2-chains and square roots of Thompson's group F

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Abstract. We study 2-generated subgroups $\langle f, g \rangle < \mathrm{Homeo}^+(I)$ such that $\langle f^2, g^2 \rangle$ is isomorphic to Thompson's group F, and such that the supports of f and g form a chain of two intervals. We show that this class contains uncountably many isomorphism types. These include examples with non-abelian free subgroups, examples which do not admit faithful actions by C^2 diffeomorphisms on 1-manifolds, examples which do not admit faithful actions by PL homeomorphisms on an interval, and examples which are not finitely presented. We thus answer questions due to Brin. We also show that many relatively uncomplicated groups of homeomorphisms can have very complicated square roots, thus establishing the behavior of square roots of F as part of a general phenomenon among subgroups of $\mathrm{Homeo}^+(I)$.

1. *Introduction*

Thompson's group F is a remarkable group of piecewise linear (abbreviated PL) homeomorphisms of the interval $I = [0, 1]$ that occurs naturally and abundantly as a group of homeomorphisms of the real line, and that has been extensively studied since the 1970s. The group F has been shown to satisfy various exotic properties, and has been proposed as a counterexample to well-known conjectures in group theory [**2**, **9**, **12**]. Among the best-known facts about Thompson's group F are the following theorems.

THEOREM 1.1. (Brin and Squier [7]) *The group F satisfies no law and contains no non-abelian free subgroups.*

THEOREM 1.2. (Ghys and Sergiescu [19]) *The group F admits a faithful action by C^∞ diffeomorphisms of the circle.*

THEOREM 1.3. (Thompson; see [12]) *The group F is finitely presented.*

THEOREM 1.4. (See [8, 10, 12, 20, 21]) *The commutator subgroup of F is an infinite simple group.*

In this paper we study a certain class of groups which we call *square roots of Thompson's group F*. These are 2-generated subgroups $\langle f, g \rangle < \mathrm{Homeo}^+(I)$ of the group of orientation-preserving homeomorphisms of the interval, which satisfy

$$\langle f^2, g^2 \rangle \cong \langle A, B \mid [A, (AB)^{-k} B(AB)^k] \quad \text{for } k \in \{1, 2\}\rangle \cong F,$$

and for which the supports supp f and supp g of f and g respectively form a 2-*chain* of intervals. That is, supp f and supp g are both open intervals, and the intersection supp $f \cap$ supp g is a proper subinterval of both supp f and supp g.

Among other things, we demonstrate that (the second part of) Theorems 1.1, 1.2, and 1.3 all fail for square roots of F. In particular, we show that there are square roots of F which contain non-abelian free subgroups, that there are square roots of F which do not admit faithful actions by C^2 diffeomorphisms on the interval, circle, or real line, and that there are uncountably many isomorphism types of square roots of F.

1.1. *Main results.* We denote the set of isomorphism classes of square roots of F by $\mathscr{S}$. The goal of this paper is to produce interesting elements of $\mathscr{S}$. Note that $\mathscr{S}$ contains F for example, since squaring the generators in the standard presentation for F as given above results in a group isomorphic to F.

In this paper we use two different finite presentations of the group F. The first presentation, mentioned above, is

$$\langle A, B \mid [A, (AB)^{-1} B(AB)], [A, (AB)^{-2} B(AB)^2]\rangle \cong F.$$

The second presentation is obtained by performing a Tietze transformation to produce generators $a = AB$, $b = B$, and is given by

$$\langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2]\rangle \cong F.$$

Next, we describe a certain subgroup P of F, which will be needed to state and prove our results. We fix two copies of F:

$$F_1 = \langle p_1, p_2 \mid [p_1 p_2^{-1}, p_1^{-1} p_2 p_1], [p_1 p_2^{-1}, p_1^{-2} p_2 p_1^2]\rangle,$$
$$F_2 = \langle q_1, q_2 \mid [q_1 q_2^{-1}, q_1^{-1} q_2 q_1], [q_1 q_2^{-1}, q_1^{-2} q_2 q_1^2]\rangle.$$

We will write P for the subgroup of $F_1 \times F_2$ generated by (p_1, q_2) and (p_2, q_1). Note that the group P is isomorphic to a subgroup of F which itself contains an isomorphic copy of F as a subgroup. The fact that P is isomorphic to a subgroup of F is an elementary exercise that we leave to the reader, and that F is isomorphic to a subgroup of P is a direct consequence of Brin's ubiquity theorem (see [5]).

We denote the free group on two generators by $\mathbf{F}_2$, and we call a group $H = \langle h_1, h_2 \rangle$ a *marked extension* of P if there exists a surjective homomorphism $H \to P$ where

$$h_1 \mapsto (p_1, q_2), \quad h_2 \mapsto (p_2, q_1).$$

Even though the map $H \to P$ may be suppressed from the notation, we always think of a marked extension of P as equipped with such a homomorphism.

A (countable) group is *left orderable* if it admits a left invariant total ordering, or equivalently if it admits a faithful action by orientation-preserving homeomorphisms of the real line (see [**15**, Proposition 1.1.8] or [**24**, Theorem 2.2.19]). Our main result is the following theorem.

THEOREM 1.5. *Let H be a marked, left orderable extension of P. Then there exists a square root of Thompson's group $G \in \mathscr{S}$ such that $H < G$.*

Since the free group $\mathbf{F}_2$ is left orderable and is naturally a marked extension of P, we immediately obtain the following corollary.

COROLLARY 1.6. *There exists a square root $G \in \mathscr{S}$ such that $\mathbf{F}_2 < G$.*

We will show that square roots of F can contain torsion-free nilpotent groups of arbitrary nilpotence degree. As a consequence of Theorem 1.5 and the Plante–Thurston theorem [**25**], we have the following result.

COROLLARY 1.7. *There exists a square root $G \in \mathscr{S}$ such that G does not admit a faithful action by C^2 diffeomorphisms on a compact 1-manifold or on the real line.*

Corollary 1.7 gives an example of a subgroup $\langle f, g \rangle < \text{Homeo}^+(I)$ which admits no faithful C^2 action on the interval, the circle, or the real line, but where $\langle f^2, g^2 \rangle$ admits a faithful C^∞ action on every 1-manifold (cf. [**19**, **22**]).

Non-abelian nilpotent groups cannot act by piecewise linear homeomorphisms on I or on S^1.

COROLLARY 1.8. *There exists a square root $G \in \mathscr{S}$ such that G does not admit a faithful action by PL homeomorphisms of a compact 1-manifold.*

Corollary 1.8 stands in contrast to the standard definition of F, which is as a group of PL homeomorphisms of the interval. Corollary 1.8 answers a question due to Brin [**6**].

In order to show that square roots of F may not be finitely presented, we prove the following result which is similar in spirit to some of the methods in [**22**].

THEOREM 1.9. *The class $\mathscr{S}$ contains uncountably many distinct isomorphism types.*

Since there are only countably many isomorphism types of finitely presented groups, we immediately obtain the following corollary.

COROLLARY 1.10. *There exists an element $G \in \mathscr{S}$ which admits no finite presentation.*

Analogues of Theorem 1.4 for square roots of F are not the primary topic of this paper, though we can give the following statement which follows immediately from the discussion of commutator subgroups of chain groups in [**22**]. Recall that the action of a group on a topological space is *minimal* if every point has a dense orbit.

PROPOSITION 1.11. *Let $G \in \mathscr{S}$ act minimally on its support. Then the commutator subgroup $[G, G]$ is simple.*

It follows from Proposition 1.11 that if $G = \langle f, g \rangle$, and if f^2 and g^2 generate a copy of Thompson's group F which acts minimally on the interior of I, then the commutator subgroup of G is simple.

1.2. *Square roots of other groups.* Essential in the discussion of square roots of F in this paper is the *dynamical realization* of F on a 2-chain of intervals, which is a dynamical setup in which F occurs naturally (see §2.1, cf. [**15**, Proposition 1.1.8]). If one abandons the dynamical framework of chains of intervals, the group-theoretic diversity phenomena witnessed by Theorems 1.5 and 1.9 become so common as to be a general feature of homeomorphism groups.

To be precise, let $H = \langle h_1, \ldots, h_n \rangle < \mathrm{Homeo}^+(I)$ be a finitely generated subgroup. An n-generated subgroup $G = \langle g_1, \ldots, g_n \rangle < \mathrm{Homeo}^+(I)$ is called a *square root of H* if

$$H \cong \langle g_1^2, \ldots, g_n^2 \rangle.$$

We note that the definition of a square root of H depends implicitly on a choice of generators for H, and is therefore really a square root of a marked group.

If $H = \langle h_1, \ldots, h_n \rangle$ is a generating set for a group H, we will define the *skew subdirect product* of H to be the subgroup of $H \times H$ generated by $\{(h_i, h_i^{-1})\}_{i=1}^n$, and we will denote this group by $\widehat{H}$.

THEOREM 1.12. *Let $\mathbb{Z} = \langle t_1, \ldots, t_{n+1} \mid t_1 = \cdots = t_{n+1} \rangle$, and let $H < \mathrm{Homeo}^+(I)$ be an n-generated group. Then there exists a square root G of $\mathbb{Z}$ such that $\widehat{H} < G$.*

COROLLARY 1.13. *There exist uncountably many isomorphism types of 3-generated subgroup of $\mathrm{Homeo}^+(I)$ such that the squares of the generators generate a cyclic group. Moreover, there exists a 3-generated subgroup of $\mathrm{Homeo}^+(I)$ such that the squares of the generators generate a cyclic group and which contains a non-abelian free group.*

THEOREM 1.14. *Let $L = \mathbb{Z} \wr \mathbb{Z}$ be the lamplighter group, equipped with standard cyclic generators of the two factors of the wreath product. Then L has uncountably many isomorphism types of (marked) square roots.*

In §4.5 we will define the notion of a formal square root of a finitely generated group. We will show that formal square roots of left orderable groups are again left orderable, and generally contain non-abelian free groups.

1.3. *Notes and references.*

1.3.1. *Remarks on context.* The bulk of the present work could just as well be a discussion of the very general setup of 2-generated subgroups of $\text{Homeo}^+(I)$ whose generators are supported on intervals J_1 and J_2, which in turn form a chain. It is well known that under suitable dynamical hypotheses (cf. §1.3.2 below), the resulting subgroup is isomorphic to F. The class $\mathscr{S}$ of square roots of F is merely the first instance of interesting algebraic behavior for such homeomorphism groups which does not follow from the properties of Thompson's group F. In particular, the results of this paper apply to higher roots of F beyond the square root.

1.3.2. *Relation to other authors' work.* To the authors' knowledge, it was Brin [6] who first asked what sorts of groups can occur as square roots of F, and in particular if square roots of F can contain non-abelian free groups, whether they can fail to be finitely presented, and whether they can fail to act by PL homeomorphisms on the interval. The main results of this paper form a natural complement to the joint work of the authors with Kim in [22]. In that paper, Kim and the authors introduced the notions of a *prechain group* and of a *chain group*. In the terminology of [22], square roots of F form a restricted subclass of 2-prechain groups, namely those which square to become 2-chain groups. The class of 2-chain groups in turn consists of just one isomorphism type (i.e. Thompson's group F). Chain groups with 'fast' dynamics also fall into very few isomorphism types (namely the Higman–Thompson groups $\{F_n\}_{n \geq 2}$), and their subgroup structure has been studied independently by Bleak, Brin, Kassabov, Moore, and Zarmesky [3] (cf. [8]). For generalities on Thompson's group F, the reader is directed to the classical Cannon–Floyd–Parry notes [12], as well as Burillo's book [10].

1.3.3. *Bi-orderability.* We briefly remark that many of the groups we construct in this paper, though they are manifestly orderable, will fail to be bi-orderable. Indeed, bi-orderable groups are known to have the *unique root property*. That is, if $f^n = g^n$ for some elements f and g in a biorderable group for some $n \neq 0$, then $f = g$ (see [15, §1.4.2]). One of the themes of this paper is the non-uniqueness of roots of homeomorphisms. Thus, the moment a given element in a left orderable group has two distinct square roots, the group cannot be bi-orderable. See, for instance, Corollary 1.13.

2. *Square roots of F*

In this section we establish the main result, after gathering some relevant preliminary facts and terminology.

2.1. *2-prechain groups and variations thereupon.* Let $\mathscr{J} = \{J_1, J_2\}$ be two non-empty open subintervals of $\mathbb{R}$. We call $\mathscr{J}$ a *chain of intervals* if $J_1 \cap J_2$ is a proper non-empty subinterval of J_1 and of J_2. See Figure 1.

If $f \in \text{Homeo}^+(\mathbb{R})$, we write $\text{supp}\, f = \{x \in \mathbb{R} \mid f(x) \neq x\}$. Let f and g satisfy $\text{supp}\, f = J_1$ and $\text{supp}\, g = J_2$. In the terminology of [22], the group $\langle f, g \rangle$ is a 2-prechain group. Note that, up to replacing f and g by their inverses, we may assume $f(x), g(x) \geq x$ for $x \in \mathbb{R}$.
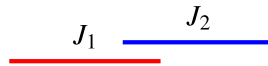
FIGURE 1. A chain of two intervals.

Writing $J_1 = (a, c)$ and $J_2 = (b, d)$ with $a < b < c < d$, we have the following basic dynamical stability result, a proof of which can be found as a special case of [**22**, Lemma 3.1].

LEMMA 2.1. *Suppose $g \circ f(b) \geq c$. Then $\langle f, g \rangle \cong F$.*

Under the dynamical hypotheses of Lemma 2.1, the group $\langle f, g \rangle$ is a chain group. There is another configuration of intervals and homeomorphisms closely related to chain groups, which naturally gives rise to F, which we will need in the sequel, and which we will describe in the next subsection.

2.2. *Nested generators for F.* A natural generating set for F emerges as homeomorphisms supported on a nested pair of intervals, satisfying elementary dynamical conditions. This will be useful in our construction to follow.

LEMMA 2.2. *Let $[a, b_1]$ and $[a, b_2]$ be compact intervals in $\mathbb{R}$ such that $b_1 < b_2$. Let f, g be homeomorphisms satisfying the following conditions.*
(1) *The supports of g and f are contained in $[a, b_1]$ and $[a, b_2]$, respectively.*
(2) *f is a decreasing map on (a, b_2).*
(3) *f, g agree on the interval $[a, f(b_1)]$.*
Then $\langle f, g \rangle \cong F$.

LEMMA 2.3. *Let $[a_1, b]$ and $[a_2, b]$ be compact intervals in $\mathbb{R}$ such that $a_1 < a_2$. Let f, g be homeomorphisms satisfying satisfying the following conditions.*
(1) *The supports of f and g are contained in $[a_1, b]$ and $[a_2, b]$, respectively.*
(2) *f is an increasing map on (a_1, b).*
(3) *f, g agree on the interval $[f(a_2), b]$.*
Then $\langle f, g \rangle \cong F$.

Proofs of Lemmas 2.2 and 2.3. The proofs of both lemmas above follow from checking that the homeomorphisms f and g in each lemma satisfy the relations

$$[fg^{-1}, f^{-1}gf] = 1, \quad [fg^{-1}, f^{-2}gf^2] = 1.$$

Since f and g do not commute, and since every proper quotient of F is abelian (see [**12**, Theorem 4.3]), they generate a group isomorphic to F.   □

2.3. *Orderable extensions of P.* We will use the following standard facts from the theory of orderable groups.

LEMMA 2.4. (See [**15**, Remark 2.1.5]) *Let $1 \to K \to G \to Q \to 1$ be an exact sequence of groups, and suppose that Q and K are left orderable. Then G admits a left ordering which agrees with any prescribed ordering on K. Moreover, any countable, left orderable group can be embedded in $\mathrm{Homeo}^+(I)$.*

The second claim of the lemma is implied by the fact that $\mathbb{R} \cong (0, 1)$. The following lemma is obvious, after the observation that $F \times F$ is left orderable, applying the Brin–Squier theorem [**7**], and Brin's ubiquity theorem [**5**].

LEMMA 2.5. *The group P is a 2-generated sub-direct product of $F \times F$. It is a left orderable group which contains no free subgroups.*

Let $\mathscr{R} < \mathbf{F}_2 = \langle A, B \rangle$ be such that $\mathbf{F}_2/\mathscr{R} \cong P$. Here, the generators A and B of $\mathbf{F}_2$ get sent to the generators (p_1, q_2) and (p_2, q_1), respectively. Note that since $P < F \times F$, we have $\mathscr{R} \neq 1$. Let $\mathscr{R}_k$ denote the kth term of the derived series of $\mathscr{R}$ and let $\mathscr{R}^k$ denote the kth term of the lower central series of $\mathscr{R}$, with the convention $\mathscr{R}_1 = \mathscr{R}^1 = \mathscr{R}$.

LEMMA 2.6. *For each $k \geq 1$, the groups*

$$S_k = \langle A, B \mid \mathscr{R}_k \rangle \quad and \quad N_k = \langle A, B \mid \mathscr{R}^k \rangle$$

are marked, left orderable extensions of P.

Proof. It is clear that for each k, the groups S_k and N_k are quotients of the free group $\mathbf{F}_2$ via the canonical map. Since $\mathscr{R}_k, \mathscr{R}^k \subset \mathscr{R}$, we have that S_k and N_k both surject to P simply by imposing the relations in $\mathscr{R}$. It therefore suffices to show that S_k and N_k are both left orderable, which since P is left orderable, reduces to showing that $\mathscr{R}/\mathscr{R}_k$ and $\mathscr{R}/\mathscr{R}^k$ are left orderable by Lemma 2.4.

Since $\mathscr{R}$ is an infinitely generated free group, these quotients are merely the universal k-step solvable and nilpotent quotients of the infinitely generated free group. We proceed by induction on k. The case $k = 1$ is trivial, and in the case $k = 2$, we obtain the group $\mathbb{Z}^\infty$ which is easily seen to be left orderable. By induction, $\mathscr{R}/\mathscr{R}_k$ (respectively, $\mathscr{R}/\mathscr{R}^k$) is left orderable, and $\mathscr{R}_k/\mathscr{R}_{k+1}$ (respectively, $\mathscr{R}^k/\mathscr{R}^{k+1}$) is again isomorphic to $\mathbb{Z}^\infty$, so the conclusion follows by applying Lemma 2.4 again. □

2.4. *Building square roots of F.* In this section we provide a recipe that produces a square root of F that contains a given group H as a subgroup, provided H is an orderable marked extension of P.

Step 1: Partition $[1, 2)$ into left closed, right open intervals $\{J_1, \ldots, J_{16}\}$ so that J_i occurs to the left of J_j in $\mathbb{R}$ whenever $i < j$. Moreover, we require that these intervals are of the same length. For ease of notation, we denote by J_X for some $X \subset \{1, \ldots, 16\}$ the union $\bigcup_{i \in X} J_i$. For example,

$$J_{\{1,\ldots,4\}} = J_1 \cup J_2 \cup J_3 \cup J_4.$$

Step 2: Construct homeomorphisms f and g of the real line that satisfy the following conditions.

(1) f and g are increasing maps on $(0, 2)$ and on $(1, 3)$, respectively, and equal the identity outside these respective intervals.
(2) f maps J_i isometrically onto J_{i+4} for $1 \leq i \leq 11$.
(3) g maps J_i isometrically onto J_{i+4} for $2 \leq i \leq 12$.
(4) The map gf^{-1} has two components of support, which are

$$[0, 1] \cup J_1, \quad J_{\{12,\ldots,16\}} \cup [2, 3].$$

It is elementary to construct homeomorphisms f and g that satisfy (1)–(3) above. If f and g satisfy (1)–(3), then that gf^{-1} is the identity on $J_{\{2,\dots,11\}}$. Hence the support of gf^{-1} is contained in

$$[0, 1] \cup J_1 \cup J_{\{12,\dots,16\}} \cup [2, 3].$$

To ensure that the components of support of gf^{-1} are precisely as stated in (4), we choose g such that it is sufficiently slow on J_1, and f so that it is sufficiently slow on $J_{\{12,\dots,16\}}$. Note that gf^{-1} is decreasing on the interior of $[0, 1] \cup J_1$ and increasing on the interior of $J_{\{12,\dots,16\}} \cup [2, 3]$.

Step 3: Let $H = \langle h_1, h_2 \rangle$ be a marked, left orderable extension of P. We identify the elements h_1 and h_2 with their dynamical realizations, both supported on the interval J_6. Here, by *dynamical realization* of a countable left orderable group H, we mean an embedding of H into $\mathrm{Homeo}^+(I)$ (see [**15**, Proposition 1.1.8] or [**24**, Theorem 2.2.19]). Define a map h_3 as

$$h_3 = g^{-1}h_2 g = f^{-1}h_2 f.$$

By definition, h_3 is supported on the interval J_{10}. Finally, we define homeomorphisms

$$\lambda_1 = h_1^{-1}h_3^{-1}f, \quad \lambda_2 = g.$$

Our goal for the rest of this section will be to demonstrate the following result.

PROPOSITION 2.7. *The group* $\langle \lambda_1, \lambda_2 \rangle$ *is a marked square root of F which contains H as a subgroup.*

The group $\langle \lambda_1, \lambda_2 \rangle$ is manifestly orderable, since it is presented as a group of orientation-preserving homeomorphisms of the interval. It is clear by our construction that λ_1^2 and λ_2^2 satisfy the dynamical condition of Lemma 2.1, and hence generate a copy of F. So it suffices to show that $H < \langle \lambda_1, \lambda_2 \rangle$.

PROPOSITION 2.8. *The elements* $\lambda_2\lambda_1^{-1}$ *and* $\lambda_1^{-1}\lambda_2$ *generate an isomorphic copy of H.*

Proof. The element $\lambda_2\lambda_1^{-1}$ has four components of support:

$$[0, 1] \cup J_1, \quad J_6, \quad J_{10}, \quad J_{\{12,\dots,16\}} \cup [2, 3].$$

Note that

$$\lambda_2\lambda_1^{-1} \restriction J_6 = h_1, \quad \lambda_2\lambda_1^{-1} \restriction J_{10} = h_3 = f^{-1}h_2 f.$$

We denote by p_2 the restriction

$$\lambda_2\lambda_1^{-1} \restriction [0, 1] \cup J_1 = gf^{-1} \restriction [0, 1] \cup J_1.$$

We denote by q_1 the restriction

$$\lambda_2\lambda_1^{-1} \restriction J_{\{12,\dots,16\}} \cup [2, 3] = gf^{-1} \restriction J_{\{12,\dots,16\}} \cup [2, 3].$$

The element $\lambda_1^{-1}\lambda_2$ has four components of support:

$$[0, 1] \cup J_{\{1,\dots,5\}}, \quad J_{10}, \quad J_{14}, \quad J_{16} \cup [2, 3].$$

Note that

$$\lambda_1^{-1}\lambda_2 \restriction J_{10} = f^{-1}h_1 f.$$

Denote by p_1 the restriction

$$\lambda_1^{-1}\lambda_2 \upharpoonright [0, 1] \cup J_{\{1,\dots,5\}} = f^{-1}g \upharpoonright [0, 1] \cup J_{\{1,\dots,5\}}.$$

Denote by q_2 the restriction

$$\lambda_1^{-1}\lambda_2 \upharpoonright J_{\{14,\dots,16\}} \cup [2, 3].$$

First observe that the restrictions on J_{10} are

$$\lambda_1^{-1}\lambda_2 \upharpoonright J_{10} = f^{-1}h_1 f \upharpoonright J_{10}, \quad \lambda_2\lambda_1^{-1} \upharpoonright J_{10} = h_3 \upharpoonright J_{10} = f^{-1}h_2 f \upharpoonright J_{10}.$$

It follows that this restriction to J_{10} corresponds to the isomorphism

$$H \to \langle \lambda_1^{-1}\lambda_2 \upharpoonright J_{10}, \lambda_2\lambda_1^{-1} \upharpoonright J_{10}\rangle,$$

given by

$$h_2 \mapsto \lambda_2\lambda_1^{-1} \upharpoonright J_{10}, \quad h_1 \mapsto \lambda_1^{-1}\lambda_2 \upharpoonright J_{10},$$

since these restrictions generate a dynamical realization of H on J_{10}.

Next observe that

$$\lambda_2\lambda_1^{-1} \upharpoonright J_6 = h_1 \upharpoonright J_6, \quad \lambda_1^{-1}\lambda_2 \upharpoonright J_6 = \mathrm{id} \upharpoonright J_6.$$

Since H is a marked extension of P, every relation in H is necessarily a product of commutators. It follows that the abelianization of H is $\mathbb{Z}^2$. We then have that this restriction to J_6 corresponds to the quotient

$$H \to \langle \lambda_1^{-1}\lambda_2 \upharpoonright J_6, \lambda_2\lambda_1^{-1} \upharpoonright J_6\rangle,$$

given by

$$h_2 \mapsto \lambda_2\lambda_1^{-1} \upharpoonright J_6, \quad h_1 \mapsto \lambda_1^{-1}\lambda_2 \upharpoonright J_6.$$

which is a homomorphism whose kernel is the normal closure of h_1 in H.

Next, we observe that by construction, the maps p_1, p_2 and q_1, q_2 satisfy the dynamical conditions described in Lemmas 2.2 and 2.3, respectively. Define

$$j_1 = \sup(J_1), \quad j_2 = \inf(J_{14}).$$

By construction, we have

$$p_1(j_1) = \lambda_1^{-1}\lambda_2(j_1) = f^{-1}g(j_1) = g(f^{-1}(j_1)) < 1$$

and

$$p_1 \upharpoonright [0, 1] = \lambda_1^{-1}\lambda_2 \upharpoonright [0, 1] = f^{-1} \upharpoonright [0, 1] = \lambda_2\lambda_1^{-1} \upharpoonright [0, 1] = p_2 \upharpoonright [0, 1].$$

It follows that

$$\langle p_1, p_2\rangle \cong \langle p_1, p_2 \mid [p_1 p_2^{-1}, p_1^{-1}p_2 p_1], [p_1 p_2^{-1}, p_1^{-2}p_2 p_1^2]\rangle \cong F.$$

Next, observe that by construction we have

$$q_1(j_2) = \lambda_2\lambda_1^{-1}(j_2) = gf^{-1}(j_2) = f^{-1}(g(j_2)) > 2$$

and

$$q_2 \upharpoonright [2, 3] = \lambda_1^{-1}\lambda_2 \upharpoonright [2, 3] = g \upharpoonright [2, 3] = \lambda_2\lambda_1^{-1} \upharpoonright [2, 3] = q_1 \upharpoonright [2, 3].$$

It follows that

$$\langle q_1, q_2 \rangle \cong \langle q_1, q_2 \mid [q_1 q_2^{-1}, q_1^{-1} q_2 q_1], [q_1 q_2^{-1}, q_1^{-2} q_2 q_1^2] \rangle \cong F.$$

In particular, the subgroup of $\langle p_1, p_2 \rangle \times \langle q_1, q_2 \rangle$ generated by the elements (p_1, q_2) and (p_2, q_1) is isomorphic to P.

Now we claim that the map

$$h_2 \mapsto \lambda_2 \lambda_1^{-1}, \quad h_1 \mapsto \lambda_1^{-1} \lambda_2$$

extends to an embedding

$$\langle h_1, h_2 \rangle \to \langle \lambda_1, \lambda_2 \rangle.$$

This is true for the component J_{10}, where this is a dynamical realization of H. So it suffices to show that each relation in h_1 and h_2 is satisfied by the restrictions of $\lambda_1^{-1} \lambda_2$ and $\lambda_2 \lambda_1^{-1}$ on other components. As we saw before, for J_6, this via the $\mathbb{Z}$-quotient given by killing the normal closure of the generator $h_1 \in H$, which factors through the abelianization map. For the components

$$[0, 1] \cup J_{\{1, \ldots, 5\}}, \quad J_{\{12, \ldots, 16\}} \cup [2, 3],$$

the action of H is precisely as P, and since H is a marked extension of P, whence the desired conclusion. □

2.5. *Smoothability.* To construct square roots of F which are not conjugate into $\mathrm{Diff}^2(I)$ or $\mathrm{Diff}^2(\mathbb{R})$, the group of C^2 orientation-preserving diffeomorphisms of the interval and the real line respectively, we use the following result.

THEOREM 2.9. (See [**16**, **25**]) *Let* $N < \mathrm{Diff}^2(M)$ *be a finitely generated nilpotent subgroup, where M is a compact and connected 1-manifold. Then N is abelian. Moreover, any nilpotent subgroup of* $\mathrm{Diff}^2(\mathbb{R})$ *is metabelian.*

Proof of Corollary 1.7. Let $N_k = \mathbf{F}_2 / \mathscr{R}^k$ be as in Lemma 2.6. Then $\mathscr{R}/\mathscr{R}^k < N_k$. Taking a finite subset of a free generating set S for $\mathscr{R}$, we have that the image of S in N_k generates a nilpotent subgroup $\Gamma_S < N_k$. It is straightforward to check that Γ_S is a retract of $\mathscr{R}/\mathscr{R}^k$, and is therefore a non-abelian nilpotent subgroup of N_k whenever $k \geq 3$. Applying Theorems 1.5 and 2.9 gives the desired conclusion in the case where M is compact. Choosing a $k \gg 0$ such that N_k contains a nilpotent subgroup which is not metabelian, we get the desired conclusion for $\mathbb{R}$ as well. □

Corollary 1.8 similarly follows from [**16**, Theorems 1.5 and 4.1].

3. *Uncountability of $\mathscr{S}$ and infinitely presented examples*

In this section we prove Theorem 1.9. For this, we retain the notation from the previous discussion.

3.1. *Sources of uncountability.* A construction of P. Hall (sometimes attributed to B. Neumann) on the existence of uncountably many distinct isomorphism classes of 2-generated groups as outlined by de la Harpe in [**13**, part III.C.40] has the advantage that the resulting groups are all left orderable, as observed in [**22**]. We summarize the relevant conclusions here.

PROPOSITION 3.1. *There exists an uncountable class $\mathcal{N}$ of pairwise non-isomorphic groups such that if $N \in \mathcal{N}$ then N is 2-generated, left orderable, and $N^{ab} = \mathbb{Z}^2$. In particular, N can be realized as a subgroup of* Homeo$^+(\mathbb{R})$.

The reader will also find groups in the class $\mathcal{N}$ described explicitly below in the proof of Corollary 1.13.

3.2. *Equations over* Homeo$^+(\mathbb{R})$. In order to prove Theorem 1.9, we will construct an explicit orderable marked extension of P which contains a given element of $\mathcal{N}$ as a subgroup. To do this, we will need to solve equations over Homeo$^+(\mathbb{R})$.

Let $\{f_1, \ldots, f_k, g\} \subset$ Homeo$^+(\mathbb{R})$ be given, and let $w \in \mathbf{F}_n$ be a reduced word in the free group on n fixed generators, where $k < n$. An *equation* over Homeo$^+(\mathbb{R})$ is an expression of the form

$$w(f_1, \ldots, f_k, x_1, \ldots, x_{n-k}) = g.$$

A tuple $\{y_1, \ldots, y_{n-k}\} \subset$ Homeo$^+(\mathbb{R})$ is a *solution* to the equation if this expression becomes an equality after substituting y_i for x_i for each i, and interpreting the expression in Homeo$^+(\mathbb{R})$.

We will restrict out attention to the case where $n = 2$. Even here, equations may not admit solutions. A trivial example can be given by taking $f \neq g$ and setting w to be the first free generator. A slightly less trivial example can be given by taking f to be fixed point free, taking g to have at least one fixed point, and setting w to be a conjugate of the first free generator.

We will concern ourselves with a particular commutator word w with free generators s and t, so that under the map $\mathbf{F}_2 \to P$ given by $s \mapsto (p_1, q_2)$ and $t \mapsto (p_2, q_1)$, the element w lies in the kernel.

The following lemma is key in proving Theorem 1.9.

LEMMA 3.2. *Fix a group $N \in \mathcal{N}$ and let τ be the map $\tau(t) = t + 1$. There exist a homeomorphism $\kappa \in$ Homeo$^+(I)$ and a non-trivial commutator word $w \in \ker\{\mathbf{F}_2 \to P\}$ such that:*

(1) *the group $\langle \kappa, \tau \rangle$ contains N as a subgroup;*
(2) *the equation $w(\tau, x) = \kappa$ admits a solution $y \in$ Homeo$^+(\mathbb{R})$.*

We first show how Lemma 3.2 implies Theorem 1.9.

Proof of Theorem 1.9. We recall some of the notation and the construction in §2.4. We will use informal language below, since we already have a precise description in that subsection.

Given any $h_1, h_2 \in$ Homeo$^+(I)$, we can build a square root $G \in \mathcal{S}$ generated by λ_1, λ_2 such that the group $H = \langle \lambda_1^{-1} \lambda_2, \lambda_2 \lambda_1^{-1} \rangle$ satisfies the following properties.

(1) H acts as a dynamical realization of P on

$$([0, 1] \cup J_{\{1,\ldots,5\}}) \cup (J_{\{12,\ldots,16\}} \cup [2, 3]).$$

(2)   The group $\langle \lambda_2 \lambda_1^{-1} \rangle$ acts faithfully by $\mathbb{Z}$ on the interval J_6 and the element $\lambda_1^{-1} \lambda_2$ acts trivially on the interval J_6.

(3)   The element $\lambda_1^{-1} \lambda_2$ acts as h_1 on J_{10} and the element $\lambda_2 \lambda_1^{-1}$ acts as h_2 on J_{10}.

(4)   The action of H outside the above intervals is trivial.

Let τ, κ and y be the homeomorphisms of the real line from Lemma 3.2. For the rest of the proof, we fix dynamical realizations of τ, κ on J_{10} obtained from conjugating by a homeomorphism of $\mathbb{R}$ to the interior of J_{10}. We shall now denote by τ, κ, y these homeomorphisms supported on J_{10}.

We use the input $h_1 = \tau$ and let $h_2 = y$ to produce a square root G of F. Consider the subgroup K of H generated by

$$k_1 = \lambda_1^{-1} \lambda_2, \quad k_2 = w(\lambda_1^{-1} \lambda_2, \lambda_2 \lambda_1^{-1}).$$

We check the following properties.

(1) $k_1 \restriction J_{10} = \tau$ and $k_2 \restriction J_{10} = \kappa$.

(2) k_2 acts trivially outside J_{10} since $w(s, t)$ represents the identity in P under the map $s \mapsto (p_1, q_2)$ and $t \mapsto (p_2, q_1)$.

(3)   Any commutator vanishes on J_6.

(4) k_1 acts trivially on J_6 and by $\mathbb{Z}$ on

$$([0, 1] \cup J_{\{1,\ldots,5\}}) \cup (J_{\{12,\ldots,16\}} \cup [2, 3]).$$

By our assumption, $N < \langle k_1, k_2 \rangle \restriction J_{10}$. We claim that in fact $N < \langle k_1, k_2 \rangle$. This follows from the fact that the relations in N are elements of the commutator subgroup of the free group, and since $\langle k_1, k_2 \rangle$ acts by $\mathbb{Z}$ outside J_{10}. Therefore $N < G$ where G is the corresponding square root of F.

We thus obtain that if $N \in \mathscr{N}$ is given, then there is a square root $G_N \in \mathscr{S}$ which contains N as a subgroup. Since the class $\mathscr{N}$ contains uncountably many different isomorphism types and since any element of $\mathscr{S}$ is 2-generated and hence countable, the class $\{G_N \mid N \in \mathscr{N}\} \subset \mathscr{S}$ consists of uncountably many different isomorphism types. □

Proof of Lemma 3.2. We shall use the commutator word

$$w(s, t) = [w_1(s, t), w_2(s, t)],$$

where

$$w_1 = [st^{-1}, s^{-2}ts^2], \quad w_2 = t[st^{-1}, t^{-1}s^{-1}t]t^{-1}.$$

It is straightforward to check that for the map $\mathbf{F}_2 \to P$ given by $s \mapsto (p_1, q_2)$ and $t \mapsto (p_2, q_1)$, the element w lies in the kernel.

Let $\phi, \psi \in \mathrm{Homeo}^+(I)$ be given generators of N. We first choose homeomorphisms $\mu, \nu \in \mathrm{Homeo}^+(I)$ such that $\psi = [\mu, \nu^{-1}]$ and homeomorphisms $\chi, \xi \in \mathrm{Homeo}^+(I)$ such that $\phi = [\xi, \chi^{-1}]$. Such choices are possible, since every element of $\mathrm{Homeo}^+(I)$ is a commutator (see, for example, [**11**, Theorem 2.65]).

Identify I with the unit interval $[0, 1] \subset \mathbb{R}$. Recall that τ is translation by 1 on $\mathbb{R}$. We set $\kappa \in \mathrm{Homeo}^+(\mathbb{R})$ as

$$\kappa = (\tau^{-2}\psi\tau^2)(\tau^{-102}\phi\tau^{102}).$$

Intuitively, κ acts by ψ on the interval $[2, 3]$, by ϕ on the interval $[102, 103]$, and by the identity otherwise. We now verify that κ satisfies the conditions of the lemma. We set

$$y = (\tau^{-1}\mu\tau^1)(\tau^{-2}\nu\tau^2)(\tau^{-101}\chi\tau^{101})(\tau^{-102}\xi\tau^{102}).$$

Intuitively, the homeomorphism y acts by μ on $[1, 2]$, by ν on $[2, 3]$, by χ on $[101, 102]$, and by ξ on $[102, 103]$. We check that y is the solution to the equation. We proceed by analyzing the two inner commutators separately, and then considering the outer commutator.

Consider the commutator $[\tau y^{-1}, \tau^{-2}y\tau^2]$. First, note that since y^{-1} has disjoint support from $\tau^{-2}y\tau^2$, they commute, and hence

$$[\tau y^{-1}, \tau^{-2}y\tau^2] = [\tau, \tau^{-2}y\tau^2].$$

We can now easily check that the action of the resulting homeomorphism is as follows.
(1)   It acts by μ on $[2, 3]$, by $\nu\mu^{-1}$ on $[3, 4]$, and by ν^{-1} on $[4, 5]$.
(2)   It acts by χ on $[102, 103]$, by $\xi\chi^{-1}$ on $[103, 104]$, and by ξ^{-1} on $[104, 105]$.
We denote this homeomorphism by α.

Next, consider the commutator

$$[\tau y^{-1}, y^{-1}\tau^{-1}y].$$

First, note that

$$[\tau y^{-1}, y^{-1}\tau^{-1}y] = [\tau, y^{-2}][\tau^{-1}, y^{-1}].$$

It is straightforward to check that the homeomorphism resulting from the product of these commutators is as follows.
(1)   It acts by μ^{-2} on $[0, 1]$, by $\nu^{-2}\mu^3$ on $[1, 2]$, by $\nu^2\mu^{-1}\nu$ on $[2, 3]$, and by ν^{-1} on $[3, 4]$.
(2)   It acts by χ^{-2} on $[100, 101]$, by $\xi^{-2}\chi^3$ on $[101, 102]$, by $\xi^2\chi^{-1}\xi$ on $[102, 103]$, and by ξ^{-1} on $[103, 104]$.
We denote by β the homeomorphism $t[\tau y^{-1}, y^{-1}\tau^{-1}y]t^{-1}$.

Finally, we consider the homeomorphism $[\alpha, \beta]$. Observe that the supports of α and β intersect in the intervals $[2, 3]$ and $[102, 103]$. Since $\psi = [\mu, \nu^{-1}]$, we see that $[\alpha, \beta]$ acts by ψ on $[2, 3]$. Similarly, since $\phi = [\chi, \xi^{-1}]$, we have that $[\alpha, \beta]$ acts by ϕ on $[102, 103]$. It follows that $[\alpha, \beta]$ agrees with κ, whence y is a solution to the equation as claimed.

Finally, we show that $N < \langle \kappa, \tau \rangle$. Indeed, the group generated by $\tau^{-100}\kappa\tau^{100}$ and κ acts as N on the interval $[102, 103]$ and as $\mathbb{Z}$ outside this interval. Since the relations in N are elements of the commutator subgroup of the free group, it follows that this group is isomorphic to N.   □

4.   *General square root phenomena*

In this section, we pass to the completely general setup of finitely generated subgroups of $\mathrm{Homeo}^+(I)$ and address the results in §1.2.

4.1. *Roots of homeomorphisms.* We begin with a completely general construction in $\mathrm{Homeo}^+(I)$ for producing roots of homeomorphisms. The following is a well-known fact, whose proof we recall for the convenience of the reader.

LEMMA 4.1. *Let $f \in \mathrm{Homeo}^+(I)$. Then for all $n \in \mathbb{N}$, there exists an element $g = g_n \in \mathrm{Homeo}^+(I)$ such that $g^n = f$. Moreover, there are uncountably many possible choices of such a map g.*

Proof. By considering the components of the support of f separately, we may consider the case where f has no fixed points in the interval $(0, 1)$. In this case, f is topologically conjugate to the homeomorphism of $\mathbb{R} \cup \{\pm\infty\}$ given by $x \mapsto x + 1$.

We now build an nth root of f defined on all of $\mathbb{R}$ in the following manner. First, we choose arbitrary orientation-preserving homeomorphisms

$$h_m : \left[\frac{m}{n}, \frac{m+1}{n}\right] \rightarrow \left[\frac{m+1}{n}, \frac{m+2}{n}\right]$$

for $0 \leq m \leq n - 2$. Next, we inductively define homeomorphisms

$$h_m : \left[\frac{m}{n}, \frac{m+1}{n}\right] \rightarrow \left[\frac{m+1}{n}, \frac{m+2}{n}\right]$$

for all $m \in \mathbb{Z}$, such that

$$h_{k+(n-1)} \circ \cdots \circ h_k = x + 1$$

for each $k \in \mathbb{Z}$. It is clear then that the homeomorphisms h_m piece together to give a homeomorphism g of $\mathbb{R} \cup \{\pm\infty\}$, whose nth power is translation by 1. Moreover, the arbitrariness of the choices made guarantees that there are uncountably many choices for g.   □

4.2. *Free groups.* Classical results from combinatorial and geometric group theory show that there exist 2-generated groups $\langle a, b \rangle$ which are not free, but such that $\langle a^2, b^2 \rangle$ is free. Moreover, one can arrange for these groups to be left orderable, and hence to be realized as subgroups of $\mathrm{Homeo}^+(I)$. For instance, we take the braid group on three strands,

$$B_3 = \langle a, b \mid aba = bab \rangle.$$

All braid groups are left orderable (in fact the reader may consult [14] as a book dedicated entirely to this subject), and it is a standard fact that the squares of the standard braids generate a free group (see [17, Ch. 3, §5]).

4.3. *The lamplighter group.* Using square roots of F, we can produce many square roots of the lamplighter group $L = \mathbb{Z} \wr \mathbb{Z}$. Recall that

$$\mathbb{Z} \wr \mathbb{Z} \cong \mathbb{Z} \ltimes \left(\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_i\right),$$

where the natural action of $\mathbb{Z}$ is by translating the index $\mathbb{Z}_i \mapsto \mathbb{Z}_{i+1}$. The group L is naturally realized as a subgroup of $\mathrm{Homeo}^+(\mathbb{R}) < \mathrm{Homeo}^+(I)$ as follows. We choose an arbitrary homeomorphism ψ such that $\mathrm{supp}\,\psi = (0, 1) \subset \mathbb{R}$, and then we consider the group generated by ψ and τ, where as before $\tau(x) = x + 1$. It is clear that $\langle \psi, \tau \rangle \cong L$. The following result clearly implies Theorem 1.14, in light of Theorem 1.9.

THEOREM 4.2. *Let G be a left orderable marked extension of P. Then there exists a square root of L containing an isomorphic copy of G.*

Sketch of proof. Let $T \in \text{Homeo}^+(\mathbb{R})$ be given by $T(x) = x + 1/2$. Note that the intervals $(0, 1)$ and $T((0, 1))$ together form a chain of intervals.

Let G be a given left orderable marked extension of P, with distinguished generators g_1 and g_2. Let ψ be a homeomorphism supported on $(0, 1)$ satsifying the following conditions.
(1) $\psi(1/2) = 1/2$.
(2) The group $\langle \psi, T\psi T^{-1} \rangle < \text{Homeo}^+(\mathbb{R})$ contains a copy of G.

It is easy to see that such a ψ exists, since besides the two conditions above it is otherwise arbitrary, and its action on the two halves of $(0, 1)$ can be chosen independently. Thus, one may arrange that the action on $(0, 1/2)$ recovers the generator g_1 and the action on $(1/2, 1)$ recovers the generator g_2. It is clear that squaring ψ and T gives rise to a group isomorphic to the lamplighter group. If one wanted the copy of G to lie in a square root of F, one could further compose ψ with a suitably chosen increasing homeomorphism of $(0, 1)$. The reader should compare with §2.4 for the details of the latter construction. $\square$

4.4. *Square roots of $\mathbb{Z}$.* In this section we give a recipe for producing many n-generated groups of homeomorphisms of the interval, so that the squares of the generators generate a cyclic group.

Proof of Theorem 1.12. Let $\tau_1, \ldots, \tau_{n+1}$ be $n + 1$ copies of the translation $\tau(x) = x + 1$ viewed as a homeomorphism of $\mathbb{R}$, and let

$$\langle h_1, \ldots, h_n \rangle = H < \text{Homeo}^+(I)$$

be an arbitrary n-generated subgroup. We set $T_{n+1}(x) = x + 1/2$.

Lemma 4.1 constructs all possible roots τ, and we follow the construction given there. We first scale down H to be a group of homeomorphisms of $[0, 1/2]$, and we abuse notation and label the generators of H by $\{h_1, \ldots, h_n\}$. We now define T_i to be the homeomorphism $T_{n+1} \circ h_i$ on $[0, 1/2]$. The requirement $T_i^2 = \tau$ determines the values of T_i on the rest of $\mathbb{R}$.

Now let $S_i = T_{n+1}^{-1} \circ T_i$ for $1 \le i \le n$. Observe that S_i acts by h_i on each interval of the form $[k, k + 1/2]$ and by h_i^{-1} on each interval of the form $[k - 1/2, k]$, where $k \in \mathbb{Z}$. It is clear then that $\widehat{H} \cong \langle S_1, \ldots, S_n \rangle < \langle T_1, \ldots, T_{n+1} \rangle$. $\square$

Proof of Corollary 1.13. First, note that if H is free then the skew subdirect product $\widehat{H}$ is also free, which establishes the second part of the corollary. For the first part, we perform a mild modification of the Hall–Neumann groups as discussed in Proposition 3.1 (see [22] for a detailed discussion of these groups). All the Hall–Neumann groups are quotients of a single 2-generated group $\Gamma = \langle t, s_0 \rangle$, which is itself left orderable. The element of $\mathcal{N}$ are given as quotients of Γ by certain central normal subgroups $N_X < \Gamma$.

We recall the definition of Γ for the convenience of the reader, following III.C.40 of de la Harpe's book [13] (cf. [22, Lemma 5.1]). We begin with a set $S = \{s_i\}_{i \in \mathbb{Z}}$. Then we define

$$R = \{[[s_i, s_j], s_k] =\}_{i,j,k \in \mathbb{Z}} \cup \{[s_i, s_j] = [s_{i+k}, s_{j+k}]\}_{i,j,k \in \mathbb{Z}}.$$

The group Γ_0 is defined by $\langle S \mid R \rangle$, and Γ is defined as a semidirect product of $\mathbb{Z}$ with Γ_0, where the generator t of the $\mathbb{Z}$-factor acts by $t^{-1} s_i t = s_{i+1}$. One sets $u_i = [s_0, s_i]$, and if $X \subset \mathbb{Z} \setminus \{0\}$, we write $N_X = \langle \{u_i\}_{i \in X} \rangle$.

Note that the group Γ is generated by t and s_0. It is straightforward to check that the map given by $t \mapsto t^{-1}$ and $s_0 \mapsto s_0^{-1}$ extends to a well-defined automorphism of Γ, whence $\widehat{\Gamma} \cong \Gamma$. Moreover, the subgroups N_X are all stable under this automorphism of Γ. In particular, it follows that if $N \in \mathcal{N}$ is one of the Hall–Neumann groups, then $N \cong \widehat{N}$. The first claim of the corollary follows from Theorem 1.12. $\square$

4.5. *General groups.* For a general finitely generated group $H = \langle x_1, \ldots, x_n \mid R \rangle$, one can *formally take the square root* of H by setting

$$G = \langle y_1, \ldots, y_n, x_1, \ldots, x_n \mid R, x_1 = y_1^2, \ldots, x_n = y_n^2 \rangle.$$

Note that this definition depends on the presentation of H which is given. If H is given as a free group with no relations then G will be free of the same rank. However, if H is not freely presented then G can be very complicated. We will call a presentation for a group H *reduced* if x_i is non-trivial in H for each i.

THEOREM 4.3. *Let H be a left orderable finitely generated group with a reduced presentation. Then the formal square root G of H is left orderable.*

Proof. If $H = \langle x_1, \ldots, x_n \mid R \rangle$, set

$$K = \langle y, x_1, \ldots, x_n \mid R, x_1 = y^2 \rangle.$$

If we can prove that K is left orderable then the result will follow by induction on n.

To this end, note that K admits a description as an amalgamated product via

$$K = \mathbb{Z} *_{2\mathbb{Z} = \langle x_1 \rangle} H.$$

Since we can order $\mathbb{Z}$ either positively or negatively, we may assume that the isomorphism $2\mathbb{Z} \cong \langle x_1 \rangle$ is order preserving. Then, a result of Bludov and Glass [4] (cf. Bergman [1]) implies that the corresponding amalgamated product is again orderable. $\square$

Note that the assumption that the presentation for H in Theorem 4.3 is reduced was essential, since otherwise the formal square root would contain torsion. Moreover, we need not assume that H be finitely generated in Theorem 4.3, and this hypothesis could be replaced by countable generation. We include this hypothesis since nearly all groups under consideration in this paper are finitely generated.

Finally, we show that formal square roots generally contain non-abelian free groups, so that free subgroups are in some precise sense a general phenomenon in square roots of groups of homeomorphisms.

THEOREM 4.4. *Let $H = \langle x_1, \ldots, x_n \mid R \rangle$ be a reduced presentation for a non-cyclic finitely generated left orderable group, and let $K = \langle y, x_1, \ldots, x_n \mid R, x_1 = y^2 \rangle$. Then K contains a non-abelian free group.*

Thus, Theorem 4.4 implies that the formal square root of a non-cyclic group always contains non-abelian free groups.

Proof of Theorem 4.4. The result follows from general Bass–Serre theory. One can construct free subgroups explicitly using the standard theory of normal forms for amalgamated products (see [**23**, **26**] for general introductions to combinatorial group theory, and in particular [**23**, Theorem 2.6] for the normal form theorem for amalgamated products). To do this, let $z \in H \setminus \langle x_1 \rangle$, which exists since H is assumed not to be cyclic. Note that z has infinite order since H is left orderable. Consider the group $\langle z, yzy^{-1} \rangle$. An arbitrary word in these generators will be of the form

$$z^{n_1} y z^{m_1} y^{-1} \cdots z^{n_k} y z^{m_k} y^{-1},$$

where all these exponents are non-zero except possibly n_1 and m_k. This word cannot collapse to the identity since it is in normal form. It follows that the group $\langle z, yzy^{-1} \rangle$ is free. $\qquad\square$

Kassabov has pointed out to the authors that if H is given a presentation with at least three generators then the formal square root G of H surjects onto

$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z},$$

which contains a non-abelian free group.

Finally, we remark that there appears to be little general compatibility between formal square roots of groups and 'dynamical' square roots of groups, such as in our discussion of square roots of F in this paper. That is, let $G = \langle f, g \rangle \in \mathscr{S}$ be a square root of F, so that the supports of f and g form a 2-chain. Then it is never the case that G is the formal square root of F. Indeed, by Theorem 4.4, we would have that g and $h = f^{-1}gf$ generate a free group. This cannot happen, since there is an endpoint x of supp g which is fixed by both g and h, and where the germs of these two homeomorphisms at x agree. By suitably conjugating g or h, one can obtain a homeomorphism k such that supp k is contained in a neighborhood of x on which g and h agree. But then $g^{-1}kg = h^{-1}kh$, violating the fact that $\langle g, h \rangle$ is free.

If, on the other hand, f and g are fully supported homeomorphisms, then fairly easy Baire category methods as in [**18**, Proposition 4.5] show that by choosing generic square roots of f and g, one obtains a group which is isomorphic to the formal square root of $\langle f, g \rangle$.

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