points p and q, we have an induced distance:

$$d(p, q) = \inf_c L(c),$$

where the infimum is taken among all the piecewise curves connecting p and q. Generally it is not true that $d(p, q) = d(q, p)$ since the Finsler metric F is defined to be only positively homogeneous.

Let $\sigma(t)$ be a smooth curve in M with velocity field T. Let $W(t) := W^i(t)(\partial/\partial x^i)$ be a vector field along σ. We have two different covariant derivatives with respect to the Chern connection according to the reference vectors:

(1) with reference vector T:

$$D_T W := \left(\frac{dW^i}{dt} + W^j T^k (\Gamma^i_{jk})_{(\sigma, T)} \right) \frac{\partial}{\partial x^i} \bigg|_{\sigma(t)} ;$$

(2) with reference vector W:

$$D_T W := \left(\frac{dW^i}{dt} + W^j T^k (\Gamma^i_{jk})_{(\sigma, W)} \right) \frac{\partial}{\partial x^i} \bigg|_{\sigma(t)} .$$

A geodesic γ on M is a curve which locally minimizes its length. In natural coordinates, by considering the variation of arc length, a constant-speed geodesic satisfies

$$D_{\gamma'} \gamma' = 0$$

with reference vector γ'. Let $\sigma : T^1 M \to M$ be the unit tangent bundle, i.e. $T^1 M = \{(x, y) \in TM : F(x, y) = 1\}$. The geodesic flow g^t is defined on $T^1 M$, whose orbits projecting to M are unit-speed geodesics. Its generator X is a vector field on $T^1 M$.

A Jacobi field along a geodesic is a variation vector field of geodesic variation. Let $J(t)$ be a Jacobi vector field along a unit-speed geodesic $\gamma(t)$. We denote by $T(t) = \gamma'(t)$ the velocity vector field along a geodesic γ and by X the generator of the geodesic flow on $T^1 M$. Define

$$J'(t) = D_T J(t)$$

with reference vector T. Then $J(t)$ satisfies the Jacobi equation

$$J''(t) + R(J, T)T = 0,$$

where R is the curvature tensor related to the flag curvature (see (1)), i.e.

$$R(J, T)T := (T^j R^i_{j \, kl} T^l) J^k \frac{\partial}{\partial x^i}.$$

The geodesic flow g^t can be described in a very similar way to the Riemannian case, due to the following Riemannian metric defined on $T^1 M$. Recall the splitting of the second tangent bundle TTM into a horizontal subbundle $H(TM)$ spanned by $\{\delta/\delta x^j\}$ and a vertical

subbundle $V(TM)$ spanned by $\{\partial/\partial y^j\}$ (cf. [5]). There is a Riemannian metric g on the manifold T^1M called the Sasaki metric

$$g_{ij}(x, y)\, dx^i \otimes dx^j + g_{ij}(x, y)\frac{\delta y^i}{F} \otimes \frac{\delta y^j}{F}$$

such that:
(1)   the splitting $TT^1M = H(T^1M) \oplus V(T^1M) \oplus RX$ is orthogonal;
(2) g is well adapted to the pulled-back tangent bundle π^*T^1M;
(3) $g(X, X) = 1$;
(4) g is D_T invariant;
(5)   the curvature operator $R(T, \cdot)T$ is symmetric with respect to g.

It is well known that geodesic flow preserves the volume form on T^1M induced by the Sasaki metric, which is called the *Liouville measure*. Similar to the Riemannian case, the tangent space of T^1M can be identified with the space of Jacobi fields. We view the Jacobi field $J(t)$ as a section of $\pi^*TM \to T^1M$. So, by the second property above, $g(J(t), J(t)) = g_T(J, J)$, where $g_T := g_{ij}(\sigma, T)\, dx^i \otimes dx^j$. The differential of the geodesic flow can be described via Jacobi fields.

PROPOSITION 2.4. (Cf. [11, Appendix]) *Suppose that for $z \in T^1M$, $Z \in T_zT^1M$, we have the splitting $dg^tZ = H_t + V_t + a_tX$, where $H_t \in H(T^1M)$ and $V_t \in V(T^1M)$. Then there exist isometries*

$$v_X : \pi^*T^1M \to V(T^1M) \quad and \quad h_X : \pi^*T^1M \to H(T^1M)$$

such that $h_X^{-1}(H_t) = J(t)$ and $v_X^{-1}(V_t) = J'(t)$, and

$$g(dg^tZ, dg^tZ) = g(J(t), J(t)) + g(J'(t), J'(t)) + g(a_tX, a_tX).$$

2.3. *Berwald spaces.*   A Finsler space (M, F) is said to be a Berwald space if the Chern connection coefficients Γ^i_{jk} have no y dependence in natural coordinates, in other words, the Chern connection is defined directly on the underlying manifold M. As a result, the hv-curvature tensor $P^i_{j\,kl}$ vanishes and only the hh-curvature tensor remains. In this sense, Berwald spaces are just a little bit more general than Riemannian spaces.

Recall the covariant derivative of W along a curve σ on M:

$$D_TW = \left(\frac{dW^i}{dt} + W^jT^k\Gamma^i_{jk}\right)\frac{\partial}{\partial x^i}\bigg|_{\sigma(t)}.$$

So, when we deal with covariant derivative for a Berwald space, we do not have an issue of reference vector.

Ichijyō proved that for Berwald spaces all the tangent spaces are linearly isometric to a common Minkowski space. The proof below is taken from [5].

PROPOSITION 2.5. (Ichijyō, cf. [5]) *Let (M, F) be a connected Berwald space. Then its tangent spaces (T_xM, F_x) with associated Minkowski norms are linearly isometric to each other by parallel translations.*

Proof. Take any two points p, q on M and a curve σ (with velocity field T) connecting them. Let W be a parallel vector field, i.e. $D_T W = 0$ (reference vector is irrelevant here). It is linear in W since Γ^i_{jk} depends only on $\sigma(t)$. So, parallel translations define a linear mapping from $T_p M$ to $T_q M$ and it is obviously one to one. It is enough to prove that the Finsler norm is preserved under parallel translations. Recall that $F(W) = \sqrt{g_W(W, W)}$, where $g_W := g_{ij}(\sigma, W) \, dx^i \otimes dx^j$.

The almost g-compatibility criterion of the Chern connection is

$$dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = 2C_{ijs}\delta y^s.$$

Evaluate it at $(\sigma, W)(t)$ and then contract with $W^i W^j$. By Euler's theorem, the right-hand side is zero since C_{ijk} is (-1)-homogeneous in the y variable. So,

$$W^i W^j (dg_{ij} - g_{kj}\Gamma^k_{is} \, dx^s - g_{ik}\Gamma^k_{js} \, dx^s) = 0.$$

Contract this 1-form equation with the velocity of the lift (σ, W):

$$W^i W^j \left(\frac{dg_{ij}}{dt} - g_{kj}\Gamma^k_{is}T^s - g_{ik}\Gamma^k_{js}T^s \right) = 0.$$

Simplifying it, we have

$$\frac{d}{dt} g_W(W, W) = 2g_W(D_T W, W).$$

Hence, the Finsler norm is preserved under parallel translations. $\square$

Moreover, there are Riemannian isometries between two punctured tangent spaces and hence between the two unit spheres in tangent spaces. For each fixed $x \in M$, the tensor $\hat{g} := g_{ij}(x, y) \, dy^i \otimes dy^j$ defines a smooth Riemannian metric $\hat{g}_x$ on each punctured space $T_x M \backslash 0$. Hence, $T^1_x M$ also has an induced Riemannian metric, say $\dot{g}_x$.

COROLLARY 2.6. *Let (M, F) be a connected Berwald space, $p, q \in M$ and $\sigma(t)$ a curve on M connecting p and q. Parallel translations along σ induce Riemannian isometries $(T_p M \backslash 0, \hat{g}_p) \to (T_q M \backslash 0, \hat{g}_q)$ and $(T^1_p M, \dot{g}_p) \to (T^1_q M, \dot{g}_q)$.*

Proof. Let $\phi : T_p M \to T_q M$ be the parallel translation. By Proposition 2.5, it is a linear isometry between the two Minkowski spaces. Let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be the bases for $T_p M$ and $T_q M$, respectively. Then there exists a matrix $A = (a^j_i)$ such that $\phi(u_i) = a^j_i v_j$. Since ϕ is linear, it is enough to prove that $\hat{g}_W(U, V) = \hat{g}_{\phi(W)}(\phi(U), \phi(V))$ for any $U, V, W \in T_p M$.

Since ϕ preserves Minkowski norm, we have

$$\tfrac{1}{2} F^2_p(W^1, \ldots, W^n) = \tfrac{1}{2} F^2_q(a^1_j W^j, \ldots, a^n_j W^j).$$

After taking the second derivative, we have

$$g_{ij}(\sigma, W) = a^k_i g_{kl}(\sigma, \phi(W)) a^l_j. \tag{2}$$

Also, $(\phi(U))^k = a^k_i U^i$, $(\phi(V))^l = a^l_i V^i$. By (2), we have $\hat{g}_W(U, V) = \hat{g}_{\phi(W)}(\phi(U), \phi(V))$. Hence, $\phi : (T_p M \backslash 0, \hat{g}_p) \to (T_q M \backslash 0, \hat{g}_q)$ is a Riemannian isometry. $\square$

3. Berwald spaces of non-positive flag curvature

In the rest of this paper, let (M, F) be a complete connected Berwald space with flag curvature $K \leq 0$ and all the geodesics are parameterized by arc length. In this case, the Cartan–Hadamard theorem holds and $\widetilde{M}$, the universal cover of M, is diffeomorphic to $\mathbb{R}^n$. In this section, we study some properties of Berwald spaces due to non-positive flag curvature which will play important roles in the proof of the higher rank rigidity theorem.

3.1. Convexity.

On Riemannian manifolds, the non-positive curvature implies the convexity of the length of Jacobi fields $\|J(t)\|$. It follows that on the universal cover $\widetilde{M}$, the distance function $d(\gamma_1(t), \gamma_2(t))$ is convex in t. This is not true generally for Finsler spaces of non-positive flag curvature. Nevertheless, for Berwald spaces of non-positive flag curvature, convexity of the distance function follows from a result in [**14**] due to Kristály and Kozma. We repeat the proof here and the argument is used several times later, especially in the proof of Lemma 3.10, i.e. the flat strip lemma for Berwald spaces. Since the reference vector is irrelevant here in a Berwald space, we have the following nice product rule:

$$\frac{d}{dt} g_W(U, V) = g_W(D_T U, V) + g_W(U, D_T V)$$

when U or V is proportional to W.

PROPOSITION 3.1. *Let $\alpha(t)$, $\beta(t)$ be two geodesics on $\widetilde{M}$; then the distance function $d(\alpha(t), \beta(t))$ is convex in t.*

Proof. First we prove the special case when the geodesics $\alpha(t)$ and $\beta(t)$ emanate from a point p. We want to prove that $d(\alpha(T/2), \beta(T/2)) \leq \frac{1}{2} d(\alpha(T), \beta(T))$ for any $T \in \mathbb{R}$.

Let $\gamma : [0, S] \to \widetilde{M}$ be the unique geodesic connecting $\alpha(T)$ and $\beta(T)$. Define $\Sigma : [0, T] \times [0, S] \to \widetilde{M}$ by

$$\Sigma(t, s) = \exp_{\gamma(s)}\left(\left(1 - \frac{t}{T}\right) \exp_{\gamma(s)}^{-1}(p)\right).$$

Then Σ is a geodesic variation with $\Gamma(\cdot, 0) = \alpha$ and $\Gamma(\cdot, S) = \beta$. $J_s(t) := (\partial/\partial s)\Sigma(t, s)$ is a Jacobi field along the t-curve. Since the flag curvature $K \leq 0$, there is no conjugate point. $J_s(0) = 0$, so $J_s(t) \neq 0$ for $t \in (0, T]$. Let T_s be the velocity field of $\Sigma(\cdot, s)$. Using the product rule and $F(J_s) = [g_{J_s}(J_s, J_s)]^{1/2}$, we have

$$\frac{d^2}{dt^2} F(J_s) = \frac{d}{dt}\left[\frac{g_{J_s}(D_{T_s} J_s, J_s)}{F(J_s)}\right]$$

$$= \frac{g_{J_s}(D_{T_s} D_{T_s} J_s, J_s) F^2(J_s) + g_{J_s}(D_{T_s} J_s, D_{T_s} J_s) F^2(J_s) - g_{J_s}^2(D_{T_s} J_s, J_s)}{F^3(J_s)}. \quad (3)$$

By the Jacobi equation, the formula for flag curvature and the Schwarz inequality, the first term in the numerator of (3) is

$$g_{J_s}(D_{T_s} D_{T_s} J_s, J_s) = -g_{J_s}(R(J_s, T_s)T_s, J_s)$$

$$= -g_{J_s}(R(T_s, J_s)J_s, T_s)$$

$$= -K(J_s, T_s) \cdot [g_{J_s}(J_s, J_s)g_{J_s}(T_s, T_s) - g_{J_s}^2(J_s, T_s)]$$

$$\geq 0.$$

The last two terms in the numerator of (3) are also non-negative by the Schwarz inequality. Hence,

$$\frac{d^2}{dt^2} F(J_s)(t) \geq 0 \quad \text{for all } t \in (0, T].$$
(4)

$F(J_s)(t)$ is C^∞ on $(0, 1]$ and continuous on $[0, 1]$, so, by (4),

$$F(J_s)\left(\frac{T}{2}\right) \leq \frac{1}{2} F(J_s)(T).$$

Hence,

$$d\left(\alpha\left(\frac{T}{2}\right), \beta\left(\frac{T}{2}\right)\right) \leq \int_0^S F(J_s)\left(\frac{T}{2}\right) ds$$

$$\leq \frac{1}{2} \int_0^S F(J_s)(T) \, ds = \frac{1}{2} d(\alpha(T), \beta(T)).$$
(5)

We consider another special case when $\alpha(T) = \beta(T)$ for some $T > 0$. A construction similar to the one above shows that

$$F(J_s)\left(\frac{T}{2}\right) \leq \frac{1}{2} F(J_s)(0)$$

and hence

$$d\left(\alpha\left(\frac{T}{2}\right), \beta\left(\frac{T}{2}\right)\right) \leq \frac{1}{2} d(\alpha(0), \beta(0)).$$
(6)

For the general case, let γ be the geodesic segment connecting $\beta(t_1)$ and $\alpha(t_2)$ and q be the mid point of γ. Then

$$d\left(\alpha\left(\frac{t_1 + t_2}{2}\right), \beta\left(\frac{t_1 + t_2}{2}\right)\right) \leq d\left(\alpha\left(\frac{t_1 + t_2}{2}\right), q\right) + d\left(q, \beta\left(\frac{t_1 + t_2}{2}\right)\right)$$

$$\leq \frac{1}{2} d(\alpha(t_1), \beta(t_1)) + \frac{1}{2} d(\alpha(t_2), \beta(t_2)).$$
(7)

The second inequality in (7) follows from (5) and (6). As the distance function is continuous, the $\frac{1}{2}$-convexity implies convexity. □

A geodesic space with the convexity property (5) above is said to be of *non-positive curvature in the sense of Busemann* (globally in our case since $\widetilde{M}$ is simply connected). It is conjectured that Finsler manifolds of non-positive curvature in the sense of Busemann must be of Berwald type; see [14].

Non-positive curvature in the sense of Busemann is a weaker notion than *non-positive curvature in the sense of Aleksandrov*; see [13] for definitions. A Berwald space of non-positive flag curvature is not necessarily of non-positive curvature in the sense of Aleksandrov; see [14]. In fact, if a reversible Finsler manifold is of non-positive curvature in the sense of Aleksandrov, then it must be a Riemannian manifold of non-positive sectional curvature. Hence, we cannot define the Aleksandrov angle for a Berwald space of non-positive flag curvature. Nevertheless, we still can define a notion of angle in the next subsection. This angle notion is enough to play the role of angles in Riemannian manifolds in the proof of the higher rank rigidity theorem.

3.2. *A notion of angle.*   Based on the convexity property, we define a notion of angle to measure the distance between two directions. Consider a point $p \in \widetilde{M}$ and two geodesics $c_1(t)$ and $c_2(t)$ emanating from p. By convexity, the function $d(c_1(t), c_2(t))/t$ is non-decreasing in t. So, the limit

$$\lim_{t \to 0} \frac{d(c_1(t), c_2(t))}{t}$$

exists. By the triangle inequality of d, the limit is bounded by 2.

Definition 3.2. Let $u, v \in T_p^1 \widetilde{M}$ and $c_1(t)$ and $c_2(t)$ be two geodesics emanating from p such that $c_1'(0) = u$ and $c_2'(0) = v$. The angle between u and v is defined as

$$\angle_p(u, v) := \lim_{t \to 0} \frac{d(c_1(t), c_2(t))}{t}.$$

The tangent space $T_p \widetilde{M}$ equipped with Minkowski norm $F_p := F(p, \cdot)$ is a Finsler manifold if we identify $T_y(T_p \widetilde{M})$ with $T_p \widetilde{M}$ itself and provide it with the norm $F(p, \cdot)$. We also call this Finsler manifold a Minkowski space. In Minkowski space, the Finsler metric F has no x dependence in natural coordinates. It turns out that the Chern connection coefficient $\Gamma_{ij}^k = 0$ and hence the flag curvature is always zero. Moreover, the geodesics in Minkowski space are lines. In Minkowski space $(T_p \widetilde{M}, F_p)$, let $\tilde{c}_1(t)$ and $\tilde{c}_2(t)$ be the two geodesic rays emanating from the origin in the directions u and v. Then we have the following.

PROPOSITION 3.3.

$$\lim_{t \to 0} \frac{d(c_1(t), c_2(t))}{t} = \lim_{t \to 0} \frac{d(\tilde{c}_1(t), \tilde{c}_2(t))}{t}.$$

Proof. For Berwald space, the exponential map $\exp_p$ is C^∞ on $T_p \widetilde{M}$ and $d \exp_p |_p = \text{Id}$; see [**5**].

Consider $\tilde{\Sigma} : [0, 1] \times [0, 1] \to T_p \widetilde{M}$ defined by

$$\tilde{\Sigma}(s, t) = t[(1 - s)u + sv].$$

Let $\Sigma(s, t) = \exp_p(\tilde{\Sigma}(s, t))$ be defined on $\widetilde{M}$. We have

$$\lim_{t \to 0} \frac{d(c_1(t), c_2(t))}{t} \leq \lim_{t \to 0} \frac{\int_0^1 F(\Sigma(s, t), (\partial/\partial s)\Sigma(s, t)) \, ds}{t}$$

$$= \lim_{t \to 0} \int_0^1 F\left(\Sigma(s, t), \frac{1}{t} d \exp_p \Big|_{\tilde{\Sigma}(s,t)} \left[\frac{\partial}{\partial s} \tilde{\Sigma}(s, t)\right]\right) ds$$

$$= \lim_{t \to 0} \int_0^1 F(\Sigma(s, t), d \exp_p |_{\tilde{\Sigma}(s,t)}(v - u)) \, ds$$

$$= \int_0^1 F(p, v - u) \, ds$$

$$= \lim_{t \to 0} \frac{d(\tilde{c}_1(t), \tilde{c}_2(t))}{t}. \tag{8}$$

We used continuity of $d \exp_p$ and F in the third equality in (8).

Conversely, consider $\sigma : [0, 1] \times [0, 1] \to \widetilde{M}$ defined by

$$\sigma(s, t) = \exp_{c_1(t)}[s \exp_{c_1(t)}^{-1} c_2(t)].$$

Let $\tilde{\sigma}(s, t) = \exp_p^{-1}[\sigma(s, t)]$ be defined on $T_p\widetilde{M}$. We have

$$
\begin{aligned}
\lim_{t \to 0} \frac{d(c_1(t), c_2(t))}{t} &= \lim_{t \to 0} \frac{\int_0^1 F(\sigma(s, t), (\partial/\partial s)\sigma(s, t))\, ds}{t} \\
&= \lim_{t \to 0} \int_0^1 F\left(\sigma(s, t), \frac{1}{t} d \exp_p \Big|_{\tilde{\sigma}(s,t)} \left[\frac{\partial}{\partial s}\tilde{\sigma}(s, t)\right]\right) ds \\
&= \lim_{t \to 0} \int_0^1 F\left(p, \frac{1}{t}\frac{\partial}{\partial s}\tilde{\sigma}(s, t)\right) ds \\
&\geq \lim_{t \to 0} \frac{d(\tilde{c}_1(t), \tilde{c}_2(t))}{t}.
\end{aligned}
\tag{9}
$$

At the third equality in (9), we used the smoothness of $\exp_p$ (at least C^2) and teh smoothness of F (at least C^1) away from the zero section. □

COROLLARY 3.4. *Let $u, v \in T_p^1\widetilde{M}$. Then $\angle_p(u, v) = F(p, u - v)$.*

Proof. This is an immediate corollary of Proposition 3.3 since

$$\lim_{t \to 0} \frac{d(\tilde{c}_1(t), \tilde{c}_2(t))}{t} = F(p, u - v)$$

in Berwald space $\widetilde{M}$. □

COROLLARY 3.5. $d(c_1(t), c_2(t)) \geq d(\tilde{c}_1(t), \tilde{c}_2(t))$.

Proof. This follows from Proposition 3.3 and that $(1/t)d(c_1(t), c_2(t))$ is non-decreasing in t, but $(1/t)d(\tilde{c}_1(t), \tilde{c}_2(t))$ is a constant. □

Proposition 3.3 implies that the angle is indeed a metric on $T_p^1\widetilde{M}$ and it induces the usual topology on $T_p^1\widetilde{M}$ for any $p \in \widetilde{M}$. In particular, the angle metric is complete on $T_p^1\widetilde{M}$.

3.3. *Sphere at infinity.* Two geodesics γ and δ are called asymptotes if $d(\gamma(t), \delta(t)) \leq c$ for some constant c and for all $t \geq 0$. We also call $u, v \in T^1\widetilde{M}$ asymptotic if γ_u and γ_v, the geodesics with initial vectors u and v, respectively, are asymptotic.

LEMMA 3.6. *Let $c_1(t), c_2(t)$ be two distinct geodesics emanating from $p \in \widetilde{M}$. Then*

$$d(c_1(t), c_2(t)) \to \infty \quad for\ t \to \infty.$$

Proof. It follows from the convexity of the function $d(c_1(t), c_2(t))$. □

Similar to the Riemannian case, we have the following.

PROPOSITION 3.7. *Let $\gamma(t)$ be a geodesic on $\widetilde{M}$. For any $p \in \widetilde{M}$, there exists a unique geodesic starting at p and asymptotic to γ.*

Proof. Same proof as in [**9**, Proposition 1.2]. Convexity of the distance function $d(\gamma(t), \delta(t))$ is used in the uniqueness part. □

The asymptotes relation is an equivalence relation. Denote by $\widetilde{M}(\infty)$ the set of all equivalence classes. By Proposition 3.7, there exists a unique geodesic, denoted by γ_{px}, connecting $p \in \widetilde{M}$, $x \in \widetilde{M}(\infty)$. So, we can define a topology on $\widetilde{M}(\infty)$ such that the map $x \to \gamma'_{px}(0)$ is a homeomorphism between $\widetilde{M}(\infty)$ and $T^1_p\widetilde{M}$. This topology on $\widetilde{M}(\infty)$ is independent of p and we call $\widetilde{M}(\infty)$ the sphere at infinity. Moreover, for any $p \in \widetilde{M}$, $x, y \in \widetilde{M} \cup \widetilde{M}(\infty)$ ($p \neq x$, $p \neq y$), we can define $\angle_p(x, y) = \angle_p(\gamma'_{px}(0), \gamma'_{py}(0))$. The angle $\angle_p(x, y)$ depends continuously on p, x and y.

Two geodesics α and β are said to be parallel if $d(\alpha(t), \beta(t)) \leq c$ for some constant c and for all $t \in \mathbb{R}$. In fact, by convexity, for parallel geodesics α and β, $d(\alpha(t), \beta(t)) \equiv c$ for some constant c and for all $t \in \mathbb{R}$. We call u, v parallel if γ_u and γ_v are parallel geodesics. $u \parallel v$ if and only if u is asymptotic to v and $-u$ is asymptotic to $-v$.

LEMMA 3.8. *Let γ be an arbitrary geodesic in $\widetilde{M}$ and $\gamma'(0) = v$. Let $x \in \widetilde{M}(\infty)$. For any $t \geq 0$, let $w(t)$ be the unique vector at point $\gamma(t)$ such that $\gamma_{w(t)}(\infty) = x$. Then*

$$\angle_{\gamma(0)}(v, w(0)) \leq \angle_{\gamma(t)}(g^t(v), w(t))$$

for any $t > 0$.

Proof. We fix $t > 0$ and let $\alpha(s)$, $\beta(s)$ be the geodesics emanating from $\gamma(0)$ and $\gamma(t)$, respectively, such that $\alpha(\infty) = \beta(\infty) = x$. Let $s > 0$. Denote $d(t) = d(\gamma(t), \alpha(t))$, $d(s + t) = d(\gamma(s + t), \alpha(s + t))$, $d_1(s) = d(\gamma(t + s), \beta(s))$, $c(s, t) = d(\alpha(t + s), \beta(s))$.

By the triangle inequality, $d(s + t) \leq c(s, t) + d_1(s)$. Since α and β are asymptotic, the distance function between α and β (with a time shift t) is decreasing. Thus, $c(s, t) \leq d(t)$. We have

$$\frac{d(t)}{t} \leq \frac{d(s + t)}{s + t} \leq \frac{c(s, t) + d_1(s)}{s + t} \leq \frac{d(t) + d_1(s)}{s + t}.$$

Hence, $d(t)(s + t) \leq t(d(t) + d_1(s))$, which gives $d(t)/t \leq d_1(s)/s$. Let $s \to 0$; then we have

$$\frac{d(t)}{t} \leq \angle_{\gamma(t)}(g^t v, w(t)).$$

But, by convexity, $\angle_{\gamma(0)}(u, w(0)) = \lim_{r \to 0} (d(r)/r) \leq d(t)/t$. Hence,

$$\angle_{\gamma(0)}(v, w(0)) \leq \angle_{\gamma(t)}(g^t v, w(t)). \qquad \square$$

The equality holds in Lemma 3.8 if and only if the two geodesics γ and α bound an area which is totally geodesic and flat. In fact, we can also prove for any geodesic triangle in $\widetilde{M}$ that an exterior angle is larger than the corresponding interior angle by a similar proof; then Lemma 3.8 also follows from this fact and the continuity of angle functions. Lemma 3.8 will be used in the proof of the higher rank rigidity theorem.

Now we can define a (Tits) metric on $\widetilde{M}(\infty)$. Let $x, y \in \widetilde{M}(\infty)$. Choose arbitrary $p \in \widetilde{M}$ and let α and β be the two geodesics emanating from p such that $\alpha(\infty) = x$ and $\beta(\infty) = y$. Denote $d(t) = d(\alpha(t), \beta(t))$.

Definition 3.9. Define

$$d(x,\, y) = \lim_{t \to \infty} \frac{d(t)}{t}.$$

It is easy to prove that limit above is independent of the choice of p and $d(x,\, y)$ is indeed a metric on $\widetilde{M}(\infty)$.

3.4. *Flat-strip lemma.* A flat strip means a totally geodesic isometric imbedding $r : \mathbb{R} \times [0,\, c] \to \widetilde{M}$, where $\mathbb{R} \times [0,\, c]$ is a strip in a Minkowski plane.

LEMMA 3.10. *If two distinct geodesics α and β are parallel, then they bound a flat strip in $\widetilde{M}$.*

Proof. Since the geodesic $\beta(t),\, t \in \mathbb{R}$ is a convex set, there exists a unique foot point of $\alpha(0)$ on β. We may suppose that β is parameterized such that $\beta(0)$ is the foot point of $\alpha(0)$ on β. Since α and β are parallel, for any $a \in \mathbb{R}$, $d(\alpha(t),\, \beta(a+t)) \equiv c$. But $d(\alpha(0),\, \beta(a)) \geq d(\alpha(0),\, \beta(0))$, so $d(\alpha(t),\, \beta(a+t)) \geq d(\alpha(t),\, \beta(t))$ for any $a \in \mathbb{R}$, i.e. $\beta(t)$ is the foot point of $\alpha(t)$ on the geodesic β. Furthermore, it also follows that $\alpha(t)$ is the foot point of $\beta(t)$ on the geodesic α. Choose arbitrary $t_1,\, t_2 \in \mathbb{R}$ and let $\gamma(s)$ and $\delta(s)$ be the geodesics connecting $\alpha(t_1)$ and $\beta(t_1)$, $\alpha(t_2)$ and $\beta(t_2)$, respectively. Denote $d(\alpha(t),\, \beta(t)) \equiv c$. Consider $\Sigma : [t_1,\, t_2] \times [0,\, c] \to \widetilde{M}$ defined by

$$\Sigma(t,\, s) = \exp_{\gamma(s)} \left(\frac{t - t_1}{t_2 - t_1} \cdot \exp_{\gamma(s)}^{-1} \delta(s) \right).$$

Then Σ is a geodesic variation with all t-curves geodesics.

We first prove that all s-curves are also geodesics. Let $J_s(t)$ be the variation vector field; then, by the same computation as in Proposition 3.1, we have

$$\frac{d^2}{dt^2} F(J_s)(t) \geq 0 \quad \text{for all } t \in [t_1,\, t_2].$$

So, the length of the s-curves $L(\Sigma(t,\, \cdot))$ is convex in t. But $L(\Sigma(t_1,\, \cdot)) = L(\Sigma(t_2,\, \cdot)) = c$ and hence $L(\Sigma(t,\, \cdot)) = c = d(\alpha(t),\, \beta(t))$ for all $t \in [t_1,\, t_2]$. So, all s-curves are geodesics. It also follows that $(d^2/dt^2) F(J_s)(t) = 0$ for all $t \in [t_1,\, t_2]$; hence, the flag curvature $K = 0$ on $\Sigma([t_1,\, t_2] \times [0,\, c])$ for any $t_1 < t_2$.

Next we prove that $\Sigma([t_1,\, t_2] \times [0,\, c])$ is totally geodesic. It is enough to prove that the geodesic segment connecting any two points $\Sigma(t',\, s')$ and $\Sigma(t'',\, s'')$ lies in $\Sigma([t_1,\, t_2] \times [0,\, c])$. In fact, it is enough to prove that the geodesic segment $\kappa(t)$ connecting $\alpha(t_1)$ and $\beta(t_2)$ lies in $\Sigma([t_1,\, t_2] \times [0,\, c])$. Suppose that $\kappa(t)$ has speed $d(\alpha(t_1),\, \beta(t_2))/(t_2 - t_1)$. Since $\kappa(t)$ and $\alpha(t)$ are two geodesics emanating from $\alpha(t_1)$, we can construct a geodesic variation Σ_1 as in Proposition 3.1 and it follows that $L(\Sigma_1(t,\, \cdot)) \leq (t - t_1)/(t_2 - t_1) \cdot c$. Similarly, β and κ are two geodesics merging at $\beta(t_2)$ and we can construct a geodesic variation Σ_2 such that $L(\Sigma_2(t,\, \cdot)) \leq (t_2 - t)/(t_2 - t_1) \cdot c$. Hence, $L(\Sigma_1(t,\, \cdot)) + L(\Sigma_2(t,\, \cdot)) \leq (t - t_1)/(t_2 - t_1) \cdot c + (t_2 - t)/(t_2 - t_1) \cdot c = c = d(\alpha(t),\, \beta(t))$. It follows that the joining of $\Sigma_1(t,\, \cdot)$ and $\Sigma_2(t,\, \cdot)$ is the geodesic connecting $\alpha(t)$ and $\beta(t)$ and hence $\kappa(t)$ lie on this geodesic. Hence, $\kappa(t)$ lies in $\Sigma([t_1,\, t_2] \times [0,\, c])$.

Since $\Sigma([t_1,\, t_2] \times [0,\, c])$ is totally geodesic and the flag curvature $K = 0$, it follows that $\Sigma([t_1,\, t_2] \times [0,\, c])$ is an imbedded Minkowski rectangle. Since t_1 and t_2 are arbitrary, the lemma follows. $\square$

4. *Higher rank rigidity*

We shall prove the higher rank rigidity theorem for Berwald spaces in this section. Let $(\widetilde{M}, F)$ always be a complete, simply connected Berwald space with flag curvature $K \leq 0$.

Moreover, if not pointed out specifically, we suppose that $\widetilde{M}$ admits a quotient manifold M of finite volume and the flag curvature is bounded from below, $-b^2 \leq K \leq 0$. The finiteness of the volume of M guarantees the finiteness of the Liouville measure of $T^1 M$.

The volume of a Finsler manifold can be defined as follows. Since the Sasaki metric is a Riemannian metric on TM, it induces a volume form on TM:

$$dV_g = \det(g_{ij}(x, y)) \, dx^1 \cdots dx^n \, dy^1 \cdots dy^n,$$

which gives the Liouville measure on $T^1 M$ preserved by the geodesic flow. Hence, we can define a Finsler volume form on M by

$$dV_F := \sigma_F(x) \, dx^1 \cdots dx^n,$$

where $\sigma_F(x) := (\int_{B^n_x} \det(g_{ij}(x, y)) \, dy^1 \cdots dy^n)/(\mathrm{Vol}(\mathcal{B}^n))$, $B^n_x := \{y \in T_x M | F(y) \leq 1\}$ and $\mathcal{B}^n$ is a Euclidean unit ball. In such a way, we have

$$\int_{T^1 M} f \, dV_g = \int_M dV_F(x) \int_{T^1_x M} f|_{T^1_x M} \, dS^{n-1},$$

where dS^{n-1} is the Euclidean volume element on the $(n-1)$-sphere. The above-defined volume measure V_F is called the *Sasaki volume*. It was also discovered by Holmes and Thompson by a Minkowski geometry approach. Consequently, it is also called the *Holmes–Thompson volume*. There are other notions of volume such as the Busemann–Hausdorff volume. The Sasaki volume has the following advantage: it is easy to see that the Liouville measure is finite for a Finsler manifold of finite Sasaki volume. See [**16**, §5.1] for various notions of volume.

From now on, we will only consider the non-trivial case when $(\widetilde{M}, F)$ is not a Minkowski space, i.e. $\mathrm{rank}(\widetilde{M}) < \dim \widetilde{M}$. We follow the structure of [**2**, **3**, **7**]. Firstly, in §4.1, we give the definition of rank and study the properties of Jacobi fields for Berwald spaces. In §4.2, we prove that a higher rank Berwald space admits a great deal of k-flats, that is, complete, flat, totally geodesic k-dimensional submanifolds without boundary where $k = \mathrm{rank}(\widetilde{M})$. In the third subsection, we construct strong stable manifolds for regular vectors. Next in, §4.4, we construct Weyl chambers, $k-1$ first integrals and prove a closing lemma. Following that, we prove that all Weyl chambers are isometric to each other and can be extended to $\widetilde{M}(\infty)$, the boundary of $\widetilde{M}$ at infinity. In §4.5, we prove that $\widetilde{M}(\infty)$ has a structure of a Tits building. And, the higher rank rigidity theorem follows from a similar argument of Gromov in the final §4.6.

We try to keep the convention of notation in [**2**, **3**, **7**]. We mainly focus on the difference between the Berwald case and the Riemannian case, but just simply give references to [**2**, **3**, **7**] if the proof of a result can be extended *verbatim* to the Berwald case.

4.1. *Rank and more on Jacobi fields.*

As in the Riemannian case, a parallel Jacobi field $J(t)$ along a geodesic γ (with velocity field T) satisfies $D_T J = 0$ with reference vector T. But, since M is a Berwald space, the reference vector is irrelevant here.

Definition 4.1. Let $v \in T^1 M$. We define rank(v) as the dimension of the space of parallel Jacobi fields along geodesic γ_v and rank$(M) := \min_{v \in T^1 M}$ rank(v).

The operator $R(T, \cdot)T$ is symmetric and negatively semidefinite with respect to $g_T := g_{ij}(\sigma, T)\, dx^i \otimes dx^j$. Hence, the Jacobi field $J(t)$ is parallel if and only if $K(T, J) = 0$. So, the rank of a vector reflects infinitesimal flatness along the geodesic. The notions above can be extended to the universal cover space $\widetilde{M}$.

By a limit argument, rank$(w) \leq$ rank(v) for all vectors w which are in a sufficiently small neighborhood of a given vector $v \in T^1 M$. The topology of $T^1 M$ can be given by the Sasaki metric. When restricted to a fiber $T_p^1 M$, this topology coincides with the one induced by angle distance (see Definition 3.2). As in [**2**], we define a set of regular vectors $\Re := \{v : \text{rank}(w) = \text{rank}(v) \text{ for all } w \text{ sufficiently close to } v\}$ and

$$\Re_m := \{v \in \Re : \text{rank} = m\}.$$

Obviously, $\Re$ is open and dense in $T^1 M$ and $\Re_m$ is open.

A computation from the Jacobi equation using $K \leq 0$ gives the convexity of $\|J(t)\|_T = \sqrt{g_{T(t)}(J(t), J(t))}$ in t. So, we have three special classes of Jacobi fields along geodesic γ_v.

(1) $J^s(v)$: the space of stable Jacobi fields with $\|J(t)\|_T$ non-increasing on $\mathbb{R}$ and hence $\|J(t)\|_T \leq c$ for some constant c and for all $t \geq 0$.

(2) $J^u(v)$: the space of unstable Jacobi fields with $\|J(t)\|_T$ non-decreasing on $\mathbb{R}$ and hence $\|J(t)\|_T \leq c$ for some constant c and for all $t \leq 0$.

(3) $J^p(v)$: the space of parallel Jacobi fields with $\|J(t)\|_T = c$ for some constant c and for all $t \in \mathbb{R}$.

There is a fundamental difference from the Riemannian case. When we deal with Jacobi fields, even though the reference vector is irrelevant here, the fundamental tensor g_T must be evaluated at T. For example, (part of) the Rauch comparison theorem for Finsler manifolds is formulated as follows.

PROPOSITION 4.2. *Let $J(t)$ be a Jacobi field along a geodesic γ (with velocity vector T) with $J(0) = 0$ and $g_T(T, J) = 0$. If $-a^2 \leq K \leq 0$, then, for any $0 < t_1 \leq t_2$,*

$$\frac{t_2}{t_1} \leq \frac{\|J(t_2)\|_{T(t_2)}}{\|J(t_1)\|_{T(t_1)}} \leq \frac{s_a(t_2)}{s_a(t_1)},$$

where $\|\cdot\|_T := \sqrt{g_T(\cdot, \cdot)}$ and $s_a(t) = (1/a)\sinh(at)$.

The length of $J(t)$ is evaluated with reference vector T. So, if we want to estimate the distance function d from the Rauch theorem, we need to deal with the reference vector issue. Nevertheless, we have a coarse estimation of distance due to the following observation.

PROPOSITION 4.3. *There exists a uniform constant C_0 for $\widetilde{M}$ such that for all $p \in \widetilde{M}$, for all $v, w \in T_p^1 \widetilde{M}$, we have*

$$\frac{1}{C_0} \|\cdot\|_v \leq \|\cdot\|_w \leq C_0 \|\cdot\|_v.$$

Proof. Fix a point $p \in \widetilde{M}$; since $T_p^1 \widetilde{M}$ is a compact space, there exists $C_0(p)$ such that

$$\frac{1}{C_0(p)} \| \cdot \|_v \leq \| \cdot \|_w \leq C_0(p) \| \cdot \|_v$$

for all v, $w \in T_p^1 \widetilde{M}$. By Corollary 2.6, a parallel translation induces a Riemannian isometry $(T_p \widetilde{M} \setminus 0, \hat{g}_p) \to (T_q \widetilde{M} \setminus 0, \hat{g}_q)$. So, in fact, C_0 can be chosen to be independent of p. $\quad\square$

Remark 4.4. In [**10**], a Finsler manifold with the property in Proposition 4.3 is called a uniform Finsler manifold. For example, compact Finsler spaces, their cover spaces and Berwald spaces are all uniform. The advantage of a uniform Finsler manifold is that we can have a coarse estimation of the distance function by overcoming the reference vector issue partially. In particular, in the strictly negative curvature case, it works very well and many properties in Riemannian geometry can be extended to the uniform Finsler case; see [**10**].

4.2. *Construction of flats.* Suppose that $\mathrm{rank}(\widetilde{M}) \geq 2$ in the rest of this paper. We integrate parallel Jacobi fields to obtain the flats. Recall that Proposition 2.4 builds an isomorphism between $TT^1\widetilde{M}$ and the space of Jacobi fields. Denote by $\wp(v)$ the preimage of $J^p(v)$ under this isomorphism. For each $m \geq \mathrm{rank}(\widetilde{M})$, the distribution $\wp$ has dimension locally constant m on $\Re_m$. We shall prove that $\wp$ is integrable on $\Re_m$.

LEMMA 4.5. [**2**, Lemma 2.1] *The distribution $\wp$ is smooth on $\Re_m$.*

Proof. The proof of Lemma 2.1 in [**2**] can be adapted here. We just need to replace the Euclidean inner product $\langle \cdot, \cdot \rangle$ in the symmetric bilinear form by $g_T(\cdot, \cdot)$. For each vector $v \in \Re_m$ and any $S > 0$, consider the symmetric bilinear form

$$Q_S^v(X, Y) = \int_{-S}^{S} g_T(R(X, T)T, Y) \, dt,$$

where $T(t) = \gamma_v'(t)$. For a small neighborhood U of $v \in \Re_m$, there exists a large number $S > 0$ such that the nullspace of the form Q_S^w is exactly $J^p(w)$ and has constant dimension m for any $w \in U$. So, $\wp$ depends smoothly on w. $\quad\square$

Given $v \in T^1\widetilde{M}$, let

$$P(v) = \{w \in T^1\widetilde{M} : w \text{ is parallel to } v\}$$

and $F(v) = \pi(P(v))$, where $\pi : T^1\widetilde{M} \to \widetilde{M}$ is the natural projection. By the flat strip lemma (Lemma 3.10), $F(v)$ is a union of flat strips.

LEMMA 4.6. *$\wp$ is integrable on $\Re_m$, $m \geq \mathrm{rank}(\widetilde{M})$. The maximal arc-connected integral manifold through $v \in \Re_m$ is an open subset of $P(v)$.*

Proof. See [**2**, Lemma 2.2], where the Frobenius theorem is used to show the integrability. For Berwald space, we need a coarse estimation to show that the curves tangent to $\wp$ are exactly those curves consisting of parallel vectors.

(1) Let $\sigma : (-\epsilon, \epsilon) \to \Re_m$ be an integral curve of $\wp$; then $\sigma(s)$ is parallel to $\sigma(0) = v$ for all s.

In fact, consider the geodesic variation $\gamma_s(t) = (\pi \circ g^t)(\sigma(s))$ of $\gamma_0 = \gamma_v$. So, the variation vector fields

$$Y_s(t) = (d\pi \circ dg^t)\left(\frac{d}{ds}\sigma(s)\right)$$

are parallel Jacobi fields along γ_s by definition of $\wp$. Then, for any $t \in \mathbb{R}$,

$$d(\gamma_0(t), \gamma_s(t)) \le \int_0^s F(Y_u(t))\, du$$

$$\le C_0 \int_0^s \|Y_u(t)\|_T \, du = C_0 \int_0^s \|Y_u(0)\|_T \, du,$$

where C_0 is the uniform constant in Proposition 4.3 and the last equality is because $Y_u(t)$ are parallel Jacobi fields for any u. Hence, γ_0 and γ_s are parallel geodesics. So, $\sigma(s)$ is parallel to $\sigma(0) = v$ for all s.

(2) Conversely, if $\sigma(s) \in T^1\widetilde{M}$ is parallel to $\sigma(0) = v$ for all s, then $\sigma(s)$ is an integral curve of $\wp$.

In fact, consider the geodesic variation $\gamma_s(t) = (\pi \circ g^t)(\sigma(s))$ of $\gamma_0 = \gamma_v$. $\gamma_s(t)$ are parallel geodesics. Fix arbitrary s_0; then $d(\gamma_{s_0}(t), \gamma_s(t)) = c(s)$ for all t and some constant $c(s)$ dependent on s. By the Busemann–Mayer theorem,

$$F(J_{s_0}(t)) = \lim_{h \to 0^+} \frac{d(\gamma_{s_0+h}(t), \gamma_{s_0}(t))}{h} = \lim_{h \to 0^+} \frac{c(s_0 + h)}{s_0}.$$

Then we have $F(J_{s_0}(t)) \le C$ for any t and some constant C dependent on the function $c(s)$. Hence, $\|J_{s_0}(t)\|_T \le C_0 C$ for any $t \in \mathbb{R}$. This implies that $J_{s_0}(t)$ is a parallel Jacobi field along the geodesic γ_{s_0} for arbitrary fixed s_0. Hence, $\sigma(s)$ is an integral curve of $\wp$.

The rest of the argument is the same as in [**2**, Lemma 2.2]. □

Moreover, given $w \in P(v) \cap \Re_m$, it follows that $P(v)$ and $F(v)$ are smooth m-dimensional manifolds near w and $\pi(w)$, respectively (see [**2**, Lemma 2.2]). Globally, $F(v)$ is a union of flat strips and it is closed and convex. The following proposition says that if M has finite volume, these flat strips join very well to form an m-flat.

PROPOSITION 4.7. *Suppose that $\widetilde{M}$ admits a quotient manifold M of finite Finsler volume. Then, for every $v \in \Re_m$, the set of $F(v)$ is an m-flat.*

The following two lemmas are needed to prove Proposition 4.7. The first one is an analogue of [**2**, Lemma 2.4]. The notion of orthogonality is with respect to $g_T(\cdot, \cdot)$ instead of the Euclidean inner product $\langle \cdot, \cdot \rangle$ now. Keeping this in mind, the proof of the following lemma is standard. Similar properties of Jacobi fields can be found in [**5**, §5.4].

LEMMA 4.8. *Let N^* be a totally geodesic Berwald submanifold of a Berwald space N. Let γ be a geodesic of N that lies in N^* and let J be a Jacobi field in N along γ. Let J_1 and J_2 denote the components of Y tangent and orthogonal to N^*, respectively. Then J_1 and J_2 are Jacobi fields in N along γ.*

Let $M = \widetilde{M}/\Gamma$ have finite volume; so does $T^1 M$ and hence every vector in $T^1 M$ is non-wandering relative to the geodesic flow. Lifted to $T^1\widetilde{M}$, for any $w \in T^1\widetilde{M}$, there exist $\{t_n\} \to +\infty$, $w_n \to w$ and $\{\phi_n\} \subseteq \Gamma$, the deck group of M, such that

$(d\phi_n \circ g^{t_n})(w_n) \to w$ as $n \to +\infty$. The following lemma is also needed to prove Proposition 4.7. A key ingredient in the proof is Proposition 3.7 for the Berwald case and the rest is straightforward.

LEMMA 4.9. [**2**, Lemma 2.5] *Let $M = \widetilde{M}/\Gamma$ have finite volume. Let v, w be asymptotic vectors in $T^1\widetilde{M}$. Then there exist $\{s_n\} \to +\infty$, $v_n \to v$ and $\{\phi_n\} \subseteq \Gamma$ such that $(d\phi_n \circ g^{s_n})(v_n) \to w$ as $n \to +\infty$.*

Proof of Proposition 4.7. Lemma 4.9 is used to prove that for arbitrary $q \in F(v)$, $F(v)$ is an m-dimensional manifold near q. Since, as *a priori*, $F(v)$ is already closed and convex, it follows that $F(v)$ is a complete, totally geodesic m-dimensional submanifold without boundary. See [**2**, proof of Proposition 2.3]. □

Next we prove that all the regular vectors have rank all equivalent to rank($\widetilde{M}$) = k. Here we also borrow a dynamical tool. We say that $v \in T^1\widetilde{M}$ is recurrent if there exists $\{\phi_n\} \subseteq \Gamma$ and $\{t_n\} \to \infty$ such that $d\phi_n \circ g^{t_n}(v) \to v$ as $n \to +\infty$. When $\widetilde{M}$ admits a quotient manifold of finite volume, all recurrent vectors form a dense G_δ set in $T^1\widetilde{M}$.

We need the following four lemmas.

LEMMA 4.10. [**2**, Lemma 2.7] *For $v \in \mathfrak{R}$, denote $Q(v) = S_{\pi(v)}F(v)$. Then:*
(1)   *if $\mathfrak{I}$ is a dense open subset in $T^1\widetilde{M}$, then*

$$\mathfrak{R}_0 = \{v \in \mathfrak{R} : Q(v) \cap \mathfrak{I} \text{ is a dense open subset of } Q(v)\}$$

   is a dense G_δ set in $\mathfrak{R}$;
(2)   *if $\mathfrak{I}$ is a subset of full measure in $T^1\widetilde{M}$, then*

$$\mathfrak{R}_1 = \{v \in \mathfrak{R} : Q(v) \cap \mathfrak{I} \text{ has full measure in } Q(v)\}$$

   is a subset of full measure in $\mathfrak{R}$.

We remark that for the Berwald case, the topology and measure on $T^1\widetilde{M}$ above are both equivalent to the ones in the Riemannian case. Thus, the proof of Lemma 2.7 in [**2**] for the Riemannian case can be adapted here.

LEMMA 4.11. [**2**, Lemma 2.8] *Let $v \in \mathfrak{R}$ be recurrent and choose $\{\phi_n\} \subseteq I(\widetilde{M})$ and $\{t_n\} \to +\infty$ such that $d\phi_n \circ g^{t_n}(v) \to v$ as $n \to \infty$. Let $z \in \widetilde{M}(\infty)$ be arbitrary; then all cluster points of the sequence $\{\phi_n(z)\}$ lie in $F(v)(\infty)$.*

A key ingredient in the proof of the above lemma is Lemma 3.8 for the Berwald case, which is an important property of our angle notion.

The proofs of the following two lemmas are identical to those in [**2**].

LEMMA 4.12. [**2**, Lemma 2.9] *Let $x \in \widetilde{M}(\infty)$ be arbitrary and let $A = \overline{I(\widetilde{M})(x)}$, the orbit closure of the isometry group; then $\overline{I(\widetilde{M})(z)} = A$ for every $z \in A$.*

LEMMA 4.13. [**2**, Lemma 2.10] *If v, $w \in \mathfrak{R}$ are asymptotic, then $\mathrm{rank}(v) = \mathrm{rank}(w)$.*

With the four lemmas above, one can prove the following main result by adapting the proof of Theorem 2.6 in [**2**].

THEOREM 4.14. *Let $k = \mathrm{rank}(\widetilde{M})$. If $\widetilde{M}$ admits a quotient manifold of finite volume, then:*

(1) *every regular vector has rank k;*
(2) *$F(v)$ is a k-flat for every $v \in \Re$;*
(3) *every vector $v \in T^1\widetilde{M}$ is tangent to at least one k-flat.*

4.3. *Strong stable manifolds.* In this subsection, we construct strong stable manifolds for uniformly recurrent (see [**2**, Definition 3.1]) and regular vectors. At first, we prove that any strong stable Jacobi field defined below has an exponentially decreasing length. The main difference is that we must use a coarse estimation on distance functions, based on Proposition 4.3.

Definition 4.15. A stable Jacobi field Y along γ_v (with velocity field T) is called strong stable if $g_T(Y(0), Z(0)) = 0$ for any $Z \in J^p(v)$, where $g_T = g_{ij}(\gamma_v, T)$. Denote by $J^{ss}(v)$ the space of all strong stable Jacobi fields along γ_v.

For $Y \in J^{ss}(v)$, $g_T(Y(t), Z(t)) \equiv 0$ for any $Z \in J^p(v)$. The proof is the same as in [**2**, Lemma 3.3], with inner product replaced by g_T norm. Under this modification, Lemma 3.4 in [**2**] is also true for the Berwald case. Here we let $H(v)$ denote the horosphere determined by v and let $W(v) = \{v(q) : q \in H(v)\}$.

LEMMA 4.16. *Let $v \in \Re$ be a uniformly recurrent vector. Then there exist a neighborhood U of v in $W(v)$ and constants C and $\lambda = \lambda(v) > 0$ such that for every $w \in U$, $Y \in J^{ss}(w)$ and $t > 0$,*
$$\|Y(t)\|_T \leq Ce^{-\lambda t}\|Y(0)\|_T.$$

Definition 4.17. [**2**, Lemma 3.4] For $v \in \Re$ and $w \in W(v) \cap \Re \subseteq T^1\widetilde{M}$, define
$$D_v^s(\pi(w)) = \{Y(0)|Y \in J^{ss}(w)\}.$$

Recall that the Busemann function $f_\gamma : \widetilde{M} \to R$ is defined as
$$f_\gamma(p) = \lim_{t \to \infty}(d(p, \gamma(t)) - t),$$

which is a convex function. In [**10**], it is proved that f_γ is at least C^1 and $df_\gamma(Y) = g_X(X, Y)$, where $X = -\alpha'(0)$ and $\alpha(t)$ is the unique geodesic from $\pi(Y)$ to $\gamma(\infty)$. Moreover, as in [**12**], we can show that the radial vector field $X(p)$ is continuously differentiable in p. Hence, f_γ is C^2. Now, in Definition 4.17, if $Y \in J^{ss}(w)$ and $w \in W(v) \cap \Re$, then $g_w(Y(0), w) = 0$ and hence $Y(0)$ is tangent to the horosphere $H(v)$ at $\pi(w)$. Hence, $D_v^s(\pi(w))$ is a distribution on $H(v)$. Since f_γ and hence $H(v)$ is C^2, D_v^s is a C^1 distribution on an open set of $\pi(v)$ on $H(v)$. We integrate D_v^s on $H(v)$.

LEMMA 4.18. [**2**, Lemma 3.6] *If $v \in \Re$ is uniformly recurrent in $T^1\widetilde{M}$, then D_v^s is integrable in a neighborhood of $\pi(v)$ in $H(v)$.*

Proof. See [**2**, Lemma 3.6]. We only need to modify a little bit when we prove that if $\alpha(s)$ is a piecewise C^1 curve tangent to D_v^s with $\alpha(0) = p$, $\alpha(1) = q$, then $d(\gamma_{v(p)}(t), \gamma_{v(q)}(t)) \to 0$ as $t \to \infty$. This follows from the coarse estimation of the

distance function and Lemma 4.16: there is a geodesic variation $\gamma_{v(\alpha(s))}(t)$ with Jacobi fields $J_s(t)$

$$d(\gamma_{v(p)}(t), \gamma_{v(q)}(t)) \le \int_0^1 F(J_s(t))\, ds \le \int_0^1 C_0 \|J_s(t)\|_T\, ds$$
$$\le C_0 C e^{-\lambda t} \|J_s(0)\|_T \to 0 \quad \text{as } t \to \infty,$$

where C_0 is the constant in Proposition 4.3.   □

PROPOSITION 4.19. [**2**, Proposition 3.7] *For every $v \in \mathfrak{R}$, D_v^s is integrable on $\pi(W(v) \cap \mathfrak{R}) \subseteq H(v)$.*

Proof. See [**2**, Proposition 3.7]. When we construct the approximating vector fields, note that we use the fact that radial vector fields $X(q)$ are C^1 in q; and orthogonal projection is with respect to the norm g_u.   □

Denote by $H^{ss}(v)$ the integral manifold of D_v^s near $\pi(v)$ on $H(v)$. Then $H^{ss}(v)$ is an $(n-k)$-dimensional submanifold which intersects $F(v)$ transversally and orthogonally with respect to g_v, where $k = \text{rank}\,(M)$. The strong stable manifold of v is defined as

$$W^{ss}(v) = \{w \in W(v) | \pi(w) \in H^{ss}(v)\}.$$

The following proposition says that $W^{ss}(v)$ is indeed a strong stable manifold in the dynamics sense.

PROPOSITION 4.20. [**2**, Proposition 3.10] *Let $v \in \mathfrak{R}$ be uniformly recurrent. Then*

$$W^{ss}(v) = \{w \in W(v) | d(g^t v, g^t w) \to 0 \quad \text{as } t \to \infty\}$$

and, if $w \in W^{ss}(v)$, then

$$d(g^t v, g^t w) \le C d_s(v, w) e^{-\lambda t}$$

for all $t \ge 0$, where $\lambda = \lambda(v) > 0$, $C = C(w)$ is positive and continuous on $W^{ss}(v)$ and d_s is the induced distance in $W^{ss}(v)$. Moreover, $W^{ss}(v)$ is a closed connected submanifold of $T^1\widetilde{M}$ which is diffeomorphic to $\mathbb{R}^{n-k}$.

Definition 4.21. For any vector $v \in \mathfrak{R}$, the strong unstable manifold of v is defined as $W^{uu}(v) = -W^{ss}(-v)$.

4.4. *Weyl chambers.*   Before defining Weyl chambers, let us discuss the following two lemmas, which are important tools to show that all the Weyl chambers are isometric under the angle metric.

LEMMA 4.22. (Convexity lemma, [**2**, Lemma 1.5]) *Consider $v, w \in T_p^1\widetilde{M}$, $v', w' \in T_{p'}^1\widetilde{M}$. Suppose that v is asymptotic to v', w is asymptotic to w' and both $\triangle(v, w)$ and $\triangle(v', w')$ are flat triangles. Then:*
(1) $\angle(v, w) = \angle(v', w') = \theta > 0$;
(2)   *for any number α with $0 \le \alpha \le \theta$, let v_α, v'_α denote vectors at p and p' tangent to $\triangle(v, w)$, $\triangle(v', w')$ such that $\angle(v, v_\alpha) = \angle(v', v'_\alpha) = \alpha$. Then v_α and v'_α are asymptotic.*

Proof. Recall the angle notion in Definition 3.2. In flat triangles, by Proposition 3.3, the following relations are also true for the Berwald case:

$$d(\gamma_v(t), \gamma_w(t)) = t\theta \quad \text{and} \quad d(\gamma_{v'}(t), \gamma_{w'}(t)) = t\theta'.$$

Then the lemma follows from the same argument in [**2**, Lemma 1.5]. □

Denote $\lambda(v) = -[\max_{Y \in J^s(v)} \limsup_{t \to +\infty} ((\ln \|Y(t)\|_T)/t)]$. By Proposition 4.20, $\lambda(v) > 0$ if v is uniformly recurrent. Suppose that $-b^2 \le K \le 0$.

LEMMA 4.23. (Angle lemma, [**2**, Lemma 4.4]) *Let $v \in \Re$ be uniformly recurrent. Choose $\alpha \in [0, \pi/4]$ such that $0 < \alpha < \lambda(v)/b$ and let $0 < \beta < \lambda(v) - \alpha b$ be chosen arbitrarily. Then, for any $w \in T_{\pi(v)}F(v)$ with $\angle(v, w) \le \alpha$:*
(1) *for every $q \in H^{ss}(v)$, $w(q)$ is tangent to $F(v(q))$. Moreover,*

$$\angle(v, w) = \angle(v(q), w(q));$$

(2) *if $q \in H^{ss}(v)$, then*

$$d(g^t w, g^t w(q)) \le C e^{-\beta t} d(w, w(q)),$$

where $C = C(q)$ is bounded on compact subsets of $H^{ss}(v)$;
(3) *$w \in \Re$.*

Proof. The proof is the same as [**2**, Lemma 4.4]. But the following coarse estimation should be used:
(1) if $v, w \in T_p^1 \widetilde{M}$, $(1/C_1)\angle(v, w) \le d^*(v, w) \le C_1 \angle(v, w)$;
(2) $(1/C_0)F(J) \le \|J\|_T \le C_0 F(J)$,
where C_0 is the constant in Proposition 4.3. The notation is the same as in [**2**, Lemma 4.4]: for example, $w(q)$ denotes the unique vector asymptotic to w and with foot point q; and d^* is induced by the Sasaki metric on $T^1\widetilde{M}$. □

Now we define Weyl chambers for p-regular vectors. A vector $v \in T^1\widetilde{M}$ is called p-regular if there is a point $q \in F(v)$ such that $v(q)$ is regular. If v is p-regular, we set

$$A(v) = \{q \in \widetilde{M} | v(q) \text{ is } p\text{-regular}\}.$$

It can be proved that if $v \in \Re$ is uniformly recurrent, then $A(v) = \widetilde{M}$.

Definition 4.24. For a p-regular vector v, we can define two types of Weyl chambers:
(1) $\mathcal{C}(v)$ is the set of all p-regular vectors $w \in S_{\pi(v)}F(v)$ such that $A(w) = A(v)$ and $F(w(q)) = F(v(q))$ for all $q \in A(v)$;
(2) $\tilde{\mathcal{C}}(v) := \{w \in S_{\pi(v)}F(v) | w(q) \in S_q F(v(q)) \text{ for all } q \in A(v)\}$.

Clearly, $\mathcal{C}(v) \subseteq \tilde{\mathcal{C}}(v)$. Since isometries of $\widetilde{M}$ and parallel translations of $F(v)$ preserve angles, they induce isometries (with respect to angle distance) between two Weyl chambers. Moreover, by the convexity lemma (Lemma 4.22), there are isometries $\mathcal{C}(v) \to \mathcal{C}(v(q))$ and $\tilde{\mathcal{C}}(v) \to \tilde{\mathcal{C}}(v(q))$ both defined by $w \to w(q)$. As, *a priori*, $\mathcal{C}(v)$ is not necessarily open, but we can prove that $\tilde{\mathcal{C}}(v)$ is closed and convex. See [**3**, Lemma 2.5]. Furthermore, by the angle lemma (Lemma 4.23), we have the following result.

LEMMA 4.25. [**3**, Lemma 2.7] *If v is uniformly recurrent, then:*
(1) $\mathcal{C}(v)$ *contains an open neighborhood of v in $S_{\pi(v)}F(v)$;*
(2) $\tilde{\mathcal{C}}(v)$ *is the closure of $\mathcal{C}(v)$;*
(3) $\mathcal{C}(v)$ *is convex.*

When we prove that two Weyl chambers are isometric, we use limit arguments. The following two lemmas are the main tools. See [**3**, Lemmas 2.8 and 2.9]. Given a sequence of subsets X_n in a space X, denote

$$\overline{\lim}\, X_n = \{x \in X \mid x \text{ is a limit point of a sequence } x_n \in X_n\}.$$

LEMMA 4.26. [**3**, Lemma 2.8] *Let a sequence $v_n \in \mathfrak{R}$ converge to $v \in \mathfrak{R}$. Then $\overline{\lim}\, \tilde{\mathcal{C}}(v_n)$ is contained in $\tilde{\mathcal{C}}(v)$.*

LEMMA 4.27. [**3**, Lemma 2.9] *Let a sequence $v_n \in \mathfrak{R}$ converge to $v \in \mathfrak{R}$. For $\alpha > 0$, let $\mathcal{C}_\alpha(v_n)$ denote the α-interior (under angle distance) of $\mathcal{C}(v_n)$ in $S_{\pi(v_n)}F(v_n)$. Assume that for all n, $v_n \in \mathcal{C}_{\alpha_0}(v_n)$ for some $\alpha_0 > 0$. Then, for all positive $\alpha \leq \alpha_0$,*

$$\overline{\lim}\, \mathcal{C}_\alpha(v_n) \subset \mathcal{C}(v).$$

Let $\mathcal{B}$ denote the vectors which are uniformly recurrent in both positive and negative directions. The following result is important, while the proof is identical to that of [**3**, Lemma 2.10]:

LEMMA 4.28. (Rigidity lemma, [**3**, Lemma 2.10]) *Let $v \in \mathcal{B}$. Then there is an open set $U = U(v) \subset \mathfrak{R}$ containing v such that for all $w \in U$, we have $w \in \text{Int}\, \tilde{\mathcal{C}}(w)$ and $\text{Int}\, \tilde{\mathcal{C}}(w)$ is isometric to $\text{Int}\, \tilde{\mathcal{C}}(v)$.*

So, locally around a vector in $\mathcal{B}$, all Weyl chambers are isometric.

We can construct first integrals on a g^t-invariant dense open subset of T^1M, which is an analogue of [**3**, Theorem 3.7]. The proof is identical.

THEOREM 4.29. (First integrals, [**3**, Theorem 3.7]) *Let M be a quotient manifold of $\widetilde{M}$ of finite volume; then there are a g^t-invariant open and dense subset R of T^1M and $k-1$ independent C^1 first integrals*

$$\Phi_i : R \to \mathbb{R}, \quad 1 \leq i \leq k-1$$

such that each $v \in R$ has a neighborhood $R(v)$ in R with the following property: if $v^* \in SF_{R(v)}(v')$, then $\Phi_i(v^*) = \Phi_i(v')$ for all i if and only if v^* is parallel to v' in $F(v')$, where $SF_{R(v)}$ is the foliation induced by SF in $R(v)$.

Denote by $I(v)$ the level set of the first integrals $\Phi_1, \ldots, \Phi_{k-1}$ containing v. For notation, refer to [**3**, §4]. In a neighborhood of a vector in $\mathfrak{R}^*$, a dense open subset of $\mathfrak{R}$, the following Anosov-type closing lemma holds.

LEMMA 4.30. (Closing lemma, [**3**, Lemma 4.5]) *Denote $I^*(v_0) := I(v_0) \cap \mathfrak{R}^*$. For every compact $K \subset I^*(v_0)$ and any $\epsilon \in (0, 1)$, there exist $T_0 = T_0(K) > 0$ and $\delta = \delta(K, \epsilon) > 0$ with the following property: if*

$$d(d\phi(g^T v), v) < \delta$$

for some $v \in K$, $T \geq T_0$ and for some deck transformation ϕ, then there are $v' \in R \subset \mathfrak{R}$ and $T' \in [T - \epsilon, T + \epsilon]$ such that:

(1) $d(g^t v, g^t v') < \epsilon$ *for all $t \in [0, T]$;*

(2) $d\phi \circ g^{T'} v' = v'$;

(3) ϕ *is a pure translation of the flat $F(v')$.*

COROLLARY 4.31. *If M has finite volume and the flag curvature is bounded below, then vectors tangent to regular closed geodesics are dense in $T^1 M$.*

4.5. *Tits buildings at infinity.*   In [**7**], Burns and Spatzier constructed Tits buildings for $\widetilde{M}(\infty)$ using Weyl chambers (or called Weyl simplices in terms of Tits buildings) in the above subsection. They called $v \in T^1 \widetilde{M}$ *l-regular* if v is asymptotic to a regular vector. The notion of Weyl simplices can be extended to *l*-regular vectors with little modification. The first important work of Burns and Spatzier was to show that all Weyl simplices for *l*-regular vectors are isometric to each other. Hence, Weyl simplices can be constructed at $\widetilde{M}(\infty)$. See [**7**]. The results are also true for our Berwald case.

THEOREM 4.32. *All Weyl simplices for l-regular vectors are isometric. If $v \in T^1 \widetilde{M}$ is l-regular, then $\tilde{\mathcal{C}}(v)$ is a $(k - 1)$-dimensional convex subset of $T^1_{\pi(v)} \widetilde{M}$.*

Proof. See [**7**, §2]. The main tools are the rigidity lemma (Lemma 4.28) and one technical lemma, which is stated below. □

LEMMA 4.33. [**7**, Lemma 1.1] *Let γ_v be a periodic regular geodesic and ϕ be an axial isometry for γ_v. Then, for any $x \in \widetilde{M}(\infty)$, $n \geq 0$,*

$$\angle_{\gamma_v(0)}(\phi^n x, \gamma_v(\infty)) \leq \angle_{\gamma_v(0)}(x, \gamma_v(\infty)).$$

If n is large enough, equality holds if and only if $x \in F(v)(\infty)$. Any limit point of $\{\phi^n x : n \geq 0\}$ lies in $F(v)(\infty)$.

Proof. See [**7**, Lemma 1.1]. The argument works for Berwald space because of Lemma 3.8. □

Remark 4.34. In fact, we also have a parallel inequality:

$$\angle_{\gamma_v(0)}(\phi^{-n} x, \gamma_v(\infty)) \geq \angle_{\gamma_v(0)}(x, \gamma_v(\infty))$$

and the equality holds for large n if and only if $x \in F(v)(\infty)$. The idea of this fact is used in the proof of Theorem 4.32. See [**7**].

Now we can define Weyl simplices at $\widetilde{M}(\infty)$.

Definition 4.35. Let $x \in \widetilde{M}(\infty)$ with an *l*-regular geodesic representative γ_v. The Weyl simplex of x is defined as

$$C(x) := \{\gamma_w(\infty) : w \in \tilde{\mathcal{C}}(v)\}.$$

By the convexity lemma (Lemma 4.22) and Theorem 4.32, this definition does not depend on the choice of *l*-regular geodesic representative. If we introduce the Tits distance on $\widetilde{M}(\infty)$, $C(x)$ is a $(k - 1)$-dimensional convex subset.

Definition 4.36. A spherical Tits building is a simplicial complex Δ together with a family $\{\Sigma\}$ of finite subcomplexes called apartments satisfying the following axioms:

(1) Δ is thick, i.e. every codimensional-1 simplex in a top-dimensional simplex is contained in at least three top-dimensional simplices;

(2) every apartment is a Coxeter complex;

(3) any two elements of Δ belong to an apartment;

(4) if Σ and Σ' are two apartments containing both A and $A' \in \Delta$, then there is an isomorphism of Σ onto Σ' which leaves A, A' and all their faces invariant.

Now let Δ be the set consisting of the Weyl simplices in $\widetilde{M}(\infty)$ and all their intersections. The subcomplex Σ that is isomorphic to a complex Σ_F for some regular k-flat F is called an apartment if the union of its Weyl simplices is homeomorphic to a $(k-1)$-sphere. Let $\mathcal{A}$ be the collection of all apartments in Δ. In [**7**], axioms 1–4 are verified for $(\Delta, \mathcal{A})$. Similarly, it is true for the Berwald case and hence we have the following result.

THEOREM 4.37. $(\Delta, \mathcal{A})$ *is a spherical Tits building.*

We need to use the main theorem in [**6**].

THEOREM 4.38. (Cf. [**6**]) *Let Δ be an infinite, irreducible, locally connected, compact, metric, topologically Moufang building of rank at least two. Then Δ is the building of parabolic subgroups of a real simple Lie group G.*

If M is a complete Berwald space with flag curvature $-b^2 \le K \le 0$ of finite volume and rank $k \ge 2$ whose universal cover $\widetilde{M}$ is irreducible, we can prove that the spherical Tits building $(\Delta, \mathcal{A})$ at $\widetilde{M}(\infty)$ is an infinite, irreducible, locally connected, compact, metric and topologically Moufang building of rank k. See [**7**, §§3 and 4].

4.6. *Classification.* The last step is to adapt the arguments of Gromov's rigidity theorem [**4**, Ch. 4] to prove Main Theorem 1. We also use the results on locally symmetric Finsler spaces from [**8**].

Now let G be the topological automorphism group of Δ and G^0 be the connected component of the identity in G. So, G^0 is a simple non-compact real Lie group without center. Let $\Delta(G^0)$ be the topological building of parabolic subgroups of G^0. By Theorem 4.38, $\Delta(G^0)$ is isomorphic to Δ. Moreover, if we let $X = G^0/K$ be the symmetric space attached to G^0 and $\Delta(X)$ constructed at $X(\infty)$, then $\Delta(X)$ is isomorphic to $\Delta(G^0)$ and hence $\Delta(X)$ is isomorphic to Δ too.

There is a well-defined so-called Tits metric on $\widetilde{M}(\infty)$. We want to carry over this metric from $\widetilde{M}(\infty)$ to $X(\infty)$ via the isomorphism $\phi : \Delta \to \Delta(X)$ described above. Take any $x \in \widetilde{M}(\infty)$; the geometric structure of the Weyl simplex $C(x)$ can be identified with $\tilde{\mathcal{C}}(v)$, a convex subset of the $(k-1)$-sphere, for some $v \in T^1\widetilde{M}$ such that $\gamma_v(\infty) = x$. Since $\phi(C(x))$ is a Weyl simplex in $\Delta(X)$, we identify it with a Weyl simplex $\tilde{\mathcal{C}}(v^*)$ for some $v^* \in T^1 X$. Hence, there exists an isomorphism between two simplices still denoted as $\phi : \tilde{\mathcal{C}}(v) \to \tilde{\mathcal{C}}(v^*)$, i.e. if $\{v_1, v_2, \ldots, v_k\}$ are vertices of $\tilde{\mathcal{C}}(v)$, then $\{\phi(v_1), \phi(v_2), \ldots, \phi(v_k)\}$ are vertices of $\tilde{\mathcal{C}}(v^*)$. ϕ can be linearly extended to the map

$\phi : \tilde{\mathcal{C}}(v) \to \tilde{\mathcal{C}}(v^*)$. Recall that F is a Minkowski norm in $T_{\pi(v)}\tilde{M}$. Let $F^* = F \circ \phi^{-1}$ and extend it to the cone spanned by $\{\phi(v_1), \phi(v_2), \ldots, \phi(v_k)\}$ via $F^*(\lambda w) = \lambda F(w)$ for all $\lambda > 0$ and for all $w \in \tilde{\mathcal{C}}(v^*)$. By [**8**, Theorem 4.6] (and its proof), F^* can be extended to a Minkowski norm F^* on $T_{\pi(v^*)}X$ and then extended to TX by left translations. Hence, we obtain a locally symmetric Finsler space (X, F^*).

Next we adapt the argument in [**7**] to construct an isometry Φ between $(\tilde{M}, F)$ and (X, F^*). Notation is the same as in [**7**].

First we define $\Phi : \tilde{M} \to X$. Take $p \in \tilde{M}$. The geodesic symmetry σ_p defines a continuous automorphism of Δ. By [**15**, Corollary 16.2], σ_p determines an analytic involuntary isomorphism Θ_p of G^0. Θ_p induces an isometry $\theta_p : (X, F^*) \to (X, F^*)$. θ_p has order two and it has a unique fixed point p^*. Set $\Phi(p) = p^*$. The proofs of the following lemmas are identical to those in [**7**].

LEMMA 4.39. [**7**, Lemma 5.2]
(1) $\Phi : \tilde{M} \to X$ *is continuous.*
(2) *If $F \subset \tilde{M}$ is an l-regular k-flat, then $\Phi(F) \subset F^*$, where F^* is the unique k-flat in X such that $\Sigma_{F^*} = \Sigma_F$ under the isomorphism.*

Let γ be a maximally singular geodesic, that is, $\gamma(\infty)$ is a vertex of Δ. Let C_1 and C_2 be two opposite chambers in Star $\gamma(\infty)$. Then $C_1 \cap C_2 = \{\gamma(\infty)\}$. Let F_i be the l-regular k-flat through $\gamma(0)$ and C_i. Then $F_1 \cap F_2 = \gamma$. By Lemma 3.30, $\Phi(\gamma) \subset F_1^* \cap F_2^*$. Since $F_1^*(\infty) \cap F_2^*(-\infty) = \{\gamma(\infty), \gamma(-\infty)\}$, $F_1^* \cap F_2^*$ is a maximally singular geodesic in X, denoted as γ^*. Since the Tits distances on $\tilde{M}(\infty)$ and $X(\infty)$ are isometric, we have the following.

LEMMA 4.40.
(1) *If γ_1 and γ_2 are two parallel maximally singular geodesics, then γ_1^* and γ_2^* are parallel.*
(2) *If γ_1 and γ_2 are two maximally singular geodesics, then the families of geodesics parallel to γ_1^* and γ_2^* make the same angle as do those parallel to γ_1 and γ_2.*

LEMMA 4.41. [**7**, Lemma 5.3] *Let γ be a maximally singular geodesic; then $\Phi|\gamma : \gamma \to \gamma^*$ is affine.*

LEMMA 4.42. *Let F be an l-regular k-flat; then $\Phi|F : F \to F^*$ is affine.*

Since every geodesic of $\tilde{M}$ lies in an l-regular flat, for each geodesic γ, there is a constant $\lambda(\gamma)$ such that

$$d_X[\Phi(q_1), \Phi(q_2)] = \lambda(\gamma)d(q_1, q_2)$$

for any points q_1 and q_2 on γ.

LEMMA 4.43. [**7**, Lemma 5.4] *Let p be a point of an l-regular k-flat F. Then $\lambda(\gamma)$ is the same for all geodesics γ with $\gamma'(0) \in S_p F$.*

Hence, $d_X[\Phi(q_1), \Phi(q_2)] = \lambda(F)d(q_1, q_2)$ for any point $q_1, q_2 \in F$.

LEMMA 4.44. $\lambda(F)$ *is independent of F.*

Hence, $d_X(\Phi(q_1), \Phi(q_2)) = \lambda d(q_1, q_2)$ for any $q_1, q_2 \in \widetilde{M}$. And, obviously, $\lambda \neq 0$. Hence, we have an isometry between $(\widetilde{M}, F)$ and (X, F^*) up to a renormalization. Hence, we have proved Main Theorem 1.

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