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SOLVABLE LIE GROUPS DEFINABLE IN O-MINIMAL THEORIES

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Abstract In this paper, we completely characterize solvable real Lie groups definable in o-minimal expansions of the real field.

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1. Introduction

It is proved in [15] that every group definable in an o-minimal expansion of the real field is a Lie group (i.e., it can be equipped with a smooth manifold structure so that the group operations are smooth), and it is natural to ask when the opposite is true.

Question 1. What real Lie groups are Lie isomorphic to groups definable in o-minimal expansions of the real field?

A significance of the above question is that for groups definable in o-minimal structures one can use the well-developed and powerful tools of o-minimality (see [17, 19] for more details on o-minimality).

In this paper, we give a complete answer to Question 1 for solvable Lie groups. Let G be a solvable connected Lie group. Assume G is definable in an o-minimal expansion of the real field. Then, by properties of groups definable in o-minimal structures (see Fact 5.1) G contains a definable normal torsion-free subgroup H such that G/H is compact.

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First we prove (see Theorem 4.4) that every torsion-free group definable in an o-minimal structure is completely solvable. It follows then that a necessary condition for a connected solvable Lie group G to be Lie isomorphic to a group definable in an o-minimal expansion of the real field is that G contains a torsion-free completely solvable subgroup N such that G/N is compact; and we prove (see Theorem 5.4) that this condition is sufficient. Moreover, every solvable Lie group satisfying this condition is Lie isomorphic to a group definable in the structure $\mathbb{R}_{an,exp}$.

By the Levi decomposition, any connected Lie group G is the product of its solvable radical R and a semisimple subgroup S. We know by results in [12] that any semisimple Lie group is Lie isomorphic to a group definable in an o-minimal expansion of the field of reals if and only if it has a finite center: The center of a semisimple group is discrete, so any semisimple group definable in an o-minimal theory must have finite center. Now, if S is a semisimple group with finite center $Z(S)$, then $S/Z(S)$ is semisimple of trivial center, the adjoint representation is faithful, $S/Z(S)$ is isomorphic to a linear group, and by Theorem 4.3 in [12], it is algebraic, so $S/Z(S)$ is definable. But any extension of a definable group by a finite group is definable, which implies definability of S. Thus, to answer Question 1 in the whole generality one needs to understand definability of actions of semisimple groups on solvable groups.

1.1. The structure of the paper

In § 2, we recall basic facts about groups definable in o-minimal structures and Lie groups that we need in this paper.

In § 3, we prove that every completely solvable connected torsion-free real Lie group is Lie isomorphic to a group definable in the structure $\mathbb{R}_{exp}$ (see Theorem 3.6).

In § 4, we prove that every torsion-free group definable in an *arbitrary* o-minimal structure is completely solvable (see Theorem 4.4).

In § 5, we answer Question 1 for solvable Lie groups (see Theorem 5.4 there).

2. Preliminaries

2.1. Solvable and completely solvable Lie groups

In this paper, by a Lie group we always mean a real Lie group.

Fact 2.1. *For a connected solvable Lie group G the following are equivalent.*

(1) *G is torsion-free.*

(2) *G is simply connected.*

(3) *G is diffeomorphic to $\mathbb{R}^n$ for some n.*

Proof. This follows from results in Ch. 2, § 3 in [6]. All the references in this proof refer to this.

The equivalence between (2) and (3) is precisely Corollary 2.

As for the other implications, Corollary 1 states that given any solvable connected Lie group G one can find a decomposition

$$\{0\} \leqslant G_1 \leqslant G_2 \leqslant \cdots \leqslant G_n = G$$

where G_{i+1}/G_i is a one-dimensional Lie group. Any torsion element in G appears as a torsion element in G_{i+1}/G_i for some i, so G would not be contractible, which shows that (3) implies (1). Conversely, using the Euler characteristic one shows that if G is torsion-free then so is G_{i+1}/G_i in the above decomposition. So G_{i+1}/G_i is diffeomorphic to $\mathbb{R}$, as required. □

Thus, for connected solvable Lie group G we use torsion-free and simply connected interchangeably.

We now turn to completely solvable Lie groups and Lie algebras.

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra.

(1) *A flag of ideals* in $\mathfrak{g}$ is a chain

$$\mathfrak{g} = \mathfrak{g}_n > \mathfrak{g}_{n-1} > \cdots > \mathfrak{g}_0 = 0$$

such that each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$.

(2) A flag of ideals $\mathfrak{g} = \mathfrak{g}_n > \mathfrak{g}_{n-1} > \cdots > \mathfrak{g}_0 = 0$ is called *complete* if $\dim(\mathfrak{g}_i) = i$ for each $i = 0, \ldots, n$.

(3) A real Lie algebra $\mathfrak{g}$ is called *completely solvable* (also often called *split-solvable*) if it has a complete flag of ideals.

A connected Lie group is called *completely solvable* (also often called *triangular* or *split-solvable*) if its corresponding Lie algebra is.

By the functorial correspondence between simply connected Lie groups and their Lie algebras, one obtains the following alternative definition of connected torsion-free completely solvable Lie groups.

Fact 2.2. *A connected torsion-free solvable Lie group G is completely solvable if and only if there exist a sequence of subgroups*

$$G = G_n > G_{n-1} > \cdots G_0 = \{e\}$$

such that each G_i is normal in G and G_{i+1}/G_i is one-dimensional simply connected Lie group for $i < n$.

The following is a well-known example of a connected torsion-free solvable group $\widetilde{E}^0(2)$ that is not completely solvable (see [6, Ch. 2, § 6.4] and also [8, Ch. 1, § 1, Example 12(c)]).

Example 2.3. Let $\widetilde{E}^0(2)$ be the semidirect product $\widetilde{E}^0(2) = \mathbb{R}^2 \rtimes_\gamma \mathbb{R}$, where for $x \in \mathbb{R}$

$$\gamma(x) = \begin{bmatrix} \cos 2\pi x & \sin 2\pi x \\ -\sin 2\pi x & \cos 2\pi x \end{bmatrix}.$$

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The group $\widetilde{E}^0(2)$ is a connected torsion-free solvable group that is not completely solvable. It is a simply connected group with the Lie algebra of all matrices

$$\begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

2.2. Groups definable in o-minimal structures

We refer to [17] for basics on o-minimal structures.

In this paper, we need two particular o-minimal expansions of the real field.

Fact 2.4 (See [20]). *The expansion of the real field by the exponential function e^x is o-minimal. (This structure is denoted by $\mathbb{R}_{exp}$.)*

Fact 2.5 (See [18]). *The expansion of the real field by the exponential function and all restrictions of analytic functions to compact domains is o-minimal. (This structure is denoted by $\mathbb{R}_{an,exp}$.)*

We now turn to groups definable in o-minimal structures.

The following two facts are proved in [15].

Fact 2.6. *Let G be a group definable in an o-minimal structure. Then G can be equipped with a definable topological manifold structure so that the group operations are continuous. In particular, if G is definable in an o-minimal expansion of the real field then G is a Lie group.*

Using the above fact we always view groups definable in o-minimal structures as definable topological groups.

Fact 2.7. *Let G be a group definable in an o-minimal structure. If $H < G$ is a definable subgroup then H is closed in G. In particular, if G is definable in an o-minimal expansion of the real field and $H < G$ is a definable subgroup then H is a Lie subgroup of G.*

The following is Theorem 1.2 in [14].

Fact 2.8. *If G is a torsion-free group definable in an o-minimal structure then G contains a definable one-dimensional subgroup.*

The following fact follows from Corollary 2.15 in [15].

Fact 2.9. *If G is a definably connected one-dimensional group definable in an o-minimal structure then G is abelian.*

Using the following 'definable choice' for groups, proved by Edmundo (see [5, Theorem 7.2]), we always view quotients of groups definable in an o-minimal structure as definable objects.

Fact 2.10. *Let G be a group definable in an o-minimal structure and let $\{T(x) \mid x \in X\}$ be a definable family of nonempty definable subsets of G. Then there is a definable function $t : X \to G$ such that for all $x, y \in X$, we have $t(x) \in T(x)$ and if $T(x) = T(y)$ then $t(x) = t(y)$.*

In the following fact we collect properties of definable torsion-free groups that we need in this paper. For proofs we refer to [5] and also to [13].

Fact 2.11. *Let $\mathcal{M}$ be an o-minimal structure and G a torsion-free group definable in $\mathcal{M}$.*

(1) *There is a definable normal subgroup $H < G$ such that the group G/H is one-dimensional (hence abelian).*

(2) *G is definably connected and solvable. More precisely, there are definable subgroups*
$$G = G_n > G_{n-1} > \cdots > G_0 = \{e\}$$
such that G_i is normal in G_{i+1} and the group G_{i+1}/G_i is a torsion-free abelian group for $i = 0, \ldots, n-1$.

(3) *If $H < G$ is a definable normal subgroup then the group G/H is also torsion-free.*

(4) *If $\mathcal{M}$ is an expansion of a real closed field then G is definably diffeomorphic to M^n.*

Proof. (1) Is Corollary 2.12 in [13]. (2) Follows immediately by induction on the (o-minimal) dimension, and (4) is a direct consequence of (2). Finally, (3) is Corollary 2.3 in [13]. □

By analogy with simply connected Lie groups we define definably completely solvable groups.

Definition 2.2. A torsion-free group G definable in an o-minimal structure is called *definably completely solvable* if there exists a sequence of definable subgroups $G = G_n > G_{n-1} > \cdots > G_0 = \{e\}$ such that each G_i is normal in G and G_{i+1}/G_i is a one-dimensional group.

Remark 2.12. It follows from Facts 2.9 and 2.11 that in the above definition all groups G_{i+1}/G_i are abelian and torsion-free.

3. Solvable connected torsion-free Lie groups are definable in $\mathbb{R}_{exp}$

In this section, we prove that any connected torsion-free solvable Lie group of finite dimension is isomorphic (as a Lie group) to a group definable in $\mathbb{R}_{exp}$.

In order to prove this we need the following lemma about completely solvable Lie algebras over $\mathbb{R}$.

Lemma 3.1. *Let $\mathfrak{g}$ be a solvable finite-dimensional Lie algebra over $\mathbb{R}$. The following are equivalent.*

(1) *$\mathfrak{g}$ is completely solvable.*

(2) *For any $\xi \in \mathfrak{g}$ all eigenvalues of the linear operator $\mathrm{ad}(\xi)$ are in $\mathbb{R}$.*

(3) *$\mathfrak{g}$ is isomorphic to a subalgebra of the upper-triangular matrices $t_n(\mathbb{R})$ for some $n \in \mathbb{N}$.*

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Proof. For the equivalence of (1) and (2) we refer to [8, Corollary 1.30].

The implication (3) $\Rightarrow$ (1) is easy.

Thus, we only need to see that (1) implies (3). Although this implication is stated in several books, e.g., in [6, § 2], we could not find a reference for a proof of it. Here we present an argument provided to us by E. B. Vinberg in a private communication.

(1) $\Longrightarrow$ (3). Using Ado's theorem we can embed $\mathfrak{g}$ into $\mathfrak{gl}(V)$ for some finite-dimensional $\mathbb{R}$-vector space V, and we assume that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

Let $\mathfrak{g}^a$ be the minimal algebraic subalgebra of $\mathfrak{gl}(V)$ containing $\mathfrak{g}$, i.e., $\mathfrak{g}^a$ is the Lie algebra of an algebraic subgroup of $\mathrm{GL}(V)$, $\mathfrak{g}^a$ contains $\mathfrak{g}$ and is the minimal such. (It exists by a dimension argument.) By [6, Ch. 1, Theorem 6.2] we have $[\mathfrak{g}^a, \mathfrak{g}^a] = [\mathfrak{g}, \mathfrak{g}]$, in particular, $\mathfrak{g}$ is an ideal of $\mathfrak{g}^a$ and $\mathfrak{g}^a/\mathfrak{g}$ is abelian.

Let $\mathbb{G} < \mathrm{GL}(V)$ be a connected algebraic subgroup whose Lie algebra is $\mathfrak{g}^a$. Clearly $\mathbb{G}$ is defined over $\mathbb{R}$.

Let $\mathfrak{g} = \mathfrak{g}_n > \mathfrak{g}_{n-1} > \cdots > \mathfrak{g}_0 = 0$ be a complete flag of ideals in $\mathfrak{g}$. Consider the adjoint action of $\mathbb{G}$ on $\mathfrak{g}^a$. Let

$$G' = \{g \in \mathbb{G} : \mathrm{Ad}(g)(\mathfrak{g}_i) \subseteq \mathfrak{g}_i \text{ for each } i = 0, \ldots, n\}.$$

G' is an algebraic subgroup of $\mathbb{G}$ and, by [10, Ch. 2, Proposition 1.4], its Lie algebra is

$$\mathfrak{g}' = \{\xi \in \mathfrak{g}^a : [\xi, \mathfrak{g}_i] \subseteq \mathfrak{g}_i \text{ for each } i = 0, \ldots, n\}.$$

Obviously $\mathfrak{g} \subseteq \mathfrak{g}'$, so by minimality of $\mathfrak{g}^a$ we obtain $\mathfrak{g}' = \mathfrak{g}_a$ and $G' = \mathbb{G}$.

Thus, each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}^a$, and since $\mathfrak{g}^a/\mathfrak{g}$ is abelian, the chain $\mathfrak{g} = \mathfrak{g}_n > \mathfrak{g}_{n-1} > \cdots > \mathfrak{g}_0 = 0$ can be extended to a complete flag of ideals of $\mathfrak{g}^a$. Hence, $\mathfrak{g}^a$ is completely solvable. Replacing $\mathfrak{g}$ by $\mathfrak{g}^a$ if needed we assume that $\mathfrak{g}$ is the Lie subalgebra of an algebraic group $\mathbb{G}$ defined over $\mathbb{R}$.

Let $\mathbb{G}_u$ be the subset of $\mathbb{G}$ consisting of unipotent elements, and $\mathbb{T} < \mathbb{G}$ a maximal algebraic torus defined over $\mathbb{R}$ (as usual by an algebraic torus we mean a commutative algebraic group consisting of semisimple elements). Then, by [2, Theorem III.10.6], $\mathbb{G}_u$ is a normal subgroup of $\mathbb{G}$ and $\mathbb{G} = \mathbb{T} \cdot \mathbb{G}_u$ (a semidirect product).

We now consider $\mathbb{R}$-points of groups $\mathbb{G}$, $\mathbb{G}_u$ and $\mathbb{T}$. We have $\mathbb{G}(\mathbb{R}) = \mathbb{T}(\mathbb{R}) \cdot \mathbb{G}_u(\mathbb{R})$, $\mathbb{G}_u(\mathbb{R})$ is a normal subgroup of $\mathbb{G}(\mathbb{R})$, and $\mathfrak{g}$ is the Lie algebra of $\mathbb{G}(\mathbb{R})$ viewed as a Lie group.

We can write $\mathbb{T}$ as a product $\mathbb{T} = \mathbb{D} \cdot \mathbb{S}$ of algebraic groups $\mathbb{D}$, $\mathbb{S}$ defined over $\mathbb{R}$ such that $\mathbb{D}(\mathbb{R})$ is isomorphic over $\mathbb{R}$ to a product of multiplicative group $\mathbb{G}_m(\mathbb{R})$ and $\mathbb{S}(\mathbb{R})$ is compact.

We have $\mathbb{G}(\mathbb{R}) = \mathbb{S}(\mathbb{R}) \cdot \mathbb{D}(\mathbb{R}) \cdot \mathbb{G}_u(\mathbb{R})$. Let $\mathbb{H} = \mathbb{D} \cdot \mathbb{G}_u$. Then $\mathbb{H}(\mathbb{R})$ is a normal subgroup of $\mathbb{G}(\mathbb{R})$ and $\mathbb{G}(\mathbb{R}) = \mathbb{S}(\mathbb{R}) \cdot \mathbb{H}(\mathbb{R})$ with finite intersection $\mathbb{S}(\mathbb{R}) \cap \mathbb{H}(\mathbb{R})$. It is not hard to see that $\mathbb{H}$ is an $\mathbb{R}$-split-solvable group; hence, by [2, Theorem V.15.4], we can choose a basis in V so that every matrix in $H(\mathbb{R})$ is upper triangular.

Consider the adjoint representation $\mathrm{Ad} : \mathbb{G}(\mathbb{R}) \to \mathrm{GL}(\mathfrak{g})$. The differential of this representation is the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ of $\mathfrak{g}$. By the equivalence of (1) and (2), for every $\xi \in \mathfrak{g}$ all eigenvalues of $\mathrm{ad}(\xi)$ are real. Hence, by [8, Corollary 1.30], there is a basis of $\mathfrak{g}$ such that $\mathrm{ad}(\mathfrak{g})$ consists of upper-triangular matrices; hence, $\mathrm{Ad}(\mathbb{G}(\mathbb{R}))$ consists of upper-triangular matrices as well. Since the group of upper-triangular matrices does not have nontrivial compact subgroup it implies that the image of $\mathbb{S}(\mathbb{R})$ under adjoint representation is trivial, hence $\mathbb{S}(\mathbb{R})$ is in the center of $\mathbb{G}(\mathbb{R})$, in particular, it is a normal subgroup of $\mathbb{G}(\mathbb{R})$.

Let $\mathfrak{s}$ be the Lie algebra of $\mathbb{S}(\mathbb{R})$ and $\mathfrak{h}$ be the Lie algebra of $\mathbb{H}(\mathbb{R})$. Both $\mathfrak{s}$ and $\mathfrak{h}$ are ideals of $\mathfrak{g}$ and $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{h}$. We already know that (after choosing a suitable basis) H consist of upper-triangular matrices, hence $\mathfrak{h}$ also consists of upper-triangular matrices, so $\mathfrak{h}$ has a faithful representation by upper-triangular matrices. The Lie algebra $\mathfrak{s}$ is abelian, hence it also has a faithful representation by upper-triangular matrices, e.g., by diagonal matrices. Taking the direct sum of these two representations we obtain a faithful representation of $\mathfrak{g}$ by upper-triangular matrices. □

We also need the following fact due to Dixmier [4].

Fact 3.2. *Let G be a connected Lie group whose Lie algebra $\mathfrak{g}$ is completely solvable. Then the exponential map $\exp_{\mathfrak{g}} : \mathfrak{g} \to G$ is surjective. If in addition G is simply connected then $\exp_{\mathfrak{g}}$ is a diffeomorphism.*

Remark 3.3. For $n \in \mathbb{N}$ let $T_n^+(\mathbb{R})$ be the group of all upper-triangular $n \times n$-matrices with positive diagonal entries. Its Lie algebra is the algebra of all upper-triangular $t_n(\mathbb{R})$. Let $\mathrm{Exp}_n : \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be the usual matrix exponentiation

$$\mathrm{Exp}_n(M) = \sum_{i=0}^{\infty} \frac{1}{n!} M^n.$$

It follows from Fact 3.2 (and also can be seen by direct computations) that Exp_n maps $t_n(\mathbb{R})$ *diffeomorphically* onto $T_n^+(\mathbb{R})$. Also for any Lie subgroup $G < T_n^+(\mathbb{R})$ the matrix exponentiation Exp_n maps the Lie algebra $\mathfrak{g}$ of G diffeomorphically onto G.

Lemma 3.4. *Let G be a connected torsion-free completely solvable Lie group. Then G is Lie isomorphic to a Lie subgroup of $T_n^+(\mathbb{R})$ for some $n \in \mathbb{N}$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. Since $\mathfrak{g}$ is completely solvable, by Lemma 3.1, there is an embedding $\varphi : \mathfrak{g} \to t_n(\mathbb{R})$ for some $n \in \mathbb{N}$.

By the general theory of Lie groups and Lie algebras, there is a smooth group homomorphism $\Phi : G \to T_n^+(\mathbb{R})$ whose differential is φ.

By [6, § 2, Theorem 3.4(1)], the image $\Phi(G)$ is a simply connected Lie subgroup of $T_n^+(\mathbb{R})$. Since a lift of an isomorphism between Lie algebras to corresponding simply connected Lie groups is an isomorphism, Φ is a Lie isomorphism between G and $\Phi(G)$. □

Lemma 3.5. *For every $n \in \mathbb{N}$ the restriction of the map Exp_n to $t_n(\mathbb{R})$ is definable in $\mathbb{R}_{exp}$.*

Proof. Using Jordan normal form, because the diagonal and nilpotent components of each Jordan block commute, it is easy to see that the restriction of Exp_n to the set of $n \times n$-matrices whose eigenvalues are real is definable in $\mathbb{R}_{exp}$. □

Theorem 3.6. *Any connected torsion-free solvable Lie group is Lie isomorphic to a group definable in* $\mathbb{R}_{exp}$.

Proof. Let G be a connected torsion-free solvable Lie group. By Lemma 3.4, G is Lie isomorphic to some Lie group $G' \subseteq T_n^+(\mathbb{R})$. Let $\mathfrak{g}' \subseteq t_n(\mathbb{R})$ be the Lie algebra of G'. Then $G' = \mathrm{Exp}_n(\mathfrak{g}')$ by Fact 3.2, and G' is definable in $\mathbb{R}_{exp}$ by Lemma 3.5. □

4. Torsion-free solvable definable groups are completely solvable

In this section, we prove that every torsion-free group definable in *any* o-minimal structure is definably completely solvable.

We need the following result by Baro, Jaligot and Otero, which is Corollary 5.6 in [1].

Fact 4.1. *Let G be a nontrivial definably connected solvable group definable in an o-minimal structure. Then G has an infinite abelian characteristic (in particular normal) definable subgroup.*

We also need the following fact that follows from results in [9].

Fact 4.2. *Let $\mathcal{R}$ be an o-minimal expansion of a real closed field with additive group $(R,+)$ and multiplicative group $(R^{>0},\cdot)$. Then either $(R,+)$ is $\mathcal{R}$-definably isomorphic to $(R^{>0},\cdot)$, or every definable endomorphism of $(R^{>0},\cdot)$ is 0-definable.*

Proof. Theorem 4.1 in [9] asserts that if an o-minimal expansion of a real closed field is not exponential, then it is power-bounded. By [9, Theorem B], if such a field is power-bounded, then every definable endomorphisms of $(R^{>0},\cdot)$ is 0-definable, as required. □

The following lemma about definable abelian subgroups of centerless definable groups will be a key.

Lemma 4.3. *Let $\mathcal{R}$ be an o-minimal expansion of a real closed field, let G be a centerless definably connected definable subgroup of $\mathrm{GL}_n(\mathcal{R})$ for some $n \in \mathbb{N}$, and let $A < G$ be a definable torsion-free normal abelian subgroup of G. Then A is $\mathcal{R}$-definably isomorphic to a Cartesian power of the additive group of $\mathcal{R}$.*

Proof. Since A is a linear group, by [12, Proposition 3.8] it is definably isomorphic to an algebraic linear group and by [12, Fact 3.1] there are definable subgroups A_m and A_a of A such that $A = A_a \times A_m$, A_a is definably isomorphic to a Cartesian power $(R,+)^k$, and A_m is definably isomorphic to a Cartesian power $(R^{>0},\cdot)^l$. If $(R,+)$ is definably isomorphic to $(R^{>0},\cdot)$, there is nothing to prove.

Assume $(R,+)$ and $(R^{>0},\cdot)$ are not definably isomorphic. It is easy to see then that for any definable automorphism σ of A we have $\sigma(A_a) = A_a$ and $\sigma(A_m) = A_m$. Considering action of G on A by conjugation we obtain that both A_a and A_m are normal subgroups of G. By Fact 4.2 every uniformly definable family of automorphisms of A_m is finite. Since G is connected, it implies that the action of G on A_m is trivial; hence, A_m is in the center of G, but G is centerless. Thus, A_m is trivial and $A = A_a$. □

We can now prove the main result of this section.

Theorem 4.4. *Every torsion-free group definable in an o-minimal structure is completely solvable.*

Proof. By Fact 2.11, if G is a torsion-free group definable in an o-minimal structure and $N < G$ is a definable normal subgroup, then the factor group G/N is torsion-free as well. Thus, by an easy induction on dimension, it is sufficient to show that every torsion-free group G definable in an o-minimal structure has a definable one-dimensional (hence abelian) normal subgroup.

We prove an existence of a definable one-dimensional normal subgroup by induction on the dimension of G.

If $\dim(G) = 1$ then there is nothing to prove.

Let G be a torsion-free group definable in o-minimal structure $\mathcal{M}$ with $\dim(G) = k$ and we assume that every torsion-free group definable in $\mathcal{M}$ of dimension less than k has a definable normal one-dimensional subgroup. We need to show that G has a definable normal one-dimensional subgroup.

If G is abelian then we are done by Fact 2.8. Assume G is not abelian. If the center $Z(G)$ is nontrivial, then $\dim(Z(G)) < k$; hence, $Z(G)$ contains a definable normal one-dimensional subgroup A. Obviously A is also normal in G.

Thus, we may assume that G is centerless. If G is a direct product $G = G_1 \otimes G_2$ of definable proper subgroups then, by induction hypothesis, G_1 has a definable normal one-dimensional subgroup, and this subgroup is normal in G. Thus, we may assume that G is not a direct product of definable proper subgroups. It follows then from Theorems 3.1 and 3.2 in [11] that there is a real closed field R definable in $\mathcal{M}$ such that G is definably isomorphic to a subgroup of $\mathrm{GL}(n, R)$. Hence, we may assume that $\mathcal{M}$ is an expansion of a real closed field R and G is a definable subgroup of $\mathrm{GL}(n, R)$.

By Fact 4.1, G contains a nontrivial normal definable abelian subgroup A, and by Lemma 4.3 the group A is definably isomorphic to a Cartesian power $(R, +)^l$.

Let $\mathbb{P}(A)$ be the set of all definable one-dimensional subgroups of A. Since A is definably isomorphic to $(R, +)^l$ the set $\mathbb{P}(A)$ can be identified with the projective space $\mathbb{P}^l(R)$. In particular, it is definable and definably compact.

The group G acts on A by conjugation and this action induces a continuous action of G on the set $\mathbb{P}(A)$. To finish the proof of the theorem it is sufficient to show that under this action G has a fixed point in $\mathbb{P}(A)$. It will follow from the following general lemma.

Lemma 4.5. *Let H be a torsion-free group definable in an o-minimal expansion of a real closed field. Assume H acts definably and continuously on a nonempty definably compact set X. Then H has a fixed point in X.*

Proof. We will do induction on the dimension of H. For $h \in H$ and $x \in X$ we denote by $h \cdot x$ the image of x under the action of h.

Assume $\dim(H) = 1$. Then, by [16], H is definably homeomorphic to the interval $(0, 1)$. Let x be an arbitrary element of X. Since X is definably compact, the limit $\lim_{t\to 1} t \cdot x$ exists in X, and it is not hard to see that this limit is fixed by H.

Assume $\dim(H) > 1$. By Fact 2.11(2), H has a definable normal subgroup $K < H$ with $\dim(H/K) = 1$. By induction hypothesis, K has a fixed point in X. Let $X' \subseteq X$ be the subset of all points in X fixed by K. By the continuity of the action, it is a closed subset of X, and since K is a normal subgroup of H the set X' is H-invariant.

The action of H on X' induces an action of H/K on X'. Since H/K is one-dimensional, it has a fixed point in X', and this point is fixed by H. □

This finishes the proof of Theorem 4.4. □

Combining Theorems 3.6 and 4.4 we obtain a complete description of torsion-free solvable groups definable in o-minimal expansions of the real field.

Theorem 4.6. *For a connected torsion-free solvable Lie group G the following are equivalent.*

(1) *G is Lie isomorphic to a group definable in $\mathbb{R}_{exp}$.*

(2) *G is Lie isomorphic to a group definable in an o-minimal expansion of the real field.*

(3) *G is completely solvable.*

5. Extensions of compact groups by torsion-free groups

Theorem 4.6 provides a complete characterization of connected torsion-free solvable Lie groups definable in o-minimal expansions of the real field. In this section we extend it to a characterization of solvable Lie groups.

The following fact follows from [3, Propositions 2.1 and 2.2]. (Although the context in [3] is of o-minimal expansions of a real closed field, the proofs of this propositions hold in any o-minimal theory.)

Fact 5.1. *Let G be a group definable in an o-minimal structure. Then G contains a maximal normal definable torsion-free subgroup H.*

In addition, if G is solvable and $H < G$ is the maximal normal definable torsion-free subgroup then the group G/H is definably compact.

Combining the above fact with Theorem 4.4 and using Fact 2.7 we obtain the following corollary.

Corollary 5.2. *Let G be a solvable Lie group Lie isomorphic to a group definable in an o-minimal expansion of the real field. Then G contains a normal Lie subgroup H such that*

(a) *The group H is connected torsion-free and completely solvable.*

(b) *The factor group G/H is compact.*

Our goal is to show that the converse in the above corollary is also true.

We need a lemma.

Lemma 5.3. *Let G be a Lie group and $\mathcal{M}$ an o-minimal expansion of $\mathbb{R}_{an}$. Assume G is a semidirect product of a normal Lie subgroup H and a compact subgroup K. If H is Lie isomorphic to a group definable in $\mathcal{M}$ then G also is isomorphic to a group definable in $\mathcal{M}$.*

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Proof. Let $\gamma : K \to \mathrm{Aut}(H)$ be the group homomorphism given by the action of K on H by conjugations. So $G = H \rtimes_{\gamma} K$.

Any compact Lie group admits a structure of an algebraic group (see [6, Ch. 4, Corollary to Theorem 2.3]); hence, K is Lie isomorphic to a semialgebraic group K' definable in the real field. Let H' be a group definable in $\mathcal{M}$ Lie isomorphic to H. We have that G is Lie isomorphic to $H' \rtimes_{\gamma'} K'$ for some $\gamma' : K' \to \mathrm{Aut}(H')$. The group $\mathrm{Aut}(H')$ is a Lie group and the group homomorphism γ' from the compact Lie group K' into the Lie group $\mathrm{Aut}(H')$ is a real analytic map on a compact set, so it is definable in $\mathbb{R}_{an}$. The corresponding group isomorphism between $H \rtimes_{\gamma} K$ and $H' \rtimes_{\gamma'} K'$ is a Lie isomorphism as required. □

Theorem 5.4. *For a solvable Lie group G the following are equivalent.*

(1) *G is Lie isomorphic to a group definable in $\mathbb{R}_{an,exp}$.*

(2) *G is Lie isomorphic to a group definable in an o-minimal expansion of the real field.*

(3) *G contains a normal connected torsion-free completely solvable Lie subgroup H such that G/H is compact.*

Proof. Implication (1) ⇒ (2) is obvious, and (2) ⇒ (3) is Corollary 5.2. It remains to show that (3) implies (1).

Let G be a solvable Lie group with a normal connected torsion-free subgroup H such that the group $K = G/H$ is compact.

Since K is a solvable compact connected Lie group it is abelian by [7, Lemma 2.2], hence G/H is abelian and H contains the commutator subgroup G' of G. Since H is simply connected, by [10, Theorem 5.1] and [6, Ch. 2, Theorem 3.4(1)], G' is closed in H, connected and simply connected. By a theorem of Malcev (see [6, Ch. 2, Theorem 7.1]), G can be decomposed into a semidirect product $T \ltimes F$ of a torus T and a simply connected Lie subgroup F. It is not hard to see that we must have $H = F$.

By Theorem 3.6 the group H is Lie isomorphic to a group definable in $\mathbb{R}_{exp}$, and by Lemma 5.3 G is Lie isomorphic to a group definable in $\mathcal{M} = \mathbb{R}_{an,exp}$. □

Corollary 5.5. *A connected real Lie group with compact Levi subgroups is Lie isomorphic to a group definable in an o-minimal expansion of a real closed field if and only if its solvable radical R is Lie isomorphic to a group definable in $\mathbb{R}_{an,exp}$. By Fact 5.1 and Theorem 4.6 this happens if and only if the R has a maximal normal torsion-free definable subgroup H which is completely solvable.*

Proof. If G is definable, then R is definable as well, and by Theorem 5.4 R can be defined in $\mathbb{R}_{an,exp}$. Conversely, if R is definable and G has compact Levi subgroups, the maximal normal torsion-free definable subgroup H of R is a Lie subgroup of G that is a complement of any maximal compact subgroup K of G. Then G is the semidirect product of H and K and Lemma 5.3 then implies definability of G. □

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