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The strength of Engeler's lemma

JAN PASEKA[†]

Department of Mathematics, Masaryk University, Janáčkovo nám. 2a, 602 00 Brno.

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Dedicated to Klaus Keimel on the occasion of his 65th birthday

A useful separation lemma for partial cm-lattices is proved equivalent to **PIT**, the Prime Ideal Theorem. The relation of various versions of the Lemma to each other and to **PIT** is also explored.

1. Preliminaries

In Paseka (2004), we formulated a general principle, the *General Separation Lemma for Quantales* (**GSLQ**), stating that any element outside some Scott-open distributive filter S in a non-trivial quantale is below a prime element outside S. **GSLQ** for two-sided quantales is well known (Banaschewski and Erné 1993) to be equivalent to the Prime Ideal Theorem. Relevant sources in that context are, for example, Erné (2000) and the references therein.

Our concern in this paper is to introduce a more general form of **GSLQ** for partial cm-lattices and to show that this form follows from Engeler's lemma (Engeler 1959; Erné 1997). This, in turn, implies that **GSLQ** is equivalent in Zermelo–Fraenkel Set Theory to the Prime Ideal Theorem and also to the Constraint Compactness Theorem (Cowen 1998), which is an infinite version of the constraint satisfaction problems studied in computer science.

In addition, we prove that the Prime Ideal Theorem implies R-spatiality for any R-semidistributive algebraic partial cm-lattice.

All unexplained facts concerning cm-lattices and quantales can be found in Erné (1997), Rosenthal (1990) and Banaschewski and Erné (1993).

Definition 1.1.

(a) By a *partial m-semilattice* we mean a semilattice Q (arbitrary finite joins, including the non-empty one, exist) equipped with a partial multiplication $\cdot : R \to Q$ with domain $\mathrm{dom}(\cdot)=R \subseteq Q \times Q$. We shall use 1 to denote the top element of Q whenever it exists.

(b) A *partial m-lattice* is a partial m-semilattice that is also a lattice (arbitrary finite joins and finite meets, including the non-empty one, exist).

(c) A *partial cm-lattice* is a partial m-semilattice that is also a $\bigvee$-semilattice (arbitrary joins exist).

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Each quantale Q is a partial cm-lattice for any $R \subseteq Q \times Q$. Notice here that, in contrast to the case of cm-lattices (Keimel 1972; Erné 2000), our partial multiplication $\cdot$ need not be order-preserving.

Definition 1.2.

(a) An element $p \neq 1$ of a partial m-semilattice Q is said to be R-*prime* if

$$r \cdot l \leqslant p \Rightarrow r \leqslant p \text{ or } l \leqslant p$$

for all $(r, l) \in R$.

(b) Q is called R-*spatial* if each of its elements is a meet of R-primes.

Example 1.3.

1. Let Q be a (distributive) lattice, $R = Q \times Q$. The notion of an R-prime element coincides with the standard notion of a prime for lattices (Rav 1989).
2. Let Q be a cm-lattice, $R = Q \times Q$. The notion of an R-prime element coincides with the standard notion of a prime for cm-lattices (Keimel 1972; Rosický 1987).
3. Let Q be a cm-lattice, $R = \{(x, x) : x \in Q\}$. The notion of an R-prime element coincides with the standard notion of a semiprime element for cm-lattices (Keimel 1972).
4. Let Q be a quantale, $R = \mathscr{R}(Q) \times \mathscr{L}(Q)$, with $\mathscr{R}(Q)$ the set of right-sided elements of Q, and $\mathscr{L}(Q)$ the set of left-sided elements of Q. The notion of an R-prime element coincides with the standard notion of a prime for quantales (Kruml 2003).

Definition 1.4.

(a) A *partial* R-*point* of a partial m-semilattice Q is a non-zero $\vee$-preserving (including the empty join) partial map $f : Q \to \mathbf{2}$ such that $f(r \vee a) = 1 = f(l \vee a)$ implies $f(r \cdot l \vee a) = 1$ for all $a \in Q$, $(r, l) \in R$ such that $r \vee a, l \vee a, r \cdot l \vee a \in \mathrm{dom}(f)$; here $\mathbf{2} = \{0, 1\}$ is the 2-element Boolean algebra.

(b) An R-*point* is a partial R-point such that $\mathrm{dom}(f) = Q$.

(c) A *complete* R-*point* is an R-point f preserving arbitrary joins.

Note that, for a partial cm-lattice Q, there is a one-to-one correspondence between R-primes and complete R-points.

Definition 1.5. Let Q be a partial m-semilattice.

(a) A non-trivial upper subset $S \subseteq Q$ is called an R-*distributive filter* of Q if $r \vee a, l \vee a \in S$, $a \in Q, (r, l) \in R$ implies $r \cdot l \vee a \in S$.

(b) The top element 1 of Q is said to be R-*distributive* if $\{1\}$ is an R-distributive filter.

The notion of a distributive filter was also motivated by Rav's notion of a semiprime filter and Erné's notion of a distributor.

Definition 1.6. A partial m-lattice Q is said to be

(a) R-*semidistributive* if $(r \vee a) \wedge (l \vee a) \leqslant r \cdot l \vee a$ for all $a \in Q, (r, l) \in R$.

(b) R-*distributive* if $(r \vee a) \wedge (l \vee a) = r \cdot l \vee a$ for all $a \in Q, (r, l) \in R$.

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Notice that as in Kruml (2003), any R-spatial partial m-lattice Q is R-semidistributive. In particular, we have:

(i) In any R-semidistributive partial m-lattice Q, $(r, 1) \in R$ implies $r \leqslant r \cdot 1$ and $(1, l) \in R$ implies $l \leqslant 1 \cdot l$.
(ii) In any R-distributive partial m-lattice Q, $(r, 1) \in R$ implies $r \in \mathscr{R}(Q)$ and $(1, l) \in R$ implies $l \in \mathscr{L}(Q)$.

We introduce the following *General R-Separation Lemmas.*

Definition 1.7.

R-**GSLQ.** Any element outside some Scott-open R-distributive filter S in a non-trivial quantale is below an R-prime element outside S.

R-**PETQ.** Any non-trivial quantale with compact R-distributive top element has an R-prime element.

R-**MPETQ.** For any family $(Q_i)_{i \in I}$ of non-trivial quantales equipped with relations $R_i \subseteq Q_i \times Q_i$, $i \in I$, with compact R_i-distributive top elements, there is a family $(p_i)_{i \in I}$ of R_i-prime elements.

R-**GSLC.** Any element outside some Scott-open R-distributive filter S in a non-trivial partial cm-lattice is below an R-prime element outside S.

R-**PETC.** Any non-trivial partial cm-lattice with compact R-distributive top element has an R-prime element.

R-**MPETC.** For any family $(Q_i)_{i \in I}$ of non-trivial partial cm-lattices equipped with relations $R_i \subseteq Q_i \times Q_i$, $i \in I$, with compact R_i-distributive top elements, there is a family $(p_i)_{i \in I}$ of R_i-prime elements.

Clearly: R-**GSLC** implies R-**GSLQ** and R-**PETC**; R-**MPETC** implies R-**PETC** and R-**MPETQ**; R-**PETC** implies R-**PETQ**; R-**MPETQ** implies R-**PETQ**; and, R-**GSLQ** implies R-**PETQ**. Each of the above statements implies the Prime Ideal Theorem (Erné 2000; Paseka 2004).

2. Engeler's lemma implies both R-GSLC and R-MPETC

The following theorem (Engeler 1959) is well known to be equivalent to the Prime Ideal Theorem.

Theorem 2.1 (Engeler's lemma (PIT)). Let X be a non-empty set and $\mathscr{E}$ be a collection of functions from subsets of X into a two-point boolean algebra **2** such that:

1. $\mathscr{E}$ has finite character (that is, a function φ from a subset of X into **2** is in $\mathscr{E}$ if and only if for every finite $F \subseteq \mathrm{dom}(\varphi)$, φ/F is in $\mathscr{E}$).
2. For all finite subsets F of X, there is a function $\varphi \in \mathscr{E}$ whose domain is F.

Then X is the domain of some $\varphi \in \mathscr{E}$.

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Motivated by the paper Banaschewski (1985), we obtain the main result of our paper.

Theorem 2.2 (PIT). Let Q be a partial cm-lattice, $\mathrm{dom}(\cdot) = R \subseteq Q \times Q$, $S \subseteq Q$ be an R-distributive Scott-open filter, and $a \in Q - S$. Then there is an R-prime element $p \in Q$, $p \geqslant a$ such that $p \notin S$.

Proof. Let $P = Q - S$. Then P is a poset closed under the join of chains. We put, for all $x \in P$, $Q(x) = \uparrow x$ and for any $E \subseteq Q(x)$ we use $\langle E\rangle^x$ to denote the $\vee$-subsemilattice of $Q(x)$ generated by E. Note that we always have $x \in \langle E\rangle^x$, where x is the bottom element of $\langle E\rangle^x$. We also put $Q(P) = \{(x, y) : x \in P, y \in Q(x)\}$. For $Y \subseteq Q(P)$, $x \in P$ we put $Y(x) = \{y \in Q(x) : (x, y) \in Y\}$, $\pi_1(Y) = \{x \in P : (x, y) \in Y\}$.

We define

$$\begin{aligned}\mathscr{E} = \{\varphi : Y \to \mathbf{2};\quad & Y \subseteq Q(P) \text{ such that there is a (unique) extension } \overline{\varphi} : \overline{Y} \to \mathbf{2},\\ & \text{where } \overline{Y} = \bigcup\{\{x\} \times \langle Y(x)\rangle^x : x \in \pi_1(Y)\}, \overline{\varphi}(x, -) : \langle Y(x)\rangle^x \to \mathbf{2}\\ & \text{is a partial } R\text{-point for all } x \in \pi_1(Y) \text{ and } \overline{\varphi}(x, \langle Y(x)\rangle^x \cap \downarrow a) \subseteq \{0\},\\ & \overline{\varphi}(x, \langle Y(x)\rangle^x \cap S) \subseteq \{1\}\}.\end{aligned}$$

We shall show that $\mathscr{E}$ is a system of finite character. Note that $\emptyset \in \mathscr{E}$.

Let $\varphi \in \mathscr{E}$ be a function with the domain $\mathrm{dom}(\varphi)$. Let φ_0 be a finite subset of φ. Evidently, $\pi_1(\mathrm{dom}(\varphi_0)) \subseteq \pi_1(\mathrm{dom}(\varphi))$, hence $\overline{\mathrm{dom}(\varphi_0)} \subseteq \overline{\mathrm{dom}(\varphi)}$. We put $\overline{\varphi_0} = \overline{\varphi}/\overline{\mathrm{dom}(\varphi_0)}$. Then $\overline{\varphi_0}$ is an extension of φ_0 $\overline{\varphi_0} \subseteq \overline{\varphi}$, $\overline{\varphi_0}$ is finite and $\varphi_0 \in \mathscr{E}$.

Conversely, let φ be a partial function from $Q(P)$ into $\mathbf{2}$ such that for any finite subset $\varphi_0 \subseteq \varphi$, we have $\varphi_0 \in \mathscr{E}$. We have to show that $\varphi \in \mathscr{E}$. Note that clearly $\overline{\mathrm{dom}(\varphi)} = \bigcup\{\overline{\mathrm{dom}(\varphi_0)} : \varphi_0 \subseteq\subseteq \varphi\}$. We put $\overline{\varphi} = \bigcup\{\overline{\varphi_0} : \varphi_0 \subseteq\subseteq \varphi\}$. Obviously, $\overline{\varphi}$ is correctly defined, since for any $\varphi_1, \varphi_2 \subseteq\subseteq \varphi$ there is some $\varphi_3 \subseteq\subseteq \varphi$ containing both φ_1 and φ_2. Hence, for any $(x, z) \in \mathrm{dom}(\overline{\varphi_1}) \cap \mathrm{dom}(\overline{\varphi_2})$, we have $\overline{\varphi_1}(x, z) = \overline{\varphi_3}(x, z) = \overline{\varphi_2}(x, z)$. Evidently, $\overline{\varphi}$ is the unique extension of φ from $\mathrm{dom}(\varphi)$ to $\overline{\mathrm{dom}(\varphi)}$.

Let $x \in \pi_1(\mathrm{dom}(\varphi))$ and Z be a finite subset of $\langle \mathrm{dom}(\varphi)(x)\rangle^x$. Then there is a finite subset $\varphi_0 \subseteq\subseteq \varphi$ such that $Z \subseteq \langle \mathrm{dom}(\varphi_0)(x)\rangle^x$ and $\overline{\varphi_0} = \overline{\varphi}/\overline{\mathrm{dom}(\varphi_0)}$. Hence, $\overline{\varphi}(x, \bigvee Z) = \overline{\varphi_0}(x, \bigvee Z) = \bigvee_{z \in Z} \overline{\varphi_0}(x, z) = \bigvee_{z \in Z} \overline{\varphi}(x, z)$.

Similarly, let $x \in \pi_1(\mathrm{dom}(\varphi))$, $c \in Q$, $(u, v) \in R$, $u \vee c, v \vee c, u \cdot v \vee c \in \langle \mathrm{dom}(\varphi)(x)\rangle^x$ and $\overline{\varphi}(x, u \vee c) = \overline{\varphi}(x, v \vee c) = 1$. Then there is a finite subset $\varphi_0 \subseteq\subseteq \varphi$ such that $u \cdot v \vee c, u \vee c, v \vee c \in \langle \mathrm{dom}(\varphi_0)(x)\rangle^x$ and $\overline{\varphi_0}(x, u \vee c) = \overline{\varphi_0}(x, v \vee c) = 1$. This implies $\overline{\varphi_0}(x, u \cdot v \vee c) = 1$. Hence, $\overline{\varphi}(x, u \cdot v \vee c) = 1$.

Now, let $x \in \pi_1(\mathrm{dom}(\varphi))$, $u \leqslant a$ ($u \in S$) and $u \in \langle \mathrm{dom}(\varphi)(x)\rangle^x$. Then there is a finite subset $\varphi_0 \subseteq\subseteq \varphi$ such that $u \in \langle \mathrm{dom}(\varphi_0)(x)\rangle^x$. Hence, $\overline{\varphi_0}(x, u) = 0$ ($\overline{\varphi_0}(x, u) = 1$), and this implies $\overline{\varphi}(x, u) = 0$ ($\overline{\varphi}(x, u) = 1$).

We shall prove that for any finite subset F of $Q(P)$, there is a function $\varphi_F \in \mathscr{E}$ whose domain is F. If $F = \emptyset$, then $\varphi_F = \emptyset \in \mathscr{E}$. Let us assume that F is non-empty. Evidently, $\overline{F}$ is a finite non-empty subset of $Q(P)$ containing F. For all (finitely many) $x \in \pi_1(F)$, we choose a maximal element $a_F^x \in \langle F(x)\rangle^x - S$ (from a finite non-empty subset – namely $x \in \langle F(x)\rangle^x - S$) such that $u \leqslant a$, $u \in \langle F(x)\rangle^x$ implies $u \leqslant a_F^x$. We define a function $\varphi_F : F \to \mathbf{2}$ by the prescription $\varphi_F(x, u) = 0$ ($\varphi_F(x, u) = 1$) iff $u \leqslant a_F^x$ ($u \nleqslant a_F^x$) for all $(x, u) \in F$. Then the unique extension $\overline{\varphi_F}$ of φ_F to $\overline{F}$ is defined by the prescription $\overline{\varphi_F}(x, z) = 0$ ($\varphi_F(x, z) = 1$) iff $z \leqslant a_F^x$ ($z \nleqslant a_F^x$) for all $(x, z) \in \overline{F}$. Notice that

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$(x, a_F^x) \in \overline{F}$ and $(x, z) \in \overline{F} \cap S$ implies $\overline{\varphi_F}(x, z) = 1$. Hence $\overline{\varphi_F}(x, -)$ preserves finite joins, $\overline{\varphi_F}(x, \langle F(x)\rangle^x \cap \downarrow a)) \subseteq \{0\}$ and $\overline{\varphi_F}(x, \langle F(x)\rangle^x \cap S) \subseteq \{1\}$. Let $x \in \pi_1(\mathrm{dom}(\varphi_F))$, $c \in Q$, $(u, v) \in R$, $u \vee c, v \vee c, u \cdot v \vee c \in \langle \mathrm{dom}(\varphi_F)(x)\rangle^x$ and $\overline{\varphi_F}(x, u \vee c) = \overline{\varphi_F}(x, v \vee c) = 1$. By the maximality of $a_F^x \in \langle F(x)\rangle^x - S$, we have $u \vee c \vee a_F^x > a_F^x$ and $v \vee c \vee a_F^x > a_F^x$. Then $u \vee c \vee a_F^x, v \vee c \vee a_F^x \in S$. Since S is an R-distributive filter, we have $u \cdot v \vee c \vee a_F^x \in S$. This in turn implies that $1 = \overline{\varphi_F}(x, u \cdot v \vee c \vee a_F^x) = \overline{\varphi_F}(x, u \cdot v \vee c)$ by $\overline{\varphi_F}(x, a_F^x) = 0$.

$\mathscr{E}$ satisfies the assumptions of Engeler's lemma. Hence there is a map $\varphi : Q(P) \to \mathbf{2}$ such that $\mathrm{dom}(\varphi) = Q(P)$, $\varphi = \overline{\varphi}$ and, for all $x \in P$, $\varphi(x, -) : Q(x) \to \mathbf{2}$ is a $\vee$-semilattice homomorphism, $\varphi(x, Q(x) \cap \downarrow a)) \subseteq \{0\}$, $\varphi(x, Q(x) \cap S) \subseteq \{1\}$ and $c, u, v \in Q$, $(u, v) \in R$, $u \vee c, v \vee c, u \cdot v \vee c \in Q(x)$ and $\varphi(x, u \vee c) = \varphi(x, v \vee c) = 1$ implies $\varphi(x, u \cdot v \vee c) = 1$. Note that $J_x = \varphi(x, -)^{-1}(\{0\})$ is an ideal in $Q(x)$, $x \in J_x$ since $\varphi(x, -)$ preserves the bottom element.

Let $P(a) = P \cap \uparrow a$. Again, $P(a)$ is closed under the join of chains. Putting $\sigma(x) = \bigvee J_x$, $x \in P(a)$, we obtain a map $\sigma : P(a) \to P(a)$ by the Scott-openness of S. Since $x \leqslant \sigma(x)$ for each $x \in P(a)$, we have, by Bourbaki's fix point lemma, that there is an element $p \in P(a)$ such that $p = \sigma(p)$. For any such p, $J_p = \{p\}$. Clearly, $a \leqslant p$ and $p \notin S$. Let us check that p is R-prime. Let $u, v \in Q$, $(u, v) \in R$ and $u \cdot v \leqslant p$. Then $\varphi(p, u \cdot v \vee p) = 0$. Assume that $u \nleqslant p$ and $v \nleqslant p$. Hence $p < u \vee p$ and $p < v \vee p$. In particular, $\varphi(p, u \vee p) = 1$ and $\varphi(p, v \vee p) = 1$. Then $0 = \varphi(p, u \cdot v \vee p) = 1$, which is a contradiction. Hence $u \leqslant p$ or $v \leqslant p$. □

Theorem 2.3. (**PIT**) implies (R-**MPETC**).

Proof. Let $P_i = Q_i - \{1_i\}$, $i \in I$. Then P_i is a poset closed under the join of chains. We also put $Q(P, I) = \{(i, x, y) : i \in I, x \in P_i, y \in Q_i(x)\}$. For $Y \subseteq Q(P, I)$, $i \in I$, $x \in P_i$, we put $Y(i, x) = \{y \in Q_i(x) : (i, x, y) \in Y\}$ and $\pi_{1,2}(Y) = \{(i, x) : (i, x, y) \in Y\}$.

We define

$$\begin{aligned}\mathscr{E}(I) = \{\varphi : Y \to \mathbf{2}; Y \subseteq Q(P, I) \text{ such that there is a (unique) extension} \\ \widehat{\varphi} : \widehat{Y} \to \mathbf{2} \text{ here } \widehat{Y} = \bigcup\{\{i\} \times \{x\} \times \langle Y(i, x)\rangle^x : (i, x) \in \pi_{1,2}(Y)\}, \\ \widehat{\varphi}(i, x, -) : \langle Y(i, x)\rangle^x \to \mathbf{2} \text{ is a partial } R_i\text{-point and} \\ \widehat{\varphi}(i, x, x) = 0, \widehat{\varphi}(i, x, 1) = 1 \text{ for all } (i, x) \in \pi_{1,2}(Y)\}.\end{aligned}$$

Applying the same considerations as in Theorem 2.2, we can check that $\mathscr{E}(I)$ satisfies the assumptions of Engeler's lemma.

Hence there is a map $\varphi : Q(P, I) \to \mathbf{2}$ such that $\mathrm{dom}(\varphi) = Q(P, I)$, $\varphi = \widehat{\varphi}$ and, for all $i \in I$, $x \in P_i$, $\varphi(i, x, -) : Q_i(x) \to \mathbf{2}$ is a $\vee$-semilattice homomorphism, $\varphi(i, x, x) = 0$, $\varphi(i, x, 1) = 1$ and $u, v, c \in Q_i$, $(u, v) \in R_i$, $u \vee c, v \vee c, u \cdot v \vee c \in Q_i(x)$ and $\varphi(i, x, u \vee c) = \varphi(i, x, v \vee c) = 1$ implies $\varphi(i, x, u \cdot v \vee c) = 1$. Note that $J_x^i = \varphi(i, x, -)^{-1}(\{0\})$ is a non-empty ideal in $Q_i(x)$.

Let $Q = \sum_{i \in I} Q_i$ be the non-trivial sum in the category of sup-lattices, and $j_k : Q_k \to \sum_{i \in I} Q_i$ be the coproduct maps, for $k \in I$. Then the set $P = \{\bigvee_{i \in I} j_i(x_i) : x_i \in P_i, i \in I\}$ is a subposet of Q and is closed under the join of chains. Putting $\sigma(\bigvee_{i \in I} j_i(x_i)) = \bigvee_{i \in I} j_i(\bigvee J_{x_i}^i)$, we obtain a map $\sigma : P \to P$ by the compactness of Q_i. Since $x \leqslant \sigma(x)$ for each $x \in P$, we have, by Bourbaki's fix point lemma, that there is an element $p \in P$ such that $p = \sigma(p)$. Hence $p = \bigvee_{i \in I} j_i(p_i) = \bigvee_{i \in I} j_i(\bigvee J_{p_i}^i)$. Similarly, as in Theorem 2.2, any p_i is R_i-prime. □

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The following proposition is obvious and follows immediately from Theorem 2.2. Recall only that a completely prime upper set is a complement of a principal ideal.

Proposition 2.4. **PIT** holds if and only if every Scott-open R-distributive filter of a non-trivial partial cm-lattice is an intersection of completely prime R-distributive filters.

We are now going to establish spatiality of continuous (algebraic) R-semidistributive partial cm-lattices. We first prove a simple lemma.

Lemma 2.5. In any R-semidistributive partial cm-lattice Q, each Scott-open filter is a Scott-open distributive filter.

Proof. Let $S \subseteq Q$ be a Scott-open filter of Q, $r, l, a \in Q$, $(r, l) \in R$ and $r \vee a, l \vee a \in S$. Then $(r \vee a) \wedge (l \vee a) \in S$. Since S is an upper set and Q is R-semidistributive, we have $r \cdot l \vee a \in S$. □

The proofs of the following statements are mostly based on Lemma 2.5, and in the remaining parts they mimic the proofs of the corresponding quantalic versions from Paseka (2004). Hence we shall omit them.

Theorem 2.6. Given the **Principle of Countable Dependent Choice** (**CDC**) and **PIT**, every continuous R-semidistributive partial cm-lattice is R-spatial.

Assuming stable continuity or algebraicity of Q, **CDC** is redundant.

Proposition 2.7. Given **PIT**, every stably continuous R-semidistributive partial cm-lattice is R-spatial.

Proposition 2.8. Given **PIT**, every algebraic R-semidistributive partial cm-lattice is R-spatial.

Definition 2.9. A partial cm-lattice Q is said to be *conjunctive* (Paseka 2004) if for each two elements $a, b \in Q, a \nleq b$ there is an element $c \in Q$ such that $a \vee c = 1$ and $b \vee c \neq 1$.

Proposition 2.10. Given **PIT**, any compact conjunctive partial cm-lattice with compact R-distributive top is R-spatial.

Proposition 2.11. Given **PIT**, any conjunctive partial cm-lattice such that $\{1\}$ coincides with an intersection of Scott-open R-distributive filters containing 1 is R-spatial.

We conclude the paper with some comments.

Remark 2.12. Note that, as in Johnstone (1984), our arguments could be carried out in the internal logic of a Boolean topos. Regarding the spatiality properties in Theorem 2.6 and Propositions 2.7 and 2.8, they should be compared with the results in Keimel (1972) and Rosický (1987).

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