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STOCHASTIC ORDERINGS OF DISCRETE-TIME PROCESSES AND DISCRETE RECORD VALUES

FÉLIX BELZUNCE
Departamento Estadística e Investigación Operativa
Universidad de Murcia, Campus de Espinardo
30100 Espinardo, Murcia, Spain
E-mail: belzunce@um.es

EVA-MARÍA ORTEGA
Departamento Estadística, Matemáticas e Informática
Centro de Investigación Operativa
Universidad Miguel Hernández, Campus de Orihuela
03312 Orihuela, Alicante, Spain
E-mail: evamaria@umh.es

JOSÉ M. RUIZ
Departamento Estadística e Investigación Operativa
Universidad de Murcia, Campus de Espinardo
30100 Espinardo, Murcia, Spain
E-mail: jmruizgo@um.es

Recently, Pellerey, Shaked, and Zinn [6] introduced a discrete-time analogue of the nonhomogeneous Poisson process. The purpose of this article is to provide some results for stochastic comparisons of the epoch times and the interepoch times of those processes. Also, we show the relationships between these processes and discrete record values and we provide several results for discrete weak record values.

1. INTRODUCTION

Recently, Pellerey, Shaked, and Zinn [6] introduced a discrete-time analogue of the nonhomogeneous Poisson process. They considered a discrete-time process, in which, at any time i, there is a jump of size one or there is no jump. Also, the probability of a jump depends only on the calendar time i.

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Formally, let $\{B_i\}_{i=1}^{\infty}$ be a sequence of independent Bernoulli random variables such that

$$\begin{aligned} P(B_i = 1) &= r_i, \\ P(B_i = 0) &= 1 - r_i, \end{aligned} \qquad i = 1,2,\ldots,$$

and define the counting process $Z(j) = \sum_{i=1}^{j} B_i$, $j = 1,2,\ldots$.

Let

$$S_n = \min\{j : Z(j) = n\}, \qquad n = 1,2,\ldots \tag{1}$$

($S_0 \equiv 0$); that is, S_n is the discrete-time of the nth jump.

This model corresponds to the following situation. Consider a device that executes sequentially a list of tasks, numbered $1,2,\ldots$. Associated with any task j, there is a probability, r_j, of failure of the device during the execution of the task. The tasks are executed sequentially until the failure at, say, task i_1 of the device. Upon failure, the device is replaced by a similar life device, which then proceeds to execute task $i_1 + 1$ until the next failure at task i_2. Again, the device is replaced by a similar life device, which then proceeds to execute task $i_2 + 1$ and the following tasks, and so forth. Then, the sequence $S_j = i_j$, $j = 1,2,\ldots$, corresponds to the one defined in Eq. (1). The *nonhomogenous Poisson process* (NHPP) is well known in reliability because the epoch times correspond to the times of repair of a unit, which is being repaired under a minimal repair policy, that is, when the unit fails, it is restored to a working condition just prior to the failure (see Ascher and Feingold [1]). By this reason, the previously discussed discrete process is also known as the *discrete-time minimal repair process*.

For the discrete-time analogue of the NHPP, Pellerey et al. [6] found conditions under which the epoch times and the interepoch intervals of these discrete-time processes have logconcave discrete probability mass functions.

The purpose of this article is to consider two of these processes and to find conditions under which the epoch times and the interepoch intervals can be stochastically compared in several senses. It is also important to notice, as we will see in Section 2, that the epoch times of a discrete-time minimal repair process correspond to the record values associated with a proper discrete distribution, and the results can be applied in this context too. We will show that similar results can be stated for "weak" record values in the discrete case. The organization of the article is as follows. In Section 2 we consider several distributional properties of the discrete-time minimal repair process, which will be used throughout the article, and the relationship with discrete record values. We will recall the definition of discrete "weak" record values. In Sections 3 and 4 we find conditions to compare the epoch times and the interepoch intervals of those processes. Similar results will be given for "weak" record values.

Throughout the article, for any discrete random variable X, we will assume that $P(X < n) < 1$ for all $n \in \mathbb{Z}_+$. Also, we will denote equality in law by "$=_{\text{st}}$."

2. PRELIMINARIES ON DISCRETE-TIME MINIMAL REPAIR MODELS

In this section we recall and derive some properties of the discrete-time minimal repair process that will be used throughout. First, we recall the distribution of the discrete epoch time S_n. We will assume that $\prod_{j=i}^{\infty}(1 - r_j) = 0$, $i = 1,2,\ldots$; this ensures (see Pellerey et al. [6]) that the probability mass function of S_1 is given by

$$p_1 \equiv P(S_1 = 1) = r_1,$$

$$p_i \equiv P(S_1 = i) = r_i \prod_{j=1}^{i-1} (1 - r_j), \qquad i = 2,3,\ldots.$$

We will denote the distribution function by P and the corresponding probability mass function by $\{p_i\}_{i=1}^{\infty}$. This distribution function will be called the *underlying distribution* of the discrete-time minimal repair process.

The relationship between the p_i's and the r_i's is the following:

$$r_i = \frac{p_i}{\sum_{j=i}^{\infty} p_j}, \qquad i = 1,2,\ldots;$$

therefore, the values r_i's can be interpreted as the hazard rate (in discrete time) of S_1 (see Salvia and Bollinger [7] and Shaked, Shanthikumar, and Valdez-Torres [9–11]). Finally, if we denote by

$$l_i = \frac{p_i}{\sum_{j=i+1}^{\infty} p_j}, \qquad i = 1,2,\ldots,$$

$$L_1(i) = \sum_{j=1}^{i} l_j, \qquad i \geq 1$$

and, by induction,

$$L_n(i) = \begin{cases} 0 & \text{if } i < n \\ \sum_{j=1}^{i} l_j L_{n-1}(j-1) & \text{if } i \geq n, \end{cases} \qquad n \geq 2,$$

then the probability mass function of S_n for all $n \geq 2$ is given by

$$p_n(i) = p_i L_{n-1}(i-1), \qquad i \geq n, n \geq 2.$$

Next, we derive a property that will be used throughout.

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LEMMA 2.1: *For a discrete-time minimal repair process with underlying distribution P and with epoch times S_i, $i \geq 1$,*

$$\{S_i | S_1 = x_1, \ldots, S_{i-1} = x_{i-1}\} =_{\text{st}} \{X | X > x_{i-1}\}, \qquad i \geq 2, \tag{2}$$

where X is a random variable with distribution function P.

PROOF: First, we observe the following chain of equalities:

$$\begin{aligned} \{X | X > x\} &=_{\text{st}} \{S_1 | S_1 > x\} \\ &= \min\{h : Z(h) = 1 | Z(x) = 0\} \\ &= \min\left\{h : \sum_{k=x+1}^{h} B_k = 1\right\}. \end{aligned}$$

Also, we have

$$\begin{aligned} \{S_i | S_1 = x_1, \ldots, S_{i-1} = x_{i-1}\} &= \min\{h : Z(h) = i | Z(x_{i-1}) = i-1\} \\ &= \min\left\{h : \sum_{k=x_{i-1}+1}^{h} B_k = 1\right\}. \end{aligned}$$

From the two previous equalities we get the result. ■

Now from Eq. (2), we get the joint probability mass function of the random vector $(S_1, \ldots, S_n)$ as

$$\begin{aligned} P(S_i = x_i, i = 1, \ldots, n) &= P(S_1 = x_1) \prod_{j=2}^{n} P(S_j = x_j \mid S_1 = x_1, \ldots, S_{j-1} = x_{j-1}) \\ &= p_{x_1} \prod_{j=2}^{n} \frac{p_{x_j}}{\sum_{k=j-1}^{\infty} p_{x_k+1}} \\ &= \prod_{j=1}^{n-1} l_{x_j} p_{x_n} \quad \text{for all } x_1 < \cdots < x_n \in \mathbb{Z}_+. \end{aligned} \tag{3}$$

In a similar way, the joint probability mass function of the random vector of interepoch intervals times $(U_1, \ldots, U_n)$, where $U_i \equiv S_i - S_{i-1}$ for $i = 1, 2 \ldots, n$ and $S_0 \equiv 0$, is given by

$$P(U_i = x_i, i = 1, \ldots, n) = \prod_{j=1}^{n-1} l_{\sum_{k=1}^{j} x_k} p_{\sum_{k=1}^{n} x_k} \quad \text{for all } x_i \in \mathbb{Z}_+, i = 1, \ldots, n.$$

On the other hand, Eq. (3) also indicates the relationship between the epoch times of a discrete-time minimal repair process and the record values associated with a discrete distribution.

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Formally, let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables, having the same distribution as the random variable X. We denote by

$$L(1) = 1,$$
$$L(n+1) = \min\{j > L(n) : X_j > X_{L(n)}\}$$

the *record times*, and the *record values* are defined as

$$\{R_{L(n)}\}_{n=1}^{\infty}.$$

If we suppose that X is a discrete random variable taking only nonnegative integer values, such that $P(X < n) < 1$, and denoting it by $X(n) \equiv R_{L(n)}$, then the joint distribution of the first n record values is given as follows (see Nevzorov [4] and Nevzorov and Balakrishnan [5, p. 528]):

$$P(X(1) = j_1, \ldots, X(n) = j_n) = P(X = j_n) \prod_{r=1}^{n-1} \frac{P(X = j_r)}{P(X > j_r)}$$
$$\text{for all } j_1 < \cdots < j_n \in \mathbb{Z}_+.$$

Therefore, they are more equally distributed than the first n epoch times of a discrete-time minimal repair process, where $r_j = P(X = j)/P(X \geq j)$.

In the context of record values, a repetition of a record value can be considered a new record, and this makes sense when the underlying distributions are discontinuous. This leads to the notion of *weak records* (see Vervaat [14] and Stepanov [12,13]).

Formally, let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, having the same distribution as the random variable X. We denote by

$$L_w(1) = 1,$$
$$L_w(n+1) = \min\{j > L_w(n) : X_j \geq \max\{X_1, X_2, \ldots, X_{j-1}\}\}$$

the *weak record times*, and the *weak record values* are defined as

$$\{R_{L_w(n)}\}_{n=1}^{\infty}.$$

For weak record values and denoting it by $X_w(n) \equiv R_{L_w(n)}$, then the joint distribution of weak record values is given as follows (see Nevzorov and Balakrishnan [5, p. 529]):

$$P(X_w(1) = j_1, \ldots, X_w(n) = j_n) = P(X = j_n) \prod_{r=1}^{n-1} \frac{P(X = j_r)}{P(X \geq j_r)},$$
$$0 \leq j_1 \leq \cdots \leq j_n \in \{0\} \cup \mathbb{Z}_+.$$

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The discrete weak record values also satisfy the Markovian property (see Nevzorov and Balakrishnan [5, p. 529] and Vervaat [14]); that is,

$$P(X_w(n+1) = j | X_w(n) = i) = \frac{P(X = j)}{P(X \geq i)}, \qquad j \geq i.$$

3. STOCHASTIC COMPARISONS OF EPOCH TIMES AND RECORD VALUES

In this section we study conditions under which epoch times of two discrete-time minimal repair processes are ordered, in the sense of the hazard rate order and the likelihood ratio order. Similar results will be given for weak record values.

3.1. Epoch Times of Discrete-Time Minimal Repair Processes

Let us consider two discrete-time minimal repair processes with underlying distributions P and Q, and we denote by $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$ the random vector of the first n discrete epoch times. The purpose is to obtain conditions under which it is possible to compare $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$, as well as the marginal distributions.

We begin by considering the usual multivariate stochastic order. Given two random vectors $\mathbf{X}$ and $\mathbf{Y}$, we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the *multivariate stochastic order* (denoted by $\mathbf{X} \leq_{\rm st} \mathbf{Y}$) if $E[\phi(\mathbf{X})] \leq E[\phi(\mathbf{Y})]$ for all increasing functions ϕ, for which the previous expectations exist. In the univariate case, let two random variables X and Y with distribution functions F and G be given. Then $X \leq_{\rm st} Y$ if $F(x) \geq G(x)$ for all $x \in \mathbb{R}$, and it can be denoted by $F \leq_{\rm st} G$.

It is possible to prove, by construction on the same probability space, that $(S_1, \ldots, S_n) \leq_{\rm st} (T_1, \ldots, T_n)$, under the assumption that P and Q are ordered in the hazard rate order. However, under this assumption, it is possible to prove a stronger result for the multivariate hazard rate order. First, we recall the definition of the (univariate and multivariate) *hazard rate order* in the discrete case.

In order to define the multivariate hazard rate order, we need to introduce the concept of history of a random vector. Let us consider a random vector $\mathbf{X}$ of dimension n that takes values on $\mathbb{Z}_+^n$. Given $t \in \mathbb{Z}_+$, let h_t be the event

$$h_t \equiv \{\mathbf{X}_I = \mathbf{x}_I, \mathbf{X}_{\bar{I}} \geq t\mathbf{e}\},$$

where $\mathbf{X}_I$ denotes the list of components of $\mathbf{X}$ with index in I, for some $I \subseteq \{1, 2, \ldots, n\}$ and $\mathbf{x}_I < t\mathbf{e}$, where $\mathbf{e}$ is a vector of 1's and the dimension is determined from the context in which it appears. If the components of $\mathbf{X}$ denote the discrete times in which the events $A_1, A_2, \ldots, A_n$ occur and given that the set of indexes of events A_i that have occurred at time t is I, then h_t denotes the list of times, $\mathbf{X}_I$, in which the events with index in I occur, whether for the remaining components $\mathbf{X}_{\bar{I}} \geq t\mathbf{e}$. The event h_t usually is called a *history at time t* of $\mathbf{X}$.

Given a history h_t, we can define the *discrete multivariate conditional hazard rate* at time t for a component with index in $\bar{I}$ as (see Shaked et al. [9–11])

$$\lambda_{i|I}(t|t_I) \equiv P(X_i = t|h_t).$$

Also, given a discrete random variable X, the previous concept corresponds to the *discrete hazard rate* at time t defined by (see Salvia and Bollinger [7])

$$r_t = \frac{P(X = t)}{P(X \geq t)}.$$

Now we define the discrete multivariate hazard rate order. We will see later a result for the multivariate hazard rate order of $(S_1, \dots, S_n)$ and $(T_1, \dots, T_n)$; in this case, the components are strictly ordered in increasing order. If, for two random vectors $\mathbf{X} = (X_1, X_2, \dots, X_n)$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$, there cannot be ties for the components of $\mathbf{X}$ and for the components of $\mathbf{Y}$, then the definition of the multivariate hazard rate order reduces to the following (see Shaked et al. [10]).

Definition 3.1: *Let $\mathbf{X}$ and $\mathbf{Y}$ be two discrete random vectors of dimension n taking values on $\mathbb{Z}_+^n$. Let $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ and $\eta_{\cdot|\cdot}(\cdot|\cdot)$ be the multivariate conditional hazard rates of $\mathbf{X}$ and $\mathbf{Y}$, respectively. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the multivariate hazard rate order (denoted by $\mathbf{X} \leq_{\text{hr}} \mathbf{Y}$) if*

$$\lambda_{i|I\cup J}(t|\mathbf{x}_{I\cup J}) \geq \eta_{i|I}(t|\mathbf{y}_I),$$

where $\mathbf{x}_I \leq \mathbf{y}_I < t\mathbf{e}$, $\mathbf{x}_J < t\mathbf{e}$, $I \cap J = \emptyset$, and $i \in \overline{I \cup J}$.

In the univariate case, given two discrete random variables X and Y taking values on $\mathbb{Z}_+$ with discrete hazard rates r and s, then the multivariate hazard rate order reduces to

$$r_i \geq s_i \quad \text{for all } i \in \mathbb{Z}^+.$$

In the univariate case, the previous condition is equivalent to the monotonicity of the ratio of the survival functions; that is, $X \leq_{\text{hr}} Y$ if and only if

$$\frac{P(X \geq i)}{P(Y \geq i)} \text{ is decreasing in } i \in \mathbb{Z}_+.$$

Again, if X and Y have distribution functions F and G, respectively, then the hazard rate order will be denoted by $F \leq_{\text{hr}} G$. It is well known that the hazard rate order is stronger than the stochastic order; that is, $\leq_{\text{hr}} \Rightarrow \leq_{\text{st}}$.

Theorem 3.2: *For two discrete-time minimal repair processes, as above, with underlying distributions P and Q, then $P \leq_{\text{hr}} Q$ if and only if*

$$(S_1, \dots, S_n) \leq_{\text{hr}} (T_1, \dots, T_n) \quad \textit{for all } n \geq 1. \tag{4}$$

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PROOF: First we prove that $P \leq_{\text{hr}} Q$ is a sufficient condition for Eq. (4). Given that $T_1 < \cdots < T_n$, a.s., then, for a history at time l of $(T_1, \ldots, T_n)$, the set of indexes I of the epoch times that have occurred until the time point l must be of the form $I = \{1, 2, \ldots, m\}$, for some $m \geq 1$, or $I = \emptyset$.

If we denote by $\eta_{\cdot|\cdot}(l|\cdot)$ the multivariate conditional hazard rate of $(T_1, \ldots, T_n)$ at time l with $I = \{1, 2, \ldots, m\}$ and by s the discrete hazard rate for the distribution function Q, then

$$\eta_i(l|\mathbf{y}_I) = \begin{cases} P(T_i = l | T_I = \mathbf{y}_I, T_{\bar{I}} \geq l\mathbf{e}) & \\ \qquad = P(T_i = l | T_m = y_m, T_i \geq l) = s_l & \text{if } i = m+1 \\ 0 & \text{if } i > m+1, \end{cases}.$$

where the value for $i = m+1$ follows from Lemma 2.1 and for $i > m+1$ given that $T_1 < \cdots < T_n$, a.s.

In a similar way, if we denote by $\lambda_{\cdot|\cdot}(l|\cdot)$ the multivariate conditional hazard rate of $(S_1, \ldots, S_n)$ at time l for a history in the set of components $I \cup J = \{1, 2, \ldots, m, m+1, \ldots, k\}$ and by r the discrete hazard rate for the distribution function P, then

$$\lambda_i(l|\mathbf{x}_{I\cup J}) = \begin{cases} P(S_i = l | S_I = \mathbf{x}_I, S_{\overline{I\cup J}} \geq l\mathbf{e}) & \\ \qquad = P(S_i = l | S_k = x_k, S_i \geq l) = r_l & \text{if } i = k+1 \\ 0 & \text{if } i > k+1. \end{cases}$$

Given $i \in \overline{I \cup J}$ (i.e., $i > k$) if $m < k$, then

$$\lambda_i(l|\mathbf{x}_{I\cup J}) = r_l \geq 0 = \eta_i(l|\mathbf{y}_I) \quad \text{if } i = k+1,$$
$$\lambda_i(l|\mathbf{x}_{I\cup J}) = 0 = \eta_i(l|\mathbf{y}_I) \quad \text{if } i > k+1.$$

If $m = k$, since $F \leq_{\text{hr}} G$, then

$$\lambda_i(l|\mathbf{x}_{I\cup J}) = r_l \geq s_l = \eta_i(l|\mathbf{y}_I) \quad \text{if } i = m+1,$$
$$\lambda_i(l|\mathbf{x}_{I\cup J}) = 0 = \eta_i(l|\mathbf{y}_I) \quad \text{if } i > m+1.$$

So, Eq. (4) holds.

On the other hand, $P \leq_{\text{hr}} Q$ is a necessary condition taking $n = 1$ in Eq. (4). ■

Given that the hazard rate order is stronger than the stochastic order and from the fact that the stochastic order is preserved under marginalization (see Shaked and Shanthikumar [8]), we have the following corollary.

COROLLARY 3.3: *For two discrete-time minimal repair processes, as above, with underlying distributions P and Q, if* $P \leq_{\text{hr}} Q$, *then*

$$S_n \leq_{\text{st}} T_n \quad \text{for all } n \geq 1.$$

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In particular, we can get bounds for the distribution function of the epoch time of a discrete-time minimal repair process.

Next we get a stronger result related to the multivariate likelihood ratio order of two discrete-time minimal repair processes. In particular, we get the *multivariate likelihood rate order* between $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$ with stronger conditions on the underlying distributions P and Q.

First, we recall the definition of the multivariate likelihood rate order in the discrete case.

DEFINITION 3.4: *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two discrete random vectors of dimension n taking values on* $\mathbb{Z}_+^n$. *Let f and g be the joint probability density function of* $\mathbf{X}$ *and* $\mathbf{Y}$, *respectively. We say that* $\mathbf{X}$ *is less than* $\mathbf{Y}$ *in the multivariate likelihood rate order (denoted by* $\mathbf{X} \leq_{\text{lr}} \mathbf{Y}$*) if*

$$f(x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n) g(x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n) \geq f(x_1, x_2, \ldots, x_n) g(y_1, y_2, \ldots, y_n)$$

for all $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *on* $\mathbb{Z}_+^n$, *where* $\wedge$ *and* $\vee$ *denote the minimum and the maximum, respectively.*

If X and Y are discrete random variables with distribution functions F and G and probability mass functions f and g, respectively, then $X \leq_{\text{lr}} Y$ if and only if g/f is increasing, and in this case, it will be denoted by $F \leq_{\text{lr}} G$.

Finally, we observe that the likelihood rate order and the stochastic order are preserved under marginalization, but for the hazard rate order, this property does not hold.

Next we fix the following notation. Given two distribution functions P and Q, with probability mass functions p and q, respectively, we will denote

$$l_i = \frac{p_i}{\sum_{j=i+1}^{\infty} p_j}, \qquad i = 1, 2, \ldots \tag{5}$$

and

$$m_i = \frac{q_i}{\sum_{j=i+1}^{\infty} q_j} \qquad i = 1, 2, \ldots. \tag{6}$$

Also, we will consider the values

$$L_n(i) = \begin{cases} 0 & \text{if } i < n \\ \sum_{j=1}^{i} l_j L_{n-1}(j-1) & \text{if } i \geq n \end{cases} \qquad n \geq 2$$

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and

$$M_n(i) = \begin{cases} 0 & \text{if } i < n \\ \sum_{j=1}^{i} m_j M_{n-1}(j-1) & \text{if } i \geq n \end{cases} \quad n \geq 2.$$

The following lemma will be used to provide some sufficient conditions for the likelihood rate order of the underlying distributions of two discrete-time minimal repair processes. Its proof is immediate from Eqs. (5) and (6).

LEMMA 3.5: *For two discrete distribution functions P and Q, if $P \leq_{\text{hr}} Q$ and m_i/l_i is increasing in $i \geq 1$, then $P \leq_{\text{lr}} Q$.*

THEOREM 3.6: *For two discrete-time minimal repair processes, as above, with underlying distributions P and Q, if $P \leq_{\text{hr}} Q$ and m_i/l_i is increasing in $i \geq 1$, then*

$$(S_1, \dots, S_n) \leq_{\text{lr}} (T_1, \dots, T_n) \quad \text{for all } n \geq 1.$$

PROOF: For $n = 1$, the result is trivial from Lemma 3.5 and by the fact that P and Q have the same distribution as S_1 and T_1, respectively.

Let us assume that $n \geq 2$. From the definition of the multivariate likelihood rate order and Eq. (3) we have to prove that

$$\prod_{j=1}^{n-1} l_{x_j \wedge y_j} p_{x_n \wedge y_n} \prod_{j=1}^{n-1} m_{x_j \vee y_j} q_{x_n \vee y_n} \geq \prod_{j=1}^{n-1} l_{x_j} p_{x_n} \prod_{j=1}^{n-1} m_{y_j} q_{y_n},$$

for all $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$.

If we denote $E = \{i \leq n-1 : x_i \geq y_i\}$, then the last inequality becomes

$$\prod_{j \in E} l_{y_j} p_{x_n \wedge y_n} \prod_{j \in E} m_{x_j} q_{x_n \vee y_n} \geq \prod_{j \in E} l_{x_j} p_{x_n} \prod_{j \in E} m_{y_j} q_{y_n}. \tag{7}$$

From Lemma 3.5, we have that $P \leq_{\text{lr}} Q$ and, therefore,

$$p_{x_n \wedge y_n} q_{x_n \vee y_n} \geq p_{x_n} q_{y_n}. \tag{8}$$

On the other hand, from the condition that m_i/l_i is increasing in $i \geq 1$, for all $j \in E$ we have that

$$l_{y_j} m_{x_j} \geq l_{x_j} m_{y_j}. \tag{9}$$

Now, from Eqs. (8) and (9), Eq. (7) holds. ■

A consequence of the preservation property under marginalization of the likelihood rate order (see Shaked and Shanthikumar [8, Thm. 4.E.3(b)]) and the last theorem is the following corollary in the univariate case.

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COROLLARY 3.7: *For two discrete-time minimal repair processes, as above, with underlying distributions P and Q, if* $P \leq_{\rm hr} Q$ *and* m_i/l_i *is increasing in* $i \geq 1$, *then* $S_n \leq_{\rm lr} T_n$ *for all* $n \geq 1$.

However, it is possible to give a stronger result with an independent proof.

THEOREM 3.8: *For two discrete-time minimal repair processes, as above, with underlying distributions P and Q, if* $P \leq_{\rm lr} Q$ *and* $M_n(i)/L_n(i)$ *is increasing in* $i \geq n$, *then* $S_n \leq_{\rm lr} T_n$ *for all* $n \geq 1$.

PROOF: For $n = 1$, the result is trivial. Let us assume that $n \geq 2$ and let us denote by p_n and q_n the probability function of S_n and T_n, respectively. In order to prove that $q_n(i)/p_n(i)$ is increasing in $i \geq n$, we observe that

$$\frac{q_n(i)}{p_n(i)} = \frac{q_i}{p_i} \frac{M_{n-1}(i-1)}{L_{n-1}(i-1)},$$

where q_i/p_i is increasing by the hypothesis $P \leq_{\rm lr} Q$ and $M_{n-1}(i-1)/L_{n-1}(i-1)$ is increasing in $i \geq n$ by hypothesis. So, the result holds. ■

At this point, the next step is to find out if the conditions of Theorem 3.8 are weaker than those of Theorem 3.6. In order to see this and from Lemma 3.5, it would be sufficient to prove that the condition m_i/l_i is increasing in $i \geq 1$ implies that $M_n(i)/L_n(i)$ is increasing in $i \geq n$. The next lemma states this conclusion.

LEMMA 3.9: *For two discrete-time minimal repair processes with underlying distributions P and Q, if* m_i/l_i *is increasing in* $i \geq 1$, *then* $M_n(i)/L_n(i)$ *is increasing in* $i \geq n$.

PROOF: We proceed by induction in n. For $n = 1$, we have to prove that

$$\frac{M_1(i+1)}{L_1(i+1)} - \frac{M_1(i)}{L_1(i)} \geq 0. \tag{10}$$

From the definitions of M_1 and L_1 we have that

$$\frac{M_1(i+1)}{L_1(i+1)} - \frac{M_1(i)}{L_1(i)} = \frac{\sum_{j=1}^{i+1} m_j}{\sum_{j=1}^{i+1} l_j} - \frac{\sum_{j=1}^{i} m_j}{\sum_{j=1}^{i} l_j}$$

$$=_{\rm sgn} \sum_{j=1}^{i} l_j \sum_{j=1}^{i} m_j \left(m_{i+1} \sum_{j=1}^{i} l_j - l_{i+1} \sum_{j=1}^{i} m_j \right),$$

where $=_{\rm sgn}$ denotes the equality in sign.

Therefore, Eq. (10) is equivalent to the condition

$$\frac{m_{i+1}}{\sum_{j=1}^{i} m_j} - \frac{l_{i+1}}{\sum_{j=1}^{i} l_j} \geq 0;$$

that is,

$$\frac{l_i}{l_{i+1}} + \frac{l_{i-1}}{l_{i+1}} + \cdots + \frac{l_1}{l_{i+1}} \geq \frac{m_i}{m_{i+1}} + \frac{m_{i-1}}{m_{i+1}} + \cdots + \frac{m_1}{m_{i+1}}. \tag{11}$$

Now from the hypothesis we have that

$$\frac{l_j}{l_{i+1}} \geq \frac{m_j}{m_{i+1}} \quad \text{for all } 1 \leq j \leq i+1, \tag{12}$$

and then Eq. (11) holds.

Next, let us assume that the result holds for n and we will prove it for $n+1$. We have to prove that

$$\frac{M_{n+1}(i+1)}{L_{n+1}(i+1)} - \frac{M_{n+1}(i)}{L_{n+1}(i)} \geq 0 \quad \text{for } i \geq n+1. \tag{13}$$

From the definitions of M_n and L_n we have that

$$\frac{M_{n+1}(i+1)}{L_{n+1}(i+1)} - \frac{M_{n+1}(i)}{L_{n+1}(i)} =_{\text{sgn}} m_{i+1} M_n(i) \sum_{j=1}^{i} l_j L_n(j-1) - l_{i+1} L_n(i) \times \sum_{j=1}^{i} m_j M_n(j-1).$$

Therefore, Eq. (13) is equivalent to the condition

$$\frac{l_i L_n(i-1)}{l_{i+1} L_n(i)} + \frac{l_{i-1} L_n(i-2)}{l_{i+1} L_n(i)} + \cdots + \frac{l_1 L_n(0)}{l_{i+1} L_n(i)} \geq \frac{m_i M_n(i-1)}{m_{i+1} M_n(i)} + \frac{m_{i-1} M_n(i-2)}{m_{i+1} M_n(i)} + \cdots + \frac{m_1 M_n(0)}{m_{i+1} M_n(i)}. \tag{14}$$

Now from the induction hypothesis we have that

$$\frac{L_n(j)}{L_n(i)} \geq \frac{M_n(j)}{M_n(i)} \quad \text{for } n \leq j \leq i. \tag{15}$$

Therefore, from Eqs. (12) and (15), Eq. (14) holds. ∎

As we have observed in Section 2, these results provide stochastic comparisons for discrete record values; next we will provide results for discrete "weak" record values.

3.2. Discrete Weak Record Values

In this subsection we consider two sequences of discrete weak record values and provide conditions under which the first n weak record values of the two sequences can be ordered in the stochastic and likelihood ratio orders.

Let $\{X_i\}_{i=1}^{\infty}$ and $\{Y_i\}_{i=1}^{\infty}$ be two sequences of i.i.d. random variables taking on values $1,2,\ldots,$ with distribution functions P and Q, respectively, and let us denote by $\{X_w(i)\}_{i=1}^{\infty}$ and $\{Y_w(i)\}_{i=1}^{\infty}$ the corresponding sequences of discrete weak record values.

First, we observe that we have not been able to provide a result for the hazard rate as in Theorem 3.2. However, it is possible to provide the following result for the stochastic order.

Now we recall the CIS notion, which will be used in the next theorem. Given a random vector $\mathbf{X} = (X_1,\ldots,X_n)$, $\mathbf{X}$ is said to be *conditionally increasing in sequence* (CIS) (see Lehman [3]) if, for $i = 2,3,\ldots,n$,

$$(X_i|X_1 = x_1,\ldots,X_{i-1} = x_{i-1}) \leq_{\text{st}} (X_i|X_1 = x_1',\ldots,X_{i-1} = x_{i-1}')$$

whenever $x_j \leq x_j'$, $j = 1,2,\ldots,i-1$.

THEOREM 3.10: *For two sequences $\{X_w(i)\}_{i=1}^{\infty}$ and $\{Y_w(i)\}_{i=1}^{\infty}$ of discrete weak record values, as above, from distribution functions P and Q, if $P \leq_{\text{hr}} Q$, then*

$$(X_w(1),\ldots,X_w(n)) \leq_{\text{st}} (Y_w(1),\ldots,Y_w(n)) \quad \textit{for all } n \geq 1.$$

PROOF: The proof follows from Theorem 4.B.4 in Shaked and Shanthikumar [8]. Provided that $(X_w(1),\ldots,X_w(n))$ and $(Y_w(1),\ldots,Y_w(n))$ are CIS, we just need to prove that

$$X_w(1) \leq_{\text{st}} Y_w(1), \tag{16}$$

and for $i = 2,3,\ldots,n$,

$$\begin{aligned}&\{X_w(i)|X_w(1) = x_1,\ldots,X_w(i-1) = x_{i-1}\}\\&\quad \leq_{\text{st}} \{Y_w(i)|Y_w(1) = x_1,\ldots,Y_w(i-1) = x_{i-1}\} \quad \text{for all } x_j, j = 1,2,\ldots,i-1.\end{aligned} \tag{17}$$

The CIS property of $(X_w(1),\ldots,X_w(n))$ and $(Y_w(1),\ldots,Y_w(n))$ follows from the Markovian property of the sequence of weak record values (see Section 2), where the transition probabilities $P(X_w(n+1) = j|X_w(n) = i)$ are clearly increasing in i.

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Equation (16) follows from the fact that $X_w(1)$ and $Y_w(1)$ have distribution functions P and Q, respectively, and the fact that the hazard rate order is stronger than the stochastic order.

Finally, Eq. (17) follows from the previously mentioned Markovian property and, again, the fact that the hazard rate order is stronger than the stochastic order. ■

Next we consider the likelihood hazard rate order. Under arguments similar to those in Lemma 3.5 and Theorem 3.6, we can prove the following results.

LEMMA 3.11: *For two discrete distribution functions P and Q, with hazard rates r and s, respectively, if $P \leq_{\text{hr}} Q$ and s_i/r_i is increasing in $i \geq 1$, then $P \leq_{\text{lr}} Q$.*

THEOREM 3.12: *For two sequences of discrete weak record values, as above, from distribution functions P and Q, if $P \leq_{\text{hr}} Q$ and s_i/r_i is increasing in $i \geq 1$, then*

$$(X_w(1), \ldots, X_w(n)) \leq_{\text{lr}} (Y_w(1), \ldots, Y_w(n)) \quad \text{for all } n \geq 1.$$

4. STOCHASTIC COMPARISONS OF INTEREPOCH TIMES AND INTERRECORD VALUES

In this section we consider conditions under which we can give stochastic comparisons of interepoch times of two discrete-time minimal repair processes and interrecord values. First, we derive results for two discrete-time minimal repair processes and then we will give similar results (without proofs) for weak record values.

4.1. Interepoch Times of Discrete-Time Minimal Repair Processes

Again consider two discrete-time minimal repair processes with underlying distributions P and Q and denote by $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$ the random vector of the first n discrete epoch times. Let $(U_1, \ldots, U_n)$ and $(V_1, \ldots, V_n)$ be the corresponding random vectors of interepoch times; that is, $U_i \equiv S_i - S_{i-1}$ and $V_i \equiv T_i - T_{i-1}$, for $i = 1, 2, \ldots, n$, where $U_0 \equiv V_0 \equiv 0$. The purpose is to study whether we can compare in the stochastic and likelihood ratio orders the random vectors $(U_1, \ldots, U_n)$ and $(V_1, \ldots, V_n)$.

Next we give conditions under which $(U_1, \ldots, U_n)$ and $(V_1, \ldots, V_n)$ are ordered in the stochastic order. In order to give the result first, we need to give the decreasing failure rate (DFR) notion.

Let X be a discrete random variable taking values in $\{0\} \cup \mathbb{Z}_+$, with distribution function P and hazard rate r; we say that X is DFR if r is decreasing. It is well known that X is DFR if and only if $P(X > t + x | X > x)$ is increasing in $x \in \{0\} \cup \mathbb{Z}_+$ for all $t \in \{0\} \cup \mathbb{Z}_+$.

THEOREM 4.1: *For two discrete-time minimal repair processes, as above, with underlying distributions P and Q, if $P \leq_{\text{hr}} Q$ and either P or Q are DFR, then*

$$(U_1, \ldots, U_n) \leq_{\text{st}} (V_1, \ldots, V_n) \quad \text{for all } n \geq 1.$$

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PROOF: From Theorem 4.B.4 in Shaked and Shanthikumar [8], the proof follows if $(U_1,\dots,U_n)$ or $(V_1,\dots,V_n)$ are CIS, and

$$U_1 \equiv S_1 \leq_{\text{st}} V_1 \equiv T_1, \tag{18}$$

and for $i = 2,3,\dots,n$,

$$\{U_i \mid U_1 = x_1,\dots,U_{i-1} = x_{i-1}\} \leq_{\text{st}} \{V_i \mid V_1 = x_1,\dots,V_{i-1} = x_{i-1}\} \quad \text{for all } x_j, j = 1,2,\dots,i-1. \tag{19}$$

We observe that, for $i = 2,\dots,n$,

$$\{U_i \mid U_1 = x_1,\dots,U_{i-1} = x_{i-1}\} =_{\text{st}} \left\{S_1 - \sum_{j=1}^{i-1} x_j \mid S_1 > \sum_{j=1}^{i-1} x_j\right\}$$

and, similarly,

$$\{V_i \mid V_1 = x_1,\dots,V_{i-1} = x_{i-1}\} =_{\text{st}} \left\{T_1 - \sum_{j=1}^{i-1} x_j \mid T_1 > \sum_{j=1}^{i-1} x_j\right\}.$$

Then, from previous equalities, the CIS property of $(U_1,\dots,U_n)$ (or $(V_1,\dots,V_n)$) follows from the DFR property of P (or Q) and Eqs. (18) and (19) follows from the hypothesis $P \leq_{\text{hr}} Q$. ■

Again, given that the stochastic order is preserved under marginalization, we can get the comparison in the stochastic order of U_n and V_n.

COROLLARY 4.2: *For two discrete-time minimal repair processes as above, with underlying distributions P and Q, if $P \leq_{\text{hr}} Q$ and P or Q or both are DFR, then*

$$U_n \leq_{\text{st}} V_n \quad \textit{for all } n \geq 1. \tag{20}$$

Next we consider the likelihood rate order between $(U_1,\dots,U_n)$ and $(V_1,\dots,V_n)$.

THEOREM 4.3: *For two discrete-time minimal repair processes with underlying distributions P and Q, if $P \leq_{\text{hr}} Q$ and m_i/l_i is increasing in $i \geq 1$, P or Q have logconvex probability mass functions, and l_i or m_i is logconvex in i, then*

$$(U_1,\dots,U_n) \leq_{\text{lr}} (V_1,\dots,V_n) \quad \textit{for all } n \geq 1.$$

PROOF: Let us assume that the probability mass function p of P and l are logconvex functions. The proof in other cases is similar.

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Let us consider u_k, $v_k > 0$, $k = 1, \ldots, n$, and fix the following notation for $j = 1, \ldots, n$:

$$a_j = \sum_{k=1}^{j} u_k, \qquad b_j = \sum_{k=1}^{j} u_k \wedge v_k,$$

$$a'_j = \sum_{k=1}^{j} v_k, \qquad b'_j = \sum_{k=1}^{j} u_k \vee v_k,$$

and

$$c_j = a'_j - b_j = b'_j - a_j,$$

where the last equality is easy to prove and also $a_j \geq b_j$, $b'_j \geq a'_j$, and $c_j \geq 0$.

The logconvex property of l implies for $j = 1, \ldots, n-1$ that

$$l_{a_j+c_j} l_{b_j} \geq l_{a_j} l_{b_j+c_j} \tag{21}$$

and the logconvex property of p implies

$$p_{a_n+c_n} p_{b_n} \geq p_{a_n} p_{b_n+c_n}. \tag{22}$$

So, we have to prove that

$$\prod_{j=1}^{n-1} l_{b_j} p_{b_n} \prod_{j=1}^{n-1} m_{b'_j} q_{b'_n} \geq \prod_{j=1}^{n-1} l_{a_j} p_{a_n} \prod_{j=1}^{n-1} m_{a'_j} q_{a'_n}, \tag{23}$$

where q is the probability function of Q. From Eqs. (21) and (22), we have that

$$\prod_{j=1}^{n-1} l_{b_j} p_{b_n} \prod_{j=1}^{n-1} m_{b'_j} q_{b'_n} \geq \prod_{j=1}^{n-1} \frac{l_{a_j} l_{b_j+c_j}}{l_{a_j+c_j}} m_{b'_j} \frac{p_{a_n} p_{b_n+c_n}}{p_{a_n+c_n}} q_{b'_n}. \tag{24}$$

On the other hand, from the hypothesis and Lemma 3.5,

$$\begin{aligned}
\prod_{j=1}^{n-1} \frac{l_{a_j} l_{b_j+c_j}}{l_{a_j+c_j}} m_{b'_j} \frac{p_{a_n} p_{b_n+c_n}}{p_{a_n+c_n}} q_{b'_n} &= \prod_{j=1}^{n-1} \frac{m_{b'_j}}{l_{b'_j}} l_{a_j} l_{a'_j} \frac{q_{b'_n}}{p_{b'_n}} p_{a_n} p_{a'_n} \\
&\geq \prod_{j=1}^{n-1} \frac{m_{a'_j}}{l_{a'_j}} l_{a_j} l_{a'_j} \frac{q_{a'_n}}{p_{a'_n}} p_{a_n} p_{a'_n} = \prod_{j=1}^{n-1} m_{a'_j} l_{a_j} q_{a'_n} p_{a_n}.
\end{aligned}$$

Therefore, from Eq. (24) and the last inequality, Eq. (23) is true and the result holds. ■

4.2. Weak Interrecord Values

By following the same notation as in Section 3.2, we consider the random vectors $(U_w(1), \ldots, U_w(n))$ and $(V_w(1), \ldots, V_w(n))$ of interrecord values for discrete weak

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record values, where $U_w(i) \equiv X_w(i) - X_w(i-1)$ and $V_w(i) \equiv Y_w(i) - Y_w(i-1)$, for $i = 1,2,\ldots,n$, where $U_w(0) \equiv V_w(0) \equiv 0$. In this subsection we get similar results to those of Section 4.1; the proofs are similar and are therefore omitted.

THEOREM 4.4: *For two sequences of discrete weak record values, as above, with underlying distributions P and Q, if $P \le_{\text{hr}} Q$ and P or Q is DFR, then*

$$(U_w(1),\ldots,U_w(n)) \le_{\text{st}} (V_w(1),\ldots,V_w(n)) \quad \textit{for all } n \ge 1.$$

THEOREM 4.5: *For two sequences of discrete weak record values, as above, with underlying distributions P and Q and failure rates r and s, respectively, if $P \le_{\text{hr}} Q$ and s_i/r_i is increasing in $i \ge 1$, F or G have logconvex probability mass functions, and r_i or s_i is logconvex in i, then*

$$(U_w(1),\ldots,U_w(n)) \le_{\text{lr}} (V_w(1),\ldots,V_w(n)) \quad \textit{for all } n \ge 1.$$

Remark 4.6: Some of the results stated in this article can be used to also provide results of positive association for random vectors of record values or interrecord values. For example, it is well known that given a random vector **X**, **X** is said to be MTP2 (multivariate totally positive of order 2; see Karlin and Rinot [2]) if and only if $\mathbf{X} \le_{\text{lr}} \mathbf{X}$. Now, from Theorem 4.3, given a random vector of interepoch times $(U_1,\ldots,U_n)$ from a discrete-time minimal repair process with underlying distribution P and values $\{l_i\}_{i=1}^{\infty}$, defined as in Section 2, if P has a logconvex density function and l_i is logconvex, then $(U_1,\ldots,U_n)$ is MTP2. A similar result can be obtained from Theorem 4.5. By the same arguments and from Theorems 3.6 and 3.12, a random vector of epoch times of a discrete-time minimal repair process and a random vector of discrete weak record values, respectively, is always MTP2.

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