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CONTACT SURGERY GRAPHS

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Abstract

We define a graph encoding the structure of contact surgery on contact 3-manifolds and analyse its basic properties and some of its interesting subgraphs.

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1. Introduction

In an unpublished work, William Thurston defined a graph consisting of a vertex v_M for every diffeomorphism type of a closed, orientable 3-manifold M. Two vertices, v_M and $v_{M'}$, are connected by an edge if there exists a Dehn surgery between M and M'. In [21], this graph is called the *big Dehn surgery graph* and studied in various ways.

We define a directed graph encoding the structure of contact (± 1)-surgeries on contact 3-manifolds. The *contact surgery graph* Γ is the graph consisting of a vertex $v_{(M,\xi)}$ for every contactomorphism type of a contact 3-manifold (M,ξ). Whenever there exists a Legendrian knot L in (M_1,ξ_1) such that contact (-1)-surgery along L yields (M_2,ξ_2), we introduce a directed edge pointing from $v_{(M_1,\xi_1)}$ to $v_{(M_2,\xi_2)}$.

The contact surgery graph is closely tied to the properties of Stein or Weinstein cobordisms and fillings of contact manifolds since contact (-1)-surgery along a Legendrian knot in a contact 3-manifold (M,ξ), also known as Legendrian surgery, corresponds to the attachment of a Weinstein 2-handle to the symplectisation of (M,ξ). The inverse operation of a contact (-1)-surgery is called contact $(+1)$-surgery.

1.1. Properties of the contact surgery graph. In Section 2, we study the basic properties of Γ. First, we observe that the contact surgery graph Γ is connected by the work of Ding–Geiges [4] who showed that any contact 3-manifold can be constructed from the standard tight contact structure ξ_{st} on S^3 by a sequence of (± 1)-contact surgeries. However, we know from [10, 19] that a contact manifold (M,ξ) is Stein fillable if and only if there exists a directed path from $v_{(\#_k S^1\times S^2,\xi_{st})}$ to $v_{(M,\xi)}$ and thus Γ is

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not strongly connected since there exist non-Stein fillable contact manifolds. (Recall that an oriented graph is strongly connected if for any pair of vertices (v_1, v_2) there exists a path from v_1 to v_2 following the orientations of the edges.) Etnyre–Honda [15] showed that there exists a directed path from any vertex corresponding to a given overtwisted contact manifold to any other vertex, that is, Γ is strongly connected from any vertex corresponding to an overtwisted contact manifold. The question whether there exists a vertex such that Γ is strongly connected to that vertex is equivalent to the open question whether there exists a maximal element with respect to the Stein cobordism relation [32].

We show that Γ stays connected after removing an arbitrary finite collection of vertices and edges. Recall that a graph is called *k-connected* if it is still connected after removing k arbitrary vertices and *k-edge-connected* if it remains connected after removing k arbitrary edges.

THEOREM 1.1. *The contact surgery graph Γ is k-connected and k-edge-connected for any integer $k \geq 0$.*

We equip Γ with its graph metric d. Then the distance from $v_{(S^3,\xi_{\mathrm{st}})}$ to another vertex $v_{(M,\xi)}$ equals the contact (± 1)-surgery number $\mathrm{cs}_{\pm 1}(M,\xi)$, that is, the minimal number of components of a contact (± 1)-surgery link L in (S^3, ξ_{st}) describing (M, ξ) [16].

PROPOSITION 1.2. *The contact surgery graph Γ has infinite diameter, infinite indegree and infinite outdegree.*

Next, we study the existence of Euler and Hamiltonian walks and paths. We first recall the necessary definitions. A *track t* in Γ is an infinite sequence whose terms are alternately vertices and edges of Γ starting at a vertex and such that any edge in t joins the vertices preceding and following the edge in t. A *Hamiltonian walk* of Γ is a track running through any vertex of Γ at least once. A *Hamiltonian path (Eulerian path)* of Γ is a track running through any vertex (edge) of Γ exactly once. If a track is following the direction of the oriented edges, it is called a *ditrack* and then the definitions of Hamiltonian diwalks and Hamiltonian and Eulerian dipaths are obvious.

THEOREM 1.3. *The contact surgery graph Γ admits Eulerian and Hamiltonian paths and also Hamiltonian walks. However, there exists no Hamiltonian diwalk, no Hamiltonian dipath and no Eulerian dipath in Γ.*

1.2. Contact geometric subgraphs of Γ. In Section 3, we study interesting contact geometric subgraphs of Γ. We define the subgraphs Γ_{OT}, Γ_{tight}, Γ_{Stein}, Γ_{strong}, Γ_{weak} and $\Gamma_{\mathrm{c}\neq 0}$ consisting of vertices (and the corresponding edges connecting two such vertices) of Γ representing contact manifolds which are overtwisted, tight, Stein fillable, strongly fillable, weakly fillable or with a nonvanishing contact class c in Heegaard Floer homology. We analyse some of the basic properties of these subgraphs and in particular prove the following results.

THEOREM 1.4. *Γ_{OT} is strongly connected. Each of Γ_{tight}, Γ_{Stein}, Γ_{strong}, Γ_{weak} and $\Gamma_{\mathrm{c}\neq 0}$ is connected, but not strongly connected.*

THEOREM 1.5. *There exists no Hamiltonian diwalk in each of* Γ_{tight}, Γ_{Stein}, Γ_{strong}, Γ_{weak} *and* $\Gamma_{c\neq 0}$, *and thus also no Hamiltonian or Eulerian dipath. On the contrary,* Γ_{OT} *admits an Eulerian dipath and thus also a Hamiltonian diwalk.*

1.3. Topological subgraphs of Γ. In Section 4, we concentrate on topological subgraphs of the contact surgery graph. Let Γ_M denote the subgraph that consists of vertices corresponding to contact manifolds where the underlying topological manifolds are all diffeomorphic to a fixed manifold M. Similarly, we denote by $\Gamma_{(M,\mathfrak{s})}$ the subgraph of Γ_M consisting of all contact structures on a fixed manifold M lying in the same *spin*^{c} structure $\mathfrak{s}$.

THEOREM 1.6. *The connected components of* Γ_M *are given by* $\Gamma_{(M,\mathfrak{s})}$, *and thus the connected components of* Γ_M *are in bijection with* $H_1(M)$.

The *link* $\mathrm{lk}(M,\xi)$ of a contact 3-manifold (M,ξ) is defined as

$$\mathrm{lk}(M,\xi) := \mathrm{lk}(v_{(M,\xi)}) := \{v_{(N,\eta)} \mid \mathrm{d}(v_{(M,\xi)}, v_{(N,\eta)}) = 1\}.$$

In [21], it is shown that the link of S^3 in the topological surgery graph is connected and of bounded diameter. The main question still remains open: Is the link of any topological 3-manifold connected? It turns out that we can answer this question for contact 3-manifolds.

THEOREM 1.7. *The link* $\mathrm{lk}(M,\xi)$ *of any contact* 3-*manifold* (M,ξ) *is connected and of diameter less than* 4.

Conventions. Throughout this paper, we work in the smooth category. We assume all 3-manifolds to be connected, closed, oriented and smooth; all contact structures are positive and coorientable. For background on contact surgery and symplectic and Stein cobordisms, we refer to [2, 4, 6, 8, 16, 17, 20, 22, 32]. Legendrian links in (S^3,ξ_{st}) are always presented in their front projection. We choose the normalisation of the d_3-invariant as in [2, 16] which differs from the normalisations in [6, 8, 19] by 1/2. Using our normalisation, we see that contact structures on homology spheres have integral d_3-invariants (in particular, $\mathrm{d}_3(S^3,\xi_{\text{st}}) = 0$) and that the d_3-invariant is additive under connected sums.

2. Properties of the contact surgery graph

We start by discussing the basic properties of the contact surgery graph Γ.

PROOF OF PROPOSITION 1.2. Γ has infinite diameter since a single surgery can change the rank of the first homology at most by one.

To show that the outdegree of Γ is infinite, let $v_{(M,\xi)}$ be a vertex of Γ. For an even integer n, choose a Legendrian knot L in a Darboux ball in (M,ξ) with Thurston–Bennequin invariant $\mathrm{tb} = 1$ and rotation number $\mathrm{rot} = n$. Denote the contact manifold obtained by contact (-1)-surgery along L by $L(-1)$. A calculation, as for

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example in [6, 8], shows that the homology of $L(-1)$ is

$$H_1(L(-1)) = H_1(M) \oplus \mathbb{Z}_{\mu_L},$$

where the $\mathbb{Z}$-summand is generated by a meridian μ_L of L, and that the Poincaré dual of the Euler class $\mathrm{e}(L(-1))$ is given by

$$\mathrm{e}(L(-1)) = \mathrm{e}(M, \xi) + n\mu_L,$$

where $\mathrm{e}(M, \xi)$ denotes the Poincaré dual of the Euler class of (M, ξ). Thus, we get infinitely many different contact manifolds by contact (-1)-surgery from M. The same construction with contact $(+1)$-surgery along a Legendrian knot with Thurston–Bennequin invariant $\mathrm{tb} = -1$ provides the fact that the indegree is also infinite. □

Proof of Theorem 1.1. Let E be a set consisting of k-different vertices in Γ and let (M_0, ξ_0) and (M_n, ξ_n) be two contact 3-manifolds representing vertices in $\Gamma \backslash E$. We consider a path

$$(M_0, \xi_0) \to (M_1, \xi) \to (M_2, \xi_2) \to \cdots \to (M_n, \xi_n)$$

in Γ between (M_0, ξ_0) and (M_n, ξ_n). We denote by L_0 a Legendrian surgery link in (S^3, ξ_{st}) representing (M_0, ξ_0). Let K_{i+1} be a Legendrian knot in (M_i, ξ_i) such that contact (± 1)-surgery along K_i is contactomorphic to (M_{i+1}, ξ_{i+1}). By Lemma 4.7.1 in [22], we can represent K_1 as a Legendrian knot in the complement of L_0 and thus $L_1 := L \cup K_1$ is a surgery link in (S^3, ξ_{st}) for (M_1, ξ_1). By induction, we get surgery links L_i in (S^3, ξ_{st}) representing (M_i, ξ_i) describing the above path in Γ. We will construct a path from (M_0, ξ_0) to (M_n, ξ_n) in $\Gamma \backslash E$.

Let $p \geq 2$ be an integer such that no contact structure on $M_i \# L(p, 1)$ is in E for all $i = 0, \ldots, n$. (Such a p exists because E is finite.) We define a new sequence of Legendrian surgery links as follows. We set $L'_0 = L_0$ and $L'_{i+1} = L_i \sqcup U_p(-1)$, where $\sqcup$ denotes the disjoint union of knots and $U_p(-1)$ denotes the contact (-1)-surgery along a Legendrian unknot with $\mathrm{tb} = 1 - p$. It follows that L'_{i+1} represents a contact structure on $M_i \# L(p, 1)$. Finally, we set L'_{n+2} to be the disjoint union of L_n with $U_p(-1)$ together with a $(+1)$-framed meridian μ_{U_p} of U_p. Since $U_p(-1) \cup \mu_{U_p}(+1)$ yields (S^3, ξ_{st}) by [1], we see that L'_{n+2} represents (M_n, ξ_n). Thus, a path in $\Gamma \backslash E$ between (M_0, ξ_0) and (M_n, ξ_n) is constructed. The same argument shows that Γ is also k-edge connected. □

In fact, the above proof directly implies the following corollary.

Corollary 2.1. *Let E be a finite set of vertices in Γ and let (M_1, ξ_1) and (M_2, ξ_2) be two vertices in $\Gamma \backslash E$. Then their distances in the corresponding graphs are related by*

$$d_{\Gamma \backslash E}(M_1, M_2) \leq d_{\Gamma}(M_1, M_2) + 2.$$

With the above results, we study the existence of Eulerian and Hamiltonian paths and walks.

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PROOF OF THEOREM 1.3. Since Γ is connected, k-connected and k-edge-connected, for any natural number k, and of infinite degree, the main results from [13, 14, 26] immediately imply that Γ contains Eulerian and Hamiltonian paths.

Since contact (-1)-surgery preserves tightness by [29], a Hamiltonian diwalk has to start at an overtwisted contact manifold and has to run first through all overtwisted contact manifolds before reaching a tight contact manifold. However, since there exists infinitely many overtwisted contact manifolds, this is not possible. □

Using the main result of [25], we deduce the following corollary.

COROLLARY 2.2. *Γ is **biased**, that is, there exists a subset X of the vertices of Γ such that there are infinitely many edges oriented from X to its complement X^c, but only finitely many edges oriented from X^c to X.*

It would be interesting to find such a set X explicitly.

PROOF OF COROLLARY 2.2. The main result of [25] says that an oriented graph admits an Eulerian dipath based at a vertex v if and only if the graph is countable, connected, 1-coherent, v-solenoidal and unbiased. We refer to [25] for the definitions. Since the contact surgery graph Γ is connected and countable but admits no Eulerian dipath, it is enough to check that Γ is 1-coherent and v-solenoidal. First, 1-coherency is a condition of the underlying unoriented graph and since Γ admits an undirected Euler path, Γ is 1-coherent. Second, it follows that Γ is v-solenoidal for any vertex since the indegree and outdegree are infinite for any vertex. □

3. Contact geometric subgraphs of Γ

Here we study the subgraphs Γ_{OT}, Γ_{tight}, Γ_{Stein}, Γ_{strong}, Γ_{weak} and $\Gamma_{c\neq 0}$.

PROOF OF THEOREM 1.4. We first show that Γ_{OT} is strongly connected. Let (M_1,ξ_1) and (M_2,ξ_2) be two overtwisted contact manifolds. By [15], there exist a directed path p from (M_1,ξ_1) to (M_2,ξ_2) in Γ. We argue that any vertex in p corresponds to an overtwisted contact manifold. Let us assume the contrary. Since (M_2,ξ_2) is overtwisted, there exists an overtwisted contact manifold $(M_{\mathrm{OT}},\xi_{\mathrm{OT}})$ that can be obtained by contact (-1)-surgery from a tight contact manifold $(M_{\mathrm{tight}},\xi_{\mathrm{tight}})$ contradicting Wand's result which says that contact (-1)-surgery preserves tightness [29].

Next, we consider Γ_* with $* =$ tight, Stein, strong, weak or $c \neq 0$. To show that Γ_* is connected, we show that there exists an undirected path in Γ_* from (S^3,ξ_{st}) to any other contact manifold (M,ξ) with property $*$. We consider the contact $(+1)$-surgery along the Legendrian unknot with Thurston–Bennequin invariant $\mathrm{tb} = -2$ and rotation number $\mathrm{rot} = 1$ in (S^3,ξ_{st}). It is well known that the resulting contact manifold is the overtwisted contact structure ξ_1 on S^3 with normalised d_3-invariant equal to 1 [6]. Then by [15], there exists a directed path in Γ of contact (-1)-surgeries from (S^3,ξ_1) to (M,ξ). However, this path runs through at least one overtwisted contact manifold and hence it is not in Γ_*. In total, we get a surgery link L in (S^3,ξ_{st}) describing (M,ξ) with only a single contact surgery coefficient $(+1)$. Now we change the order of the surgeries and first perform all contact (-1)-surgeries and at the end, we perform the single contact $(+1)$-surgery. Since contact (-1)-surgery is known to preserve any of the properties $*$ by [10, 27, 29, 30], we get a path in Γ_* from (S^3, ξ_{st}) to (M, ξ).

The contact surgery subgraphs Γ_{tight}, Γ_{strong}, Γ_{weak} and $\Gamma_{c\neq 0}$ are not strongly connected since there exists in each of these graphs a contact manifold (M, ξ) which is not Stein fillable [12, 18] and then there cannot be a directed path from (S^3, ξ_{st}) to (M, ξ).

Finally, we show that Γ_{Stein} is not strongly connected. We show that there exists no directed path of contact (-1)-surgeries from (S^3, ξ_{st}) to $(S^1 \times S^2, \xi_{st})$. Let us assume the contrary. Then there must be a simply connected Stein cobordism from (S^3, ξ_{st}) to $(S^1 \times S^2, \xi_{st})$. We glue this Stein cobordism to the standard 4-ball filling of (S^3, ξ_{st}) to get a simply connected Stein filling (W, ω_{st}) of $(S^1 \times S^2, \xi_{st})$. However, by [10], any Stein filling of $(S^1 \times S^2, \xi_{st})$ is diffeomorphic to $S^1 \times D^3$ which is not simply connected. □

REMARK 3.1. We remark that we have shown that Γ_{OT} is a strongly connected component of Γ and we wonder what are the other strongly connected components of Γ. Using Wendl's theorem on symplectic fillings of planar contact manifolds [31], Plamenevskaya [28] deduced that any planar Stein fillable contact manifold (M, ξ) cannot be obtained from itself by a sequence of contact (-1)-surgeries. It follows that any planar Stein fillable contact manifold is its own strongly connected component.

QUESTION 3.2. Is Γ_{OT} the only nontrivial strongly connected component of Γ?

REMARK 3.3. We know that each of Γ_{OT}, Γ_{tight}, Γ_{Stein}, Γ_{strong}, Γ_{weak} and $\Gamma_{c\neq 0}$ has infinite diameter and infinite in- and outdegree. That Γ_{tight} and $\Gamma_{c\neq 0}$ have infinite indegree follows from the work of Lisca–Stipsicz [23]. As part of their main theorem, they describe infinitely many different Legendrian knots such that contact $(+1)$-surgery on them yields tight contact manifolds with nonvanishing contact class. That Γ_{strong}, Γ_{weak} and Γ_{Stein} have infinite indegree follows similarly from [3]. The other statements follow from the arguments in the proof of Proposition 1.2.

For the proof of Theorem 1.5, we need the following lemma.

LEMMA 3.4. *Let (M, ξ^M) be an overtwisted manifold and (M, ξ^M_{stab}) be its stabilisation, that is, $(M, \xi^M_{stab}) = (M, \xi^M) \# (S^3, \xi_1)$. Let (N, ξ^N) be an overtwisted contact manifold which can be obtained from (M, ξ^M_{stab}) by a single contact (-1)-surgery. Then we can obtain (M, ξ^M) by a contact (-1)-surgery from (N, ξ^N).*

PROOF OF LEMMA 3.4. Let L be a Legendrian knot in (M, ξ^M_{stab}) such that $L(-1) = (N, \xi^N)$. Let L^* be the dual surgery knot of L in (N, ξ^N). By the cancellation lemma, $L^*(+1)$ is again contactomorphic to (M, ξ^M_{stab}). Now we choose a loose Legendrian realisation L' of L^* such that if we stabilise L', once positive and once negative, we get a Legendrian knot which is formally isotopic to L^*. (This is possible since we can destabilise any loose Legendrian knot.)

We claim that $L'(-1)$ yields (M, ξ^M). Since L' is topologically isotopic to L^* and its contact framing and the contact framing of L^* differ by 2, the contact (-1)-surgery

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along L' yields topologically the same manifold as the contact $(+1)$-surgery along L^*, which is in fact M. Note that $L'(-1)$ is overtwisted since L' is a loose knot. Then a straightforward computation in a local model, as in [7], shows that the homotopical invariants of the contact structures agree. Thus by Eliashberg's classification of contact structures [9], it follows that the contact structures are contactomorphic. □

Proof of Theorem 1.5. First we remark that by following the proof of Theorem 1.1, we conclude that each of the subgraphs Γ_{OT}, Γ_{tight}, Γ_{Stein}, Γ_{strong}, Γ_{weak} and $\Gamma_{\mathrm{c}\neq 0}$ is k-connected and k-edge-connected for any natural number k. Thus, each of these subgraphs admits Eulerian and Hamiltonian paths and also Hamiltonian walks. For the directed paths, we conclude as in the proof of Theorem 1.3 that Γ_* does not admit a Hamiltonian diwalk, a Hamiltonian dipath and a Eulerian dipath for $* =$ tight, Stein, strong, weak or $\mathrm{c} \neq 0$.

However, we show that an Eulerian dipath exists in Γ_{OT}. As in the proof of Corollary 2.2, it follows that Γ_{OT} is 1-coherent and ν-solenoidal. By the main result of [25], it is therefore enough to show that Γ_{OT} is unbiased. Assume there exists a subset X of the vertices of Γ_{OT} such that there are infinitely many edges oriented from X to its complement X^c, but only finitely many edges oriented from X^c to X. As a first step, we show that there exists, for any $i \in \mathbb{N}$, a Legendrian knot K_i in an overtwisted contact manifolds (M_i, ξ_i) in X such that all M_i are pairwise nondiffeomorphic and the contact (-1)-surgery $K_i(-1)$ along K_i lies in X^c. (This step will not use the overtwistedness.)

By assumption, we know that there are infinitely many edges pointing out of X. For any $i \in \mathbb{N}$, we choose some Legendrian knot K_i' in an overtwisted contact manifold (M, ξ) in X such that $K_i'(-1)$ lies in X^c. We show that we can obtain infinitely many of the $K_i'(-1)$ by a contact (-1)-surgery from infinitely many different manifolds in X. For that, we first observe that we can get an overtwisted contact structure ξ_{OT} on $M\#L(i+1,1)$ by a single contact $(+1)$-surgery along a Legendrian unknot U_i in a Darboux ball in $M\backslash K_i$. Since only finitely many edges point into X, we conclude that an infinite subset of the $(M\#L(i+1,1), \xi_{\mathrm{OT}})$ are elements of X again. Performing a contact (-1)-surgery along K_i in $(M\#L(i+1,1), \xi_{\mathrm{OT}})$ yields $K_i(-1)\#L(i+1,1)$ with an overtwisted contact structure. Then cancelling the contact $(+1)$-surgery along U_i by a contact (-1)-surgery along a push-off of U_i yields $K_i(-1)$. Thus we can conclude that there exists an infinite family of Legendrian knots K_i in different overtwisted contact manifolds (M_i, ξ_i) in X such that $K_i(-1)$ are elements of X^c.

In the next step, which only works for overtwisted contact manifolds, we show that we can get back from the $K_i(-1)$ to X by contact (-1)-surgeries. We write (M_i, ξ_i) as $(M, \xi_i')\#(S^3, \xi_1)$. By Lemma 3.4, there exists a contact (-1)-surgery along a Legendrian knot in $K_i(-1)$ yielding (M_i, ξ_i'), and from (M_i, ξ_i'), we can get back to (M_i, ξ_i) by another contact (-1)-surgery. Thus we have constructed an infinite family of edges pointing from X^c into X contradicting the assumption and finishing the proof of Theorem 1.5. □

Next, we study the difference of the distance functions.

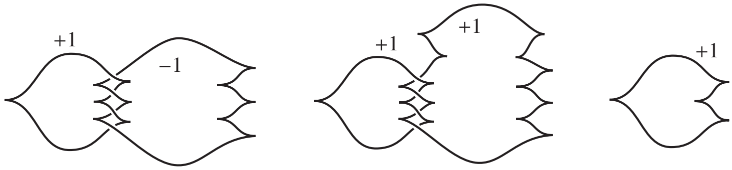

FIGURE 1. Contact surgery diagrams of contact structures on S^3: left, (S^3, ξ_{-1}); centre, (S^3, ξ_0); right, (S^3, ξ_1) [6, 16].

THEOREM 3.5. *Given two overtwisted contact manifolds* (M_1, ξ_1) *and* (M_2, ξ_2)*:*

(1) *the distance between* (M_1, ξ_1) *and* (M_2, ξ_2) *in* Γ_{OT} *is at most* 2 *larger than their distance in* Γ*;*

(2) *the minimal lengths of directed paths from* (M_1, ξ_1) *to* (M_2, ξ_2) *agree in* Γ *and* Γ_{OT}*.*

PROOF. (1) Let p be a minimal (undirected) path between (M_1, ξ_1) and (M_2, ξ_2) in Γ. If every vertex in p corresponds to an overtwisted contact manifold, the distances in Γ and Γ_{OT} agree. In general, however, the path p might run through tight contact manifolds. To prevent this, we choose an ordered surgery link L in (M_1, ξ_1) such that contact (± 1)-surgery in that given order along L corresponds to the path p. We denote by ξ_0 the overtwisted contact structure on S^3 with vanishing normalised d_3-invariant. A 2-component surgery diagram of (S^3, ξ_0) is shown in Figure 1 (centre). We add this surgery diagram in a Darboux ball in the exterior of L to the surgery link, where we first perform the surgeries along the two new Legendrian knots and afterwards the surgeries corresponding to p. Then any contact manifold in the path p is replaced by a connected sum with (S^3, ξ_0). By Eliashberg's classification of overtwisted contact structures [9], it follows that we have constructed a new path p' from (M_1, ξ_1) to (M_2, ξ_2) in Γ_{OT} of length 2 larger than the length of p.

(2) By Wand's theorem [29], any directed path in Γ between two overtwisted contact manifolds cannot run through a tight contact manifold (see the proof of Theorem 1.4). □

4. Topological subgraphs of Γ

Before we prove Theorem 1.6, we first discuss the corresponding result for S^3.

LEMMA 4.1. Γ_{S^3} *is connected.*

PROOF. Recall that, by the work of Eliashberg, ξ_{st} is the unique tight contact structure on S^3 [11] and the overtwisted contact structures are in one-to-one correspondence to the homotopy classes of tangential 2-plane fields [9], which are on homology spheres in bijection with the integers via their normalised d_3-invariants [19]. We denote the unique overtwisted contact structure on S^3 with normalised d_3-invariant equal to n by ξ_n.

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Contact surgery diagrams for all contact structures on S^3 were explicitly described in [6]. A contact surgery diagram of ξ_1 is given by the contact $(+1)$-surgery along the Legendrian unknot with Thurston–Bennequin invariant $\mathrm{tb} = -2$ and rotation number $\mathrm{rot} = 1$, shown on the right of Figure 1. The contact (±1)-surgery diagram along the 2-component link, shown on the left of Figure 1, represents ξ_{-1}. The disjoint union of two surgery diagrams describes a connected sum of the underlying contact manifolds. Since the d_3-invariant behaves additively under the connected sum, we get contact surgery diagrams of all contact structures on S^3 by taking appropriate disjoint unions of the contact surgery diagrams of ξ_1 and ξ_{-1}.

It follows that we can get (S^3, ξ_1) by a single contact $(+1)$-surgery from (S^3, ξ_{st}) and that there exists a contact $(+1)$-surgery on (S^3, ξ_k) yielding (S^3, ξ_{k+1}) and thus conversely a contact (-1)-surgery from (S^3, ξ_k) to (S^3, ξ_{k-1}). □

PROOF OF THEOREM 1.6. First we discuss the case that M is a homology sphere. Then we can get any overtwisted contact structure on M from a fixed contact structure on M by connected summing with overtwisted contact structures of S^3 [6]. It follows that any two overtwisted contact structures on M can be connected in Γ_M. Now let ξ_{tight} be some tight contact structure on M. Then there exists a single contact $(+1)$-surgery along a Legendrian unknot in (M, ξ_{tight}) yielding $(M, \xi_{\text{tight}})\#(S^3, \xi_1)$ which is overtwisted, and thus it follows that Γ_M is connected.

If the underlying manifold is not a homology sphere, it gets slightly more complicated since the classification of tangential 2-plane fields is more involved. As a further invariant, we have the *spin*^{c} structure of a contact structure. However, it is known that for a given contact structure with *spin*^{c} structure $\mathfrak{s}$, we can get any other overtwisted contact structure with the same *spin*^{c} structure $\mathfrak{s}$ by connected summing with the overtwisted contact structures on S^3 [6]. Thus, we can apply the same argument as in the homology sphere case to deduce that $\Gamma_{(M,\mathfrak{s})}$ is connected.

It remains to show that there is no edge connecting two different *spin*^{c} structures on M. For that, we use Gompf's Γ-invariant which classifies *spin*^{c} structures [19]. Let $\mathfrak{s}$ be a *spin*^{c} structure on M and ξ be a contact structure inducing $\mathfrak{s}$. Let L_0 in (M, ξ) be a Legendrian knot such that contact $(+1)$- or contact (-1)-surgery along L_0 yields another contact structure ξ' on M. We want to show that ξ' induces the same *spin*^{c} structure $\mathfrak{s}$. For that, we choose a spin structure $\mathfrak{t}$, describe (M, ξ) by a contact surgery diagram along a Legendrian link $L = L_1 \cup \cdots \cup L_n$ in (S^3, ξ_{st}) and present L_0 as a knot in the exterior of L. To show that ξ and ξ' induce the same *spin*^{c} structures, it is enough to compute that Gompf's Γ invariants of ξ and ξ' with respect to $\mathfrak{t}$ agree. We present $\mathfrak{t}$ via a characteristic sublink $(L_j)_{j\in J}$, $J \subset \{1, \dots, n\}$, of L. Since the homologies of M and $L_0(\pm1)$ agree, we deduce that μ_{L_0} is nullhomologous in $L_0(\pm1)$ and that J is also a characteristic sublink of the surgery diagram $L_0 \cup L$ of $L_0(\pm1)$. Then we can use the formula for computing Gompf's Γ-invariant from [16] to compute

$$\Gamma(\xi', \mathfrak{t}) - \Gamma(\xi, \mathfrak{t}) = \frac{1}{2}\Big(\sum_{i=0}^{n} r_i\mu_i + \sum_{j\in J}(Q'\mu)_j\Big) - \frac{1}{2}\Big(\sum_{i=1}^{n} r_i\mu_i + \sum_{j\in J}(Q\mu)_j\Big) = 0,$$

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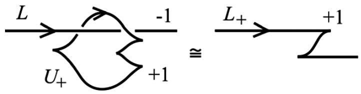

FIGURE 2. From a contact (-1)-surgery to a contact $(+1)$-surgery.

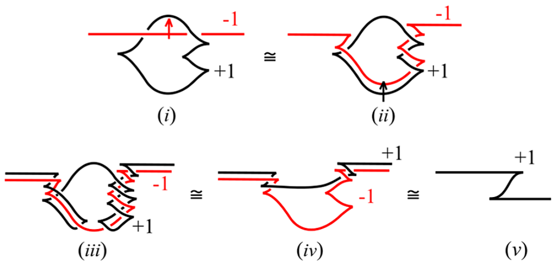

FIGURE 3. Proof of Lemma 4.2 via two handle slides and a lantern destabilisation.

where r_i denotes the rotation number and μ_i the meridian of L_i, Q' and Q denote the linking matrices of $L_0 \cup L$ and L, and all noncancelling terms are multiples of $\mu_0 = 0$. □

We now turn to the proof of Theorem 1.7. For the proof, we need the following lemma.

LEMMA 4.2. *Let L be a Legendrian knot in some contact* 3*-manifold* (M, ξ) *and U be a Legendrian meridian of L with* $\mathrm{tb} = -1$. *Let* $L_\pm$ *denote the positive/negative stabilisation of L (similarly for U). Then contact* (-1)*-surgery along L followed by contact* $(+1)$*-surgery along* $U_\pm$ *is contactomorphic to the contact* $(+1)$*-surgery along* $L_\pm$ *(see Figure 2), that is,* $L(-1) \cup U_\pm(+1) = L_\pm(+1)$.

PROOF. We relate the two contact surgery diagrams in Figure 2 by first performing two handle slides [1, 2, 5] followed by a lantern destabilisation [24] (see also [16]). We first slide the red knot L over U_+, as in Figure 3(i). Then, we slide U_+ over the red one as in Figure 3(ii), where the handle slides are indicated via the arrows. After isotoping Figure 3(iii) to get Figure 3(iv), we apply the lantern destabilisation to get Figure 3(v). □

PROOF OF THEOREM 1.7. Let (N, ξ_N) be a contact manifold in the link of (M, ξ). We construct a path of length one or two in $\mathrm{lk}(M, \xi)$ from (N, ξ_N) to $(M, \xi)\#(S^3, \xi_1)$. (We recall that $(M, \xi)\#(S^3, \xi_1)$ can be obtained from (M, ξ) by a single contact $(+1)$-surgery along a Legendrian unknot with $\mathrm{tb} = -2$ in a standard Darboux ball in (M, ξ).)

First, we consider the case in which we can obtain (N, ξ_N) by a contact $(+1)$-surgery along a Legendrian knot K in (M, ξ). In [1] it is shown that performing a contact $(+1)$-surgery K followed by a contact $(+1)$-surgery along a Legendrian meridian U of K with $\mathrm{tb} = -1$ corresponds to a negative stabilisation of (M, ξ) and thus yields $(M, \xi)\#(S^3, \xi_1)$.

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The case in which (N, ξ_N) arises as contact (-1)-surgery from (M, ξ) can be reduced to the first case by applying Lemma 4.2 once. □

The proof of Theorem 1.7 directly implies the following corollary.

COROLLARY 4.3.

(1) *If (N, ξ) is in the link of (M, ξ), then*

$$d_{\mathrm{lk}(M,\xi)}((N, \xi), (M, \xi)\#(S^3, \xi_1)) \leq 2.$$

(2) *If (N, ξ) can be obtained from (M, ξ) by a single contact $(+1)$-surgery, then (N, ξ) can be obtained from $(M, \xi)\#(S^3, \xi_1)$ by a single contact (-1)-surgery.*

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