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ARTICLES

LEARNABILITY OF E–STABLE EQUILIBRIA

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If private sector agents update their beliefs with a learning algorithm other than recursive least squares, expectational stability or learnability of rational expectations equilibria (REE) is not guaranteed. Monetary policy under commitment, with a determinate and E-stable REE, may not imply robust learning stability of such equilibria if the RLS speed of convergence is slow. In this paper, we propose a refinement of E-stability conditions that allows us to select equilibria more robust to specification of the learning algorithm within the RLS/SG/GSG class. E-stable equilibria characterized by faster speed of convergence under RLS learning are learnable with SG or generalized SG algorithms as well.

Keywords: Adaptive Learning, Expectational Stability, Stochastic Gradient, Speed of Convergence

1. INTRODUCTION

Adaptive learning and expectational stability (E-stability) arise naturally in self-referential macroeconomic models. The literature on adaptive learning assumes that economic agents act as econometricians who run recursive regressions using historical data to inform their decisions. The asymptotic outcome of adaptive learning algorithms may be consistent with rational expectations. Evans and Honkapohja (2001) provide the methodology and derive the conditions under which recursive learning dynamics converges to rational expectations equilibria (REE). If economic agents use recursive least squares (RLS) learning to update their expectations of the future (or learn adaptively), then only E-stable REE can be the asymptotic outcomes of a real–time learning process. Equilibria, stable under a particular form of adaptive learning, are also called *learnable*.[1] Hence, E-stability is a necessary condition for RLS learnability.

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Evans and Honkapohja (2001) also draw attention to the lack of general results on stability and convergence of different learning algorithms. Barucci and Landi (1997) and Heinemann (2000) demonstrate that E-stability may not be a sufficient condition for learnability if agents use adaptive algorithms other than RLS. Barucci and Landi (1997) show that an alternative learning mechanism, namely, stochastic gradient (SG) learning, converges to REE but under conditions different from RLS learning.[2]

Furthermore, Giannitsarou (2005) provides examples, with a lagged endogenous variable, of E-stable equilibria that are not learnable under SG learning. Evans et al. (2010, Sec. 4) discuss additional conditions that ensure E-stable equilibria are learnable under SG and generalized stochastic gradient (GSG) learning. They propose the GSG algorithm as an extension to model learning in the presence of uncertainty and parameter drift in agents' beliefs. In this context, the algorithm is a maximally robust learning rule. In addition, and more important, the authors illustrate that given a particular weighting matrix the conditions for GSG–stability are closely related to E-stability, and equivalent in a new Keynesian model of monetary policy with alternative interest rate rules.

Evans and Honkapohja (2003) and Bullard (2006), among others, establish the E-stability criterion as a minimum requirement for the design of meaningful monetary policy. E-stability of the resulting REE is a desirable property of any monetary policy rule, claim Bullard and Mitra (2007); in effect, equilibria ought to be learnable under RLS. Recently Tetlow and von zur Muehlen (2009) posed a problem of policy design in a world populated with agents who might learn using a misspecified model of the economy. They state that an equilibrium that is learnable for a wide range of possible specifications, even at a potential cost of welfare losses, is a valuable property of a monetary policy rule. Robustifying policy in this way ensures that learnability is achieved on the transition path to REE, without compromising convergence.

In this paper, we focus on the properties of E-stable equilibria which facilitate such a design problem allowing learnability under both RLS and GSG.[3] We propose a refinement of E-stability conditions that selects equilibria more robust to specification of the learning algorithm within the RLS/SG/GSG class. We show that the (mean–dynamics) speed of convergence under RLS learning is an important component of such a refinement. E-stable equilibria, characterized by a faster RLS speed of convergence, tend to remain learnable under SG or GSG algorithms as well. The mean-dynamics speed of convergence, discussed in Ferrero (2007), can also have consequences for welfare, and is related to the asymptotic behavior of the agents' beliefs as demonstrated by Marcet and Sargent (1995).

We extend E-stability requirements using two additional criteria. First, we require that REE be learnable under a broad set of learning algorithms of the RLS/SG/GSG class. In some sense, this allows us to choose a subset of REE with properties such that they remain learnable even if agents' learning process is misspecified asymptotically relative to RLS. Second, the speed of convergence under RLS should be fast not only to aid learnability, but also to ensure a fairly quick return of agents' beliefs toward the REE even after a small disturbance or deviation.[4] We confirm that the two additional criteria are related and can be met simultaneously.

2. E-STABILITY AND LEARNABILITY REVISITED

It has been well established in Evans and Honkapohja (2001) and elsewhere that the convergence of the RLS algorithm is closely related to E-stability. The equilibrium is said to be E-stable if a stationary point $\overline{\Phi}$ of the following ordinary differential equation (ODE) is asymptotically stable:

$$\frac{d\Phi}{d\tau} = T(\Phi) - \Phi. \tag{1}$$

$\overline{\Phi}$ corresponds to the rational expectations equilibrium of a forward–looking model. T is the mapping from perceived law of motion (PLM) to actual law of motion (ALM), and Φ is a vector of the parameters of interest. The differential equation (1) governs the behavior of the approximating, or "mean," dynamics in continuous "notional" or meta-time.[5] In a univariate setting, its equilibrium point is asymptotically stable if the Jacobian of (1) evaluated at $\overline{\Phi}$,

$$J = DT(\Phi)|_{\Phi=\overline{\Phi}} - I,$$

has only eigenvalues with negative real parts.[6]

If, instead of using RLS, economic agents rely on SG learning, the convergence of the mean dynamics of the learning process is governed by the following ODE:

$$\frac{d\Phi}{d\tau} = M(\Phi) \cdot [T(\Phi) - \Phi], \tag{2}$$

where $M(\Phi)$ is a symmetric and positive definite matrix of second moments of the state variables used by agents in forming their forecasts.

The RE equilibrium $\overline{\Phi}$ is still a stationary point of (2). It is learnable if $\overline{\Phi}$ is the locally asymptotically stable equilibrium of the ODE (2), which obtains when all eigenvalues of $M(\overline{\Phi}) \cdot J$ have negative real parts. Barucci and Landi (1997) first provided a proof of this result. It is important to note that the conditions that establish the analogue of the E-stability condition in this case are different from those obtained under RLS.

If the agents update their beliefs with a GSG learning algorithm instead, learnability is related to the negative real parts of all eigenvalues of the matrix

$$\Gamma M(\overline{\Phi}) \cdot J, \tag{3}$$

where we explicitly restrict our attention to weighting (symmetric) positive definite matrices Γ such that $\Gamma M(\overline{\Phi})$ is also arbitrary symmetric and positive definite. The class of such matrices includes $\Gamma = I$ (classic SG) and $\Gamma = M(\overline{\Phi})^{-1}$ (GSG asymptotically equivalent to RLS), any linear (convex) combination of them, or such a matrix Γ that has the same set of eigenvectors as $M(\overline{\Phi})$ (GSG more generally)—for example, $\Gamma = M^{\alpha}$, $\alpha \in \mathbf{R}$.[7] This fact is well documented and illustrated in Evans et al. (2010). In some sense, we claim there is a single basis in which the representations of both linear transformations (Γ and M) are diagonal. In particular, any matrix that starts from diagonalization of $M(\overline{\Phi}) = P \Delta P^T$, where P is a nonsingular matrix containing the eigenvectors of M, and uses $\Gamma = P \tilde{\Delta} P^T$, where $\tilde{\Delta}$ is an arbitrary diagonal matrix with positive diagonal, or a linear combination of such a Γ with the identity matrix, will yield a matrix in the class of interest here: symmetric positive definite.

The problem of a correspondence between E-stability and GSG-stability for all such Γ is, therefore, equivalent to the following linear algebraic problem: Given a matrix J with all its eigenvalues to the left of the imaginary axis, can we guarantee that no eigenvalue of $\Gamma M(\overline{\Phi}) \cdot J$ has positive real parts? This problem is well known and is referred to as H–stability, and was discussed earlier in Carlson (1968), Arrow (1974), Johnson (1974a), and Johnson (1974b). A sufficient condition for H-stability that is easy to check exists: matrix J is H-stable if its symmetric part, $\frac{1}{2}(J + J^T)$, is stable. Such a matrix is called negative quasidefinite. It is rather difficult to interpret this condition meaningfully from an economic point of view. Again, Evans et al. (2010) provide an economic example and an extended discussion of GSG learning and H-stability.

Although the convergence of the adaptive learning algorithms has been extensively studied, the transition along the learning path toward the equilibrium REE of interest is less well understood. Our starting point of reference is the results in Benveniste et al. (1990) and Marcet and Sargent (1995), who first identified the behavior of the speed of convergence (how fast or slowly agents' beliefs approach a REE point) and analyzed the asymptotic properties of the fixed point under RLS learning. For the purposes of this paper, we use the term "speed of convergence" to mean the minimum absolute value of the real parts of the eigenvalues of the linearized E-stability ODE. This value governs the speed of convergence of the mean dynamics under RLS learning.

In the linearized E-stability ODE $\frac{d\Phi}{d\tau} = J \cdot \Phi$, where all eigenvalues of J are distinct and have only negative real parts, the solution will be given as a linear combination of terms of the form $C_i \cdot e^{\lambda_i \cdot t}$, where λ_i are the eigenvalues of J and C_i are arbitrary constants. In the long run, the solution is dominated by the term that corresponds to $\hat{\lambda} = \min_i | \operatorname{Re}(\lambda_i) |$, the minimum absolute value of the real part of the eigenvalue.[8] In the context of adaptive learning, this speed of convergence, $\hat{\lambda}$, determines how fast the approximating mean dynamics described by the ODE in (1) approaches the REE asymptotically. Under standard decreasing-gain RLS learning, the speed of convergence is time-varying, subsiding as time evolves, and changes along the mean dynamics with the parameter estimates of the perceived law of motion, $\hat{\Phi}$.

The behavior along the transition path and the importance of short-run deviations away from the REE were illustrated by Evans and Honkapohja (1993) and Marcet

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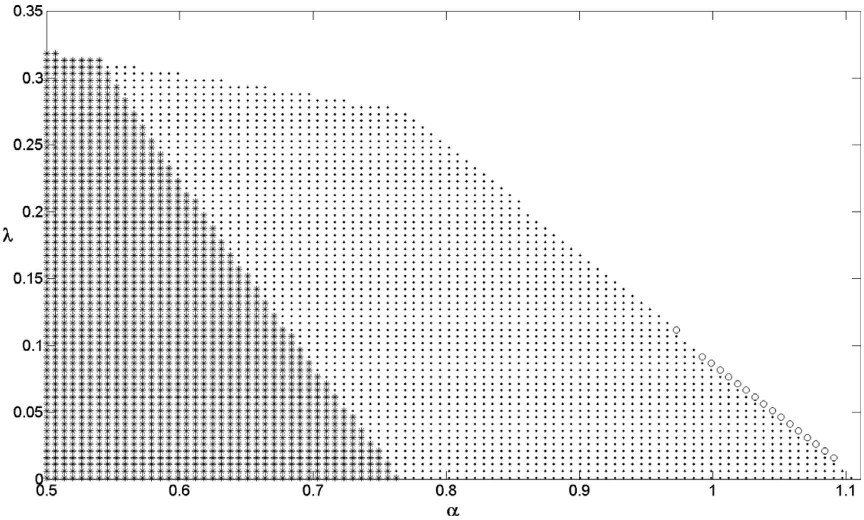

FIGURE 1. SG- and H-stability. The asterisks represent E-stable equilibria for which the sufficient condition of H-stability is satisfied. The dots show SG-stable equilibria that do not satisfy negative quasidefiniteness. The empty circles are SG-unstable equilibria.

and Sargent (1995). Ferrero (2007) further argued that the speed of convergence can be considered an important policy variable in the design of monetary policy. An open question is how important the RLS speed of convergence is for learnability under alternative learning algorithms.

The concepts of GSG-learning stability for all Γ considered in this paper and the speed of convergence under RLS appear distinct and far apart. However, it turns out that there is a close connection between the two. Consider, for example, the model in Sections 2 and 3 of Giannitsarou (2005). The reduced form of this univariate model is given by

$$y_t = \lambda y_{t-1} + \alpha E_t^* y_{t+1} + \gamma w_t,$$
$$w_t = \rho w_{t-1} + u_t.$$

In this model, $|\rho| < 1$ and $u_t \sim N(0, \sigma_u^2)$. The equilibrium of the model, with the same parameter values as in the paper, $\gamma = \sigma_u = 0.5$, $\rho = 0.9$, is E-stable, and therefore learnable under RLS. Both eigenvalues are real for all values of (α, λ) for which the solution $\overline{\Phi}_-$ is stationary and E-stable.

The E-stability ODE for this model is given by equation (1), where the mapping T is defined by equation (5) of Giannitsarou (2005, p. 277), and the vector Φ is two-dimensional. Figures 1 and 2 summarize the negative quasidefiniteness for the corresponding Jacobian and the speed of convergence of the mean dynamics, respectively, as functions of the parameters α and λ. These figures clearly indicate

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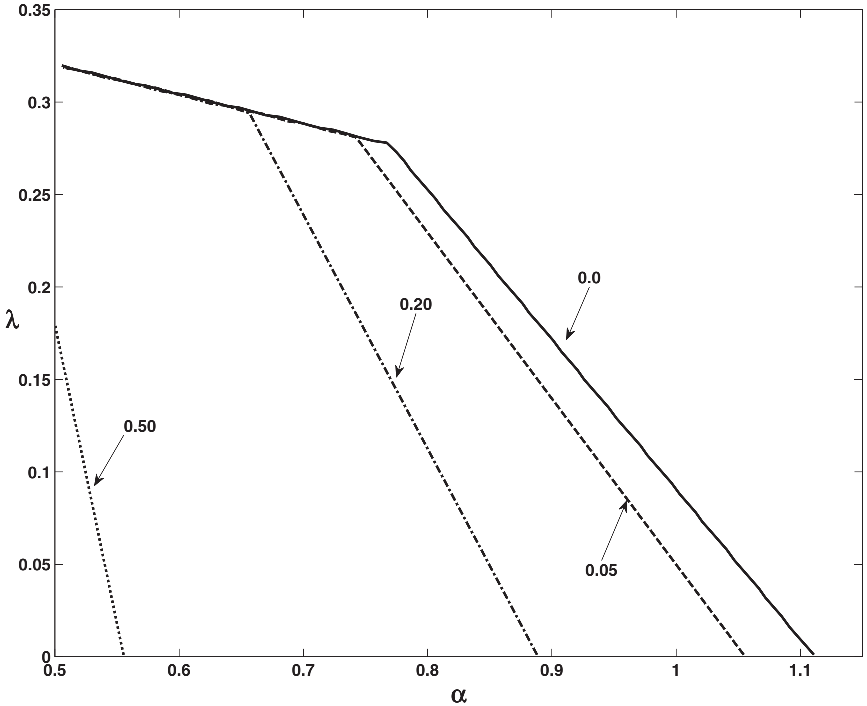

FIGURE 2. The RLS mean dynamics speed of convergence. The arrows point to the contour levels of the speed of convergence.

that negative quasidefiniteness obtains in regions of the parameter space where the speed of convergence is higher. Figure 2 is a contour plot of the speed of convergence, which increases toward the lower left corner of the graph, a region where the negative quasidefiniteness of the Jacobian is also observed.

In the region of the parameter space where the Jacobian is not negative quasidefinite, we expect that a matrix Γ might exist such that $\Gamma M(\overline{\Phi}) \cdot J$ was not stable. Therefore, the GSG learning algorithm that corresponds to this Γ does not result in an approximating dynamics converging to the REE. This conjecture stands correct: Giannitsarou (2005) shows that the equilibrium achieved under the SG learning algorithm (for which Γ equals the identity matrix) cannot be learned for a small set of parameter values. Figure 2 illustrates that for these parameter values the equilibrium is SG-unstable, and the speed of convergence is close to zero. On the other hand, for parameter values that correspond to a negative quasidefinite Jacobian, the speed of convergence is high and is never less than 0.35. Given that the negative quasidefiniteness is a sufficient condition for H-stability, we are guaranteed that GSG learning with any choice of Γ induces a learnable equilibrium. In particular, SG–learning will always converge for this set of parameter values.

This analysis indicates that there appears to exist a close relationship between SG-learning stability and the speed of convergence of the mean dynamics under RLS learning. The paper studies the nature of this relationship and addresses the following questions:

(i) Why is SG-instability associated with a lower speed of convergence under RLS?
(ii) In contrast, why do conditions that guarantee fast convergence to REE also seem to ensure SG- and GSG-stability for any Γ?
(iii) Are the answers to (i) and (ii) general and applicable enough to a wide variety of self-referential models?

In what follows, we provide a two-dimensional geometric interpretation of the case where E-stability holds but GSG-stability is not achieved (i.e., a matrix J is stable but not H-stable) and relate this finding to the speed of convergence of expectations to their REE values under RLS learning.

3. A GEOMETRIC INTERPRETATION OF LEARNABILITY

Provided J is a stable matrix, when is $\Omega \cdot J$ stable? We propose a simple two-dimensional geometric approach to answer this question. To preview our results: we study the eigenvalues of a matrix J, but not its components, and then relate those values to the speed of convergence of the mean dynamics under recursive least-squares learning. These findings have an intuitive and meaningful interpretation in a wide variety of adaptive learning models.

Now suppose that the 2×2 matrix J has only eigenvalues with negative real parts. This matrix is the Jacobian of the ODE in (1) for some adaptive learning model. J is asymptotically stable; therefore, the equilibrium associated with the model is E-stable and learnable under RLS.

The eigenvalue problem of J can be written as

$$J \cdot V = V \cdot \Lambda, \tag{4}$$

where V is a matrix with columns containing the eigenvectors of J, and Λ is diagonal with the corresponding eigenvalues λ_i on the main diagonal. If the eigenvectors are linearly independent, the matrix J can be diagonalized as $J = V\Lambda V^{-1}$.

Learnability of the equilibrium with GSG learning is determined by the eigenvalues of $\Omega \cdot J$. For our purposes, Ω is assumed to be symmetric and positive definite and thus can be written as $\Omega = P\Delta P^T$, where Δ is diagonal with main elements the eigenvalues of Ω.[9] The eigenvalue problem for $\Omega \cdot J$ can be written as:

$$P\Delta P^T \cdot V\Lambda V^{-1} \cdot \widetilde{V} = \widetilde{V} \cdot \widetilde{\Lambda}, \tag{5}$$

where the columns of $\widetilde{V}$ are the eigenvectors of $\Omega \cdot J$ and $\widetilde{\Lambda}$ is a diagonal matrix with the eigenvalues of $\Omega \cdot J$ as entries. Next premultiply (5) by P^{-1} and define

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$\overline{V} = P^{-1}\widetilde{V}$ to get

$$\Delta \cdot P^T V \Lambda V^{-1} P \cdot \overline{V} = \Delta \widetilde{J} \cdot \overline{V} = \overline{V} \cdot \widetilde{\Lambda}. \quad \textbf{(6)}$$

It is clear that the matrix $\widetilde{J} = P^T V \Lambda V^{-1} P$ has the same eigenvalues as J, i.e., the values on the main diagonal of Λ. Geometrically, if J represents a linear mapping in a two–dimensional space, then $\widetilde{J}$ represents the same mapping, after rotation, in new coordinates, given by the two orthogonal eigenvectors of Ω.

We work in the new coordinates and replace the problem of seeking conditions on the eigenvalues and eigenvectors of J such that $\Omega \cdot J$ has an eigenvalue with a positive real part (i.e., turns unstable) with the equivalent problem concerning $\widetilde{J}$ and $\Delta\widetilde{J}$. To fix notation, let us order δ_1 and δ_2, the (positive) eigenvalues of Ω, so that the following is always true: $(\delta_2/\delta_1) > 1$. The real eigenvalues of J and $\widetilde{J}$ are $-\lambda_1$ and $-\lambda_2$, ordered so that $(\lambda_2/\lambda_1) > 1$. Denote the eigenvectors of $\widetilde{J}$ corresponding to $-\lambda_1$ and $-\lambda_2$ as $v_1 = (v_{11}, v_{21})^T$ and $v_2 = (v_{12}, v_{22})^T$. Define $\Upsilon = v_{22}/v_{21} v_{11}/v_{12}$. The main focus of our analysis is on the real eigenvalue case, but for completeness we also discuss the possiblity of complex eigenvalues. The proofs are relegated to Appendices A and B, respectively.

PROPOSITION 1. *Let $\lambda_{1,2}$ be real. The matrix $\Omega \cdot J$ has an eigenvalue with a positive real part and thus J is not H-stable if and only if the following conditions hold:*

(i) $0 < \Upsilon < 1$,

(ii) $\frac{\lambda_2}{\lambda_1} > \frac{1}{\Upsilon}$, *and*

(iii)

$$\frac{\delta_2}{\delta_1} > \frac{\left(\frac{\lambda_2}{\lambda_1} - \Upsilon\right)}{\left(\frac{\lambda_2}{\lambda_1}\Upsilon - 1\right)}. \quad \textbf{(7)}$$

Proof. See Appendix A. ∎

COROLLARY 1. *Let $\lambda_{1,2}$ be real. If either $\Upsilon < 0$ or $\Upsilon < \frac{\lambda_1}{\lambda_2}$, the matrix J is H-stable and the equilibrium is learnable for any GSG learning algorithm.*

Proof. See Appendix A. ∎

Given this choice of ordering of the eigenvalues of J, the speed of convergence is equal to $|\lambda_1|$. If the speed of convergence, $\hat{\lambda}$, is very low, the ratio (λ_2/λ_1) will tend to be extremely large. At the same time, higher $\hat{\lambda}$ will typically lead to smaller values of the ratio (λ_2/λ_1). This therefore implies that the higher speed of convergence facilitates stability under learning for GSG algorithms. It is also possible to observe lower values of this ratio when both eigenvalues are similar in magnitude while the speed of convergence is relatively small. But for a range of the parameter set of interest we rarely observe this, and geometrically, the eigenvectors associated with these eigenvalue configurations are close to orthogonal. In this case, the conditions required in Propostion 1 are impossible to satisfy, and learning stability is achieved. We emphasize that a higher $\hat{\lambda}$ plays an essential role in ensuring stability in the class of RLS/SG/GSG algorithms, as confirmed in our numerical analysis. We also give an intuitive explanation for this requirement.

Proposition 1 constructs a counterexample of a matrix Ω such that $\Omega \cdot J$ is not stable. This means that agents who update their beliefs adaptively with the corresponding GSG algorithm cannot learn the REE, even though it is E-stable. A necessary condition for E-stable REE not to be learnable under a GSG learning algorithm for some Γ, as defined in this paper, is a positive Υ less than one. Geometrically, $0 < \Upsilon < 1$ implies that, after rotation into the system of coordinates defined by the eigenvectors of Ω, the two eigenvectors of $\widetilde{J}$ are in the same quadrant. When the eigenvectors of J are close to being collinear, a larger set of matrices Ω will satisfy this geometric condition. This leads to a larger set of GSG algorithms such that the REE is not learnable.[10]

It may be impossible to satisfy the assumptions of Proposition 1 if the eigenvectors are close to being orthogonal. In this case Υ is either too small (less than λ_1/λ_2) or too large (greater than 1) when it is positive. If they are exactly orthogonal, the positive Υ equals 0 or approaches ∞. The necessary (and sufficient) condition described in (7) shows that, for a given Υ, instability of $\Omega \cdot J$ occurs when (λ_2/λ_1) or (δ_2/δ_1) or both are sufficiently large. What does this imply? Or when can this occur? The ratio (λ_2/λ_1) will be large if $|\lambda_1|$ is very small, i.e., the RLS speed of convergence is low. Increasing $|\lambda_1|$ will facilitate GSG learnability as the condition in Proposition 1 becomes more difficult to fulfill. For sufficiently large $|\lambda_1|$, the ratio (λ_2/λ_1) is such that Corollary 1 confirms that the REE is learnable for any GSG algorithm.

Moreover, higher values of (δ_2/δ_1) imply that the eigenvalues of Ω are highly unbalanced. If agents update their beliefs using SG learning, this means that Ω is the covariance matrix of regressors with highly unequal diagonal terms. For example, the scaling invariance issue with SG identified by Evans et al. (2010) is more severe. The economic agents, who learn adaptively, are less likely to be using an SG algorithm when the (co-)variances of the regressors are extremely unequal. Therefore, the speed-of-convergence criterion directly ensures that the set of equilibria learnable by agents using algorithms within the RLS/SG/GSG class is sufficiently large. This result indicates how the speed of convergence plays an essential role in leading to (G)SG instability in general: the slower you approach the REE, the more likely you are to observe (G)SG instability, as exhibited in our simulations.

Figure 3 plots the necessary and sufficient condition (7) for three values of Υ: 0.3, 0.1, and 0.05 (where the last value exhibits eigenvectors of $\widetilde{J}$ close to being orthogonal). The condition is satisfied in the area of the figure located above and to the right of the corresponding line. If v_1 and v_2 are almost collinear,

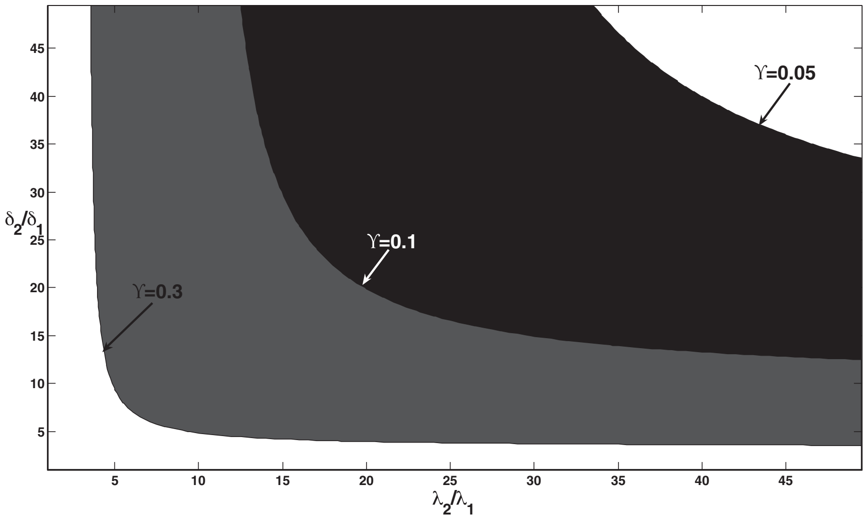

FIGURE 3. The necessary and sufficient conditions of Proposition 1. The white area in the upper right corner is where (7) holds for $\Upsilon \geq 0.05$. The black area: (7) holds for $0.10 \geq \Upsilon \geq 0.05$. The gray area: (7) holds for $0.3 \geq \Upsilon \geq 0.10$.

GSG-instability can be achieved for relatively mild ratios of the eigenvalues $(\frac{\lambda_2}{\lambda_1})$ and $(\frac{\delta_2}{\delta_1})$.

Turning to the results in Giannitsarou (2005), we show that points in the parameter space for which the equilibrium is not learnable under SG satisfy the conditions of Proposition 1. In this case, the corresponding values of $(\frac{\lambda_2}{\lambda_1})$ and $(\frac{\delta_2}{\delta_1})$ are extreme and lie in the vicinity of 112 and 5.5×10^5, respectively. Such a high degree of imbalance in the matrix Ω emerges because Υ is very close to zero (the eigenvectors of J are almost orthogonal) throughout the whole parameter space. For these parameters the speed of convergence is low (of order of magnitude 10^{-3}), and hence the ratio of the eigenvalues of J is very large. It is clear from this example that as the parameters of the model change, so do the associated eigenvalues. Hence, Corollary 1 (and Proposition 1) establishes a link between GSG-stability conditions and the speed of convergence, $\hat{\lambda}$, both of which are influenced by the agents' estimates of the model parameters. We attempt to quantify this intuition via simulations in the next section.

For completeness, we next turn to the case where the eigenvalues of the stability matrix are complex. To fix notation, assume that $\widetilde{J}$ has two complex eigenvalues: $\nu \pm i\mu$, $\nu < 0$, and two complex eigenvectors $w_1 \pm i w_2$, where w_1 equals $(w_{11}, w_{21})^T$ and w_2 is $(w_{12}, w_{22})^T$. Define $\widetilde{W} = w_{11}w_{12} + w_{21}w_{22}$ and $|W| = w_{11}w_{22} - w_{12}w_{21}$. The following proposition provides the necessary and sufficient condition for the instability of the matrix $\Omega \cdot J$:

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PROPOSITION 2. *Let $\lambda_{1,2}$ be complex. The matrix $\Omega \cdot J$ has an eigenvalue with a positive real part, and thus J is not H-stable, if and only if the following condition holds:*

$$\frac{\mu}{|\nu|}\frac{\widetilde{W}}{|W|}\frac{\left(\frac{\delta_2}{\delta_1}-1\right)}{\left(\frac{\delta_2}{\delta_1}+1\right)} > 1. \tag{8}$$

Proof. See Appendix B. ■

Similarly, Proposition 2 demonstrates that a smaller ratio of the eigenvalues of Ω is conducive to the stability of $\Omega \cdot J$. Hence, the corresponding GSG learning algorithm generates a convergent dynamics. A higher speed of convergence, larger $|\nu|$, makes the necessary and sufficient condition described in (8) harder to fulfill, and thus increases the set of parameters for which equilibria are learnable. Orthogonality of w_1 and w_2 means that the condition in Proposition 2 cannot be satisfied. The intuition is similar to that for the real eigenvalue case: in both cases orthogonality of eigenvectors (real case) or their real and imaginary components (complex case) ensures that GSG-learning instability is impossible.

Furthermore, Propositions 1 and 2 construct examples and state that if the spectrum of the Jacobian J is located farther away from the instability region (at least one eigenvalue with a positive real part), it is harder to find a set of Ω matrices (associated with) J—premultiplication by a (symmetric) positive definite matrix-that will lead to instability. Corollary 1 allows us to determine the robustness of the learning rule against such "disturbances." We establish that the speed of convergence facilitates this robustness property in the class of GSG algorithms considered here.

In this sense, Propositions 1 and 2 indicate that the second criterion we impose on all desirable REE—the high speed of convergence under RLS learning—is in accordance with the first criterion; that is, the REE are learnable under a range of learning algorithms within the RLS/SG/GSG class. To illustrate further the alignment of these two criteria and the way in which they modify selection of monetary policy rules, we study a standard model of monetary policy under commitment with learning.

4. MONETARY POLICY UNDER COMMITMENT

4.1. The Model Environment

Following Evans and Honkapohja (2006), we start with a standard two-equation new Keynesian (NK) model:

$$x_t = -\varphi\left(i_t - \widehat{\pi}_{t+1}\right) + \widehat{x}_{t+1}, \tag{9a}$$

$$\pi_t = \lambda x_t + \beta\widehat{\pi}_{t+1} + u_t. \tag{9b}$$

Here x_t and π_t express the output gap and inflation in period t, and all variables with circumflexes denote private sector expectations. i_t is the nominal interest rate,

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in deviation from its long-run steady state. The parameters φ and λ are positive and have the standard interpretation, and the discount factor is $0 < \beta < 1$. Our main interest is in the learning behavior of private sector agents, and we maintain the assumption that expectations may not be rational. We also assume the presence of only one shock to illustrate the results in this paper, and disregard the influence of the demand shock in the IS equation (9). The cost-push shock in (9b) is given by $u_t = \rho u_{t-1} + \epsilon_t$ where $\epsilon_t \sim \text{iid}(0, \sigma_\epsilon^2)$ is independent white noise. In addition, $|\rho| < 1$.

We consider the expectations-based interest rate policy rule under commitment, using the timeless perspective solution

$$i_t = \phi_L x_{t-1} + \phi_\pi \widehat{\pi}_{t+1} + \phi_x \widehat{x}_{t+1} + \phi_u u_t. \tag{10}$$

The optimal values of the policy rule parameters, based on a standard loss function, are given in Evans and Honkapohja (2006, p. 26, equation (15)) (notice that the coefficient ϕ_g is assumed to be zero in our specification). Here we do not restrict our attention to optimal monetary policy. We fix the values of the policy parameters, ϕ_L and ϕ_u, at their optimal level, and treat the other two policy parameters as choice variables of the policy response of the monetary authority.

Under the assumed policy rule the model can be written as

$$\begin{aligned} y_t = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} 1 - \varphi\phi_x & \varphi(1-\phi_\pi) \\ \lambda(1-\varphi\phi_x) & \lambda\varphi(1-\phi_\pi) + \beta \end{bmatrix} \begin{bmatrix} \widehat{x}_{t+1} \\ \widehat{\pi}_{t+1} \end{bmatrix} \\ &+ \begin{bmatrix} -\varphi\phi_L & 0 \\ -\varphi\lambda\phi_L & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} -\varphi\phi_u \\ 1 - \lambda\varphi\phi_u \end{bmatrix} u_t \\ y_t &= A\widehat{y}_{t+1} + CE^T y_{t-1} + Bu_t, \end{aligned} \tag{11}$$

where $E = (1, 0)^T$.

The MSV solution of this system can be expressed in the following way, with c and b both being vectors such that $c = (c^x, c^\pi)^T$, $b = (b^x, b^\pi)^T$:

$$y_t = cE^T y_{t-1} + bu_t. \tag{12}$$

Using the method of undetermined coefficients we find the REE solution, where c^x solves the cubic equation

$$c^x = -\varphi\phi_L + (1 - \varphi\phi_x)(c^x)^2 + \frac{\lambda\varphi(1-\phi_\pi)(c^x)^2}{1 - \beta c^x}.$$

The rest of the solution is provided in

$$c^\pi = \frac{\lambda c^x}{1 - \beta c^x}$$

and

$$b = [I - A(cE^T + \rho I)]^{-1} B.$$

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Next we turn to the conditions under which REE are learnable. We check for determinacy of the RE solution using the conditions derived in Evans and Honkapohja (2006, p. 35), which are not reproduced here.

4.2. Determinacy and E-Stability: The Minimum Requirements for Desirable RE Equilibria

Discussing E-stability, we follow the rest of the literature in assuming that the MSV solution obtained in the preceding is the PLM used by the private sector agents in the model. Let us rewrite the model as

$$\begin{aligned} y_t &= A\widehat{y}_{t+1} + CE^T y_{t-1} + Bu_t, \\ u_t &= \rho u_{t-1} + \epsilon_t. \end{aligned}$$

Calculate $\widehat{y}_{t+1}$, the nonrational expectation of the model (12), as

$$\begin{aligned} \widehat{y}_{t+1} &= E_t^*[cE^T y_t + bu_{t+1}] \\ &= [A(cE^T)(cE^T) + CE^T]y_{t-1} + [A(cE^T + \rho I)b + B]u_t. \end{aligned}$$

Then the T–map for the problem becomes

$$T(b, c) = [A(cE^T + \rho I)b + B, (c^T E)Ac + C].$$

This allows us to compute the (4×4) Jacobian matrix

$$J = \begin{bmatrix} A(\overline{c}E^T + \rho I) - I_2 & (E^T\overline{b})A \\ 0 & A(\overline{c}E^T + E^T\overline{c} \cdot I) - I_2 \end{bmatrix}.$$

4.3. SG-Stability

The matrix of the second moments of the state variables in the model used by the agents to forecast inflation and the output gap is obtained from a two-variable VAR written as

$$\begin{bmatrix} x_t \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} c^x & b^x \\ 0 & \rho \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \epsilon_{t+1}.$$

Thus, to study the SG-stability of the model, we examine the eigenvalues of $\Omega \cdot J$, which are now

$$\Omega \cdot J = (M \otimes I_2) \cdot J.$$

The Kronecker product appears in the expression because we have a multivariate model; Evans et al. (2010, Sect. 3), provide further details. Although it would be straightforward to include an intercept in the learning model [as in Evans et al. (2010)], we have omitted it here to reduce dimensionality, and it is not essential for our purposes.

We also study GSG learning stability, in which the agents are updating their beliefs about the parameters in the model by making small errors around the outcomes achieved under RLS learning. We adopt Γ in (3) to be the inverse of a second moments matrix $\widetilde{M}$, which is compatible with the same model evaluated at parameter values obtained within the neighborhood of the calibrated parameters used in the simulation analysis. We assume that these alternative learning algorithms model agents' uncertainty about the second moments of the state variables, which they need to know in order to use GSG learning as asymptotically equivalent to RLS learning. GSG-learning stability of the equilibria under this assumption is achieved when all eigenvalues of the matrix $[(\widetilde{M}^{-1} \cdot M) \otimes I_2] \cdot J$ have negative real parts. We note that the (Cholesky) transformation of variables proposed by Evans et al. (2010) delivers the same E-stability ODE as running generalized SG with $\Gamma = M^{-1}$. However, along the transition to the asymptotic equilibrium, the dynamics in these two cases will be different. Whereas the variable transformation in Evans et al. (2010) requires knowledge of the covariance matrix of the regressors, we allow for agents' mistakes regarding its estimates.

Propositions 1 and 2 cannot be stated in the four-dimensional case we have specified in the model, and therefore we analyze the stability of the system via simulations. Still, we expect that the main finding presented in the Propositions remains valid in the higher dimensions as well, namely that the lower speed of convergence of the mean dynamics under RLS will be associated with higher incidence of GSG-instability.

5. LEARNING INSTABILITY AND EQUILIBRIA: DISCUSSION

To analyze the link between GSG-stability for all Γ and the speed of convergence under RLS, we resort to simulations of the simple NK model under commitment for different values of the expectations-based policy rule parameters. We take this example because it gives robust learning stability, as argued in Evans and Honkapohja (2003). We do not perform an exhaustive study of the possible monetary policy rules in this model. To be more precise, we keep the parameters ϕ_L and ϕ_u at their optimal values derived in Evans and Honkapohja (2006), but vary ϕ_π and ϕ_x in a sufficiently broad range. The theoretical results on expectation-based policy rules under commitment, namely determinacy and E-stability of the REE, for any parameter values, were derived only for optimal policy by Evans and Honkapohja (2006). Therefore, we proceed to check every point for determinacy and E-stability (i.e., we check the eigenvalues of J for a negative real part). In addition, we calculate the speed of convergence of the mean dynamics under RLS learning, as described in Sect. 2, and check for convergence of the SG learning algorithm, by evaluating the eigenvalues of $(M \otimes I_2) \cdot J$.

We calibrate our model using the following parameter values. They are the same as in the Clarida et al. (1999) calibration: $\beta = 0.99$, $\varphi = 4$, and $\lambda = 0.075$, also used in Evans and Honkapohja (2003). We assume different values for the persistence of the cost-push shock, $\rho = 0.90$ (commonly used in the literature)

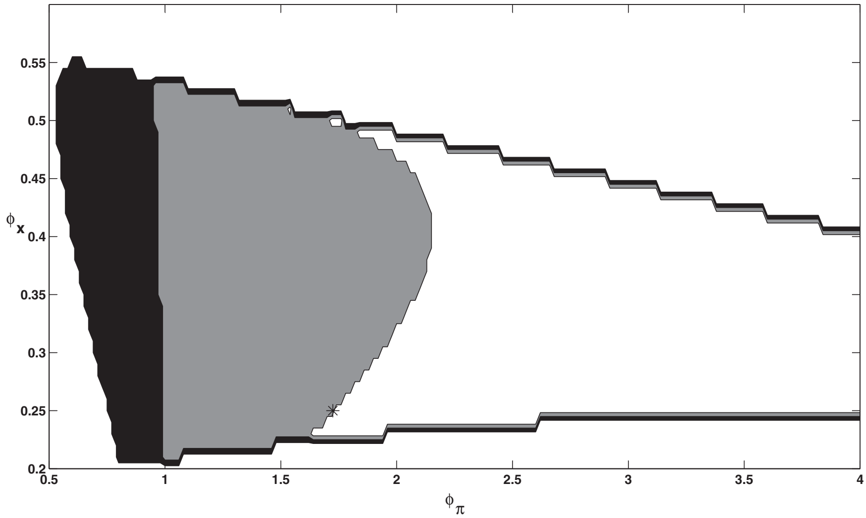

FIGURE 4. Determinacy and SG–stability: Monetary policy under commitment ($\rho = 0.9$). All points within the outer contour are E-stable equilibria. The black area is indeterminate. The gray area is determinate but SG-unstable. The white area is determinate and SG-stable. The asterisk represents the optimal monetary policy under commitment.

and $\rho = 0.60$. We also perform sensitivity analysis and robustness checks for various combinations of parameters other than ρ, e.g., the various permutations of the parameter space used in Evans and McGough (2010). We vary the number of simulation runs and confirm that the results are not altered.

The results of our simulations with $\rho = 0.90$ are presented in Figure 4. It illustrates, for every pair of the policy parameters (ϕ_π, ϕ_x), whether the resulting REE is determinate, E-stable, and SG-stable.[11] We only plot the E-stable points. The black area represents all indeterminate equilibria. We see that the standard Taylor principle applies [see Llosa and Tuesta (2009) for the theoretical derivations]. The points satisfying the Taylor principle are further split into SG-stable (white area) and SG-unstable (gray area).[12] SG-instability is concentrated in areas where ϕ_π is relatively low; as evident, more active monetary policy is associated with SG-stability.

How does this result fare against our Propositions 1 and 2, which associate the robustness of learning stability under alternative algorithms with the higher speed of convergence under RLS? In Figure 5 we see the link clearly. The association is shown by plotting contour levels of the speed of convergence for the same values of (ϕ_π, ϕ_x). All SG-unstable points have low convergence speed. When we move to more active monetary policy under commitment, both the speed of convergence and the robustness of SG learnability increase.

To compare our results with those of Evans and Honkapohja (2006), we plot a black asterisk at the point corresponding to the optimal monetary policy for

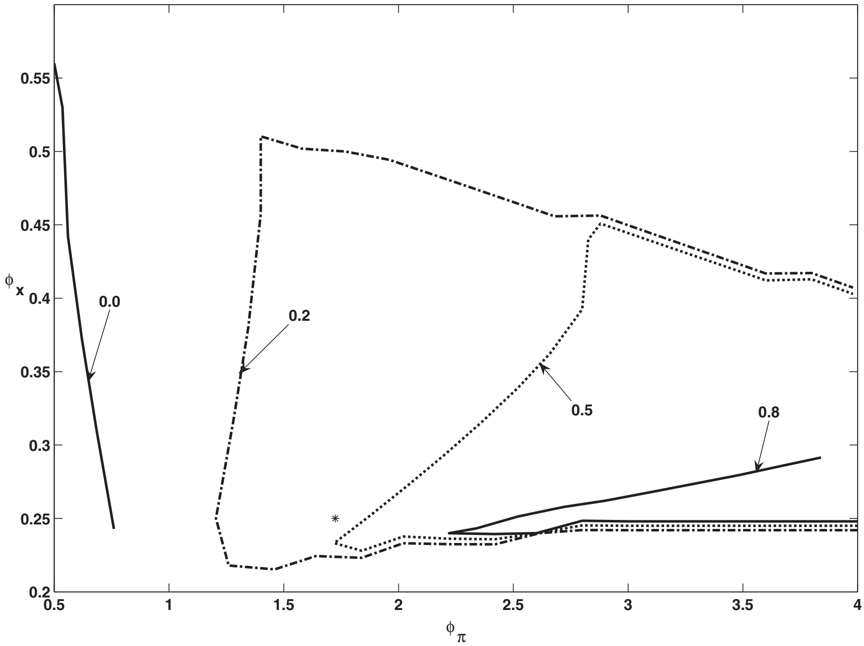

FIGURE 5. The RLS mean dynamics speed of convergence: Monetary policy under commitment ($\rho = 0.9$). The arrows point to the contour levels of the convergence speed. The asterisk represents the optimal monetary policy under commitment.

our calibrated values.[13] As expected, this policy delivers determinate and E-stable REE; however, notice that this policy is very close to both SG–stability and E-stability boundaries. This proximity raises the issue of the robustness of the optimal monetary policy if the agents are making small mistakes in their learning process. Evans and McGough (2010) study optimal monetary policy in NK models with inertia. They show that such a policy typically is located near the boundary of a set in the space of policy parameters where an E-stable and determinate equilibrium obtains. Small mistakes in calculating the policy parameters thus could lead to E-instability, indeterminacy, or both. We consider a forward-looking model, with lagged endogenous variable, where inertia is introduced through monetary policy under commitment. We show that even if we select policy parameters well inside the E-stable and determinate region, the outcome may turn unstable when the learning algorithm adopted by agents is SG, or GSG, which is not asymptotically equivalent to RLS.

We now perform the following experiment to study the robustness of the optimal or near-optimal monetary policy. We assume that the agents update their beliefs by running not RLS but instead a GSG learning algorithm. If the agents knew exactly the second moments matrix M associated with the parameter values (including the optimal monetary policy parameters) of the model, they would run a GSG that

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used M^{-1} as a weighting matrix. This GSG algorithm would be asymptotically equivalent to RLS, delivering determinate and E-stable REE, as explained in Evans et al. (2010).

In the experiment, we assume that agents face uncertainty regarding the second moments matrix. Given that the agents are *learning* second-order moments, such as correlations between future inflation and past output gap and the cost-push shock (and, therefore, they do not know them, at least away from the REE achieved asymptotically in infinite time), it seems natural to assume that their knowledge of other second-order moments is limited as well. Thus economic agents take the deep, structural parameters of the model to be somewhere in the neighborhood of the "true" parameter vector θ that we use in simulations. The agents would believe in $\widetilde{\theta}$ and use it to compute the second moments matrix $\widetilde{M}(\widetilde{\theta})$.[14] Then the agents would tend to use the matrix $\widetilde{M}^{-1}$ as a weighting matrix in updating their beliefs. Hence, $\widetilde{M}^{-1} = \Gamma$ in equation (3). The condition for the convergence of this real–time learning process, as we explained, is given by all eigenvalues of the matrix $[(\widetilde{M}^{-1} \cdot M) \otimes I_2] \cdot J$ with negative real parts.

We draw realizations of agents' beliefs about the parameters, $\widetilde{\theta}$, from a distribution that is centered at the true parameters θ. The range of the distribution is comparable to the prior distributions usually found in the literature on estimated DSGE models, for example, Milani (2007). We nest the true RLS learning in this procedure because it could be argued that SG-learning is too different from the RLS [for example, it is not scale-invariant; see Evans et al. (2010)] to be a realistic description of any actual updating process. Then we check whether this GSG algorithm is learnable or not. By repeating this procedure 1,000 times, we obtain an estimate of the probability of obtaining GSG instability for a given parameter pair, (ϕ_π, ϕ_x).[15,16]

The simulation exercise confirms the results described in Propositions 1 and 2. Comparison of Figures 4 through 6 shows that, for the SG-unstable points with a low RLS (mean dynamics) speed of convergence, we generally observe high incidence of GSG learning instability with imperfect knowledge of the second moments matrix. For the lowest speed still consistent with determinate and E-stable REE, we observe up to 60% probability of GSG instability. The optimal monetary policy (black asterisk) is associated with about 20% probability of becoming GSG-unstable. This result lends support to the finding in Evans and McGough (2010), where optimal monetary policy is shown to be rendered E–unstable or indeterminate by even small mistakes committed by adaptive agents.

The probability of observing a divergent GSG algorithm measures only how likely it is to find parameter draws such that the agents' misperceptions become strong enough to lead to expectational instability. How far should these mistaken perceptions be from the "truth" in order to generate a divergent algorithm? To answer this question, we take the matrix $\widetilde{M}^{-1}M$ and evaluate its largest eigenvalue.[17] We take this value as a measure of the mismatch between the "true" second moments matrix M and the agents' erroneous beliefs $\widetilde{M}$. We further consider the minimum of this measure over those among the 1,000 realizations

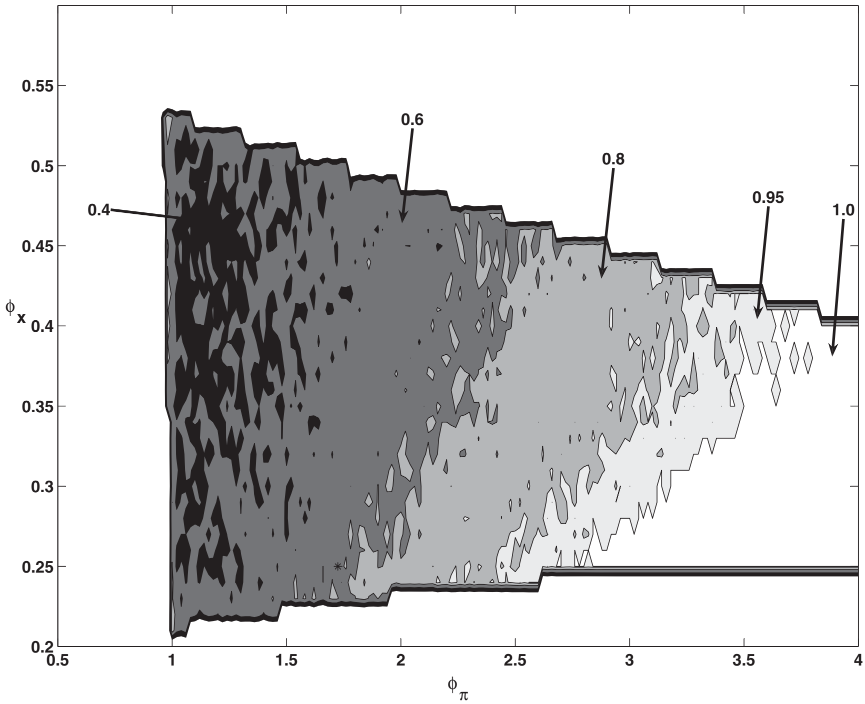

FIGURE 6. The probability of GSG stability: Monetary policy under commitment ($\rho = 0.9$). All points within the outer contour are determinate and E-stable equilibria. The probability of GSG stability is at least as large as the arrows indicate.

that lead to divergent GSG algorithms, and plot their contour levels in Figure 7. The lower intensity of gray depicted in the right area of the figure is associated with the higher contour levels. For example, the white area represents the strongest misperceptions about the true second moments matrix. The darkest area corresponds to the least mismatch of perceptions that allows a divergent GSG algorithm.[18]

The results of this exercise point in the same direction. More active monetary policy precludes a divergent GSG learning algorithm, because the necessary mismatch of beliefs is stronger (the lighter areas in Figure 7). For points in the (ϕ_π, ϕ_x) space that correspond to almost zero probability of observing GSG-instability, the mismatch measure equals 10 or higher, with the few unstable points exhibiting a very large mismatch of beliefs.[19]

The simulations depicted in Figures 4 through 7 support the conditions established in Propositions 1 and 2. According to Corollary 1, the higher the RLS speed of convergence, the harder it is to generate a divergent GSG-type algorithm. The higher speed of convergence corresponds to SG-learning stability and a lower probability of finding a second-moments matrix, $\widetilde{M}$, with bigger misperceptions

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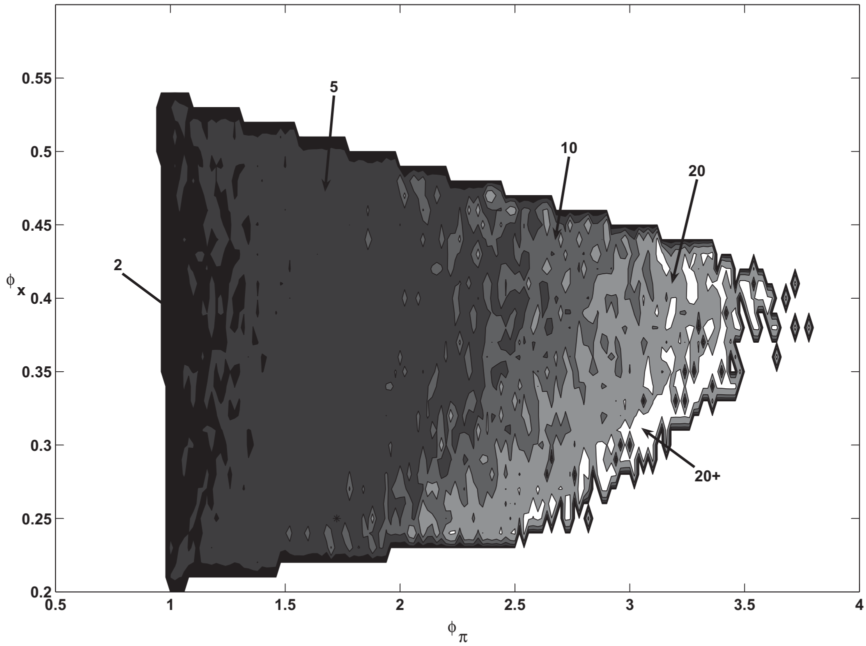

FIGURE 7. The minimum measure of misperceptions of GSG instability: Monetary policy under commitment ($\rho = 0.9$). All points within the outer contour are determinate and E-stable equilibria. The minimum measure of misperceptions of beliefs to get GSG instability is at least as large as the arrows indicate.

necessary to generate a divergent GSG algorithm. Thus, the two "refinements" to the concept of E-stability that we propose, the higher mean dynamics RLS speed of convergence and the greater robustness within a class of RLS/SG/GSG learning algorithms, do not need to generate trade-offs and can be satisfied simultaneously.

We summarize the results for $\rho = 0.9$ in Table 1. We also perform some sensitivity analysis of our simulation results. Table 2 presents the results for the same set of parameter values, but changes to $\rho = 0.6$. In the presence of a highly persistent cost-push shock, a temporary increase in inflation might result in increased inflation expectations that would remain elevated for a prolonged period of time, and induce actual inflation persistence as well. The convergence speed under adaptive learning decreases in the persistence of the shock, and thus any initial deviation in either actual inflation or expected inflation takes longer to die out. If the shock persistence is *lower* than in the baseline model calibration, in accordance with Propositions 1 and 2 and the results of this section, we expect higher RLS convergence speeds to lead to a larger area of SG-stability and a lower probability of finding a GSG-unstable algorithm.

In Tables 1 and 2, the speed of convergence is a decreasing function of the cost-push shock persistence.[20] The area of SG-instability disappears completely

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TABLE 1. Simulations results: Monetary policy under commitment ($\rho = 0.9$)

	SG-stab	Speed	GSG-stab Prob	Min Dist
		$\phi_x = 0.25$		
$\phi_\pi = 1.5$	–	0.33	0.73	2.71
$\phi_\pi = 2.0$	+	0.56	0.83	4.15
$\phi_\pi = 2.5$	+	0.80	0.93	16.00
		$\phi_x = 0.35$		
$\phi_\pi = 1.5$	–	0.25	0.64	2.17
$\phi_\pi = 2.0$	–	0.38	0.74	3.59
$\phi_\pi = 2.5$	+	0.48	0.84	6.56
		$\phi_x = 0.45$		
$\phi_\pi = 1.5$	–	0.22	0.70	3.79
$\phi_\pi = 2.0$	–	0.33	0.71	5.10
$\phi_\pi = 2.5$	+	0.43	0.76	5.72

Note: SG-stab: SG–stability. Speed: RLS convergence speed. GSG-stab prob: The probability of GSG–stability. Min dist: The minimum measure of misperceptions necessary to get GSG instability.

for values of persistence less than 0.6 [Table 2, column (3)]. For $\rho = 0.6$, the probability of GSG-stability is everywhere above .99, and it becomes essentially unity for monetary policy with $\phi_\pi > 1.2$, which is less active than the optimal under REE. In the baseline calibration with persistence shock 0.9, only a Taylor rule with ϕ_π as high as 3.5 guarantees GSG-stability.[21]

Finally, we comment on the known examples in this context where E-stability and SG-stability conditions are found to be equivalent. This is true in the univariate,

TABLE 2. Simulations results: Monetary policy under commitment ($\rho = 0.6$)

	SG-stab	Speed	GSG-stab prob	Min dist
		$\phi_x = 0.25$		
$\phi_\pi = 1.5$	+	0.59	1	N/A
$\phi_\pi = 2.0$	+	0.77	1	N/A
$\phi_\pi = 2.5$	+	0.97	1	N/A
		$\phi_x = 0.35$		
$\phi_\pi = 1.5$	+	0.51	1	N/A
$\phi_\pi = 2.0$	+	0.60	1	N/A
$\phi_\pi = 2.5$	+	0.68	1	N/A
		$\phi_x = 0.45$		
$\phi_\pi = 1.5$	+	0.49	1	N/A
$\phi_\pi = 2.0$	+	0.57	1	N/A
$\phi_\pi = 2.5$	+	0.64	1	N/A

Note: SG-stab: SG–stability. Speed: RLS convergence speed. GSG-stab prob: The probability of GSG–stability. Min dist: The minimum measure of misperceptions necessary to get GSG instability.

purely forward-looking, cobweb-type models discussed in Evans and Honkapohja (2001, p. 37). The Jacobian J of the E-stability ODE is given by $(\alpha - 1)$ times the identity matrix. Premultiplying J by a positive definite $M(\Phi)$ therefore cannot affect stability, and thus SG-stability is equivalent to E-stability. In a multivariate extension of the cobweb model studied by Evans and Honkapohja (1998), the eigenvectors of $\Omega \cdot J$ can be expressed as $m \otimes j$, where m is some eigenvector of the second moments matrix M and j is an eigenvector of J. Because the eigenvectors of a symmetric matrix are orthogonal, certain eigenvectors of $\Omega \cdot J$ are orthogonal as well. We conjecture that a variant of the geometric condition stated in Corollary 1 continues to hold in higher dimensions; namely, the equivalent orthogonality condition is inconsistent with $\Omega \cdot J$ instability whereas J is stable. The equivalence of E-stability and SG-stability conditions for univariate and multivariate cobweb-type models agrees with our results.

6. CONCLUSION

Whereas under recursive least-squares learning the dynamics of linear and some nonlinear models converge to E-stable rational expectations equilibria, recent examples argue that E-stability is not a sufficient condition for SG-stability. We establish that there is a close relationship between the learnability of E-stable equilibria and the speed of convergence of the RLS learning algorithm. In the 2×2 case, we give conditions that ensure that a higher mean dynamics speed of convergence implies learnability under a broad set of learning algorithms of the RLS/SG/GSG class. This is a refinement of the set of E-stable REE with properties such that learnability is achieved even if agents' learning is misspecified asymptotically relative to RLS.

Furthermore, we quantify the significance of the RLS speed of convergence for learnability under alternative learning algorithms. Evans and Honkapohja (2006) show that optimal monetary policy under commitment, specifically for their "expectations-based rule," leads to expectational stability in private agents' learning. We provide evidence that such an E-stable REE might fail to obtain its GSG learning stability if agents used a misspecified second-order moments matrix of the model parameters. For the lowest speeds of convergence consistent with determinate and E-stable REE, we observe up to a 60% probability of GSG instability. If the agents use an algorithm other than RLS, the optimal monetary policy under commitment is also associated with an approximately 20% probability of being subject to expectational instability.

NOTES

1. The possible convergence of learning processes and the E-stability criterion of REE dates back to DeCanio (1979) and Evans (1985). Marcet and Sargent (1989) first showed the conditions for convergence in a learning context using stochastic approximation techniques.

2. Evans and Honkapohja (1998) find that E-stability and SG-stability are the same in a multivariate cobweb-type model.

3. In the literature, the concepts of E-stability and learnability are often used interchangeably. Following Giannitsarou (2005), we consider learnability a broader concept and distinguish between E-stability, which is related to learnability under RLS, and learnability under different learning algorithms.

4. Ferrero (2007) examines the welfare consequences of slow adjustment of inflationary expectations to their REE values.

5. There are technical conditions other than the stability of the approximating mean dynamics that ought to be satisfied for convergence of the real-time dynamics under learning; see Evans and Honkapohja (2001, Chap. 6, Sects. 6.2, 6.3). We assume that these conditions are always satisfied, and claim that learnability is obtained when the equilibrium is stable under the approximating mean dynamics.

6. The exceptional cases where we observe eigenvalues with a zero real part do not typically arise in economic models.

7. Evans et al. (2010, p. 246) choose to transform variables using the Cholesky transformation, and hence allow *any* matrix Γ to have the same eigenvectors as M. Diagonal matrices, as in their Proposition 5, p. 247, also have the same eigenspace spanned by coordinate unit vectors.

8. With RLS learning, the mean dynamics ODE that governs estimates of the second moments matrix of regressors, R, is given by $dR/d\tau = M(\Phi) - R$. The Jacobian of this part of the mean-dynamics ODE equals $-I$, with all eigenvalues equal to -1; see Evans and Honkapohja (2001, pp. 234–235).

9. The eigenvectors of a symmetric matrix are orthogonal, and so $P^{-1} = P^T$. In addition if we adopt the approach of Evans et al. (2010) and state our geometric conditions based on transformed variables, such that $M = I$, the decomposition of Ω will also allow the proposed form in the paper. Refer to Mangnus and Neudecker (1999, Theorem 23, p. 22).

10. Note that the angle between the eigenvectors of J is preserved under the rotation into the orthogonal coordinate system determined by the eigenvectors of Ω. Therefore, we use collinearity of the eigenvectors of J and $\widetilde{J}$ interchangeably.

11. We also check the sufficient condition for H-stability (symmetric part of J stable), but in the range of our calibration and policy parameters we found no points that satisfied the condition. This further shows how restrictive the negative quasidefiniteness of a matrix proves to be.

12. We do not track SG-stability for indeterminate REEs.

13. To derive the optimal policy values used in our simulations, we assume a relative weight equal to 0.02 on the output gap.

14. We do not think that agents who learn adaptively and "know" exactly a wrong second moments matrix is an assumption that is any more restrictive than assuming they are endowed with perfect knowledge of M. Using this procedure, we intend to generate "perturbed" second moments matrices with similar correlation structure to the true one.

15. We assume that the agents keep parameters (ϕ_π, ϕ_x) the same but recalculate (ϕ_L, ϕ_u).

16. The exact values of the probability of obtaining a GSG algorithm that delivers a learnable equilibrium depend on the assumed distribution of $\widetilde{\theta}$. We are only interested in the direction in the parameter space in which this probability increases or decreases.

17. When agents use RLS learning, this eigenvalue is equal to one.

18. This exercise is in the spirit of Tetlow and von zur Muehlen (2009), who model agents with RLS learning but allow them to commit errors in the ALM. They study the minimum pertubation such that the resulting algorithm is divergent.

19. In the simple case of a constant and i.i.d. shock, the value of 10 means that agents perceive the shock as being 10 times more volatile than in "reality," which is indeed a severe misperception.

20. Compare the entries for the same ϕ_π and ϕ_x in Tables 1 and 2.

21. Simulations that illustrate the sensitivity of the results to the shock persistence are not reported, but are available upon request.

22. The case of a noninvertible $\mathcal{N}$ is not generic, and we do not consider it here.

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APPENDIX A: LEARNING INSTABILITY: THE REAL EIGENVALUES OF J (OR $\widetilde{J}$)

First we examine the real eigenvalues case. We investigate the conditions under which $\Delta\widetilde{J}$ has an eigenvalue with a positive real part, and is therefore unstable. Because $\left|\Delta\widetilde{J}\right| = |\Delta| \cdot \left|\widetilde{J}\right| > 0$, instability can only appear if the trace of $\Delta\widetilde{J}$, denoted by $\mathrm{Tr}(\Delta\widetilde{J})$, is strictly positive. Write the matrix of the eigenvectors of $\widetilde{J}$ and its inverse[22] as

$$\mathcal{N} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \ \mathcal{N}^{-1} = \frac{1}{|\mathcal{N}|}\begin{bmatrix} v_{22} & -v_{12} \\ -v_{21} & v_{11} \end{bmatrix};$$

then the matrix $\widetilde{J}$ can be written as

$$\widetilde{J} = \mathcal{N}\Lambda\mathcal{N}^{-1} = \frac{1}{|\mathcal{N}|}\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \cdot \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \cdot \begin{bmatrix} v_{22} & -v_{12} \\ -v_{21} & v_{11} \end{bmatrix}.$$

We can establish that the diagonal elements of $\widetilde{J}$ are given by

$$\widetilde{J}_{11} = \frac{\lambda_2 v_{12} v_{21} - \lambda_1 v_{11} v_{22}}{|\mathcal{N}|}$$

and

$$\widetilde{J}_{22} = \frac{\lambda_1 v_{12} v_{21} - \lambda_2 v_{11} v_{22}}{|\mathcal{N}|}.$$

The trace of the $\Delta\widetilde{J}$ is thus equal to

$$\mathrm{Tr}(\Delta\widetilde{J}) = \delta_1 \frac{\lambda_2 v_{12} v_{21} - \lambda_1 v_{11} v_{22}}{|\mathcal{N}|} + \delta_2 \frac{\lambda_1 v_{12} v_{21} - \lambda_2 v_{11} v_{22}}{|\mathcal{N}|} > 0. \qquad \textbf{(A.1)}$$

The condition (A.1) is equivalent to

$$\delta_1 \frac{\lambda_2 v_{12} v_{21} - \lambda_1 v_{11} v_{22}}{|\mathcal{N}|} + \delta_2 \frac{\lambda_1 v_{12} v_{21} - \lambda_2 v_{11} v_{22}}{|\mathcal{N}|} > 0,$$

$$\frac{\delta_2}{\delta_1} \frac{\lambda_1 v_{12} v_{21} - \lambda_2 v_{11} v_{22}}{|\mathcal{N}|} - \frac{\lambda_1 v_{11} v_{22} - \lambda_2 v_{12} v_{21}}{|\mathcal{N}|} > 0,$$

$$\frac{\delta_2}{\delta_1} \frac{v_{12} v_{21} - \frac{\lambda_2}{\lambda_1} v_{11} v_{22}}{|\mathcal{N}|} - \frac{v_{11} v_{22} - \frac{\lambda_2}{\lambda_1} v_{12} v_{21}}{|\mathcal{N}|} > 0.$$

Now select the direction of the eigenvectors so that $v_{12}v_{21} > 0$, let $\Upsilon = \frac{v_{22}}{v_{21}}\frac{v_{11}}{v_{12}} = \frac{v_{22}}{v_{12}} / \frac{v_{21}}{v_{11}}$, and observe that $|\mathcal{N}|$ equals $v_{21}v_{12}\,(\Upsilon - 1)$. The trace condition (A.1) becomes

$$\frac{\frac{\delta_2}{\delta_1} v_{12} v_{21}}{v_{21} v_{12}\,(\Upsilon - 1)}\left(1 - \frac{\lambda_2}{\lambda_1}\Upsilon\right) - \frac{v_{12} v_{21}}{v_{21} v_{12}\,(\Upsilon - 1)}\left(\Upsilon - \frac{\lambda_2}{\lambda_1}\right) > 0$$

or

$$\frac{1}{\Upsilon - 1}\left[\frac{\delta_2}{\delta_1}\left(1 - \frac{\lambda_2}{\lambda_1}\Upsilon\right) - \left(\Upsilon - \frac{\lambda_2}{\lambda_1}\right)\right] > 0. \quad \textbf{(A.2)}$$

When $\Upsilon < 0$, this expression is clearly negative. Thus, learning instability requires that both eigenvectors of J after rotation into the coordinates defined by the eigenvectors of Ω are located in the same quadrant of the plane. This condition is impossible to meet if the two eigenvectors are orthogonal.

When $\Upsilon > 1$, the term in the square brackets in (A.2) is negative: it is a decreasing function of λ_2/λ_1, δ_2/δ_1, and Υ, reaching its maximum of 0 for $\lambda_2/\lambda_1 = \delta_2/\delta_1 = \Upsilon = 1$. Therefore, the whole expression (A.2) is negative for $\Upsilon > 1$.

When $0 < \Upsilon < 1$, learning instability requires that

$$\frac{\delta_2}{\delta_1}\left(1 - \frac{\lambda_2}{\lambda_1}\Upsilon\right) < \Upsilon - \frac{\lambda_2}{\lambda_1} < 0.$$

This is possible only if $1 - \frac{\lambda_2}{\lambda_1}\Upsilon < 0$ or $\frac{\lambda_2}{\lambda_1} > \frac{1}{\Upsilon} > 1$, in which case the preceding condition can be rewritten as

$$\frac{\delta_2}{\delta_1} > \frac{\Upsilon - \frac{\lambda_2}{\lambda_1}}{1 - \frac{\lambda_2}{\lambda_1}\Upsilon} = \frac{\frac{\lambda_2}{\lambda_1} - \Upsilon}{\frac{\lambda_2}{\lambda_1}\Upsilon - 1}.$$

APPENDIX B: LEARNING INSTABILITY: THE COMPLEX EIGENVALUES OF J (OR $\widetilde{J}$)

In this case, the eigenvalues of $\widetilde{J}$ are given by $\nu \pm i\mu$, $\nu < 0$, and the corresponding eigenvectors are $w_1 \pm i w_2$. Following the same steps as in the real roots case, write

$$W = \begin{bmatrix} w_{11} + i w_{12} & w_{11} - i w_{12} \\ w_{21} + i w_{22} & w_{21} - i w_{22} \end{bmatrix},$$

$$W^{-1} = \frac{1}{2|W|}\begin{bmatrix} w_{22} + i w_{21} & -(w_{12} + i w_{11}) \\ w_{22} - i w_{21} & -(w_{12} - i w_{11}) \end{bmatrix},$$

$$|W| = w_{11}w_{22} - w_{12}w_{21},$$

$$W \Lambda W^{-1} = \frac{1}{2|W|}\begin{bmatrix} w_{11} + i w_{12} & w_{11} - i w_{12} \\ w_{21} + i w_{22} & w_{21} - i w_{22} \end{bmatrix} \cdot \begin{bmatrix} \nu + i\mu & 0 \\ 0 & \nu - i\mu \end{bmatrix} \cdot$$

$$\times \begin{bmatrix} w_{22} + i w_{21} & -(w_{12} + i w_{11}) \\ w_{22} - i w_{21} & -(w_{12} - i w_{11}) \end{bmatrix}$$

$$= \frac{1}{2|W|}\begin{bmatrix} (\nu + i\mu)(w_{11} + i w_{12}) & \overline{(\nu + i\mu)}\,\overline{(w_{11} + i w_{12})} \\ (\nu + i\mu)(w_{21} + i w_{22}) & \overline{(\nu + i\mu)}\,\overline{(w_{21} + i w_{22})} \end{bmatrix} \cdot$$

$$\times \begin{bmatrix} w_{22} + i w_{21} & -(w_{12} + i w_{11}) \\ \overline{w_{22} + i w_{21}} & -\overline{(w_{12} + i w_{11})} \end{bmatrix}.$$

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The overline in the expressions denotes a complex conjugate. Finally, the two diagonal elements of $\widetilde{J}$ can be written as

$$\widetilde{J}_{11} = \frac{\mathrm{Re}\left[(\nu + i\mu)\,(w_{11} + i w_{21})\,(w_{22} + i w_{21})\right]}{|W|},$$

$$\widetilde{J}_{22} = -\frac{\mathrm{Re}\left[(\nu + i\mu)\,(w_{12} + i w_{22})\,(w_{21} + i w_{11})\right]}{|W|},$$

which reduce to

$$\widetilde{J}_{11} = \nu - \mu\frac{w_{11}w_{12} + w_{21}w_{22}}{|W|},$$

$$\widetilde{J}_{22} = \nu + \mu\frac{w_{11}w_{12} + w_{21}w_{22}}{|W|}.$$

The trace of $\Delta\widetilde{J}$ then is given by

$$\mathrm{Tr}(\Delta\widetilde{J}) = \nu\,(\delta_2 + \delta_1) + \mu\frac{w_{11}w_{12} + w_{21}w_{22}}{|W|}\,(\delta_2 - \delta_1) > 0 \qquad \textbf{(B.1)}$$

and should be positive for the instability to occur.

Let $\widetilde{W} = w_{11}w_{12} + w_{21}w_{22}$ and recall that ν is negative. Then (B.1) is equivalent to

$$\mu\frac{w_{11}w_{12} + w_{21}w_{22}}{|W|}\,(\delta_2 - \delta_1) > -\nu\,(\delta_2 + \delta_1)\,,$$

$$\frac{\mu}{|\nu|}\,\frac{\widetilde{W}}{|W|}\,\frac{\frac{\delta_2}{\delta_1} - 1}{\frac{\delta_2}{\delta_1} + 1} > 1.$$

This expression allows us to evaluate and relate the speed of convergence, the real part of the eigenvalues in this case, and the conditions for learning instability. It is easy to show that if $w_1 \perp w_2$, then $\widetilde{W}/|W| = 0$, making (8) impossible to satisfy.

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