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Lower Assouad Dimension of Measures and Regularity

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Abstract

In analogy with the lower Assouad dimensions of a set, we study the lower Assouad dimensions of a measure. As with the upper Assouad dimensions, the lower Assouad dimensions of a measure provide information about the extreme local behaviour of the measure. We study the connection with other dimensions and with regularity properties. In particular, the quasi-lower Assouad dimension is dominated by the infimum of the measure's lower local dimensions. Although strict inequality is possible in general, equality holds for the class of self-similar measures of finite type. This class includes all self-similar, equicontractive measures satisfying the open set condition, as well as certain "overlapping" self-similar measures, such as Bernoulli convolutions with contraction factors that are inverses of Pisot numbers.

We give lower bounds for the lower Assouad dimension for measures arising from a Moran construction, prove that self-affine measures are uniformly perfect and have positive lower Assouad dimension, prove that the Assouad spectrum of a measure converges to its quasi-Assouad dimension and show that coincidence of the upper and lower Assouad dimension of a measure does not imply that the measure is s-regular.

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1. *Introduction*

The upper and lower Assouad dimensions of a metric space provide quantitative information about the extreme local geometry of the set. The analogous notion of the Assouad dimensions of a measure also quantifies, in some sense, the extreme local behaviour of the measure. These dimensions were extensively studied by Käenmäki et al., in [12] and [13], and Fraser and Howroyd, in [5], where they were called the upper and lower regularity

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dimensions. It was shown that the upper Assouad dimension of a measure is finite if and only if the measure is doubling, while the lower Assouad dimension is positive if and only if the measure is uniformly perfect. Käenmäki et al. focused their investigations on doubling measures supported on uniformly perfect complete metric spaces, whereas Fraser and Howroyd computed the upper Assouad dimension for a large class of examples, as well as establishing links to other notions of regularity. As many interesting measures are not doubling, such as is typically the case for self-similar measures that fail the open set condition, the weaker notion of the quasi-Assouad dimension of a measure is more appropriate and was studied in [**10**]. There it was shown, for example, that self-similar measures that are sufficiently regular (but not necessarily satisfying the open set condition), not only have finite quasi-upper Assouad dimension, but in fact this dimension coincides with the maximal local dimension of the measure.

In this paper, we investigate the lower Assouad dimension for measures and introduce the quasi-lower Assouad dimension. The (quasi-) lower Assouad dimension of a measure is easily seen to be dominated by the (quasi-) lower Assouad dimension of the support of the measure. It is also dominated by the infimum of the lower local dimensions (and hence the Hausdorff dimension) of the measure. Although these dimensions are equal for self-similar measures satisfying the strong separation condition, in general all the aforementioned inequalities can be strict. We give various examples to show this. We also give an example to show that equality of the upper and lower Assouad dimensions does not imply s-regularity of the measure. In analogy with what was shown for sets in [**2**] and [**4**], we prove that the quasi-lower and quasi-upper Assouad dimensions of measures can be recovered from the Assouad dimension spectrum of a measure under the assumption that the measure is quasi-doubling, i.e., has finite quasi-upper Assouad dimension. These results can all be found in Sections 2 and 6. In the Appendix, we simplify the proof given in [**2**] that the quasi-lower Assouad dimension of a doubling metric space is the limit of the dimension spectrum and remove their assumption that the metric space is uniformly perfect.

In Section 3 we establish a lower bound on the lower Assouad dimension for uniformly perfect measures and show that certain Moran constructions, such as self-similar and self-affine measures, have positive lower Assouad dimension. For these sets, we give a lower bound for the dimension in terms of the parameters of the Moran construction. We also calculate the (quasi-) lower Assouad dimension of Bedford–McMullen carpets.

In Section 4 we prove the equality of the quasi-lower Assouad dimension with the infimum of the set of lower local dimensions for self-similar measures of finite type. This class of measures includes equicontractive, self-similar measures satisfying the open set condition, as well as certain measures that only satisfy the weak separation condition, such as Bernoulli convolutions with contraction factor the inverse of a Pisot number. Our proof is constructive; we exhibit a sequence of points such that the lower local dimension of the measure at these points tends to the quasi-lower Assouad dimension of the measure.

A measure is said to be L^p-improving if it acts by convolution as a bounded map from L^2 to L^p for some $p > 2$. It is known that L^p-improving measures have positive Hausdorff dimension, thus it is natural to ask if they must also have positive lower Assouad dimension. In Section 5, examples are given to show that even the quasi-lower Assouad dimension of an L^p-improving measure can be zero, although its local dimensions must be bounded away from zero. In fact, we show that there exist measures whose Fourier transform is p-summable for some $p < \infty$, with zero quasi-lower Assouad dimension.

2. *Definitions and basic properties of the lower Assouad type dimensions*

2·1. *Assouad dimensions of sets*

Given a compact metric space X, we write $N_r(E)$ for the least number of sets of diameter at most r that are required to cover $E \subseteq X$. Given $\delta > 0$, let

$$\overline{h}(\delta) = \inf\left\{\alpha : (\exists c, c_2 > 0)(\forall 0 < r \le R^{1+\delta} \le c_1) \sup_{x\in E} N_r(B(x, R)\cap E) \le c_2\left(\frac{R}{r}\right)^{\alpha}\right\},$$

$$\underline{h}(\delta) = \sup\left\{\alpha : (\exists c_1, c_2 > 0)(\forall 0 < r \le R^{1+\delta} \le c_1) \inf_{x\in E} N_r(B(x, R)\cap E) \ge c_2\left(\frac{R}{r}\right)^{\alpha}\right\}.$$

The upper Assouad and lower Assouad dimensions of E are given by

$$\overline{\dim}_{\mathrm{A}}\ E = \overline{h}(0),\ \underline{\dim}_{\mathrm{A}}\ E = \underline{h}(0),$$

while the quasi-upper Assouad and quasi-lower Assouad dimensions are given by

$$\overline{\dim}_{\mathrm{qA}}\ E = \lim_{\delta\to 0} \overline{h}(\delta),\ \underline{\dim}_{\mathrm{qA}}\ E = \lim_{\delta\to 0} \underline{h}(\delta).$$

2·2. *Assouad dimensions of measures*

By a measure we will mean a Borel probability measure on X with compact support. The analogue of the upper Assouad and lower Assouad dimensions for measures was studied in [**5**], [**12**] and [**13**] (where they were called upper and lower regularity dimensions). The analogue of the quasi-upper Assouad dimension for measures was introduced in [**10**]. This paper is primarily concerned with the (quasi)-lower Assouad dimension for measures.

Given a measure μ and $\delta \ge 0$, set

$$\overline{H}(\delta) = \inf\left\{s : (\exists c_1, c_2 > 0)(\forall 0 < r \le R^{1+\delta} \le c_1) \sup_{x\in\mathrm{supp}\,\mu} \frac{\mu(B(x, R))}{\mu(B(x, r))} \le c_2\left(\frac{R}{r}\right)^{s}\right\}$$

and

$$\underline{H}(\delta) = \sup\left\{s : (\exists c_1, c_2 > 0)(\forall 0 < r \le R^{1+\delta} \le c_1) \inf_{x\in\mathrm{supp}\,\mu} \frac{\mu(B(x, R))}{\mu(B(x, r))} \ge c_2\left(\frac{R}{r}\right)^{s}\right\}.$$

Definition 1. The **upper Assouad** and **lower Assouad dimensions** of μ are given by

$$\overline{\dim}_{\mathrm{A}}\ \mu = \overline{H}(0),\ \underline{\dim}_{\mathrm{A}}\ \mu = \underline{H}(0).$$

The **quasi-upper Assouad** and **quasi-lower Assouad dimension** of μ are given by

$$\overline{\dim}_{\mathrm{qA}}\ \mu = \lim_{\delta\to 0} \overline{H}(\delta),\ \underline{\dim}_{\mathrm{qA}}\ \mu = \lim_{\delta\to 0} \underline{H}(\delta).$$

Remark 2. We note that these dimensions are known under various names and many different notations are in common use. The upper Assouad dimension is often referred to as *the* Assouad dimension, the lower Assouad dimension sometimes simply as lower dimension, and the measure theoretic versions as the upper and lower regularity dimensions. We have opted to use a bar to denote upper or lower Assouad dimension instead of $\dim_A$ and $\dim_L$, (as $\dim_L$ is sometimes used to refer to the Lyapunov dimension of a measure).

2·3. *Relationships between these dimensions*

It is clear from the definitions that

$$0 \leq \underline{\dim}_{\mathrm{A}}\, E \leq \underline{\dim}_{\mathrm{qA}}\, E \leq \overline{\dim}_{\mathrm{qA}}\, E \leq \overline{\dim}_{\mathrm{A}}\, E \leq \infty$$

and

$$0 \leq \underline{\dim}_{\mathrm{A}}\, \mu \leq \underline{\dim}_{\mathrm{qA}}\, \mu \leq \overline{\dim}_{\mathrm{qA}}\, \mu \leq \overline{\dim}_{\mathrm{A}}\, \mu \leq \infty.$$

It was shown in [5] and [10] that

$$\overline{\dim}_{\mathrm{A}}\, \mu \geq \overline{\dim}_{\mathrm{A}}\, \mathrm{supp}\, \mu \quad \text{and} \quad \overline{\dim}_{\mathrm{qA}}\, \mu \geq \overline{\dim}_{\mathrm{qA}}\, \mathrm{supp}\, \mu.$$

It is known that $\overline{\dim}_{\mathrm{A}}\, \mu < \infty$ if and only if μ is doubling, meaning there is a constant $C > 0$ such that

$$\mu(B(x, R)) \geq C\mu(B(x, 2R)) \text{ for all } x, R. \tag{2·1}$$

See [5] for a proof.

Recall that the lower local dimension of μ at x is defined as

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x, r))}{\log r},$$

with the upper local dimension, $\overline{\dim}_{\mathrm{loc}}\mu(x)$, defined similarly but with lim sup replacing lim inf. Fraser and Howroyd in [5] also showed that

$$\overline{\dim}_{\mathrm{qA}}\, \mu \geq \sup_{x\in \mathrm{supp}\, \mu} \{\overline{\dim}_{\mathrm{loc}}\mu(x)\}.$$

Similar relations hold for the (quasi-)lower Assouad dimensions.

PROPOSITION 3. (i) *If μ is a doubling measure, then*

$$\underline{\dim}_{\mathrm{A}}\, \mu \leq \underline{\dim}_{\mathrm{A}}\, \mathrm{supp}\, \mu \quad \textit{and} \quad \underline{\dim}_{\mathrm{qA}}\, \mu \leq \underline{\dim}_{\mathrm{qA}}\, \mathrm{supp}\, \mu.$$

(ii) *For any measure μ,*

$$\underline{\dim}_{\mathrm{A}}\, \mu \leq \underline{\dim}_{\mathrm{qA}}\, \mu \leq \inf_{x\in \mathrm{supp}\, \mu} \{\underline{\dim}_{\mathrm{loc}}\mu(x)\} \leq \dim_H \mu.$$

(iii) *If μ is a self-similar measure associated with an IFS that satisfies the strong separation condition, then*

$$\underline{\dim}_{\mathrm{A}}\, \mu = \inf_x \{\dim_{\mathrm{loc}}\, \mu(x)\}.$$

Proof. (i) The fact that $\underline{\dim}_{\mathrm{A}}\, \mu \leq \underline{\dim}_{\mathrm{A}}\, \mathrm{supp}\, \mu$ was observed in [12]. To see that $\underline{\dim}_{\mathrm{qA}}\, \mu \leq \underline{\dim}_{\mathrm{qA}}\, \mathrm{supp}\, \mu$, let $t = \underline{\dim}_{\mathrm{qA}}\, \mathrm{supp}\, \mu$ and C be the doubling constant of (2·1). For any $\varepsilon > 0$ and suitable $\delta > 0$, there are $x_i \in \mathrm{supp}\, \mu$, $R_i \to 0$ and $r_i \leq R_i^{1+\delta}$ such that $N_{r_i}(B(x_i, R_i) \cap \mathrm{supp}\, \mu) \leq (R_i/r_i)^{t+\varepsilon}$. Together with the doubling property, this implies

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$$\mu(B(x_i, 2R_i) \cap \operatorname{supp} \mu) \leq C^{-1} \mu(B(x_i, R_i) \cap \operatorname{supp} \mu)$$
$$\leq C^{-1} N_{r_i}(B(x_i, R_i) \cap \operatorname{supp} \mu) \max_{y \in B(x_i, R_i)} \mu(B(y, r_i) \cap \operatorname{supp} \mu)$$
$$\leq C^{-1} (R_i/r_i)^{t+\varepsilon} \mu(B(y_i, r_i) \cap \operatorname{supp} \mu)$$

for a suitable $y_i \in B(x_i, R_i)$. Now $B(y_i, R_i) \subseteq B(x_i, 2R_i)$ and thus

$$\frac{\mu(B(y_i, R_i) \cap \operatorname{supp} \mu)}{\mu(B(y_i, r_i) \cap \operatorname{supp} \mu)} \leq \frac{\mu(B(x_i, 2R_i) \cap \operatorname{supp} \mu)}{\mu(B(y_i, r_i) \cap \operatorname{supp} \mu)} \leq C^{-1} (R_i/r_i)^{t+\varepsilon}.$$

That suffices to show $\underline{\dim}_{\mathrm{qA}} \mu \leq t$. A similar argument shows $\underline{\dim}_{\mathrm{A}} \mu \leq \underline{\dim}_{\mathrm{A}} \operatorname{supp} \mu$.

(ii) The only new statement here is the inequality $\underline{\dim}_{\mathrm{qA}} \mu \leq \inf_x \{\underline{\dim}_{\mathrm{loc}} \mu(x)\}$ and this follows in the same manner as [**10**, proposition 2·4].

(iii) The proof of this is essentially the same as given in [**5**, theorem 2·4] for the fact that $\overline{\dim}_{\mathrm{A}} \mu = \sup_x \{\dim_{\mathrm{loc}} \mu(x)\}$.

Remark 4. In [**10**, proposition 4·2] it is shown that if $\overline{\dim}_{\mathrm{qA}} \mu < \infty$, then for each $\varepsilon > 0$ there is a constant $c > 0$ such that $\mu(B(x, R)) \geq c R^{\varepsilon} \mu(B(x, 2R))$ for all x, R. The reader can check that this weaker condition suffices to ensure $\underline{\dim}_{\mathrm{qA}} \mu \leq \underline{\dim}_{\mathrm{qA}} \operatorname{supp} \mu$.

Remark 5. Strict inequalities are possible between all these dimensions. Indeed, in [**10**, example 2·3] it is explained how to construct examples with $\overline{\dim}_{\mathrm{qA}} \operatorname{supp} \mu < \overline{\dim}_{\mathrm{A}} \operatorname{supp} \mu < \overline{\dim}_{\mathrm{qA}} \mu < \overline{\dim}_{\mathrm{A}} \mu$ and $\overline{\dim}_{\mathrm{qA}} \operatorname{supp} \mu < \overline{\dim}_{\mathrm{qA}} \mu < \overline{\dim}_{\mathrm{A}} \operatorname{supp} \mu < \overline{\dim}_{\mathrm{A}} \mu$. It is easy to modify these to produce analogous examples for the lower Assouad dimensions. In particular, one can have $\underline{\dim}_{\mathrm{A}} \mu = 0$, $\overline{\dim}_{\mathrm{A}} \mu = \infty$, but $0 < \underline{\dim}_{\mathrm{qA}} \mu < \overline{\dim}_{\mathrm{qA}} \mu < \infty$. Below we give an example where $\underline{\dim}_{\mathrm{qA}} \mu < \inf_x \{\underline{\dim}_{\mathrm{loc}} \mu(x)\}$. Another example is Example 29. In Section 4 we prove that the equality does hold for a large class of self-similar measures, which need not satisfy the open set condition.

Example 6. A measure μ on $\mathbb{R}$ with $\underline{\dim}_{\mathrm{qA}} \mu = 0$ and $\inf_x \{\underline{\dim}_{\mathrm{loc}} \mu(x)\} = 1$: we construct a probability measure μ with support [0, 1] defined iteratively on the dyadic intervals. Label the dyadic intervals of length 2^{-n} (step n) from left to right as $I_n^{(i)}$, $i = 1, \ldots, 2^n$, so $I_n^{(1)}$, $I_n^{(2)}$ are the two descendants of $I_{n-1}^{(1)}$, for example. Let $\{n_j\}$ be an integer sequence with $n_{j+1} \geq 3n_j$. Choose a sequence $1/2 \leq q_j \uparrow 1$ and put $t_j = q_j^{-n_j} 2^{-(1+n_j)}$. Assuming μ has been defined on the dyadic intervals of step $n-1$, we define μ on the dyadic intervals of step n in the following fashion:

$$\mu(I_n^{(1)}) = t_j \mu(I_{n-1}^{(1)}) \quad \text{and} \quad \mu(I_n^{(2)}) = (1 - t_j) \mu(I_{n-1}^{(1)}) \quad \text{if} \quad n = n_j,$$
$$\mu(I_n^{(1)}) = q_j \mu(I_{n-1}^{(1)}) \quad \text{and} \quad \mu(I_n^{(2)}) = (1 - q_j) \mu(I_{n-1}^{(1)}) \quad \text{if} \quad n = n_j + 1, \ldots, 2n_j.$$

All other dyadic intervals of step n will have measure 1/2 that of their parent interval.

We even have $\underline{h}(1/2) = 0$, and thus $\underline{\dim}_{\mathrm{qA}} \mu = 0$, because

$$\frac{\mu(B(0, 2^{-n_j}))}{\mu(B(0, 2^{-2n_j}))} = \frac{1}{q_j^{n_j}} = \left(\frac{2^{-n_j}}{2^{-2n_j}} \right)^t$$

for $t = -\log q_j / \log 2$ and $t \to 0$ as $q_j \to 1$.

To see that $\underline{\dim}_{\mathrm{loc}} \mu(x) \geq 1$ for all $x \in \operatorname{supp} \mu$, we first consider $x \neq 0$. Choose N_0, depending on x such that $x > 4 \cdot 2^{-N_0}$. If $2^{-(n+1)} < r \leq 2^{-n}$ for $n \geq N_0$, then $B(x, r)$ is contained in

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the union of four consecutive dyadic intervals of length 2^{-n}, none of which intersect the two left-most intervals of step N_0. Thus

$$\mu(B(x,r)) \leq 4\cdot 2^{N_0-n} \max\left(\mu(I_{N_0}^{(i)}) : i \geq 3\right) = C2^{N_0-n},$$

so

$$\frac{\log \mu(B(x,r))}{\log r} \geq \frac{\log C2^{N_0-n}}{\log 2^{-(n+1)}} \to 1 \text{ as } n \to \infty.$$

Finally, consider $x = 0$. The choice of t_j ensures that $\mu(B(0, 2^{-n})) \leq 2^{-n}$ for all n and that certainly implies $\underline{\dim}_{\text{loc}}\mu(0) \geq 1$. That completes the proof.

2·4. *Lower dimension and regularity*

A measure μ is called s-regular if there exists a uniform constant $c > 0$ such that

$$c^{-1}r^s \leq \mu(B(x,r)) \leq cr^s$$

for all $x \in \operatorname{supp}\mu$ and $0 < r < \operatorname{diam}\operatorname{supp}\mu$. It is easy to show from the definitions that if μ is s-regular then $\underline{\dim}_{\mathrm{A}}\,\mu = \overline{\dim}_{\mathrm{A}}\,\mu = s$, see e.g. [12] and [13]. However, it is not true that coinciding lower and upper Assouad dimension implies s-regularity, as the following example illustrates.

Example 7. Let M_v be the collection of triadic intervals labelled by finite words on the letters $\{0, 1, 2\}$. We construct a finite measure μ on $[0, 1]$ as follows:

$$\begin{aligned}\mu(M_v) &= (k+1)3^{-(k+1)} \text{ if } v = 1^{(k)}0 \text{ or } v = 1^{(k)}2,\\ \mu(M_{1^{(k)}jv}) &= \frac{k+1}{3^{k+1+l}} \text{ if } j \in \{0,2\} \text{ and } v \in \{0,1,2\}^l,\\ \mu(M_{1^{(k)}}) &= 2\sum_{i=k+1}^{\infty}\frac{i}{3^i} = 3^{-k}(3/2+k).\end{aligned} \tag{2·2}$$

One can easily check that μ is well defined and upon normalizing by $\mu([0, 1]) = 2\sum_{i=0}^{\infty}(i+1)/3^{i+1} = 3/2$, we obtain a probability measure.

We now estimate the ratio between any triadic interval and its descendants. Consider M_v and M_{vw} for $v \in \{0, 1, 2\}^k$ and $w \in \{0, 1, 2\}^l$, where $l \geq 1$. If $v \neq 1^{(k)}$, then $\mu(M_v)/\mu(M_{vw}) = 3^l$, using (2·2). If, however, $v = 1^{(k)}$, then

$$\frac{k+1}{3^{k+l}} = \mu(M_{v0^{(l)}}) \leq \mu(M_{vw}) \leq \mu(M_{1^{(k+l)}}) = 3^{-(k+l)}(3/2+(k+l)). \tag{2·3}$$

Note also that for $j \in \{0, 2\}$ and $k \geq 1$,

$$\frac{\mu(M_{1^{(k)}})}{\mu(M_{1^{(k-1)}j})} = \frac{3^{-k}(3/2+k)}{k3^{-k}} = \frac{3/2+k}{k} \leq \frac{5}{2}. \tag{2·4}$$

The inequalities (2·3) and (2·4) show that neighbouring triadic intervals of the same length differ by at most a factor of $5/2$.

Now let $J \subseteq I \subseteq [0, 1]$ be intervals. Write k and l for the unique integers satisfying $3^{-(k-1)} \leq \operatorname{diam} I \leq 3^{-(k-2)}$ and $3^{-(k+l-1)} \leq \operatorname{diam} J \leq 3^{-(k+l-2)}$. Thus I contains a triadic interval of length 3^{-k} and is contained within 10 intervals of length 3^{-k}. Analogously, J

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contains an interval of length $3^{-(k+l)}$ and is contained in 10 intervals of the same length. We can therefore find M_v and M_{vw}, $v \in \{0, 1, 2\}^k$, $w \in \{0, 1, 2\}^l$ such that

$$\mu(M_v) \le \mu(I) \le \left(\frac{5}{2}\right)^{10} \mu(M_v) \quad \text{and} \quad \mu(M_{vw}) \le \mu(J) \le \left(\frac{5}{2}\right)^{10} \mu(M_{vw}),$$

and hence

$$\frac{\mu(I)}{\mu(J)} \sim \frac{\mu(M_v)}{\mu(M_{vw})}$$

where $\sim$ denotes uniform comparability. But

$$\frac{\mu(M_v)}{\mu(M_{vw})} \le \max\left\{3^l,\ \frac{3^{-k}(3/2+k)}{(k+1)3^{-(k+l)}}\right\} = 3^l\frac{3/2+k}{k+1} \le \frac{3}{2}3^l$$

and

$$\frac{\mu(M_v)}{\mu(M_{vw})} \ge \min\left\{3^l,\ \frac{3^{-k}(3/2+k)}{3^{-(k+l)}(3/2+(k+l))}\right\} = 3^l\frac{3/2+k}{3/2+(k+l)} \ge 3^l\frac{5/2}{5/2+l}.$$

So

$$\left(\frac{5}{2}\right)^{11} 3^l \ge \frac{\mu(I)}{\mu(J)} \ge \left(\frac{5}{2}\right)^{-10} \frac{5/2}{5/2+l}3^l.$$

Further, $(\operatorname{diam} I)/(\operatorname{diam} J) \sim 3^l$ and so for every $\delta > 0$ there exists $C > 0$ such that

$$C\frac{\operatorname{diam} I}{\operatorname{diam} J} \ge \frac{\mu(I)}{\mu(J)} \ge C^{-1}\left(\frac{\operatorname{diam} I}{\operatorname{diam} J}\right)^{1-\delta}.$$

In particular this holds for $I = B(x, R)$ and $J = B(x, r)$ and so the upper and lower Assouad dimension of μ is 1. But $\mu(B(1/2, 3^{-k})) = \mu(M_{1^{(k)}}) = 3^{-k}(3/2+k)$ and there is no constant $K > 0$ such that $\mu(B(x, r)) \le Kr$, so μ is not 1-regular. Since it cannot be s-regular for any $s \ne 1$, the measure μ is not s-regular for any $s \ge 0$.

3. *Uniformly perfect measures*

Analogous to the metric space properties, it is known that a measure has positive lower Assouad dimension if and only if it is uniformly perfect, c.f. [**12**]. We exhibit a general Moran type construction of a measure that has positive lower Assouad dimension and give a lower bound on the lower Assouad dimension in terms of the Moran construction data. We show that many commonly considered fractal measures satisfy the construction constraints. In particular, self-affine measures are seen to have positive lower Assouad dimension, and hence are uniformly perfect, as long as they are not a degenerate point mass.

3·1. *Characterising positive lower Assouad dimension*

Definition 8. Let μ be a compactly supported Borel probability measure. If there exist positive constants c, γ such that

$$\mu(B(x, R) \setminus B(x, cR)) \ge \gamma\mu(B(x, R)) \tag{3·1}$$

for all $x \in \operatorname{supp} \mu$ and $R \le \operatorname{diam}(\operatorname{supp} \mu)$, we say that μ is **uniformly perfect**[1].

[1]This condition is also known as "inverse doubling".

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Of course (3·1) is equivalent to the statement that

$$\frac{\mu(B(x,R))}{\mu(B(x,cR))} \geq (1-\gamma)^{-1}. \tag{3·2}$$

We have opted to state our definition to mirror the metric space definition of uniformly perfect, which states that a metric space is uniformly perfect if for every centred ball, the annulus must be non-empty. From the definition of uniformly perfect for measures it is immediate that the support of a uniformly perfect measure must also be uniformly perfect. However, the converse may not be true; it is possible to construct a measure which is not uniformly perfect, but supported on an uniformly perfect set.; c.f., Example 6 where the measure μ has support equal to $[0, 1]$.

THEOREM 9. *Let μ be a compactly supported Borel probability measure. Then $\underline{\dim}_{\mathrm{A}}\,\mu > 0$ if and only if μ is uniformly perfect. More precisely, if μ is uniformly perfect with positive constants c, γ as in (3·1), then $\underline{\dim}_{\mathrm{A}}\,\mu \geq \log(1-\gamma)/\log c$.*

Proof. First, assume μ is uniformly perfect. Let c, γ be as (3·1). For $r < R$, choose n such that $c^{n-1}R > r \geq c^n R$. Without loss of generality, $n \geq 2$ and repeatedly applying (3·2) gives

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \geq \frac{\mu(B(x,R))}{\mu(B(x,c^{n-1}R))} \geq \frac{\mu(B(x,R))}{\mu(B(x,cR))}\frac{\mu(B(x,cR))}{\mu(B(x,c^2R))}\cdots\frac{\mu(B(x,c^{n-2}R))}{\mu(B(x,c^{n-1}R))}$$
$$\geq (1-\gamma)^{-(n-1)} \geq (1-\gamma)(1-\gamma)^{\log(R/r)/\log c} = (1-\gamma)\left(\frac{R}{r}\right)^{\frac{\log(1-\gamma)}{\log c}}.$$

Thus $\underline{\dim}_{\mathrm{A}}\,\mu \geq \log(1-\gamma)/\log c > 0$.

The other direction is straightforward and follows directly from the definition.

3·2. *Moran constructions*

Let $\Lambda = \{1, \ldots, N\}$ be a finite alphabet with $2 \leq N < \infty$ letters and write Λ^k for all words of length k, Λ^* for the collection of all finite words including the empty word ε_0, and $\Lambda^{\mathbb{N}}$ for all infinite words. A countable subset $S \subseteq \Lambda^*$ is called a *section* if for every long enough word $w \in \Lambda^*$ there exists $u \in S$ and $v \in \Lambda^*$ such that $w = uv$, i.e., every long enough word has an ancestor in S. A section S is *minimal* if no proper subset of S is a section.

For every word $v \in \Lambda^*$, let $M_v \subset X$ be an arbitrary set satisfying the following conditions:

(a) $M_{vw} \subseteq M_v$ for all $v, w \in \Lambda^*$;
(b) $\max_{v\in\Lambda^k} \operatorname{diam} M_v \to 0$ as $k \to \infty$;
(c) $\operatorname{diam}(M_{vj}) \geq C_1 \operatorname{diam}(M_v)$ for all $v \in \Lambda^*$, $j \in \Lambda$, and $C_1 > 0$ not depending on v and j;
(d) for every $v \in \Lambda^*$ there exist $i, j \in \Lambda$ such that $d(M_{vi}, M_{vj}) \geq C_2 \operatorname{diam}(M_v)$, where $C_2 > 0$ does not depend on v, i, j and $d(A, B)$ denotes the distance of sets A and B.

Finally, let M be the lim sup set of $\{M_v\}$:

$$M = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\bigcup_{v\in\Lambda^k} M_v.$$

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Let m be a weight function on the collection $\{M_v : v \in \Lambda^*\}$ satisfying the following conditions:

(A) $m(M_{\varepsilon_0}) = 1$;
(B) $m(M_v) = \sum_{i=1}^{N} m(M_{vi})$;
(C) $m(M_{vi}) \le C_3 m(M_v)$ for some uniform $0 < C_3 < 1$.

For $E \subseteq X$, let $\Lambda^*(E)$ be the collection of all words $v \in \Lambda^*$ such that $M_v \cap E \neq \emptyset$. We let μ be the measure[2] induced by the weight function. In other words, writing S for the collection of all minimal sections of Λ^*, the measure μ is given by

$$\mu(E) = \inf_{A \in S} \left\{ \sum_{v \in A'} m(M_v) \ : \ A' = A \cap \Lambda^*(E) \right\}, \tag{3·3}$$

for all $E \subseteq X$. In particular, our conditions give $\operatorname{supp} \mu = M$.

LEMMA 10. *The set function μ, as constructed above, is an outer measure.*

Proof. Clearly, $\Lambda^*(\emptyset) = \emptyset$ and thus $\mu(\emptyset) = 0$. For monotonicity, let $D \subseteq E \subseteq M$ and observe that for every $\epsilon > 0$ there exists a section A_ε such that

$$\mu(E) \le \sum_{v \in A'_\varepsilon} m(M_v) \le \mu(E) + \varepsilon,$$

where $A'_\varepsilon = A_\varepsilon \cap \Lambda^*(E)$. Now $D \subseteq E$ and so $\Lambda^*(D) \subseteq \Lambda^*(E)$. Therefore,

$$\mu(D) \le \sum_{v \in A \cap \Lambda^*(D)} m(M_v) \le \sum_{v \in A'_\varepsilon} m(M_v) \le \mu(E) + \varepsilon.$$

Since ε was arbitrary we obtain the required $\mu(D) \le \mu(E)$.

Finally, for countable subadditivity, let E_i, $i \in \mathbb{N}$, be a sequence of subsets of M. Let $\varepsilon > 0$ be arbitrary and define $\varepsilon_i = \varepsilon/2^i$. Let A_i be a section such that

$$\mu(E_i) \le \sum_{v \in A'_i} m(M_v) \le \mu(E_i) + \varepsilon_i,$$

where $A'_i = A_i \cap \Lambda^*(E_i)$. Let $B'' = \bigcup A'_i$ and let $B' \subseteq B''$ be a minimal subset, meaning that if $v \in B'$, then there does not exist non-empty $w \in \Lambda^*$ such that $vw \in B'$. Note that $\bigcup A_i$ is a countable section, though not necessarily minimal, and must contain a minimal section B that contains B'.

We now show that if $v \in B$ and $M_v \cap \bigcup E_i \neq \emptyset$, then $v \in B'$. So assume that for some $v \in B$ there exists $x \in M_v \cap \bigcup E_i$. Then there exists j such that $x \in E_j$ and a coding $vw \in \Lambda^{\mathbb{N}}$ such that $\bigcap_{i=1}^{\infty} M_{(vw)|_i} = x$. Since A_j is a section there must exist k such that $(vw)|_k \in A_j$. Further, as $x \in M_{vw|_k}$ we have $(vw)|_k \in \Lambda^*(\{x\}) \subseteq \Lambda^*(E_j)$ and so $(vw)|_k \in A'_j$ and $(vw)|_k \in B''$. Since B' is a minimal section it must contain $(vw)|_l$ for some $l \le k$. We cannot have $l > |v|$ as then $(vw)|_l$ has the ancestor v in B and B is not minimal. Further, we

[2]Strictly speaking, μ is an outer measure, as proven in Lemma 10. We will consider μ as a set function, and when using properties of measures we will assume, without further mention, measurability of the sets being considered.

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cannot have $l < |v|$ for then $v \in B$ has an ancestor in B, again breaking minimality. Hence $l = |v|$ and $v = (vw)|_l \in B'$, as required.

We can now bound the measure of $\bigcup E_i$:

$$\mu\left(\bigcup E_i\right) \leq \sum_{v \in B'} m(M_v) \leq \sum_{v \in B''} m(M_v) \leq \sum_{i \in \mathbb{N}} \sum_{v \in A'_i} m(M_v)$$
$$\leq \sum_{i \in \mathbb{N}} (\mu(E_i) + \varepsilon_i) = \sum_{i \in \mathbb{N}} \mu(E_i) + \varepsilon.$$

Letting $\varepsilon \to 0$ gives the required subadditivity.

Using this construction and Theorem 9 we can prove the following theorem[3].

THEOREM 11. *Let Λ be a finite alphabet and M_v, $v \in \Lambda^*$ and m satisfy the conditions above. Then*

$$\underline{\dim}_{\mathrm{A}}\, \mu \geq \frac{\log(1 - C_3)}{\log C_1} > 0$$

and hence μ is uniformly perfect.

Proof. Let $x \in M$ and $R > 0$ be arbitrary. We define

$$\mathcal{C} = \{v \in \Lambda^* : \operatorname{diam}(M_v) \leq C_1 R,\ \operatorname{diam}(M_{v^-}) > C_1 R,\ M_v \subseteq B(x, R)\}.$$

Note that, by definition, $\bigcup_{v \in \mathcal{C}} M_v \subseteq B(x, R)$. Let m be large enough that $C_1^m + C_1 < 1$. Choose n so large such that $e^{-n} \leq C_1^m$, so any $w \in \Lambda^*$ for which $M_w \cap B(x, e^{-n}R) \neq \emptyset$ and diam $M_w \leq e^{-n}R$ must have an ancestor w' such that diam $M_{w'} \leq C_1 R$ but diam $M_{w'^-} > C_1 R$. Therefore

$$d(x, y) < e^{-n}R + C_1 R \leq (C_1^m + C_1)R < R$$

for all $y \in M_{w'}$ and $M_{w'} \subseteq B(x, R)$. Hence, $w' \in \mathcal{C}$ and, in particular, every word in

$$\mathcal{B} = \{v \in \Lambda^* : \operatorname{diam}(M_v) \leq e^{-n}R,\ \operatorname{diam}(M_{v^-}) > e^{-n}R,\ M_v \cap B(x, e^{-n}R) \neq \emptyset\}$$

must have an ancestor in $\mathcal{C}$. Let k be the maximal integer such that $C_2 C_1^{k+2} R > 3e^{-n}R$ and temporarily fix $v \in \mathcal{C}$. Note that for all $1 \leq j \leq k$, there exist two words $\alpha, \beta \in \Lambda^j$ such that diam $M_{v\alpha}$, diam $M_{v\beta} > e^{-n}R$ and further that $d(M_{v\alpha}, M_{v\beta}) > 3e^{-n}R$. Hence, at most one of $M_{v\alpha}, M_{v\beta}$ can intersect $B(x, e^{-n}R)$ and for every j there exists at least one $w_j \in \Lambda$ such that $M_{vz_1z_2\ldots z_{j-1}w_j} \cap B(x, e^{-n}R) = \emptyset$ where $z_1 \neq w_1$, $z_2 \neq w_2$, etc. Since $m(M_w) < C_3 m(M_{w^-})$, we further get

$$m\left(\bigcup_{i \in \Lambda \setminus w_1} M_{vi}\right) \leq (1 - C_3) m(M_v)$$

[3] Independently, Rossi and Shmerkin [17, section 4·2] also proved that a similar Moran construction is uniformly perfect.

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and, inductively, for $W = (\Lambda \setminus w_1) \times (\Lambda \setminus w_2) \times \cdots \times (\Lambda \setminus w_{k_i})$,

$$m\left(\bigcup_{u \in W} M_{vu}\right) \leq (1 - C_3)^k m(M_v).$$

Observe that by construction $M_{vw} \cap B(x, e^{-n}R) = \emptyset$ for all $v \in \mathcal{C}$ and $w \notin W$. Further, every word in $\mathcal{B}$ must have an ancestor in $\mathcal{C} \times W$ and so $\mu(B(x, e^{-n}R)) \leq \sum_{vw \in \mathcal{C} \times W} m(M_{vw})$. Also, note that $\mu(B(x, R)) \geq \sum_{v \in \mathcal{C}} m(M_v)$ since $M_{vu} \subseteq B(x, R)$ for all $v \in \mathcal{C}$ and $u \in \Lambda^*$. Hence,

$$\frac{\mu(B(x, R))}{\mu((B(x, e^{-n}R))} \geq \frac{\sum_{v \in \mathcal{C}} m(M_v)}{\sum_{vw \in \mathcal{C} \times W} m(M_{vw})} \geq \frac{\sum_{v \in \mathcal{C}} m(M_v)}{(1 - C_3)^k \sum_{v \in \mathcal{C}} m(M_v)} \geq (1 - C_3)^{-k}$$

and we can take γ to be $1 - (1 - C_3)^k$. Now k is maximal and

$$C_1^{-k} < \frac{C_1^2 C_2}{3} \frac{R}{e^{-n}R} = \frac{C_1^2 C_2}{3e^{-n}}.$$

So,

$$k \geq \frac{\log\left(C_1^2 C_2/3\right) - \log e^{-n}}{\log(1/C_1)} - 1 = \frac{n}{\log(1/C_1)} + \frac{\log\left(-C_1^2 C_2/3\right)}{\log(1/C_1)} - 1.$$

We now apply Theorem 9 to get

$$\underline{\dim}_{\mathrm{A}}\, \mu \geq \frac{\log(1 - C_3)^{-k}}{\log e^n} \geq \frac{-\log(1 - C_3)}{n}\left(\frac{n}{\log(1/C_1)} + \frac{\log\left(C_1^2 C_2/3\right)}{\log(1/C_1)} - 1\right).$$

Since n was arbitrary, taking n large gets the required bound on the lower Assouad dimension and thus the measure is uniformly perfect.

This result can be applied to a variety of measures. For instance, suppose we are given an iterated function system (IFS) of similarities $\{S_j\}_{j=1}^N$ on $\mathbb{R}^d$ and probabilities $\{p_j\}_{j=1}^N$, with $p_j > 0$ and $\sum_{i=1}^N p_j = 1$. The self-similar set associated with the IFS is the unique non-empty compact set K such that $K = \bigcup_{j=1}^N S_j(K)$ which, without loss of generality, can be assumed to be contained in $[0, 1]^d$. We will further assume that K is not a singleton and thus perfect. The self-similar measure μ is the unique probability measure satisfying

$$\mu = \sum_{j=1}^N p_j(\mu \circ S_j^{-1}).$$

Given $v = (v_j)_{j=1}^n \in \{1, \ldots, N\}^n$, we let $S_v = S_{v_1} \circ S_{v_2} \circ \cdots \circ S_{v_n}$. If we put $M_v = S_v([0, 1]^d)$, then the collection of sets $\{M_v\}$ satisfies the first two requirements of the Moran set construction above. Condition (d) may not be satisfied, but by taking iterates of the IFS it is eventually satisfied. If we also define the weight function m by $m(M_{vj}) = p_j m(M_v)$, then the three conditions on the weight function are also fulfilled. The self-similar measure is the measure μ arising from the weight function m as in (3·3). Consequently, applying Theorem 9 we obtain

COROLLARY 12. *The lower Assouad dimension of any non-degenerate self-similar measure is positive.*

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Remark 13. This formula for the lower bound on the dimension is by no means sharp. For an IFS that satisfies the strong separation condition and contraction factors r_j, the approach gives $\log(1-\max p_i)/\log(\min r_i)$. The same methods as used in [5, theorem 2·4] for the upper Assouad dimension show that the actual value of the lower Assouad dimension is $\min(\log p_i/\log r_i)$, the same as the minimal lower local dimension (see [1]).

One can further extend Corollary 12 to equilibrium Gibbs measures and quasi-Bernoulli measures on self-conformal sets in general. A quasi-Bernoulli measure μ on the symbolic space $\{1,\ldots,N\}^{\mathbb{N}}$ is any probability measure that satisfies

$$c^{-1} \le \frac{\mu([v_1,\ldots,v_k])}{\mu([v_1,\ldots,v_l])\mu([v_{l+1},\ldots,v_k])} \le c$$

for all $v_i \in \{1,\ldots,N\}$ and $1 \le l \le k$, where

$$[v_1,\ldots,v_k] = \{w \in \{1,\ldots,N\}^{\mathbb{N}} : w_i = v_i \text{ for all } 1 \le i \le k\}$$

and $c > 0$ is a uniform constant.

Self-conformal sets satisfy the bounded distortion condition and expressing them as such a Moran construction is straightforward, see e.g. [14]. Similarly, the conditions on the mass functions are easily seen to be satisfied.

COROLLARY 14. *The lower Assouad dimension of the push-forward of a quasi-Bernoulli measure onto non-degenerate self-conformal sets is positive.*

3·3. *Self-affine measures*

The Moran construction detailed above is very flexible and also encompasses self-affine measures. Showing this needs some extra work and our approach here is similar to that of Xie, Jin and Sun [19] who proved that self-affine sets are uniformly perfect. The approach relies chiefly on the following easy lemma that only uses basic linear algebra. This lemma appears in a slightly different form as [19, lemma 2·1], but for self-containment we have chosen to include its proof.

LEMMA 15. *Let $E = \{e_1,\ldots,e_d\}$ be an orthonormal basis of $\mathbb{R}^d$, let A, B be $d \times d$ matrices of which A is invertible. Then there exists a constant $\alpha_A > 0$ depending only on A and d such that*

$$\max_{e\in E}\{|BA\,e|\} \ge \alpha_A \|B\|,$$

where $\|B\|$ denotes the operator norm of B acting as a linear transformation on $\mathbb{R}^d$.

Proof. First note that there exists $x_0 = \sum_{i=1}^d c_i e_i$ for some scalars c_i with $\sum_i |c_i| = 1$, such that $\|BA\| = |BAx_0|$. By linearity,

$$\|BA\| = |BAx_0| = |c_1 BAe_1 + \cdots + c_d BAe_d| \le d \max_{1\le i\le d} |BAe_i|. \qquad (3·4)$$

Thus the submuliplicativity of the matrix norm $\|.\|$ gives

$$\max_{e\in E}\{|BA\,e|\} \ge d^{-1}\|BA\| = \frac{\|BA\|\|A^{-1}\|}{d\|A^{-1}\|} \ge \frac{\|B\|}{d\|A^{-1}\|}.$$

Letting $\alpha_A = (d\|A^{-1}\|)^{-1}$ completes the proof.

Let $f_i(x) = A_i x + t_i$, $i = 1, \ldots, N$, be affine maps such that A_i is non-singular and $\|A_i\| < 1$ for all i. The self-affine set associated with $\{f_i\}$ is the unique compact set K that satisfies

$$K = \bigcup_{i=1}^{N} f_i(K).$$

We will assume that the attractor is not a singleton, which amounts to at least two f_i, f_j having distinct fixed points, and prove

THEOREM 16. *Let μ be the push-forward of a quasi-Bernoulli measure on the self-affine set F. If F is not a singleton, the measure μ has positive lower dimension and thus is uniformly perfect.*

Proof. Without loss of generality we can assume that K is not contained in any proper subspace of $\mathbb{R}^d$, redefining the affine maps with projections otherwise. So there exist $y_1, \ldots, y_d \in K$ such that $\{y_1, \ldots, y_d\}$ is linearly independent. Let Y be the linear transformation that maps e_i onto y_i. As this is a change of basis, Y must be invertible.

Let $B_R = B(0, R)$ be the closed ball of radius R and choose R large enough such that $f_i(B_R) \subset B_R$ for all i. Then, $f_j(f_i(B_R)) \subset f_j(B_R)$, and generally $f_{vw}(B_R) \subset f_v(B_R)$ for all non-empty $v, w \in \{1, \ldots, N\}^*$. Observe that the composition of affine maps is itself affine and diam $f_{v_1 \ldots v_k}(B_1) = 2\|A\|$, where $A = A_{v_1} \ldots A_{v_k}$ is the linear component of $f_{v_1 \ldots v_k}$. Let K be the minimal integer such that $(\max_i\{\|A_i\|\})^K < \alpha_Y/(6R)$ where α_Y is as in Lemma 15. Let $\Lambda = \{1, \ldots, N\}^K$ and define $M_\emptyset = B_R$ and $M_v = f_v(B_R)$, bearing in mind that a word $v \in \Lambda^k$ is of length $k \cdot K$. This definition clearly satisfies (a) and (b) in the Moran construction definition.

For (c) we note that for all $v \in \Lambda^k$ and $j \in \Lambda$,

$$\begin{aligned} \operatorname{diam} M_{vj} &= \operatorname{diam}(A_{v_1} \cdots A_{v_{(kK)}} A_{j_1} \cdots A_{j_K}(B_R)) = 2R \left\| A_{v_1} \cdots A_{v_{(kK)}} A_{j_1} \cdots A_{j_K} \right\| \\ &\geq 2R \left\| A_{v_1} \cdots A_{v_{(kK)}} \right\| \left\| (A_{j_1} \cdots A_{j_K})^{-1} \right\|^{-1} \\ &= \left\| (A_{j_1} \cdots A_{j_K})^{-1} \right\|^{-1} \operatorname{diam}(A_{v_1} \cdots A_{v_{(kK)}}(B_R)) \\ &= \left\| (A_{j_1} \cdots A_{j_K})^{-1} \right\|^{-1} \operatorname{diam} M_v. \end{aligned}$$

Since Λ is finite and all A_i are invertible, there exists a constant

$$C_1 = \min_{v \in \Lambda} \left\| (A_{v_1} \cdots A_{v_K})^{-1} \right\|^{-1} > 0$$

such that (c) is satisfied.

Finally we check (d). Let $v \in \Lambda^k$ and recall that Y maps the basis E onto a linearly independent set of points in K. Using Lemma 15 we obtain,

$$\max_{e \in E}\{|A_{v_1} \cdots A_{v_{(kK)}} Y e|\} \geq \alpha_Y \left\| A_{v_1} \cdots A_{v_{(kK)}} \right\| = \frac{\alpha_Y}{2R} \operatorname{diam} M_v.$$

But then

$$\operatorname{diam} f_v(YE) \geq \frac{\alpha_Y}{2R} \operatorname{diam} M_v$$

and as $Ye_i = y_i$, we have $YE \subset F$ and $f_v(YE) \subset K$. Thus there exist two points in $M_v \cap F$ that are at least $(\alpha_Y/(2R))$ diam M_v apart. Since these two points must be contained in M_{vi}

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and M_{vj}, respectively, and diam M_{vi}, diam $M_{vj} \leq (\alpha_Y/(6R))$ diam M_v we must have $i \neq j$, and further

$$d(M_{vi}, M_{vj}) \geq (\alpha_Y/(3R)) \operatorname{diam} M_v.$$

Thus Condition (d) is satisfied.

Letting $m(M_v) = \mu([v])$ for all $v \in \Lambda^*$ gives the correct measure on $M = F$. Checking the conditions on the weight function is straightforward and left to the reader.

Remark 17. It was observed by Käenmäki and Lehrbäck, see [**12**, lemma 3·1], that any doubling measure supported on a uniformly perfect metric space has positive lower dimension. Our results above show that there are many measures with positive lower Assouad dimension that are, in general, far from doubling.

3·4. *Lower dimension of Bedford–McMullen carpets*

One example of a class of self-affine measures are the pushforward measures given by a Bernoulli probability measure on Bedford–McMullen carpets. In this subsection we compute the exact lower Assouad dimension of these measures. The result is analogous to the upper Assouad dimension for sponges given in [**5**] and due to its similarity we will only give a brief sketch of its proof.

Let $2 \leq m < n$ be integers and consider maps of the form $f_i(x) = Ax + t_i$, where $1 \leq i \leq N$, A is the diagonal matrix $A = \operatorname{Diag}(1/m, 1/n)$ and $t_i = [a_i/m\; b_i/n]^\top$ for some integers $0 \leq a_i < m$ and $0 \leq b_i < n$. The attractor of the IFS $\{f_1, \ldots, f_N\}$ is known as a Bedford–McMullen carpet. If there exists $\varepsilon > 0$ such that all $f_i([-\varepsilon, 1+\varepsilon]^2)$ are pairwise disjoint, we say that the iterated function system satisfies the very strong separation condition.

Given $p_i > 0$ such that $\sum p_i = 1$, let μ be the pushforward measure of the Bernoulli measure on $\{1, \ldots, N\}^{\mathbb{N}}$ under the IFS. The lower Assouad dimension of this self-affine measure is characterised by finding a minimising column. We write

$$p_{\text{col}}(i) = \sum_{\substack{j \in \{1,\ldots,N\} \\ a_j = a_i}} p_j$$

for the measure of the column containing $f_i([0, 1])$, that is

$$p_{\text{col}}(i) = \mu([a_i/m, (a_i+1)/m] \times [0, 1]).$$

THEOREM 18. *Let μ be the self-affine measure of Bedford–McMullen type with associated probabilities p_i and contractions f_i. If the very strong separation condition holds, then*

$$\underline{\dim}_{\mathrm{A}}\, \mu = \min_{1 \leq j \leq N} \frac{-\log p_{\text{col}}(j)}{\log m} + \min_{1 \leq i \leq N} \frac{\log p_{col}(i)/p_i}{\log n}. \tag{3·5}$$

Furthermore, $\underline{H}(t) = \underline{\dim}_{\mathrm{A}}\, \mu$ for small enough t and so $\underline{\dim}_{\mathrm{qA}}\, \mu = \underline{\dim}_{\mathrm{A}}\, \mu$.

Proof. The key idea to establishing this dimension result are “approximate squares”, see [**5**] for details. Heuristically, an approximate square is a collection of words such that the corresponding set has uniformly comparable base and height, i.e. is ‘almost’ a square. We will construct approximate squares below and check that they give rise to the dimension formula. The details that allow us to transition from nested approximate squares to balls are

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based on the very strong separation condition and contained in [5]; we decided to omit them for brevity.

Let $k_1(R)$ and $k_2(R)$ be the unique integers such that $m^{-k_1(R)} \le R < m^{-k_1(R)+1}$ and $n^{-k_2(R)} \le R < n^{-k_2(R)+1}$. Due to the common diagonal structure of the linear part of f_i, the image $f_v([0,1]^2)$ will be a rectangle aligned with the first and second coordinate. If v has length $k_2(R)$, the rectangle $f_v([0,1]^2)$ will have height in $(n^{-1}R, R]$. Similarly, for any word of length $k_1(R)$, the corresponding rectangle will have base in $(m^{-1}R, R]$. Given $0 < r < R < 1$ let $v_r \in \Lambda^{k_2(r)}$ and $w_r \in \Lambda^*$ such that $v_r w_r \in \Lambda^{k_1(r)}$ and consider the set

$$Q_r = \bigcup_{w \in \Lambda^*} \{f_{v_r w}([0,1]^2) \,:\, v_r w \in \Lambda^{k_1(r)} \text{ and } a_{(v_r w)_i} = a_{(v_r w_r)_i} \text{ for all } 1 \le i \le k_1(r)\},$$

that is the set of all images of words that have v_r as the ancestor (whose rectangle has height comparable to r) such that each rectangle associated with $v_r w$ has base comparable to r and the horizontal translations all agree so all $f_{v_r w}([0,1]^2)$ align in the same column as $f_{v_r w_r}([0,1]^2)$. Therefore Q_r must have height and base comparable to r and is a (generic) approximate square. Its parent approximate square of size R is denoted by Q_R and is the set given by

$$Q_R = \bigcup_{w \in \Lambda^*} \{f_{v_R w}([0,1]^2) \,:\, v_R w \in \Lambda^{k_1(R)} \text{ and } a_{(v_R w)_i} = a_{(v_r w_r)_i} \text{ for all } 1 \le i \le k_1(R)\},$$

where $v_R \in \Lambda^{k_2(R)}$ is the parent word of v_r.

As mentioned above, it is sufficient to check $\mu(Q_R)/\mu(Q_r)$ for all arbitrary approximate squares of the above form. Their measures are

$$\mu(Q_R) = \prod_{i=1}^{k_2(R)} p_{(v_r w_r)_i} \prod_{i=k_2(R)+1}^{k_1(R)} p_{\mathrm{col}}((v_r w_r)_i)$$

and

$$\mu(Q_r) = \prod_{i=1}^{k_2(r)} p_{(v_r w_r)_i} \prod_{i=k_2(r)+1}^{k_1(r)} p_{\mathrm{col}}((v_r w_r)_i).$$

Notice that by definition we must either have

$$k_2(R) < k_1(R) < k_2(r) < k_1(r) \quad \text{or} \quad k_2(R) < k_2(r) < k_1(R) < k_1(r). \tag{3·6}$$

In the first case we get

$$\begin{aligned}
\frac{\mu(Q_R)}{\mu(Q_r)} &= \prod_{i=k_2(R)+1}^{k_1(R)} p_{\mathrm{col}}((v_r w_r)_i)/p_{(v_r w_r)_i} \left(\prod_{i=k_1(R)+1}^{k_2(r)} p_{(v_r w_r)_i} \prod_{i=k_2(r)+1}^{k_1(r)} p_{\mathrm{col}}((v_r w_r)_i) \right)^{-1} \\
&\ge \left(\min_{1 \le j \le N} \frac{p_{\mathrm{col}}(j)}{p(j)} \right)^{k_1(R)-k_2(R)-1} \left(\min_{1 \le j \le N} p(j)^{-1} \right)^{k_2(r)-k_1(R)-1} \\
&\quad \times \left(\min_{1 \le j \le N} p_{\mathrm{col}}(j)^{-1} \right)^{k_1(r)-k_2(r)-1} \\
&\ge C \left(\min_{1 \le j \le N} \frac{p_{\mathrm{col}}(j)}{p(j)} \right)^{k_1(R)-k_2(R)} \left(\min_{1 \le j \le N} p_{\mathrm{col}}(j)^{-1} \right)^{k_1(r)-k_2(r)},
\end{aligned}$$

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for some $C > 0$. Note that $k_1(t) - k_2(t) \sim \log(1/t)$ and so the lower bound increases to infinity as $R \to 0$ and $r \to 0$, irrespective of R/r. On the other hand, in the second case, we obtain

$$\begin{aligned}
\frac{\mu(Q_R)}{\mu(Q_r)} &= \left(\prod_{i=k_2(R)+1}^{k_1(R)} p_{\mathrm{col}}((v_r w_r)_i)\right)\left(\prod_{i=k_2(R)+1}^{k_2(r)} p_{(v_r w_r)_i} \prod_{i=k_2(r)+1}^{k_1(r)} p_{\mathrm{col}}((v_r w_r)_i)\right)^{-1} \\
&= \left(\prod_{i=k_2(R)+1}^{k_2(r)} p_{\mathrm{col}}((v_r w_r)_i)\right)\left(\prod_{i=k_2(R)+1}^{k_2(r)} p_{(v_r w_r)_i} \prod_{i=k_1(R)+1}^{k_1(r)} p_{\mathrm{col}}((v_r w_r)_i)\right)^{-1} \\
&= \prod_{i=k_2(R)+1}^{k_2(r)} p_{\mathrm{col}}((v_r w_r)_i)/p_{(v_r w_r)_i} \prod_{i=k_1(R)+1}^{k_1(r)} p_{\mathrm{col}}((v_r w_r)_i)^{-1} \\
&\geq C\left(\min_{1\leq j\leq N} \frac{p_{\mathrm{col}}(j)}{p_j}\right)^{k_2(r)-k_2(R)} \left(\min_{1\leq j\leq N} p_{\mathrm{col}}(j)^{-1}\right)^{k_1(r)-k_1(R)} \\
&= C\left(\min_{1\leq j\leq N} \frac{p_{\mathrm{col}}(j)}{p_j}\right)^{\log(R/r)/\log n} \left(\min_{1\leq j\leq N} p_{\mathrm{col}}(j)^{-1}\right)^{\log(R/r)/\log m} = C\left(\frac{R}{r}\right)^s
\end{aligned}$$

for some uniform $C > 0$ and s as in (3·5). This shows that $\underline{\dim}_{\mathrm{A}}\, \mu \geq s$.

Lastly, the second behaviour in (3·6) occurs when $r > R^{1+\delta}$, where $1 + \delta = \log n/\log m$. Therefore there exists a word such that this minimum is achieved and we obtain $\underline{H}(t) \leq s$ and $\underline{H}(t)$ is constant for $0 < t < \delta$.

4. *The lower Assouad dimension for self-similar measures of finite type*

4·1. *Finite type measures*

In this section, we will prove that for a class of self-similar measures on $\mathbb{R}$, called finite type, the lower Assouad dimension coincides with the minimal lower local dimension of the measure (Theorem 25). Many interesting self-similar measures that fail the open set condition are of finite type, such as Bernoulli convolutions with Pisot contractions. We begin by explaining what is meant by finite type.

Assume we are given an IFS of similarities, $S_j(x) = r_j x + d_j : \mathbb{R} \to \mathbb{R}$ for $j = 1, \ldots, N$, where $N \geq 2$ and $0 < |r_j| < 1$, and probabilities $\{p_k\}_{j=1}^N$. By rescaling and translation, there is no loss in assuming the convex hull of the self-similar set K is [0, 1]. We let μ denote the self-similar measure, $\mu(E) = \sum_{j=1}^N p_j \mu(S_j^{-1}(E))$.

Given any integer n and $v = (v_j)_{j=1}^n \in \{1, \ldots, N\}^n$, we let $v^- = (v_1, \ldots, v_{n-1})$, $r_v = \prod_{i=1}^n r_{v_i}$ and $p_v = \prod_{j=1}^n p_{v_j}$. Put

$$\lambda = \min_{j=1,\ldots,N} |r_j|$$

and

$$\Lambda_n = \{v \in \{1, \ldots, N\}^* : |r_v| \leq \lambda^n \text{ and } |r_{v^-}| > \lambda^n\}.$$

The notion of finite type was introduced by Ngai and Wang in [**15**]. The definition we will use is slightly less general, but is simpler and includes all the examples in $\mathbb{R}$ that we are aware of.

Definition 19. Assume $\{S_j\}$ is an IFS of similarities. The words $v, w \in \Lambda_n$ are said to be neighbours if $S_v(0, 1) \cap S_w(0, 1) \neq \emptyset$. Denote by $\mathcal{N}(v)$ the set of all neighbours of v. We say that $v \in \Lambda_n$ and $w \in \Lambda_m$ have the same neighbourhood type if there is a map $f(x) = \pm\lambda^{n-m}x + c$ such that

$$f \circ S_v = S_w \text{ and } \{f \circ S_u : u \in \mathcal{N}(v)\} = \{S_t : t \in \mathcal{N}(w)\}.$$

The IFS is said to be of **finite type** if there are only finitely many neighbourhood types. Any associated self-similar measure is also said to be of **finite type**.

It was shown in [16] that an IFS of finite type satisfies the weak separation condition, but not necessarily the open set condition. For instance, the IFS given by $S_j(x) = \pm\rho^{-n_j}x + b_j$ where ρ is a Pisot number[4], $n_j \in \mathbb{N}$ and $b_j \in \mathbb{Q}[\rho]$, was shown to be of finite type in [15, theorem 2·9], but fails the open set condition. The Bernoulli convolutions with contraction factors that are inverses of Pisot numbers are self-similar measures associated with an IFS of this form. As integers are also Pisot numbers, the self-similar measures coming from an IFS $\{S_j(x) = x/d + j(d-1)/d\}_{j=0}^{m-1}$, for integer $d \geq 3$, such as m-fold convolutions of the uniform Cantor measure on the Cantor set of ratio $1/d$, are another class of finite type measures.

Definition 20. For each positive integer n, let $h_1, \ldots, h_{s_n}$ be the collection of elements of the set $\{S_v(0), S_v(1) : v \in \Lambda_n\}$, listed in increasing order. Set

$$\mathcal{F}_n = \{[h_j, h_{j+1}] : 1 \leq j \leq s_n - 1 \text{ and } (h_j, h_{j+1}) \cap K \neq \emptyset\}.$$

Elements of $\mathcal{F}_n$ are known as the **net intervals of level** n.

For each $\Delta \in \mathcal{F}_n$, $n \geq 1$, there is a unique element $\widehat{\Delta} \in \mathcal{F}_{n-1}$ which contains Δ, called the *parent* (of *child* Δ). Given $\Delta = [a, b] \in \mathcal{F}_n$, we denote the *normalised length* of Δ by $\ell_n(\Delta) = \lambda^{-n}(b - a)$. By the *neighbour set* of Δ we mean the ordered tuple

$$V_n(\Delta) = ((a_1, L_1), (a_2, L_2), \ldots, (a_j, L_j)),$$

where for each i there is some $v \in \Lambda_n$ such that $\lambda^{-n}r_v = L_i$ and $\lambda^{-n}(a - S_v(0)) = a_i$. Suppose $\Delta \in \mathcal{F}_n$ has parent $\widehat{\Delta}$. If $\widehat{\Delta}$ has multiple children with the same normalized length and neighbourhood set as Δ, order them from left to right as $\Delta_1, \Delta_2, \ldots, \Delta_T$. Let $t_n(\Delta) \in \{1, \ldots, T\}$ be the integer t such that $\Delta_t = \Delta$.

Definition 21. The **characteristic vector** *of* $\Delta \in \mathcal{F}_n$ is defined to be the triple

$$\mathcal{C}_n(\Delta) = (\ell_n(\Delta), V_n(\Delta), t_n(\Delta)).$$

A very important fact, shown in [9, theorem 2·7], is that an IFS of finite type admits only finitely many characteristic vectors. The characteristic vectors are of fundamental importance because, as we will see, we can obtain key information about the local behaviour of any associated self-similar measure from them.

[4] A Pisot number is an algebraic number greater than one, all of whose Galois conjugates are strictly less than one in modulus. An example is the golden mean.

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By the **symbolic representation** of a net interval $\Delta \in \mathcal{F}_n$ we mean the $(n+1)$-tuple $(\mathcal{C}_0(\Delta_0), \ldots, \mathcal{C}_n(\Delta_n))$, where $\Delta_0 = [0, 1]$, $\Delta_n = \Delta$, and for each $j = 1, \ldots, n$, Δ_{j-1} is the parent of Δ_j. Similarly, for each $x \in K = \operatorname{supp} \mu$ the symbolic representation of x will be the sequence of characteristic vectors $[x] = (\mathcal{C}_0(\Delta_0), \mathcal{C}_1(\Delta_1), \ldots)$ where $x \in \Delta_n \in \mathcal{F}_n$ for each n and Δ_{j-1} is the parent of Δ_j. Conversely, every sequence of characteristic vectors $(\gamma_0, \gamma_1, \ldots)$ where $\gamma_0 = \mathcal{C}_0(\Delta_0)$ and γ_j is the parent of γ_{j+1} is the symbolic representation of a unique $x \in K$. We will write $\Delta_n(x)$ for a net interval of level n containing x.

By a **path** we mean a segment of a symbolic representation. A **loop class** is a set of characteristic vectors $\mathcal{L}$ with the property that given any $\chi, \psi \in \mathcal{L}$ there is some finite path η in $\mathcal{L}$ so that (χ, η, ψ) is a path (in $\mathcal{L}$).

Definition 22. Let $\Delta = [a, b] \in \mathcal{F}_n$ and let $\widehat{\Delta} = [c, d] \in \mathcal{F}_{n-1}$ denote its parent net interval. Assume $V_n(\Delta) = ((a_1, L_1), \ldots, (a_I, L_I))$ and $V_{n-1}(\widehat{\Delta}) = ((c_1, M_1), \ldots, (c_J, M_J))$. The **primitive transition matrix,** denoted

$$T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)),$$

is the $I \times J$ matrix (T_{ij}) which encapsulates information about the relationship between the $(c_i, M_i) \in V_{n-1}(\widehat{\Delta})$ and $(a_j, L_j) \in V_n(\Delta)$. To be precise, let $\sigma_j \in \Lambda_{n-1}$ be such that $\lambda^{-n+1}(c - S_\sigma(0)) = c_i$ and $\lambda^{-n+1} r_\sigma = M_i$. Let $\mathcal{T}_{i,j}$ be the set of all ω such that $\sigma\omega \in \Lambda_n$, $\lambda^{-n}(a - S_{\sigma_i\omega}(0)) = a_j$ and $\lambda^{-n} r_{\sigma_i\omega} = L_j$. Notice that $\mathcal{T}_{i,j}$ depends only on S_{σ_i} (or equivalently on c_i and M_i), and not on the choice of σ_i. We define $T_{i,j} = \sum_{\omega \in \mathcal{T}_{i,j}} p_\omega$ where the empty sum is taken to be 0.

Given a path $(\gamma_J, \gamma_{J+1}, \ldots, \gamma_N)$, we write $T(\gamma_J, \gamma_{J+1}, \ldots, \gamma_N)$ for the product

$$T(\gamma_J, \gamma_{J+1}, \ldots, \gamma_N) = T(\gamma_J, \gamma_{J+1}) T(\gamma_{J+1}, \gamma_{J+2}) \cdots T(\gamma_{N-1}, \gamma_N).$$

For brevity we write $\left\| (T_{ij}) \right\| = \sum_{i,j} \left| T_{ij} \right|$ and note the following critical fact proven in [9, section 3·2].

LEMMA 23. *There are constants $a, b > 0$ such that whenever Δ_n is a net interval of level n with symbolic representation $(\gamma_0, \gamma_1, \ldots, \gamma_n)$, then*

$$a \left\| T(\gamma_0, \gamma_1, \ldots, \gamma_n) \right\| \le \mu(\Delta_n) \le b \left\| T(\gamma_0, \gamma_1, \ldots, \gamma_n) \right\|.$$

This lemma is useful because the lower Assouad dimensions for self-similar measures of finite type can be deduced from the knowledge of the measure of net intervals, as we see next.

LEMMA 24. *If μ is of finite type and $\underline{\dim}_A \mu \le d$, then for each $\varepsilon > 0$ there are $x_i \in$* supp μ *and net intervals $\Delta_{N_i}(x_i) \supseteq \Delta_{n_i}(x_i)$ with $n_i - N_i \to \infty$ such that*

$$\frac{\mu(\Delta_{N_i}(x_i))}{\mu(\Delta_{n_i}(x_i))} < \lambda^{(d+\varepsilon)(N_i - n_i)}. \tag{4·1}$$

Proof. Suppose the statement above is false. As there are only finitely many characteristic vectors, all normalized lengths of net intervals are comparable. Thus we may choose $c > 0$ so that $\operatorname{diam}(\Delta_n) \ge c\lambda^n$ for all net intervals Δ_n of level n. Given any net interval, Δ_n, of level

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n, we will write Δ_n^R and Δ_n^L for the adjacent net intervals of level n to the right and left of Δ_n respectively, should these exist.

Let $x \in \operatorname{supp} \mu$ and consider $r < R$ where $R/r \to \infty$. Choose N, n such that $3\lambda^N \le R < 3\lambda^{N-1}$ and $c\lambda^{n+1} < r \le c\lambda^n$. Then $n - N \to \infty$ and

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \ge \frac{\mu(B(x, 3\lambda^N))}{\mu(B(x, c\lambda^n))}.$$

As all net intervals of level n have diameter between $c\lambda^n$ and λ^n, for any $x \in \operatorname{supp} \mu$ we have

$$B(x, 3\lambda^N) \cap \operatorname{supp} \mu \supseteq (\Delta_N(x) \cup \Delta_N^R \cup \Delta_N^L) \cap \operatorname{supp} \mu$$

and

$$B(x, c\lambda^n) \cap \operatorname{supp} \mu \subseteq (\Delta_n(x) \cup \Delta_n^R \cup \Delta_n^L) \cap \operatorname{supp} \mu.$$

Thus

$$\mu(B(x, 3\lambda^N)) \ge \max\{\mu(\Delta_N(x)), \mu(\Delta_N^R), \mu(\Delta_N^L)\},$$

while

$$\mu(B(x, c\lambda^n)) \le 3 \max\{\mu(\Delta_n(x)), \mu(\Delta_n^R), \mu(\Delta_n^L)\}.$$

First, suppose $\mu(B(x, c\lambda^n)) \le 3\mu(\Delta_n(x))$. Since we are assuming (4·1) fails,

$$\frac{\mu(B(x, 3\lambda^N))}{\mu(B(x, c\lambda^n))} \ge \frac{\mu(\Delta_N(x))}{3\mu(\Delta_n(x))} \ge \frac{1}{3}\lambda^{(d+\varepsilon)(N-n)} \ge C\left(\frac{R}{r}\right)^{d+\varepsilon}$$

for a suitable constant C, independent of x, R, r.

Otherwise, without loss of generality, $\mu(B(x, c\lambda^n)) \le 3\mu(\Delta_n^L)$. Notice that Δ_n^L is either a child of $\Delta_N(x)$ or Δ_N^L. If $\Delta_n^L \subseteq \Delta_N(x)$ and we let $y \in \Delta_n^L \cap \operatorname{supp} \mu$, then $\Delta_N(x) = \Delta_N(y)$ and $\Delta_n^L = \Delta_n(y)$, so we have

$$\frac{\mu(B(x, 3\lambda^N))}{\mu(B(x, c\lambda^n))} \ge \frac{\mu(\Delta_N(y))}{3\mu(\Delta_n(y))} \ge \frac{1}{3}\lambda^{(d+\varepsilon)(N-n)} \ge C\left(\frac{R}{r}\right)^{d+\varepsilon}.$$

If, instead $\Delta_n^L \subseteq \Delta_N^L$ the arguments are similar, just take y to be the right endpoint of Δ_n^L and then $\Delta_N^L = \Delta_N(y)$ and $\Delta_n^L = \Delta_n(y)$.

Consequently,

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \ge C\left(\frac{R}{r}\right)^{d+\varepsilon}$$

for all $x \in \operatorname{supp} \mu$ and $r < R$ with $R/r \to \infty$ and that implies $\underline{\dim}_A \mu \ge d + \varepsilon$; a contradiction.

4·2. *Lower Assouad dimension for measures of finite type*

We are now ready to prove the main result of this section.

THEOREM 25. *If μ is any self-similar measure of finite type, then*

$$\underline{\dim}_A \mu = \inf\{\underline{\dim}_{loc}\mu(x) : x \in \operatorname{supp} \mu\}.$$

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Proof. Throughout the proof C will denote a positive constant that may change from one occurrence to another. Let $d = \inf_x\{\underline{\dim}_{\mathrm{loc}}\mu(x)\}$ and assume for a contradiction that $\underline{\dim}_{\mathrm{A}}\,\mu < d$, say $\underline{\dim}_{\mathrm{A}}\,\mu < d - 3\varepsilon$ for $\varepsilon > 0$. We will show that this implies the existence of points which have local dimension strictly less than d.

By Lemma 24, there are $x_i \in \operatorname{supp}\mu$ and $N_i < n_i$ such that $n_i - N_i \to \infty$ and

$$\frac{\mu(\Delta_{N_i}(x_i))}{\mu(\Delta_{n_i}(x_i))} < \lambda^{(d-2\varepsilon)(N_i-n_i)} \text{ for all } i.$$

It follows from Lemma 23 that there is a constant C such that if $\Delta_{n_i}(x_i)$ has symbolic representation $(\gamma_0, \gamma_1^{(i)}, \ldots, \gamma_{n_i}^{(i)})$, then

$$C\lambda^{(d-2\varepsilon)(N_i-n_i)} \geq C\frac{\mu(\Delta_{N_i}(x_i))}{\mu(\Delta_{n_i}(x_i))} \geq \frac{\left\|T(\gamma_0, \gamma_1^{(i)}, \ldots, \gamma_{N_i}^{(i)})\right\|}{\left\|T(\gamma_0, \gamma_1^{(i)}, \ldots, \gamma_{n_i}^{(i)})\right\|}$$

$$\geq \frac{\left\|T(\gamma_0, \gamma_1^{(i)}, \ldots, \gamma_{N_i}^{(i)})\right\|}{\left\|T(\gamma_0, \gamma_1^{(i)}, \ldots, \gamma_{N_i}^{(i)})\right\| \left\|T(\gamma_{N_i}^{(i)}, \ldots, \gamma_{n_i}^{(i)})\right\|}.$$

Thus

$$\left\|T(\gamma_{N_i}^{(i)}, \ldots, \gamma_{n_i}^{(i)})\right\| \geq C\lambda^{(d-2\varepsilon)(n_i-N_i)}. \tag{4·2}$$

The path, $(\gamma_{N_i}^{(i)}, \ldots, \gamma_{n_i}^{(i)})$, can be rewritten as $(\chi_0^{(i)}, \sigma_1^{(i)}, \chi_1^{(i)}, \ldots, \sigma_{k_i}^{(i)})$, where for each $j \geq 1$, $\sigma_j^{(i)}$ is a path in a distinct maximal loop class $L_j^{(i)}$, $\chi_j^{(i)}$ is a minimal length path joining the last letter of $\sigma_j^{(i)}$ (a characteristic vector in L_j^i) to the first letter of $\sigma_{j+1}^{(i)}$ (a characteristic vector in L_{j+1}^i), and $\chi_0^{(i)}$ is a path from the first letter of $\gamma_{N_i}^{(i)}$ to the first letter of $\sigma_1^{(i)}$.

The finite type property ensures that there are only finitely many maximal loop classes and only finitely many characteristic vectors in each loop class. Hence there can only be finitely many of these minimal joining paths $\chi_j^{(i)}$ over all i, j. Thus $\sup_{i,j}\|T(\chi_j^{(i)})\|$ is bounded and $\sup_i\{\sup_j \operatorname{length}(\chi_j^{(i)})\} \leq \sup_i A^{(i)} < \infty$. Since it is not possible to return to a maximal loop class after leaving it, the numbers k_i are bounded, say by k. Hence there is a constant C such that

$$\left\|T(\gamma_{N_i}^{(i)}, \ldots, \gamma_{n_i}^{(i)})\right\| \leq \prod_{j=0}^{k_i-1}\left\|T(\chi_j^{(i)})\right\| \prod_{j=1}^{k_i}\left\|T(\sigma_j^{(i)})\right\| \leq C^k \prod_{j=1}^{k_i}\left\|T(\sigma_j^{(i)})\right\|. \tag{4·3}$$

Let $l_j^{(i)}$ denote the length of the path $\sigma_j^{(i)}$. Then

$$\sum_{j=1}^{k_i} l_j^{(i)} \leq n_i - N_i = \sum_{j=1}^{k_i} l_j^{(i)} + \sum_{j=0}^{k_{i-1}} \operatorname{length}(\chi_j^{(i)}) \leq \sum_{j=1}^{k_i} l_j^{(i)} + kA^{(i)}, \tag{4·4}$$

so $\sum_{j=1}^{k_i} l_j^{(i)} \to \infty$ as $i \to \infty$. Putting together these observations we see that for large enough $n_i - N_i$, (4·2) gives

$$\frac{\log\left\|T(\gamma_{N_i}^{(i)}, \ldots, \gamma_{n_i}^{(i)})\right\|}{n_i - N_i} \geq \frac{(d-2\varepsilon)(n_i-N_i)\log\lambda + \log C}{n_i - N_i}$$

$$\geq (d-2\varepsilon)\log\lambda - \frac{\varepsilon}{2}|\log\lambda|,$$

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while (4·3-4·4) imply

$$\frac{\log\left\|T(\gamma_{N_i}^{(i)},\ldots,\gamma_{n_i}^{(i)})\right\|}{n_i-N_i}\le\frac{\log C^k+\log\prod_{j=1}^{k_i}\left\|T(\sigma_j^{(i)})\right\|}{\sum_{j=1}^{k_i}l_j^{(i)}}$$

$$\le\frac{\sum_{j=1}^{k_i}\log\left\|T(\sigma_j^{(i)})\right\|}{\sum_{j=1}^{k_i}l_j^{(i)}}+\frac{\varepsilon}{2}|\log\lambda|.$$

Hence

$$\frac{\sum_{j=1}^{k_i}\log\left\|T(\sigma_j^{(i)})\right\|}{\sum_{j=1}^{k_i}l_j^{(i)}}\ge(d-\varepsilon)\log\lambda,\ \text{ for large } i, \tag{4·5}$$

and that implies that $\log\left\|T(\sigma_j^{(i)})\right\|\ge(d-\varepsilon)l_j^{(i)}$ for some $j=j_i$. There is no loss of generality in assuming $j_i=1$. Thus

$$\left\|T(\sigma_1^{(i)})\right\|\ge\lambda^{l_1^{(i)}(d-\varepsilon)}. \tag{4·6}$$

We will now construct $x\in\operatorname{supp}\mu$ with $\underline{\dim}_{\mathrm{loc}}\mu(x)\le d-\varepsilon/2$ by constructing a symbolic representation from the symbolic representations of a suitable subsequence of the (x_i). We will rely on the fact that there will be a subsequence of (the symbolic representations for) x_i and index j_i such that all $\sigma_{j_i}^{(i)}$ belong to the same loop class and their lengths are unbounded in i.

As there are only finitely many maximal loop classes, there must be some subsequence such that all $\sigma_1^{(i)}$ (for i in the subsequence) belong to the same maximal loop class.

Suppose $\sup_i \ell_1^{(i)}<\infty$. As there are finitely many characteristic vectors, there can be only finitely many paths of length at most $\sup_i \ell_1^{(i)}$ and hence $\sup_i\left\|T(\sigma_1^{(i)})\right\|<\infty$. Consider again inequality (4·3) with this additional information

$$\left\|T(\gamma_{N_i}^{(i)},\ldots,\gamma_{n_i}^{(i)})\right\|\le C^k\prod_{j=1}^{k_i}\left\|T(\sigma_j^{(i)})\right\|\le C^{k+1}\prod_{j=2}^{k_i}\left\|T(\sigma_j^{(i)})\right\|.$$

Since

$$\sum_{j=2}^{k_i}l_j^{(i)}\le n_i-N_i=\ell_1^{(i)}+\sum_{j=2}^{k_i}l_j^{(i)}+\sum_{j=0}^{k_{i-1}}\operatorname{length}(\chi_j^{(i)})\le\sum_{j=2}^{k_i}l_j^{(i)}+\ell_1^{(i)}+kA^{(i)}$$

and $kA^{(i)}+\ell_1^{(i)}$ is bounded over i, the same reasoning as used to deduce (4·6) shows that for some further subsequence and index $j_i\in\{2,\ldots,k\}$, which we can assume without loss of generality is 2, we have

$$\left\|T(\sigma_2^{(i)})\right\|\ge\lambda^{l_2^{(i)}(d-\varepsilon)}$$

with all $\sigma_2^{(i)}$ belonging to the same maximal loop class.

If $\sup_i \ell_2^{(i)}<\infty$, we repeat the argument. As there are only finitely many maximal loop classes, we must eventually find a subsequence of the indices i and index j such that the

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paths $\sigma_j^{(i)} = \rho_i$, all are in the same maximal loop class Λ, their lengths $\ell^{(i)} = \ell_j^{(i)} \to \infty$ as $i \to \infty$ and

$$\|T(\rho_i)\| \geq \lambda^{\ell_j^{(i)}(d-\varepsilon)}.$$

We will now 'stitch' these paths together to obtain the $x \in \operatorname{supp} \mu$ required for the contradiction. For each pair of characteristic vectors, χ, ψ, in Λ, choose a path $\eta_{\chi,\psi}$ (in Λ) with first letter χ and last letter ψ. Choose, also, a path η_ψ from γ_0 to each $\psi \in \Lambda$. Let $\mathcal{S}$ denote the finite set consisting of the chosen paths $\eta_{\chi,\psi}, \eta_\psi$. Since a transition matrix contains a non-zero entry in each column, there is some constant $c_0 > 0$ such that $\|T(\eta, \sigma)\| \geq c_0 \|T(\sigma)\|$ for all $\eta \in \mathcal{S}$ and all admissible paths σ (meaning, (η, σ) is a path). Choose i_1 such that

$$\frac{|\log c_0|}{\ell^{(i_1)}} < \frac{\varepsilon |\log \lambda|}{2}$$

and select a path $\nu_1 \in \mathcal{S}$ joining γ_0 to the path ρ_{i_1}.

Next, as ν_1, ρ_{i_1} are fixed and $\mathcal{S}$ is finite, we can choose $c_1 > 0$ such that

$$\left\|T(\nu_1, \rho_{i_1}, \eta, \sigma)\right\| \geq c_1 \|T(\sigma)\|$$

for all admissible paths $\eta \in \mathcal{S}$ and σ. Then choose $i_2 > i_1$ such that

$$\frac{|\log c_1|}{\ell^{(i_2)}} < \frac{\varepsilon |\log \lambda|}{2}.$$

As ρ_{i_1} and ρ_{i_2} belong to the same maximal loop class Λ, there is some path ν_2 joining the last letter of ρ_{i_1} to the first letter of ρ_{i_2}. Having found such a path, choose $c_2 > 0$ so

$$\left\|T(\nu_1, \rho_{i_1}, \nu_2, \rho_{i_2}, \eta, \sigma)\right\| \geq c_2 \|T(\sigma)\| \tag{4·7}$$

for all admissible paths $\eta \in \mathcal{S}$ and σ. Repeat this procedure to construct $\nu_j, \rho_{i_j}, j = 1, 2, \ldots$ and then let x be the element of $\operatorname{supp} \mu$ with symbolic representation

$$[x] = (\nu_1, \rho_{i_1}, \nu_2, \rho_{i_2}, \ldots).$$

It only remains to verify that $\underline{\dim}_{\text{loc}} \mu(x) \leq d - \varepsilon/2$. Towards this, let $\mathcal{M}_n(x) = \mu(\Delta_n(x)) + \mu(\Delta_n^R) + \mu(\Delta_n^L)$. As was essentially observed in [8, theorem 2·6],

$$\underline{\dim}_{\text{loc}} \mu(x) = \liminf_n \frac{\log \mathcal{M}_n(x)}{n \log \lambda}.$$

If $n = \sum_{j=1}^J (\text{length}(\nu_j) + \ell^{(i_j)})$, then $\Delta_n(x) = (\nu_1, \rho_{i_1}, \nu_2, \rho_{i_2}, \ldots, \rho_{i_J})$. Thus (4·7) yields

$$\mathcal{M}_n(x) \geq \mu(\Delta_n(x)) \geq C \left\|T(\nu_1, \rho_{i_1}, \nu_2, \rho_{i_2}, \ldots, \rho_{i_J})\right\| \geq C c_{J-1} \left\|T(\rho_{i_J})\right\|$$

and so

$$\frac{\log \mathcal{M}_n(x)}{n \log \lambda} \leq \frac{\log C + \log c_{J-1} + \log \left\|T(\rho_{i_J})\right\|}{n \log \lambda}.$$

Recall that $\|T(\rho_i)\| \geq \lambda^{\ell^{(i)}(d-\varepsilon)}$, hence as $n \geq \ell^{(i_J)}$

$$\frac{\log \left\|T(\rho_{i_J})\right\|}{n \log \lambda} \leq (d - \varepsilon).$$

Furthermore, the choice of i_J ensures that

$$\left|\frac{\log c_{J-1}}{n\log\lambda}\right| \le \left|\frac{\log c_{J-1}}{\ell^{(i_J)}\log\lambda}\right| < \frac{\varepsilon}{2}.$$

Consequently, for large enough n of this form,

$$\frac{\log \mathcal{M}_n(x)}{n\log\lambda} \le \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + d - \varepsilon \le d - \frac{\varepsilon}{2}.$$

That proves $\underline{\dim}_{\text{loc}}\mu(x) \le d - \varepsilon/2$ as we claimed, contradicting the initial assumption of the proof that $d = \inf_x\{\underline{\dim}_{\text{loc}}\mu(x)\}$.

Remark 26. Although we know from Example 6 that this result is not true for all measures, it would be interesting to know if was true for all self-similar measures.

5. L^p*-improving results*

A measure μ on $[0, 1]^d$ is said to be L^p-improving if there is some $p > 2$ so that $\mu * f \in L^p$ whenever $f \in L^2$. An application of the open mapping theorem implies that in this case there is a constant C such that $\|\mu * f\|_p \le C\,\|f\|_2$ for all $f \in L^2$. The Hausdorff-Young inequality shows that any measure μ whose Fourier transform $\widehat{\mu} \in \ell^q$ for some $q < \infty$ is L^p-improving. The uniform Cantor measures on Cantor sets with ratios of dissection bounded away from zero are also L^p-improving (but their transforms need not tend to zero) [3]. Conversely, a point mass measure is not L^p-improving since it acts as an isometry on the L^p spaces.

It is known that if μ is L^p-improving, then the Hausdorff and energy dimensions of μ are positive [11]. It is natural to ask if a similar statement can be made about the lower Assouad dimension of μ. In this section, we will show that while it is true that $\inf_x\{\underline{\dim}_{\text{loc}}\mu(x)\} > 0$ for an L^p-improving measure, it is not necessary for $\underline{\dim}_{\text{qA}}\,\mu > 0$, or even for $\underline{\dim}_{\text{qA}}\,\text{supp}\,\mu > 0$.

PROPOSITION 27. *If* $\mu : L^2([0, 1]^d) \to L^p([0, 1]^d)$ *for* $p > 2$*, then* $\underline{\dim}_{\text{loc}}\mu(x) \ge d(1/2 - 1/p)$ *for every* $x \in \text{supp}\,\mu$*.*

Proof. Suppose this is not true, say $\underline{\dim}_{\text{loc}}\mu(x) = \varepsilon$ for some $\varepsilon < d(1/2 - 1/p)$. Then for any $\delta > 0$ there are $r_n \to 0$ such that $\mu(B(x, r_n)) \ge r_n^{\varepsilon+\delta}$. Let $f_n = 1_{B(x,2r_n)}$, so $\|f_n\|_2 \sim \sqrt{r_n^d}$. Note that if $z \in B(0, r_n)$ and $t \in B(x, r_n)$, then $z - t \in B(x, 2r_n)$ so $\mu * f_n(z) \ge \mu(B(x, r_n))$. Hence for some constants C_1, C_2, C_3 (independent of n) and all r_n,

$$C_1 r_n^{d/2} \ge C_2\,\|f_n\|_2 \ge \|\mu * f_n\|_p \ge \mu(B(x, r_n))m(B(0, r_n))^{1/p} \ge C_3 r_n^{\varepsilon+\delta} r_n^{d/p}.$$

But this is impossible as $\varepsilon + d/p + \delta < d/2$ for small $\delta > 0$.

We will give two examples to see this does not extend to the quasi-lower Assouad dimension.

Example 28. A set $E \subseteq [0, 1]$ with $\underline{\dim}_{\text{qA}}\,E = 0$ and a measure μ supported on E that is L^p-improving: In [18, theorem 2], Salem proved that the Fourier transform of the uniform Cantor measure supported on suitable random Cantor sets is almost surely in ℓ^p for some

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$p < \infty$. Such a measure is L^p-improving. We will show that we can construct a suitable Cantor set so that its quasi-lower Assouad dimension is zero.

We will follow the notation of Salem's paper. To begin, choose a rapidly growing sequence $\{n_j\}$ and put $m_j \sim n_j \log 3/\log\log n_j$. For $k \in \Lambda_j = \{n_j, \ldots, n_j + m_j\}$, $j = 1, 2, \ldots$, put $a_k = 1/\log k$, $b_k = 2/\log k$, and otherwise put $a_k = 1/3 + 1/\log k$, $b_k = 1/3 + 2/\log k$. The sequence $\{n_j\}$ should be sufficiently sparse that $n_{j+1} \gg n_j + m_j$ and $\prod_{k=1}^{n_j} a_k \geq 3^{-n_j}$. Now construct a random Cantor set with ratio of dissection at step k equal to ξ_k where ξ_k is chosen uniformly over the interval $[a_k, b_k]$. We have $b_k - a_k = 1/\log k$ and $(\log\log k)/k \to 0$. Furthermore, it is easy to see that $\liminf(a_1 \cdots a_n)^{1/n} > 0$. Consequently, it follows from [18] that if μ is the associated (random) uniform Cantor measure, then almost surely $\widehat{\mu} \in \ell^p$ for some $p < \infty$.

It only remains to check that for all such random Cantor sets E, we have $\underline{\dim}_{\mathrm{qA}} E = 0$. This is also easy to verify. Just take R to be the length of the Cantor intervals at step n_j in the construction, r the length of the Cantor intervals at step $n_j + m_j$ and x to be an endpoint of a step n_j interval. Then there is a $\delta > 0$ such that $r \leq R^{1+\delta}$. As well, $N_r(B(x, R)) = 2^{m_j}$, while $R/r \geq (\log n_j)^{m_j}$.

Example 29. An L^p-improving measure μ with $\underline{\dim}_{\mathrm{qA}}\, \mu = 0$ and $\underline{\dim}_{\mathrm{A}}\, \mathrm{supp}\, \mu > 0$: By modifying Salem's construction in [18, theorem 2] we can also give an example of an L^p-improving measure of quasi-lower Assouad dimension zero, whose support has positive lower dimension.

We will put $a_k = 1/4$, $b_k = 1/4 + 1/\log k$ and construct the random Cantor sets with ratio of dissection ξ_k at step k, as before. Certainly all such sets will have positive lower Assouad dimension. The random measure μ_ω will be the weak* limit of the measures

$$\mu_\omega^{(N)} = \prod_{k=1}^{N} \left(p_k \delta_0 + (1 - p_k)\delta_{\xi_1 \cdots \xi_{k-1}(1-\xi_k)}\right),$$

where $p_k = 1/j$ if $k = n_j + 1, \ldots, 2n_j$ and $p_k = 1/2$ otherwise. Again, $\{n_j\}$ will be a very rapidly growing sequence with $n_{j+1} \gg 2n_j$. Note that

$$\left|\widehat{\mu_\omega^{(N)}}(n)\right| \leq \prod_{\substack{k=1 \\ k \notin \{n_j+1, \ldots, 2n_j\}}}^{N} |\cos(\pi n \xi_1 \cdots \xi_{k-1}(1 - \xi_k)|.$$

Let $\varepsilon_s = 1/\pi \int_0^\pi |\cos x|^s\, dx$ and temporarily fix integer n. Take $N = N(n) = \lfloor \log|n| / \log 3 \rfloor$ so

$$|n|\, a_1 \cdots a_{N-1}/\log N = |n|\, 3^{N-1} \geq n^2.$$

By the same reasoning as in Salem's argument,

$$\int_0^1 \left|\widehat{\mu_\omega^{(N)}}(n)\right|^s d\omega \leq \prod_{\substack{k-1 \\ k \notin \{n_j+1, \ldots, 2n_j\}}}^{N} \left(1 + \frac{1}{k^2}\right) \varepsilon_s^{N - M_N},$$

where M_N is the number of indices from the sets $\{n_j + 1, \ldots, 2n_j\}$ that are at most N. If $\{n_j\}$ is sufficiently sparse and $N \in (n_J, n_{J+1}]$, then one can easily check that $N - M_N \geq N/6$, thus

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$$\int_0^1 \left|\widehat{\mu_\omega^{(N)}}(n)\right|^s d\omega \le C\varepsilon_s^{\log|n|/(6\log 3)}$$

for a universal constant C. Since $\varepsilon_s \to 0$, we can choose it so small that $\varepsilon_s^{\log|n|/(6\log 3)} \le n^{-2}$. For this choice of s,

$$\sum_{n=-\infty}^{\infty}\int_0^1 |\widehat{\mu_\omega}(n)|^s\, d\omega \le \sum_{n=-\infty}^{\infty}\int_0^1 \left|\widehat{\mu_\omega^{(N)}}(n)\right|^s d\omega \le C\sum_{n=-\infty}^{\infty} n^{-2} < \infty.$$

Consequently, the series $\sum_{n=-\infty}^{\infty}\int_0^1 |\widehat{\mu_\omega}(n)|^s\, d\omega$ converges and hence $\widehat{\mu_\omega} \in \ell^s$ for a.e. ω. Any such μ_ω is L^p-improving.

To see that $\underline{\dim}_{\mathrm{qA}}\,\mu = 0$, consider R the length of a Cantor interval of step $n_j + 1$, x its right hand endpoint and r the length of a Cantor interval of step $2n_j$. Then $R/r \ge 4^{n_j}$ while

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} = \left(1 - \frac{1}{j}\right)^{-n_j},$$

from which it follows that $\underline{\dim}_{\mathrm{qA}}\,\mu = 0$. Being L^p-improving, $\inf_x\{\underline{\dim}_{\mathrm{loc}}\mu(x)\} > 0$ and hence is not equal to $\underline{\dim}_{\mathrm{qA}}\,\mu$.

Remark 30. The fact that there are measures μ with zero quasi-lower Assouad dimension, but $\widehat{\mu} \in \ell^p$ for some $p < \infty$ is surprising in light of the general principle that one cannot have a measure small in both its time and frequency domains.

6. *The Assouad spectrum and quasi-Assouad dimensions of measures*

In [6] and [7], Fraser and Yu introduced the notion of the Assouad spectrum of a bounded set $E \subseteq \mathbb{R}^d$. These are the functions

$$\theta \mapsto \overline{\dim}_{\mathrm{A}}^{=\theta} E = \inf\left\{s : (\exists c)(\forall 0 < R \le 1)\ \sup_{x\in E} N_{R^{1/\theta}}(B(x,R)\cap E) \le c\left(R^{1-1/\theta}\right)^s\right\}$$

and

$$\theta \mapsto \underline{\dim}_{\mathrm{A}}^{=\theta} E = \sup\left\{s : (\exists c)(\forall 0 < R \le 1)\ \sup_{x\in E} N_{R^{1/\theta}}(B(x,R)\cap E) \ge c\left(R^{1-1/\theta}\right)^s\right\}$$

for $\theta \in (0,1)$, which differ from the previously considered Assouad dimensions by fixing the relationship of r and R. In this section, we study the corresponding notion for measures on fairly general metric spaces.

Definition 31. The **upper** and **lower Assouad spectrum** of the measure μ are the functions defined on $(0,1)$ by

$$\theta \mapsto \overline{\dim}_{\mathrm{A}}^{=\theta}\,\mu = \inf\left\{s : (\exists c)\ (\forall 0 < R \le 1)\ \sup_{x\in\operatorname{supp}\mu}\frac{\mu(B(x,R))}{\mu(B(x,R^{1/\theta}))} \le c\left(R^{1-1/\theta}\right)^s\right\}$$

and

$$\theta \mapsto \underline{\dim}_{\mathrm{A}}^{=\theta}\,\mu = \sup\left\{s : (\exists c)\ (\forall 0 < R \le 1)\ \inf_{x\in\operatorname{supp}\mu}\frac{\mu(B(x,R))}{\mu(B(x,R^{1/\theta}))} \ge c\left(R^{1-1/\theta}\right)^s\right\}.$$

This fixes the relationship of r and R as $r = R^{1/\theta}$. Another way to define the spectrum is by only requiring an upper bound, i.e. we set $r \leq R^{1/\theta}$. These "less than or equal" spectra will be denoted by $\overline{\dim}_{\mathrm{A}}^{\leq\theta}$ and $\underline{\dim}_{\mathrm{A}}^{\leq\theta}$. Note that we have already defined this notion when introducing the quasi-Assouad dimension and $\overline{\dim}_{\mathrm{A}}^{\leq\theta}\,\mu = \overline{H}(1/\theta - 1)$ and $\underline{\dim}_{\mathrm{A}}^{\leq\theta}\,\mu = \underline{H}(1/\theta - 1)$. Clearly, for $\psi \leq \theta$ we have

$$\overline{\dim}_{\mathrm{qA}}\,\mu \geq \overline{\dim}_{\mathrm{A}}^{\leq\theta}\,\mu \geq \overline{\dim}_{\mathrm{A}}^{=\psi}\,\mu,$$
$$\underline{\dim}_{\mathrm{qA}}\,\mu \leq \underline{\dim}_{\mathrm{A}}^{\leq\theta}\,\mu \leq \underline{\dim}_{\mathrm{A}}^{=\psi}\,\mu.$$

In [4] it was shown that $\overline{h}(1/\theta - 1) = \sup_{0<\psi\leq\theta} \overline{\dim}_{\mathrm{A}}^{=\psi}\,E$ for subsets of $\mathbb{R}^d$, although the same proof holds for any doubling metric space E, i.e. spaces where $\overline{\dim}_{\mathrm{A}}\,E < \infty$. Consequently,

$$\lim_{\theta\to 1} \overline{\dim}_{\mathrm{A}}^{=\theta}\,E = \limsup_{\theta\to 1} \overline{\dim}_{\mathrm{A}}^{=\theta}\,E = \overline{\dim}_{\mathrm{qA}}\,E.$$

The corresponding result was later proved for the quasi-lower Assouad dimension in [2] (with the additional assumption that the space E was uniformly perfect). It is straightforward to obtain the analogous result for doubling measures, that is measures μ for which $\overline{\dim}_{\mathrm{A}}\,\mu < \infty$. But this is a stringent condition for measures. However, it is possible to obtain the same conclusion for measures which only satisfy the weaker (quasi-doubling) condition, $\overline{\dim}_{\mathrm{qA}}\,\mu < \infty$ and this we do in Theorem 32 below. The general scheme of the proof is essentially the same as in [4], but new technical complications arise. Examples of such measures include equicontractive, self-similar measures that are regular, meaning the probabilities associated with the right and left-most similarities are equal and minimal. These measures are typically not doubling if they fail the open set condition. For a proof that such measures are quasi-doubling and specific examples of quasi-doubling, but not doubling, measures, we refer the reader to [10].

THEOREM 32. *Suppose μ is a probability measure and $\overline{\dim}_{\mathrm{qA}}\,\mu < \infty$. Let $\theta \in (0, 1)$.*
(i) *Then*

$$\overline{\dim}_{\mathrm{A}}^{\leq\theta}\,\mu = \sup_{0<\psi\leq\theta} \overline{\dim}_{\mathrm{A}}^{=\psi}\,\mu \quad \textit{and} \quad \underline{\dim}_{\mathrm{A}}^{\leq\theta}\,\mu = \inf_{0<\psi\leq\theta} \underline{\dim}_{\mathrm{A}}^{=\psi}\,\mu.$$

(ii) *Moreover,* $\lim_{\theta\to 1} \underline{\dim}_{\mathrm{A}}^{=\theta}\,\mu = \underline{\dim}_{\mathrm{qA}}\,\mu$ *and* $\lim_{\theta\to 1} \overline{\dim}_{\mathrm{A}}^{=\theta}\,\mu = \overline{\dim}_{\mathrm{qA}}\,\mu$.

We remark that, in particular, the quasi-Assouad dimensions of a doubling measure can be recovered from the limiting behaviour of the Assouad spectrum.

The proof will proceed as follows: We first prove an elementary technical result, followed by the proof of part (i) of the theorem. We will then show that for quasi-doubling measures, the maps $\theta \mapsto \underline{\dim}_{\mathrm{A}}^{=\theta}$ or $\overline{\dim}_{\mathrm{A}}^{=\theta}$ are continuous for all $\theta \in (0, 1)$. Lastly, this fact will be used in proving part (ii) of the theorem.

LEMMA 33. *Let $0 < \beta < \theta < 1$ and assume $\log\theta/\log\beta \notin \mathbb{Q}$. Let*

$$L = \{m\log\beta + n\log\theta : m, n \in \mathbb{N}\} = \{y_j\}_{j=1}^{\infty},$$

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where y_j is ordered decreasingly (to $-\infty$). Then $\lim_{j\to\infty} y_j - y_{j+1} = 0$. Furthermore, assume (θ_i) is a sequence tending to 0 with $\theta_i > 0$. Then, given small $\eta > 0$, there is an index i_0 such that for all $i \geq i_0$ there exist positive integers m, n such that

$$\frac{\eta}{2\theta_i} \leq \frac{1}{\theta_i} - \frac{1}{\theta^n \beta^m} \leq \frac{4\eta}{\theta_i}. \tag{6·1}$$

Proof. The fact that $y_j - y_{j+1} \to 0$ is well known, so we will only prove the second statement.

Fix small $0 < \eta < 1$ and choose J such that $|y_j - y_{j+1}| < \eta$ for all $j \geq J$. Choose i_0 such that $\log 1/\theta_i \geq |y_J| + 1$ for all $i \geq i_0$. Temporarily fix such an i and choose the maximal index k such that $\log 1/\theta_i > |y_k| + \eta$. Note that $k \geq J$. As k is maximal, it must be that either $|y_{k+1}| \geq \log 1/\theta_i$ or $0 < \log 1/\theta_i - |y_{k+1}| < \eta$. In either case, the fact that $|y_k - y_{k+1}| < \eta$ ensures that

$$\eta < \log 1/\theta_i - |y_k| < 2\eta.$$

It is now a routine calculation to see that if $y_k = m \log \beta + n \log \theta$, then $e^{\eta} < \beta^m \theta^n / \theta_i < e^{2\eta}$. Thus for all $i \geq i_0$, we have

$$\frac{\eta}{2\theta_i} \leq \frac{\eta e^{-2\eta}}{\theta_i} \leq \frac{\eta}{\beta^m \theta^n} < \frac{1}{\theta_i} - \frac{1}{\beta^m \theta^n} < \frac{4\eta}{\beta^m \theta^n} \leq \frac{4\eta}{\theta_i}.$$

Proof of Theorem 32. (i). First, consider the upper Assouad spectrum. There is no loss of generality in assuming $s = \overline{\dim}_{\mathrm{A}}^{\leq\theta} \mu > 0$ for otherwise $\overline{\dim}_{\mathrm{A}}^{=\psi} \mu = 0$ for all $\psi \leq \theta$ as well. Fix $0 < \varepsilon < s$ and obtain $x_i \in \operatorname{supp} \mu$, $R_i \to 0$ and $r_i = R_i^{1/\theta_i} \leq R_i$, with $\theta_i \leq \theta$ and

$$\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta_i}))} \geq \left(\frac{R_i}{R_i^{1/\theta_i}}\right)^{s-\varepsilon}.$$

Without loss of generality, we can assume $\theta_i \to \psi$ where $\psi \in [0, \theta]$ and that the convergence is monotonic.

Case 1. We will first assume $\psi > 0$. If (θ_i) is a decreasing sequence (so $R_i^{1/\theta_i} \geq R_i^{1/\psi}$), then we have

$$\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\psi}))} \geq \frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta_i}))} \geq \left(\frac{R_i}{R_i^{1/\theta_i}}\right)^{s-\varepsilon}.$$

As $1 - 1/\theta_i \to 1 - 1/\psi$, it follows that

$$\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\psi}))} \geq R_i^{(1-1/\psi)\left(\frac{1-\theta_i}{1-\psi}\right)(s-\varepsilon)} \geq R_i^{(1-1/\psi)(s-\varepsilon/2)}$$

if i is sufficiently large. That implies $\overline{\dim}_{\mathrm{A}}^{=\psi} \mu \geq s - \varepsilon/2$ and as $\varepsilon > 0$ was arbitrary we deduce that $\overline{\dim}_{\mathrm{A}}^{=\psi} \mu \geq s$. Thus $\sup_{0<\psi\leq\theta} \overline{\dim}_{\mathrm{A}}^{=\psi} \mu = s$.

Otherwise, we can assume (θ_i) increases to $\psi \leq \theta < 1$. Choose $n_i \in \mathbb{N}$ so that

$$2^{-(n_i+1)} < R_i^{1/\psi} \leq 2^{-n_i}$$

(where we choose a subsequence of $\{R_i\}$, if necessary, to ensure the sequence $\{n_i\}$ is strictly increasing) and define a function g on $\mathbb{N}$ by $g(n) = R_i^{1/\theta_i} 2^{n_i}$ if $n_i \leq n < n_{i+1}$. Then $\log R_i \sim n_i$ and

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$$\log 1/2 + \left(\frac{1}{\theta_i} - \frac{1}{\psi}\right) \log R_i \le \log g(n_i) \le \left(\frac{1}{\theta_i} - \frac{1}{\psi}\right) \log R_i .$$

Hence if $n_i \le n < n_{i+1}$,

$$\frac{|\log g(n)|}{n} \le \frac{|\log g(n_i)|}{n_i} \to 0 \text{ as } n \to \infty \text{ (equivalently, } i \to \infty).$$

As proven in [10, proposition 4·2], the assumption that $\overline{\dim}_{\mathrm{qA}}\,\mu < \infty$ ensures that for each $q > 1$ there is a constant c such that for all i,

$$\mu(B(x_i, R_i^{1/\theta_i})) = \mu(B(x_i, g(n_i)2^{-n_i})) \ge cq^{-n_i}\mu(B(x_i, 2^{-n_i})) \ge cq^{-n_i}\mu(B(x_i, R_i^{1/\psi})).$$

Thus

$$\begin{aligned}\left(\frac{R_i}{R_i^{1/\theta_i}}\right)^{s-\varepsilon} &\le \frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta_i}))} = \frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\psi}))}\frac{\mu(B(x_i, R_i^{1/\psi}))}{\mu(B(x_i, R_i^{1/\theta_i}))} \\ &\le \frac{q^{n_i}}{c}\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\psi}))}.\end{aligned}$$

As

$$\left(R_i^{1/\psi}\right)^{\log q/\log 2} \le \left(2^{-n_i}\right)^{\log q/\log 2} = q^{-n_i},$$

that shows

$$\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\psi}))} \ge cR_i^{(1-1/\theta_i)(s-\varepsilon)}q^{-n_i} \ge cR_i^{(1-1/\theta_i)(s-\varepsilon)}R_i^{\log q/(\psi \log 2)} = cR_i^{(1-1/\psi)t_i}$$

where

$$t_i = \left(\frac{1-1/\theta_i}{1-1/\psi}\right)(s-\varepsilon) + \frac{\log q}{(\psi - 1)\log 2}.$$

Since $\theta_i \to \psi \in (0, 1)$ as $i \to \infty$,

$$t_i \to s - \varepsilon - \frac{\log q}{(1-\psi)\log 2}.$$

As $q > 1$ and $\varepsilon > 0$ are arbitrary, we again deduce that $\overline{\dim}_{\mathrm{A}}^{=\psi} \ge s$ and that gives the desired result.

Case 2. Now suppose $\psi = 0$. We will make use of Lemma 33 and choose $\beta \in (0, \theta)$ such that $\log\theta/\log\beta$ is irrational. Suppose for a contraction that

$$\max\{\overline{\dim}_{\mathrm{A}}^{=\theta}, \overline{\dim}_{\mathrm{A}}^{=\beta}\} \le s - 3\varepsilon.$$

For all small enough R and $x \in \operatorname{supp}\mu$ we have for $\gamma = \theta, \beta$,

$$\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\gamma}))} \le \left(\frac{R_i}{R_i^{1/\gamma}}\right)^{s-2\varepsilon}. \tag{6·2}$$

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Fix $\eta > 0$ small, to be specified later. Choose $m, n \in \mathbb{N}$ as in (6·1) with this choice of η. By repeated application of (6·2) and a telescoping argument, we see that

$$\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta^n\beta^m}))} \le \left(\frac{R_i}{R_i^{1/\theta^n\beta^m}}\right)^{s-2\varepsilon}.$$

The second part of the claim yields that

$$\frac{1}{\theta_i}(1-\eta/2) \ge \frac{1}{\theta^n\beta^m}$$

for all i sufficiently large. Thus $R_i^{1/\theta_i} \le \left(R_i^{1/(\theta^n\beta^m)}\right)^{1/(1-\eta/2)}$. It follows that if $d = \overline{\dim}_{\mathrm{qA}}\,\mu$, $\varepsilon > 0$ and η is sufficiently small, there is a constant c such that

$$\frac{\mu(B(x_i, R_i^{1/\theta^n\beta^m}))}{\mu(B(x_i, R_i^{1/\theta_i}))} \le c\left(\frac{R_i^{1/\theta^n\beta^m}}{R_i^{1/\theta_i}}\right)^{d+\varepsilon}.$$

Thus

$$\begin{aligned}\left(\frac{R_i}{R_i^{1/\theta_i}}\right)^{s-\varepsilon} \le \frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta_i}))} &\le \frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta^n\beta^m}))}\,\frac{\mu(B(x_i, R_i^{1/\theta^n\beta^m}))}{\mu(B(x_i, R_i^{1/\theta_i}))}\\ &\le c\left(\frac{R_i}{R_i^{1/\theta^n\beta^m}}\right)^{s-2\varepsilon}\left(\frac{R_i^{1/\theta^n\beta^m}}{R_i^{1/\theta_i}}\right)^{d+\varepsilon},\end{aligned}$$

and this implies that if we put

$$\begin{aligned}t_i &= \left(\frac{1}{\theta_i}-1\right)(s-\varepsilon) + (1-\frac{1}{\theta^n\beta^m})(s-2\varepsilon) + \left(\frac{1}{\theta^n\beta^m}-\frac{1}{\theta_i}\right)(d+\varepsilon)\\ &= -\varepsilon + (s-\varepsilon)\left(\frac{1}{\theta_i}-\frac{1}{\theta^n\beta^m}\right) + \frac{\varepsilon}{\theta^n\beta^m} + \left(\frac{1}{\theta^n\beta^m}-\frac{1}{\theta_i}\right)(d+\varepsilon),\end{aligned}$$

then

$$cR_i^{t_i} \ge 1 \text{ for all large } i. \tag{6·3}$$

Using the bounds from (6·1) we deduce that for small enough η,

$$\begin{aligned}t_i &\ge -\varepsilon + (s-\varepsilon)\eta/(2\theta_i) + \varepsilon(1-4\eta)/\theta_i - 4\eta(d+\varepsilon)/\theta_i\\ &= -\varepsilon + \frac{1}{\theta_i}((s-\varepsilon)\eta/2 + \varepsilon(1-4\eta) - 4\eta(d+\varepsilon)) \ge -\varepsilon + \varepsilon/(2\theta_i) \to \infty\end{aligned}$$

as $i \to \infty$. But that means $R_i^{t_i} \to 0$ and hence (6·3) cannot be satisfied for all large i (whatever the choice of constant c). This proves the result for the upper Assouad spectrum.

We now turn to the proof for the lower Assouad spectrum. Let $s = \underline{\dim}_{\mathrm{A}}^{\le\theta}\,\mu \le \overline{\dim}_{\mathrm{qA}}\,\mu < \infty$. Fix $\varepsilon > 0$ and obtain $x_i \in \operatorname{supp}\mu$, $R_i \to 0$ and $r_i = R_i^{1/\theta_i} \le R_i$, with $\theta_i \le \theta$ and

$$\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta_i}))} \le \left(\frac{R_i}{R_i^{1/\theta_i}}\right)^{s+\varepsilon}.$$

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As before, without loss of generality we can assume $\theta_i \to \psi$ where $\psi \in [0, \theta]$ and that the convergence is monotonic.

Case 1. We will first assume $\psi > 0$. If (θ_i) is an increasing sequence (so $R_i^{1/\theta_i} \leq R_i^{1/\psi}$), then, similar to the first step in the upper Assouad spectrum argument, we have

$$\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\psi}))} \leq \frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta_i}))} \leq \left(\frac{R_i}{R_i^{1/\theta_i}}\right)^{s+\varepsilon} \leq \left(\frac{R_i}{R_i^{1/\psi}}\right)^{s+2\varepsilon}$$

for large enough i. That implies $\underline{\dim}_{\mathrm{A}}^{=\psi} \mu \leq s + 2\varepsilon$ and hence $\inf_{0<\psi\leq\theta} \underline{\dim}_{\mathrm{A}}^{=\psi} \mu = s$.

Now suppose (θ_i) decreases to ψ. As each $\theta_i \leq \theta < 1$, the same is true for ψ and $R_1^{1/\theta_i} \geq R_i^{1/\psi}$. In a similar fashion to the second step in case 1 above, we choose n_i so that $2^{-(n_i+1)} < R_i^{1/\theta_i} \leq 2^{-n_i}$ and define g by $g(n) = R_i^{1/\psi} 2^{n_i}$ if $n_i \leq n < n_{i+1}$. As before, one can easily check that $\log g(n)/n \to 0$, hence the fact that $\overline{\dim}_{\mathrm{qA}} \mu < \infty$ implies that for any fixed $q > 1$ and suitable constant c we have

$$\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\psi}))} \leq cq^{n_i} \left(\frac{R_i}{R_i^{1/\theta_i}}\right)^{s+\varepsilon} \leq c R_i^{-\log q/(\theta_i \log 2)} R_i^{(1-1/\theta_i)(s+\varepsilon)} = R_i^{(1-1/\psi)t_i}$$

for

$$t_i = \frac{-\log q}{\theta_i \log 2(1-1/\psi)} + \frac{(1-1/\theta_i)(s+\varepsilon)}{1-1/\psi} \to \frac{\log q}{\log 2(1-\psi)} + s + \varepsilon.$$

Since $q > 1$ and $\varepsilon > 0$ are arbitrary, we deduce that $\inf_{0<\psi\leq\theta} \underline{\dim}_{\mathrm{A}}^{=\psi} \mu = s$.

Case 2. Now suppose $\psi = 0$. Choose $0 < \beta < \theta < 1$ with $\log\theta/\log\beta \notin \mathbb{Q}$ and suppose for a contradiction that

$$\min\{\underline{\dim}_{\mathrm{A}}^{=\theta} \mu, \underline{\dim}_{\mathrm{A}}^{=\beta} \mu\} \geq s + 3\varepsilon.$$

Then for all small enough R, $x \in \operatorname{supp} \mu$ and $\gamma = \theta, \beta$ we have

$$\frac{\mu(B(x, R))}{\mu(B(x, R^{1/\gamma}))} \geq R^{(1-1/\gamma)(s+2\varepsilon)}$$

Fix $\eta > 0$ small, to be specified later and choose $m, n \in \mathbb{N}$ as in (6·1) with this choice of η. A telescoping argument gives

$$\frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta^n\beta^m}))} \geq \left(\frac{R_i}{R_i^{1/\theta^n\beta^m}}\right)^{s+2\varepsilon}.$$

Since $1/\theta_i \geq 1/(\theta^n\beta^m)$, $R^{1/\theta_i} \leq R_i^{1/\theta^n\beta^m}$ and therefore

$$\left(\frac{R_i}{R_i^{1/\theta_i}}\right)^{s+\varepsilon} \geq \frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta_i}))} \geq \frac{\mu(B(x_i, R_i))}{\mu(B(x_i, R_i^{1/\theta^n\beta^m}))} \geq \left(\frac{R_i}{R_i^{1/\theta^n\beta^m}}\right)^{s+2\varepsilon}.$$

Equivalently, $1 \geq cR_i^{t_i}$ for all large i where

$$t_i = \left(\frac{1}{\theta_i} - 1\right)(s+\varepsilon) + \left(1 - \frac{1}{\theta^n\beta^m}\right)(s+2\varepsilon).$$

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But the properties of θ and β ensure that for small enough η, $t_i \le \varepsilon - \varepsilon/(2\theta_i) \to -\infty$ and that's a contradiction.

This completes the proof of the lower Assouad spectrum result and thus part (i).

LEMMA 34. *Assume* $\overline{\dim}_{\mathrm{qA}}\,\mu < \infty$. *Then for each* $\theta \in (0, 1)$ *the functions* $\theta \mapsto \underline{\dim}_{\mathrm{A}}^{=\theta}\,\mu$ *and* $\theta \mapsto \overline{\dim}_{\mathrm{A}}^{=\theta}\,\mu$ *are continuous.*

Proof. We will give the complete proof for the continuity of the lower Assouad spectrum and leave the analogous proof of the continuity of the upper Assouad spectrum to the reader.

Fix $\theta \in (0, 1)$ and let $t = \underline{\dim}_{\mathrm{A}}^{=\theta}\,\mu$. We proceed by contradiction. If $\theta \mapsto \underline{\dim}_{\mathrm{A}}^{=\theta}\,\mu$ is not continuous at θ, then there is some $\varepsilon > 0$ and a sequence $\theta_j \to \theta$ such that

$$|\,\underline{\dim}_{\mathrm{A}}^{=\theta}\,\mu - \underline{\dim}_{\mathrm{A}}^{=\theta_j}\,\mu| \ge 3\varepsilon \text{ for all } j.$$

Suppose that there is a subsequence such that $\theta_j \to \theta$ and $\underline{\dim}_{\mathrm{A}}^{=\theta_j}\,\mu \ge t + 3\varepsilon$ for all j. Then for each j there is some $R(j) > 0$ such that for all $R \le R(j)$ and for each $x \in \operatorname{supp}\mu$ we have

$$\frac{\mu(B(x, R))}{\mu(B(x, R^{1/\theta_j}))} \ge R^{(1-1/\theta_j)(t+2\varepsilon)}. \tag{6·4}$$

If a further subsequence satisfies $\theta_j \ge \theta$ for all j, then fix small $\delta > 0$ and choose j such that $\left|1 - 1/\theta_j\right| \ge (1-\delta)\,|1 - 1/\theta|$. Since $R^{1/\theta_j} \ge R^{1/\theta}$, we have

$$\frac{\mu(B(x, R))}{\mu(B(x, R^{1/\theta}))} \ge \frac{\mu(B(x, R))}{\mu(B(x, R^{1/\theta_j}))} \ge R^{(1-1/\theta_j)(t+2\varepsilon)} \ge R^{(1-1/\theta)(t+2\varepsilon)(1-\delta)} \ge R^{(1-1/\theta)(t+\varepsilon)},$$

for all x and $R \le R(j)$, provided we choose δ small enough, and that contradicts the assumption that $t = \underline{\dim}_{\mathrm{A}}^{=\theta}\,\mu$.

So assume that $\theta_j \le \theta$ for all j. Since $t = \underline{\dim}_{\mathrm{A}}^{=\theta}\,\mu$, we can choose a sequence x_j and $R_j \le R(j)$, $R_j \to 0$, such that

$$\frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta}))} \le R_j^{(1-1/\theta)(t+\varepsilon/4)}. \tag{6·5}$$

By passing to a further subsequence, if necessary, we can assume there is a sequence of integers, (n_j), strictly increasing to infinity, such that

$$2^{-(n_j+1)} < R_j^{1/\theta} \le 2^{-n_j}.$$

Define

$$g(n) = R_j^{1/\theta_j} 2^{n_j} \text{ if } n \in [n_j, n_{j+1}).$$

As in the proof of the first part of theorem, $\log g(n)/n \to 0$, thus by the quasi-doubling property of μ, (the assumption that $\overline{\dim}_{\mathrm{qA}}\,\mu < \infty$) for each $q > 1$ there is a constant C_q such that

$$\begin{aligned}\mu(B(x_j, R_j^{1/\theta})) &\le \mu(B(x_j, 2^{-n_j})) \le C_q q^{n_j}\mu(B(x_j, g(n_j)2^{-n_j})) = C_q q^{n_j}\mu(B(x_j, R_j^{1/\theta_j}))\\ &\le C_q R_j^{-\frac{\log q}{\theta \log 2}}\mu(B(x_j, R_j^{1/\theta_j})).\end{aligned}$$

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Choose $q > 1$ such that $\log q/(\theta \log 2) \leq (1/\theta - 1)\varepsilon/4$. With this fixed choice of q we see that

$$\mu(B(x_j, R_j^{1/\theta})) \leq C_q R_j^{(1-1/\theta)\varepsilon/4} \mu(B(x_j, R_j^{1/\theta_j})).$$

Together with (6·5) we deduce that

$$R_j^{(1-1/\theta)(t+\varepsilon/4)} \geq \frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta}))} = \frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta_j}))} \frac{\mu(B(x_j, R_j^{1/\theta_j}))}{\mu(B(x_j, R_j^{1/\theta}))} \qquad (6\cdot6)$$

$$\geq \frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta_j}))} C_q^{-1} R_j^{-(1-1/\theta)\varepsilon/4}.$$

Now choose j_1 such that for all $j \geq j_1$,

$$\frac{1 - 1/\theta_j}{1 - 1/\theta}(t + 2\varepsilon) \geq t + \varepsilon.$$

Combining (6·4) and (6·6) gives

$$R_j^{(1-1/\theta)(t+\varepsilon)} \leq R_j^{(1-1/\theta_j)(t+2\varepsilon)} \leq \frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta_j}))} \leq C_q R_j^{(1-1/\theta)(t+\varepsilon/2)},$$

for all $j \geq j_1$. But since C_q is fixed, the outer most inequalities clearly cannot hold for all $R_j \to 0$. This contradiction shows that we cannot have a sequence $\theta_j \to \theta$ such that $\underline{\dim}_{\mathrm{A}}^{=\theta_j} \mu \geq t + 3\varepsilon$ for all j.

Otherwise, there must be a subsequence such that $\theta_j \to \theta$ and $\underline{\dim}_{\mathrm{A}}^{=\theta_j} \mu \leq t - 3\varepsilon$ for all j. In this case, there must be $x_j \in \operatorname{supp} \mu$ and a decreasing sequence $R_j \to 0$ such that

$$\frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta_j}))} \leq R_j^{(1-1/\theta_j)(t-2\varepsilon)}, \qquad (6\cdot7)$$

for all j.

If $\theta_j \leq \theta$, then as $R_j^{1/\theta_j} \leq R_j^{1/\theta}$,

$$\frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta}))} \leq \frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta_j}))} \leq R_j^{(1-1/\theta_j)(t-2\varepsilon)} \leq R_j^{(1-1/\theta)(t-\varepsilon)},$$

for j sufficiently large (R_j small) and that contradicts the assumption that $t = \underline{\dim}_{\mathrm{A}}^{=\theta} \mu$.

So we may assume $\theta_j \geq \theta$. The arguments are similar to the case $\theta_j \leq \theta$ above. Without loss of generality, there is a strictly increasing sequence (n_j) satisfying

$$2^{-(n_j+1)} < R_j^{1/\theta_j} \leq 2^{-n_j}.$$

Put

$$g(n) = R_j^{1/\theta} 2^{n_j} \text{ if } n \in [n_j, n_{j+1}).$$

The quasi-doubling property of μ ensures that for each $q > 1$ there is a constant $c_q > 0$ such that

$$\mu(B(x_j, R_j^{1/\theta})) = \mu(B(x_j, g(n_j)2^{-n_j})) \geq c_q q^{-n_j} \mu(B(x_j, 2^{-n_j})) \geq c_q q^{-n_j} \mu(B(x_j, R_j^{1/\theta_j})) \geq c_q R_j^{\frac{\log q}{\theta_j \log 2}} \mu(B(x_j, R_j^{1/\theta_j})).$$

Hence from (6·7),

$$R_j^{(1-1/\theta_j)(t-2\varepsilon)} \geq \frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta_j}))} = \frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta}))} \frac{\mu(B(x_j, R_j^{1/\theta}))}{\mu(B(x_j, R_j^{1/\theta_j}))} \geq c_q R_j^{\frac{\log q}{\theta_j \log 2}} \frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta}))},$$

so that

$$\frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta}))} \leq c_q^{-1} R_j^{(1-1/\theta_j)(t-2\varepsilon)} R_j^{\frac{-\log q}{\theta_j \log 2}} = c_q^{-1} R_j^{(1-1/\theta)\left((t-2\varepsilon)\left(\frac{1-1/\theta_j}{1-1/\theta}\right) - \frac{\log q}{\theta_j (1-1/\theta)\log 2}\right)}.$$

Choose j_2 such that for all $j \geq j_2$ we have $(t-2\varepsilon)\left(1-1/\theta_j\right)/(1-1/\theta) \leq t - \varepsilon$ and $\left|\theta_j(1-1/\theta)\right| \geq |\theta - 1|/2$ and then choose q sufficiently close to 1 so that $2\log q/(|\theta-1|\log 2) \leq \varepsilon/2$. We conclude that for $j \geq j_2$,

$$\frac{\mu(B(x_j, R_j))}{\mu(B(x_j, R_j^{1/\theta}))} \leq c_q^{-1} R_j^{(1-1/\theta)(t-\varepsilon/2)}$$

and, again, this contradicts the assumption that $t = \underline{\dim}_{\mathrm{A}}^{=\theta} \mu$.

That completes the proof for the continuity of the lower Assouad spectrum.

We are now ready to complete the proof of the theorem.

Proof of Theorem 32. (ii). It follows directly from the first part of the theorem that

$$\liminf_{\theta \to 1} \underline{\dim}_{\mathrm{A}}^{=\theta} \mu = \underline{\dim}_{\mathrm{qA}} \mu \text{ and } \limsup_{\theta \to 1} \overline{\dim}_{\mathrm{A}}^{=\theta} \mu = \overline{\dim}_{\mathrm{qA}} \mu.$$

Furthermore, an immediate consequence of the factorization

$$\frac{\mu(B(x, R))}{\mu(B(x, R^{1/\theta}))} = \frac{\mu(B(x, R))}{\mu(B(x, R^{1/\theta^{1/n}}))} \frac{\mu(B(x, R^{1/\theta^{1/n}}))}{\mu(B(x, R^{1/\theta^{2/n}}))} \cdots \frac{\mu(B(x, R^{1/\theta^{(n-1)/n}}))}{\mu(B(x, R^{1/\theta}))}$$

is that $\underline{\dim}_A^{=\theta^{1/n}} \mu \leq \underline{\dim}_{\mathrm{A}}^{=\theta} \mu$ and $\overline{\dim}_A^{=\theta^{1/n}} \mu \geq \overline{\dim}_{\mathrm{A}}^{=\theta} \mu$ for all $n \in \mathbb{N}$.

These observations, together with the continuity result of the previous lemma, allow one to use the same argument as given directly after in [**4**, lemma 3·1] to show that $\liminf_{\theta \to 1} \underline{\dim}_{\mathrm{A}}^{=\theta} \mu = \lim_{\theta \to 1} \underline{\dim}_{\mathrm{A}}^{=\theta} \mu$ and similarly for the upper Assouad spectrum.

That completes the proof of Theorem 32.

We can use Theorem 32 to state an analogue of the notion of uniformly perfect for the quasi-lower Assouad dimension.

COROLLARY 35. *Suppose μ is doubling. If there exists $t > 0$ such that for every $\delta < 0$, all $x \in \operatorname{supp} \mu$ and all sufficiently small R,*

$$\mu(B(x, R) \setminus B(x, R^{1+\delta})) \geq (1 - R^{\delta t})\mu(B(x, R)),$$

then $\underline{\dim}_{\mathrm{qA}} \mu \geq t$.

Proof. The hypothesis of the corollary is equivalent to the statement

$$\frac{\mu(B(x, R))}{\mu(B(x, R^{1+\delta}))} \geq R^{-\delta t}.$$

Thus $\overline{H}(1/\delta - 1) \geq t$ for all $\delta > 0$ and hence $\underline{\dim}_{\mathrm{A}}^{\leq\theta} \mu \geq t$ for all $\theta > 0$. As μ is doubling, $\underline{\dim}_{\mathrm{qA}} \mu = \inf_{\theta > 0} \underline{\dim}_{\mathrm{A}}^{\leq\theta} \mu \geq t$.

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Appendix A. Lower Spectrum for Sets

In [2] it was shown that $\underline{\dim}_{\mathrm{A}}^{\leq\theta} := \underline{h}(1/\theta - 1) = \inf_{0 \leq \psi \leq \theta} \underline{\dim}_{\mathrm{A}}^{=\psi}$ under the assumption that the metric space is doubling and uniformly perfect. In this section we will give a shorter proof of this fact that does not require the uniformly perfect assumption.

To begin, we note that if a metric space X is doubling, then there is a constant c such that for all x, r, R and subsets $E \subseteq X$,

$$N_r(B(x, R) \cap E) \leq N_{16r}(B(x, R) \cap E) \sup_y N_r(B(y, 16r)) \leq cN_{16r}(B(x, r) \cap E).$$

For any subset F, let $M_r(F)$ be the maximal number of disjoint balls of radius r, centred in F. Since we have

$$N_{16r}(F) \leq M_{4r}(F) \leq N_{4r}(F) \leq M_r(F) \leq N_r(F) \leq cN_{16r}(F),$$

we can replace the covering numbers in the definition of the Assouad spectrum and dimensions with packing numbers.

We will also require the following observation:

LEMMA 36. *For $0 < r = r_k < r_{k-1} < \cdots < r_1 < R$,*

$$M_r(B(x, R)) \geq M_{r_1}(B(x, R - r_1)) \inf_{y_1} M_{r_2}(B(y_1, r_1 - r_2)) \cdots \inf_{y_k} M_r(B(y_1, r_{k-1} - r)).$$

Proof. Let $t_1 = \inf_{y_1} M_{r_2}(B(y_1, r_1 - r_2))$ and suppose $\{B(x_j, r_1) : j = 1, \ldots, J\}$ is a set of disjoint balls with centres in $B(x, R - r_1)$ (and thus contained in $B(x, R)$). There are at least t_1 disjoint balls centred in each $B(x_j, r_1 - r_2)$ with radius r_2. These balls are each contained in the (disjoint) sets $B(x_j, r_1)$, so that all $J \cdot t$ balls are disjoint. Hence if $k = 2$ ($r_2 = r$), then we have produced $J \cdot t_1$ disjoint balls of radius r centred in $B(x, R)$ and that proves the result in this case. If $k > 2$ we repeat the construction.

THEOREM 37. *Let E be a doubling metric space. Then for any $\theta \in (0, 1)$,*

$$\underline{\dim}_{\mathrm{A}}^{\leq\theta} E = \inf_{0<\psi\leq\theta} \underline{\dim}_{\mathrm{A}}^{=\psi} E.$$

Further,

$$\underline{\dim}_{\mathrm{qA}} E = \lim_{\theta\to 1} \underline{\dim}_{\mathrm{A}}^{=\psi} E. \tag{A·1}$$

Remark 38. The analogous result was proved for the Assouad dimension in [4] for subsets of $\mathbb{R}^d$, but the same proof applies in any doubling metric space.

Proof. Our argument is similar to the proof of the corresponding result for the lower spectrum of measures. Let $s = \underline{\dim}_{\mathrm{A}}^{\leq\theta} E$. This dimension is finite since the metric space E is doubling and hence its upper Assouad dimension is finite. Fix $\varepsilon > 0$ and obtain $x_i \in E$, $R_i \to 0$ and $\theta_i \leq \theta$ such that

$$M_{R^{1/\theta_i}}(B(x_i, R_i)) \leq R_i^{(1-1/\theta_i)(s+\varepsilon)}. \tag{A·2}$$

Without loss of generality, we can assume $\theta_i \to \psi$ where $\psi \in [0, \theta]$ and that the convergence is monotonic.

Case 1. We will first assume $\psi > 0$. If (θ_i) is an increasing sequence, then $R^{1/\theta_i} \leq R^{1/\psi}$ and thus

$$N_{R^{1/\psi}}(B(x_i, R_i)) \leq N_{R^{1/\theta_i}}(B(x_i, R_i)) \leq R_i^{(1-1/\theta_i)(s+\varepsilon)} \leq R_i^{(1-1/\psi)(s+\varepsilon/2)} \text{ for large } i.$$

Otherwise, (θ_i) decreases to ψ. As each $\theta_i \leq \theta < 1$, the same is true for ψ and furthermore, $R_1^{1/\theta_i} \geq R_i^{1/\psi}$. Let D be the upper Assouad dimension of E. For small enough R_i we have

$$N_{R^{1/\psi}}(B(x_i, R_i^{1/\theta_i})) \leq R_i^{(1/\theta_i-1/\psi)(D+\varepsilon)},$$

hence

$$\begin{aligned} N_{R^{1/\psi}}(B(x_i, R_i)) &\leq N_{R^{1/\theta_i}}(B(x_i, R_i)) N_{R^{1/\psi}}(B(x_i, R_i^{1/\theta_i})) \\ &\leq R_i^{(1-1/\theta_i)(s+\varepsilon)+(1/\theta_i-1/\psi)(D+\varepsilon)}. \end{aligned}$$

Since $\theta_i \to \psi$, and $\varepsilon > 0$ was arbitrary, we again deduce that $\overline{\dim}_{\mathrm{A}}^{=\psi} E = s$.

Case 2. $\psi = 0$. As in the proof of Theorem 32, choose $0 < \beta < \theta < 1$ with $\log\theta/\log\beta \notin \mathbb{Q}$ and suppose for a contradiction that

$$\min\{\underline{\dim}_{\mathrm{A}}^{=\theta}(E), \underline{\dim}_{\mathrm{A}}^{=\beta}(E)\} \geq s + 3\varepsilon.$$

This inequality implies that for all x, small enough R and $\gamma = \theta, \beta$,

$$M_{R^{1/\gamma}}(B(x, R)) \geq R^{(1-1/\gamma)(s+5\varepsilon/2)} \tag{A·3}$$

Let $\eta > 0$ and choose n, m as in Claim 33. Appealing to Lemma 36 we see that

$$M_{R_i^{1/\theta^n\beta^m}}(B(x_i, R_i)) \geq M_{R_i^{1/\theta}}(B(x_i, R_i - R_i^{1/\theta}) \cdot A_{i,n} \cdot B_{i,m}, \tag{A·4}$$

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where

$$A_{i,n} = \prod_{k=1}^{n-1} \inf_{y_k} M_{R_i^{1/\theta^{k+1}}}(B(y_k, R_i^{1/\theta^k} - R_i^{1/\theta^{k+1}})$$

and

$$B_{i,m} = \prod_{k=1}^{m} \inf_{z_k} M_{R_i^{1/\theta^n \beta^k}}(B(z_k, R_i^{1/\theta^n \beta^{k-1}} - R_i^{1/\theta^n \beta^k}).$$

The doubling condition combined with property (A·3) implies that there is a constant $c > 0$ such that whenever $1 \le \alpha \le 2$, $\gamma = \theta$ or β and R is small enough (depending only on ε, γ and c), then

$$M_{\alpha R^{1/\gamma}}(B(x, R)) \ge c M_{R^{1/\gamma}}(B(x, R)) \ge c R^{(1-1/\gamma)(s+5\varepsilon/2)} \ge \left(\frac{R}{R^{1/\gamma}}\right)^{s+2\varepsilon}.$$

Since

$$R_i^{1/\theta^{j-1}} - R_i^{1/\theta^j} \le R_i^{1/\theta^j} \le 2(R_i^{1/\theta^{j-1}} - R_i^{1/\theta^j})^{1/\theta}$$

and

$$R_i^{1/\theta^n \beta^{j-1}} - R_i^{1/\theta^n \beta^j} \le R_i^{1/\theta^n \beta^j} \le 2(R_i^{1/\theta^n \beta^{j-1}} - R_i^{1/\theta^n \beta^j})^{1/\beta},$$

it follows that if we simplify the notation by putting

$$P_{i,j} = R_i^{1/\theta^{j-1}} - R_i^{1/\theta^j} \text{ and } Q_{i,j} = R_i^{1/\theta^n \beta^{j-1}} - R_i^{1/\theta^n \beta^j},$$

then

$$M_{R_i^{1/\theta^n \beta^m}}(B(x_i, R_i)) \ge \left(\prod_{j=1}^{n} \frac{P_{i,j}}{P_{i,j}^{1/\theta}} \prod_{j=1}^{m} \frac{Q_{i,j}}{Q_{i,j}^{1/\beta}}\right)^{s+2\varepsilon}.$$

It is helpful to isolate the first term of the numerator together with the last term of the denominator and then pair up the remaining terms giving the expression

$$M_{R_i^{1/\theta^n \beta^m}}(B(x_i, R_i))$$
$$\ge \left(\frac{R_i - R_i^{1/\theta}}{(R_i^{1/\theta^n \beta^{m-1}} - R_i^{1/\theta^n \beta^m})^{1/\beta}} \prod_{j=2}^{n} \frac{P_{i,j}}{P_{i,j-1}^{1/\theta}} \frac{R_i^{1/\theta^n} - R_i^{1/\theta^n \beta}}{(R_i^{1/\theta^{n-1}} - R_i^{1/\theta^n})^{1/\theta}} \prod_{j=2}^{m} \frac{Q_{i,j}}{Q_{i,j-1}^{1/\beta}}\right)^{s+2\varepsilon}.$$

Using a Taylor series expansion for $(1-x)^{1/\theta}$ for x near 0, one can check that

$$\frac{P_{i,j}}{P_{i,j-1}^{1/\theta}}, \ \frac{Q_{i,j}}{Q_{i,j-1}^{1/\beta}} \text{ and } \frac{R_i^{1/\theta^n} - R_i^{1/\theta^n \beta}}{(R_i^{1/\theta^{n-1}} - R_i^{1/\theta^n})^{1/\theta}} \ge 1.$$

Hence we deduce that

$$M_{R_i^{1/\theta^n \beta^m}}(B(x_i, R_i)) \ge \left(\frac{R_i - R_i^{1/\theta}}{(R_i^{1/\theta^n \beta^{m-1}} - R_i^{1/\theta^n \beta^m})^{1/\beta}}\right)^{s+2\varepsilon} \ge \left(\frac{R_i}{2R_i^{1/\theta^n \beta^m}}\right)^{s+2\varepsilon} \quad (A·5)$$

once R_i is small enough. But since $R_i^{1/\theta_i} \le R_i^{1/\theta^n \beta^m}$ we have

$$M_{R_i^{1/\theta_i}}(B(x_i, R_i)) \ge M_{R_i^{1/\theta^n \beta^m}}(B(x_i, R_i)).$$

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Combining these observations with (A·2) gives the inequality

$$R_i^{(1-1/\theta_i)(s+\varepsilon)} \geq c R_i^{(1-1/\theta^n \beta^m)(s+2\varepsilon)}.$$

But as we saw in the conclusion of the proof of Theorem 32, it is not possible for this inequality to be true for R_i tending to zero and that contradiction completes the first part of the proof.

To see that (A·1) holds, assume $\underline{\dim}_{\mathrm{A}}^{=\theta} = t$. Repeat the argument starting at (A·4) with $m = 0$ to obtain

$$M_{R^{1/\theta^n}}(B(x, R)) \geq c \left(\frac{R}{R^{1/\theta^n}} \right)^{t-\varepsilon}$$

as in (A·5), for all $x \in E$ and $R > 0$ small enough. This shows $\underline{\dim}_A^{=\theta^n} E \geq t - \varepsilon$ for all $\varepsilon > 0$ and so $\underline{\dim}_A^{=\theta^n} E \geq \underline{\dim}_{\mathrm{A}}^{=\theta} E$. According to [6, theorem 3·10] the function $\underline{\dim}_{\mathrm{A}}^{=\theta} E$ is continuous for $\theta \in (0, 1)$. Following the argument found in [4, section 3·2], $\lim_{\theta \to 1} \underline{\dim}_{\mathrm{A}}^{=\theta} E$ exists and hence $\lim_{\theta \to 1} \underline{\dim}_{\mathrm{A}}^{=\theta} E = \liminf_{\theta \to 1} \underline{\dim}_{\mathrm{A}}^{=\theta} E = \inf_{\theta \in (0,1)} \underline{\dim}_{\mathrm{A}}^{=\theta} E$.

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