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RANKIN–SELBERG CONVOLUTIONS OF NONCUSPIDAL HALF-INTEGRAL WEIGHT MAASS FORMS IN THE PLUS SPACE

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Dedicated to Professor Hidenori Katsurada on the occasion of his 65th birthday.

Abstract. The author gives the analytic properties of the Rankin–Selberg convolutions of two half-integral weight Maass forms in the plus space. Applications to the Koecher–Maass series associated with nonholomorphic Siegel–Eisenstein series are given.

§1. Introduction

1.1 Main objects of this paper

Let k be an odd integer. Let a_d (resp. b_d) be the dth Fourier coefficient of a weight $-k/2$ Maass cusp form in the plus space on $\Gamma_0(4)$ with the eigenvalue $1/4-\rho^2$ (resp., $1/4-\kappa^2$) of the hyperbolic Laplacian. The Rankin–Selberg convolution is defined by $(\delta=\pm)$

$$\sum_{\substack{d\in\mathbf{Z},\delta d>0\\ d\equiv 0,(-1)^{(k+1)/2} \pmod 4}} \frac{a_d\overline{b_d}}{|d|^{s-1}}. \tag{1}$$

We include the case of Eisenstein series, for which the treatment turns out to be more difficult. Since the Fourier coefficients of the weight $-k/2$ Eisenstein series are described by a certain quadratic L-function, the Rankin–Selberg convolution involves three complex variables s, σ and η in the following way $(\delta=\pm)$;

$$\sum_{d\in\mathbf{Z},\delta d>0} \frac{L_{-d}(\sigma-1)L_{-d}(\eta-1)}{|d|^{s-(\sigma+\eta)/2+(3/2)}}. \tag{2}$$

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The L-function $L_D(s)$ here is defined for any discriminant $D \neq 0, D \equiv 0, 1 \pmod 4$ by

$$L_D(s) := L(s, \chi_K) \sum_{a|f} \mu(a)\chi_K(a)a^{-s}\sigma_{1-2s}(f/a), \tag{3}$$

where f is defined by $D = d_K f^2$ with the discriminant d_K of $K = \mathbf{Q}(\sqrt{D})$, χ_K is the Kronecker symbol, μ is the Möbius function, $\sigma_s(n) = \sum_{d|n} d^s$. We define $L_D(s) := 0$ for $D \equiv 2, 3 \pmod 4$, and use the convention that $L_{f^2}(s) = \zeta(s) \sum_{a|f} \mu(a)a^{-s}\sigma_{1-2s}(f/a)$ when $D = f^2$.

In this paper, we provide meromorphic continuations and vector functional equations of the Dirichlet series (1) and (2). Our main motivation comes from a certain Dirichlet series (called by Koecher–Maass series, KMDS for short) associated with nonholomorphic Siegel–Eisenstein series of even degree [1, 12, 37]. Also, in the case of Maass cusp forms, the Dirichlet series (1) arise in the study of the variance of arithmetic measures [18]. Moreover, by the work of Wen [39] and Chinta and Gunnells [6], the Dirichlet series (2) are closely related to the quadratic A_3-Weyl group multiple Dirichlet series and the Shintani zeta function of Bhargava cubes. First of all, we give more precise descriptions about these appearances.

(A) *Koecher–Maass series of Siegel–Eisenstein series.* Denote by $H_2 := \{Z = X + iY \in M_2(\mathbf{C}) : \ {}^tZ = Z, Y = \Im Z > O\}$ the Siegel half-space of degree 2. For any even integer k and $\sigma \in \mathbf{C}$ such that $2\Re(\sigma) + k > 3$, the degree 2 nonholomorphic Siegel–Eisenstein series of weight k is defined by $(Z \in H_2)$

$$E_{2,k}(Z, \sigma) := \sum_{\{C,D\}} \det(CZ + D)^{-k}|\det(CZ + D)|^{-2\sigma}.$$

Here the sum is taken over all nonassociated coprime symmetric pairs of degree 2. It has a Fourier expansion $(Z = X + iY \in H_2)$

$$E_{2,k}(Z, \sigma) = \sum_{T \in L_2} b(T, k + 2\sigma)\xi(Y, T, \sigma + k, \sigma)e^{2\pi i tr(TX)},$$

where the sum is over the set $L_2 := \left\{T = \begin{pmatrix} a & b/2 \\ b/2 & d \end{pmatrix} ; a, b, d \in \mathbf{Z}\right\}$ of all half-integral symmetric matrices of size two, $b(T, \sigma)$ is the Siegel series and $\xi(Y, T, \alpha, \beta)$ is the confluent hypergeometric function. Kaufhold [15] gave the formula

(4)
$$b(T,\sigma)=\frac{1}{\zeta(\sigma)\zeta(2\sigma-2)}\sum_{d|e(T)} d^{2-\sigma}L_{-(\det\ 2T)/d^2}(\sigma-1)\quad \text{for }\det T\neq 0,$$

where $e(T)=\text{g.c.d}(n,r,m)$ for $T=\begin{pmatrix} n & r/2\\ r/2 & m\end{pmatrix}$. Let us define

$$L_2^+ := \{T\in L_2;\text{positive definite}\}\,,\quad L_2^- := \{T\in L_2;\text{indefinite}\}\,,$$
$$(L_2^-)' := \{T\in L_2;\text{indefinite},\ -\det(2T)\neq\square\}\,,$$
$$E(T) := \{U\in \mathrm{SL}_2(\mathbf{Z});\ {}^tUTU =: T[U]=T\}\quad \text{for }T\in L_2,$$
$$\mu(T) := \int_{E(T)\backslash S_T} ds_T\quad \text{for }T\in (L_2^-)'.$$

Here for $T=\begin{pmatrix} a & b/2\\ b/2 & d\end{pmatrix}\in L_2^-$, we put

$$S_T := \{\tau=u+iv\ ;\ v>0,\ a(u^2+v^2)+bu+d=0\}$$

and ds_T is the line element on S_T induced from the hyperbolic line element $ds^2=(du^2+dv^2)/v^2$. Note that $\mu(T)$ is not finite, if $-\det(2T)=\square$, that is, if $-\det(2T)$ is a square of a natural number. Böcherer's computation in [3] yields then

$$\sum_{T\in L_2^+/\mathrm{SL}_2(\mathbf{Z})}\frac{b(T,\sigma)}{\sharp E(T)(\det\ T)^s}$$
$$=\frac{2^{2s-1}}{\pi}\frac{\zeta(2s+\sigma-2)}{\zeta(\sigma)\zeta(2\sigma-2)}\sum_{d>0}L_{-d}(1)L_{-d}(\sigma-1)d^{-s+1/2},$$

$$\sum_{T\in(L_2^-)'/\mathrm{SL}_2(\mathbf{Z})}\frac{\mu(T)b(T,\sigma)}{|\det\ T|^s}$$
$$=2^{2s+1}\frac{\zeta(2s+\sigma-2)}{\zeta(\sigma)\zeta(2\sigma-2)}\sum_{d>0,d\neq\square}L_d(1)L_d(\sigma-1)d^{-s+1/2}$$

for $s,\sigma\in\mathbf{C}$ with $\Re s\gg 0$. Here the sums extend over all representatives modulo the action $T\to {}^tUTU =: T[U]$ of $\mathrm{SL}_2(\mathbf{Z})$ on the each set L_2^+, $(L_2^-)'$. We find that these explicit forms are similar to the series in (2).

(B) *The variance of arithmetic measures.* The Eisenstein series on $\mathrm{SL}_2(\mathbf{Z})$ is defined as usual by

$$E(\tau,\sigma):=\frac{1}{2}\sum_{\substack{c,d\in\mathbf{Z}\\(c,d)=1}}\frac{v^{\sigma}}{|c\tau+d|^{2\sigma}},\qquad \tau=u+iv,\, v>0,\, \Re(\sigma)>1. \tag{5}$$

For any discriminant $-\Delta\neq 0$, one has [36, 43]

$$|\Delta|^{\sigma/2}\cdot\frac{\zeta(\sigma)}{\zeta(2\sigma)}L_{-\Delta}(\sigma)$$
$$=\begin{cases} 2^{\sigma+1}\displaystyle\sum_{T\in L_2^+(\Delta)/\mathrm{SL}_2(\mathbf{Z})}\frac{E(\tau_T,\sigma)}{\sharp E(T)}, & \Delta>0,\\ \dfrac{\Gamma(\sigma)}{\Gamma(\sigma/2)^2}\displaystyle\sum_{T\in L_2^-(\Delta)/\mathrm{SL}_2(\mathbf{Z})}\int_{E(T)\backslash S_T}E(\tau,\sigma)\,ds_T, & \Delta<0,\quad -\Delta\neq\square.\end{cases}$$

Here $L_2^+(\Delta)$ and $L_2^-(\Delta)$ are defined by

$$L_2^+(\Delta):=\{T\in L_2^+;-\det(2T)=-\Delta\},$$
$$L_2^-(\Delta):=\{T\in L_2^-;-\det(2T)=-\Delta\},$$

we put $\tau_T:=(-b+i\sqrt{\Delta})/(2a)$ for $T=\left(\begin{smallmatrix}a & b/2\\ b/2 & d\end{smallmatrix}\right)\in L_2^+(\Delta)$, and S_T, ds_T are the same as in **(A)** for $T=\left(\begin{smallmatrix}a & b/2\\ b/2 & d\end{smallmatrix}\right)\in L_2^-(\Delta)$. The integral path of $I_T(\sigma)=\int_{E(T)\backslash S_T}E(\tau,\sigma)\,ds_T$ can be taken explicitly as follows; if T is primitive ($e(T)=1$), there exists $M=\begin{pmatrix}\frac{t-bu}{2} & -du\\ au & \frac{t+bu}{2}\end{pmatrix}\in E(T)$ associated to the solution $(t,u)\in\mathbf{N}^2$ of $t^2-|\Delta|u^2=4$, where $(t+u\sqrt{|\Delta|})/2$ takes the minimum among them. As in **(A)**, S_T is the geodesic semi-circle connecting $\alpha'=(-b-\sqrt{|\Delta|})/(2a)$ and $\alpha=(-b+\sqrt{|\Delta|})/(2a)$. For any fixed $z_0\in S_T$, the integral is taken along the line S_T from z_0 to $Mz_0:=(\frac{t-bu}{2}z_0-du)/(auz_0+\frac{t+bu}{2})\in S_T$. If $e(T)=e$, the integral $I_T(\sigma)$ is defined by $I_T(\sigma)=I_{e^{-1}T}(\sigma)$.

These quantities are the Fourier coefficients of a real analytic Eisenstein series of half-integral weight on $\Gamma_0(4)$. We refer the reader in particular to Duke–Imamoḡlu [9] and Ibukiyama and Saito [13]. See also Section 2.6 of the present paper.

On the other hand, when $E(\tau,\sigma)$ is replaced by any Maass cusp form $\varphi(\tau)$ on $\mathrm{SL}_2(\mathbf{Z})$, Katok and Sarnak [14] showed that the corresponding quantities are also the Fourier coefficients a_Δ of a Maass cusp form of 1/2 weight on $\Gamma_0(4)$, namely

$$a_{-\Delta} = \begin{cases} 4\sqrt{\pi}|\Delta|^{-3/4} \displaystyle\sum_{T \in L_2^+(\Delta)/\mathrm{SL}_2(\mathbf{Z})} \frac{\varphi(\tau_T)}{\sharp E(T)}, & \Delta > 0, \\ |\Delta|^{-3/4} \displaystyle\sum_{T \in L_2^-(\Delta)/\mathrm{SL}_2(\mathbf{Z})} \int_{E(T)\backslash S_T} \varphi(\tau)\, ds_T, & \Delta < 0. \end{cases}$$

One can include the case $-\Delta = \square$ (see [4, 14] for S_T in this case), and see Theorem A1 in Biro [4, p. 92] for the modified proportional constant.

The Dirichlet series (1) have been used in the study of the variance of arithmetic measures in Luo *et al.* [18]. As another context, Sato [31] studied this type of Dirichlet series from his theory of prehomogeneous zeta functions.

(C) *Shintani zeta function of Bhargava cubes, the quadratic* A_3*-Weyl group multiple Dirichlet series.* There is another context for the series similar to (2). For any discriminant D and integer m, put

$$A(D, m) := \sharp\{\lambda (\mathrm{mod}\ m); \lambda^2 \equiv D (\mathrm{mod}\ m)\}.$$

Wen [39] established a relation between the quadratic A_3-Weyl group multiple Dirichlet series $Z_{\mathrm{WMDS}}(s_1, s_2, w)$ and the partial Shintani zeta function $Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s_1, s_2, w)$ of PVS of $2 \times 2 \times 2$ cubes, and moreover gave an explicit form for $\Re(w) \gg 0$, $\Re(s_1) \gg 0$, $\Re(s_2) \gg 0$ as

$$Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s_1, s_2, w) = \sum_{\substack{D \\ \text{odd discriminant}}} |D|^{-w} \sum_{m,n \geqslant 1} \frac{B(D, m, n)}{m^{s_1} n^{s_2}},$$

$$B(D, m, n) = \sum_{d | f, m, n} d A\left(\frac{D}{d^2}, \frac{4m}{d}\right) A\left(\frac{D}{d^2}, \frac{4n}{d}\right).$$

Here as in (3), $D = d_K f^2$, $K = \mathbf{Q}(\sqrt{D})$, d_K is its discriminant, and f is the conductor. Using $A(D, 4m) = 2 \cdot \sharp\{\lambda \pmod{2m}; \lambda^2 \equiv D \pmod{4m}\}$ and the identity (cf. Proposition 3 [41] p. 130, Proposition 10.16 [2] p. 168)

$$L_D(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{m=1}^{\infty} \frac{\sharp\{\lambda (\mathrm{mod}\ 2m); \lambda^2 \equiv D (\mathrm{mod}\ 4m)\}}{m^\sigma},$$

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the result can be rewritten as

$$Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s_1, s_2, w) = 4\frac{\zeta(2s_1)\zeta(2s_2)}{\zeta(s_1)\zeta(s_2)}\zeta(2w + s_1 + s_2 - 1) \times \sum_{\substack{D \\ \text{odd discriminant}}} |D|^{-w} L_D(s_1) L_D(s_2).$$

We refer the reader to [39] for the precise definitions of $Z_{\mathrm{WMDS}}(s_1, s_2, w)$ and $Z^{\mathrm{odd}}_{\mathrm{Shintani}}(s_1, s_2, w)$. □

There are at least three possible approaches to study analytic properties concerning the above type of Dirichlet series: Theory of Weyl group multiple Dirichlet series, theory of zeta functions attached to prehomogeneous vector spaces, and the Rankin–Selberg method. In this paper, we work out the third mentioned approach. Undoubtedly, doing so has merits in order to clarify its potential and to recognize the limitation of this approach. Also, the first and second approaches do not seem to be applicable for cusp forms case defined in (1) except for some special cases, and it seems that any detailed treatment of (2) has not been worked out yet.

The value $L_{-d}(\sigma)$ is the dth Fourier coefficient of a certain real analytic Eisenstein series of half-integral weight [9, 10, 13, 19, 35, 40]. Even so, to apply the Rankin–Selberg method, we must take into account the following obstacles as partially explained in [5, p. 7].

(a) Rankin–Selberg method for two Eisenstein series. The obstacle here is that both of them are not of rapid decay. Some problems also occur due to the existence of several cusps.
(b) To pick up each half sum $\sum_{d>0}$ and $\sum_{d<0}$ from the full sum $\sum_{d\neq 0}$. In fact, only the full sum arises naturally from the Mellin transform of nonholomorphic modular forms.
(c) To study $\int_0^\infty v^{s-2} W_{\alpha,\rho}(v) W_{\alpha,\kappa}(v) dv$. Here $W_{\alpha,\mu}(v)$ is the Whittaker function.
(d) To get a simple gamma factor matrix in the functional equation.

Applying several ideas due to Pitale [29], Müller [27] and Zagier [42] together with some additional analysis, we can overcome all of these obstacles and can establish Theorem 1 given in the next Section 1.2.

1.2 Main results of this paper

Suppose that $k \equiv 1 \pmod 4$ and $(s, \sigma, \eta) \in \mathbf{C}^3$ with $\Re s \gg 0$. For any sign $\delta = +$ or $\delta = -$, the 3 variables Dirichlet series (2) with shifted parameters

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are

$$S^{\delta}(s,k,\sigma,\eta) := C_{k,\sigma,\eta}\cdot \sum_{d\in\mathbf{Z},\delta d>0} \frac{L_{-d}(\sigma-\frac{k+1}{2})L_{-d}(\eta-\frac{k+1}{2})}{|d|^{s-(\sigma+\eta-k)/2+1}}$$

$$\times \begin{cases}\Gamma(\frac{\sigma-k}{2})^{-1}\Gamma(\frac{\eta-k}{2})^{-1}, & \delta=+,\\ \Gamma(\frac{\sigma}{2})^{-1}\Gamma(\frac{\eta}{2})^{-1}, & \delta=-,\end{cases} \tag{6}$$

where we put

$$C_{k,\sigma,\eta} := \frac{2^{2k+3-2\sigma-2\eta}\pi^{(\sigma+\eta-k)/2}}{\zeta(2\sigma-k-1)\zeta(2\eta-k-1)}.$$

Of course the variables σ,η should not be in the set of poles of each summand, for example, the poles of $C_{k,\sigma,\eta}$ and $L_{f^2}(\sigma-(k+1)/2)L_{f^2}(\eta-(k+1)/2)$, although we indicated only $\Re s \gg 0$. The same remark should also be applied to the Dirichlet series defined in Section 1.3 ((11) for example). A region of absolute convergence of this Dirichlet series is given in Section 3.2.

THEOREM 1. *The Dirichlet series $S^{\pm}(s,k,\sigma,\eta)$ with $k\equiv 1 \pmod 4$ can be meromorphically continued to the whole $(s,\sigma,\eta)\in\mathbf{C}^3$. More precisely, the function defined by*

$$S^{\delta}(s,k,\sigma,\eta)\cdot\zeta^*(2s)\Gamma(s)^{-2}\cdot s(s-1)(s-1/2)$$
$$\times\ z(k,\sigma,\eta)\prod_{j=1}^{4}\{(s+\alpha_j-1)(s-\alpha_j)\} \tag{7}$$

is holomorphic for all $(s,\sigma,\eta)\in\mathbf{C}^3$, where $\delta=\pm$ and

$$z(k,\sigma,\eta) = (\sigma-(k+2)/2)(\sigma-(k+3)/2)\zeta(2\sigma-k-1)$$
$$\times(\eta-(k+2)/2)(\eta-(k+3)/2)\zeta(2\eta-k-1), \tag{8}$$

$$\alpha_1=\frac{\sigma+\eta-k}{2},\quad \alpha_2=\frac{\sigma-\eta}{2}+1,$$
$$\alpha_3=\frac{-\sigma+\eta}{2}+1,\quad \alpha_4=2-\frac{\sigma+\eta-k}{2}, \tag{9}$$

and $\zeta^*(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$. They satisfy the vector functional equation

$$\begin{pmatrix} S^+(s,k,\sigma,\eta) \\ S^-(s,k,\sigma,\eta) \end{pmatrix} = \frac{\pi^{2s-1}\varphi(s)}{D_{\rho,\kappa}(s)} \begin{pmatrix} \mathcal{V}^+_{-k/4,\rho,\kappa}(s) & \mathcal{V}^-_{-k/4,\rho,\kappa}(s) \\ \mathcal{V}^-_{k/4,\rho,\kappa}(s) & \mathcal{V}^+_{k/4,\rho,\kappa}(s) \end{pmatrix} \times \begin{pmatrix} S^+(1-s,k,\sigma,\eta) \\ S^-(1-s,k,\sigma,\eta) \end{pmatrix}.$$

Here we put

$$\rho = \sigma/2 - k/4 - 1/2, \quad \kappa = \eta/2 - k/4 - 1/2,$$
$$\varphi(s) = \zeta^*(2-2s)/\zeta^*(2s),$$

and $D_{\rho,\kappa}(s)$, $\mathcal{V}^{\pm}_{\alpha,\rho,\kappa}(s)$ are as in Lemma 1 below. □

LEMMA 1. *Let $\alpha \in \mathbf{R}$, $s, \rho, \kappa \in \mathbf{C}$ and $\mathcal{J} = \{\pm t_1, \pm t_2\}$ with $t_1 = \rho + \kappa$, $t_2 = \rho - \kappa$ be a multiset. The functions $D_{\rho,\kappa}(s)$, $\mathcal{V}^{\pm}_{\alpha,\rho,\kappa}(s)$ in Theorem 1 are given explicitly as follows:*

$$D_{\rho,\kappa}(s) = \Gamma(s)^{-2} \prod_{t\in\mathcal{J}} \Gamma(s+t),$$

$$\mathcal{V}^-_{\alpha,\rho,\kappa}(s) = E(\alpha,\rho,\kappa)\sin(2\pi s)\left(\prod_{t\in\mathcal{J}} \sin\pi(s+t)\right)^{-1},$$

$$\mathcal{V}^+_{\alpha,\rho,\kappa}(s) = \pi\sin(\pi s)\{\cos(\pi s)\cos\pi(s+2\alpha) + \cos(\pi t_1)\cos(\pi t_2)\} \times \left(\prod_{t\in\mathcal{J}} \sin\pi(s+t)\right)^{-1}.$$

Here $E(\alpha,\rho,\kappa)$ is defined by

$$E(\alpha,\rho,\kappa) = \frac{-\pi^3}{\Gamma(\frac{1}{2}+\alpha+\rho)\Gamma(\frac{1}{2}+\alpha-\rho)\Gamma(\frac{1}{2}+\alpha+\kappa)\Gamma(\frac{1}{2}+\alpha-\kappa)}.$$ □

The case of cusp forms is easier to treat. Let k be any odd integer. Let $f(\tau) \in \mathcal{F}^+(\Gamma_0(4), \chi, 1/4-\rho^2, -k/2)$ (resp. $g(\tau) \in \mathcal{F}^+(\Gamma_0(4), \chi, 1/4-\kappa^2, -k/2)$) be a Maass cusp form of weight $-k/2$ in the plus space on $\Gamma_0(4)$ with the eigenvalue $1/4-\rho^2$ (resp. $1/4-\kappa^2$) of the Laplacian having the Fourier expansion

$$f(\tau) = \sum_{\substack{d\in\mathbf{Z}, d\neq 0 \\ d\equiv 0,(-1)^{(k+1)/2} \pmod 4}} a_d W_{-\mathrm{sgn}(d)k/4,\rho}(4\pi|d|v)e(du),$$

$$g(\tau) = \sum_{\substack{d\in\mathbf{Z}, d\neq 0 \\ d\equiv 0,(-1)^{(k+1)/2} \pmod 4}} b_d W_{-\mathrm{sgn}(d)k/4,\kappa}(4\pi|d|v)e(du).$$

Here $\tau = u + iv$, $v > 0$. See Section 2 for the precise definitions about these terminologies.

For any sign $\delta = +$ or $\delta = -$ and $\Re(s) \gg 0$, we associate

$$R^{\delta}(s, f, g) := \sum_{\substack{d \in \mathbf{Z}, \delta d > 0 \\ d \equiv 0,\ (-1)^{(k+1)/2} \pmod 4}} \frac{a_d \overline{b_d}}{|d|^{s-1}}.$$

See [27, p. 66] for a region of absolute convergence of this Dirichlet series.

THEOREM 2. *Let k be any odd integer. Let $f(\tau) \in \mathcal{F}^{+}(\Gamma_0(4), \chi, 1/4 - \rho^2, -k/2)$, $g(\tau) \in \mathcal{F}^{+}(\Gamma_0(4), \chi, 1/4 - \kappa^2, -k/2)$ be Maass cusp forms in the plus space. The Dirichlet series $R^{\delta}(s, f, g)$ with $\delta = \pm$ can be meromorphically continued to the whole $s \in \mathbf{C}$. They satisfy the vector functional equation*

$$\begin{pmatrix} R^{+}(s, f, g) \\ R^{-}(s, f, g) \end{pmatrix} = \frac{\pi^{2s-1}\varphi(s)}{D_{\rho,\overline{\kappa}}(s)} \begin{pmatrix} \mathcal{V}^{+}_{-k/4,\rho,\overline{\kappa}}(s) & \mathcal{V}^{-}_{-k/4,\rho,\overline{\kappa}}(s) \\ \mathcal{V}^{-}_{k/4,\rho,\overline{\kappa}}(s) & \mathcal{V}^{+}_{k/4,\rho,\overline{\kappa}}(s) \end{pmatrix} \begin{pmatrix} R^{+}(1-s, f, g) \\ R^{-}(1-s, f, g) \end{pmatrix},$$

where $\varphi(s)$ and $D_{\rho,\kappa}(s)$, $\mathcal{V}^{\pm}_{\alpha,\rho,\kappa}(s)$ are as in Theorem 1 and Lemma 1.

General real weight case can be treated as done in [21, 27], while arithmetically interesting examples occur in integral and half-integral weight cases. In integral weight case, it is expected that the Rankin–Selberg convolution associated to Eisenstein series can be described in terms of the Riemann zeta function and Dirichlet L-function. Hence, its analytic properties seems to be well known. By the way, in view of our prior motivation **(A)** and the facts **(B)**, **(C)**, we restrict ourself to the half-integral weight case with the plus condition in this paper.

1.3 Applications to KMDS

KMDS for degree $n \geqslant 4$. It is well known that the Koecher–Maass series (KMDS for short) associated with any holomorphic Siegel modular form has a meromorphic continuation and a functional equation. But, the approaches taken in this holomorphic case due to Maass, Koecher and Arakawa have not been worked out successfully for nonholomorphic Siegel modular case yet. In fact, Maass [20, p. 307] raised the question "whether it is possible to attach Dirichlet series by means of integral transforms to the nonanalytic

Eisenstein series" and also said that "already in the case degree is two difficulties come up which show that one can not proceed in the usual way". This problem has been considered by Suzuki [37], Arakawa [1], Ibukiyama and Katsurada [12].

Let $n \geqslant 4$ and k be both even. In Ibukiyama–Katsurada's explicit description [12] of the KMDS for nonholomorphic Siegel–Eisenstein series of degree n and weight k, the following Dirichlet series appear as a nontrivial factor ($s, \sigma \in \mathbf{C}$ with $\Re s \gg 0$);

$$\Omega_n^+(s,\sigma) := G_n^+(s,\sigma) \sum_{d\in\mathbf{Z}, (-1)^{(n/2)+1}d>0} \frac{L_{-d}(\frac{n}{2})L_{-d}(2\sigma+k-\frac{n}{2})}{|d|^{s-\sigma-(k/2)+(1/2)}},$$

$$\Lambda_n^-(s,\sigma) := G_n^-(s,\sigma) \sum_{d\in\mathbf{Z}, (-1)^{(n/2)}d>0} \frac{L_{-d}(\frac{n}{2})L_{-d}(2\sigma+k-\frac{n}{2})}{|d|^{s-\sigma-(k/2)+(1/2)}},$$

$$G_n^\delta(s,\sigma) := \pi^{-2s}\zeta(2s)\Gamma(s+\delta t_1)\Gamma(s+\delta t_2) \quad (\delta=\pm),$$

where we put $t_1 := \sigma + k/2 - 1/2$, $t_2 := n/2 - k/2 - \sigma$. A direct application of Theorem 1 implies that

THEOREM 3. *The Dirichlet series $\Omega_n^+(s,\sigma)$ and $\Lambda_n^-(s,\sigma)$ can be meromorphically continued to the whole $(s,\sigma) \in \mathbf{C}^2$. They satisfy the functional equations*

$$\Omega_n^+(s,\sigma) = \Omega_n^+(1-s,\sigma),$$

$$\Lambda_n^-(s,\sigma) = \Lambda_n^-(1-s,\sigma) - 2(-1)^{k/2}\frac{\cos(\pi\sigma)\cos(\pi s)}{\cos\pi(s-\sigma)\sin\pi(s+\sigma)}\frac{G_n^-(1-s,\sigma)}{G_n^+(1-s,\sigma)}\Omega_n^+(1-s,\sigma).$$

Theorem 3 was first proved in [23], then used in [12] to get a simpler functional equations of the KMDS than those proved in [1, 37].

KMDS for degree 2. For any sign $\delta = +$ or $\delta = -$, put

$$G_2^\delta(s,\sigma) := \pi^{-2s}\zeta(2s)\Gamma\left(s+\delta\cdot\frac{\sigma-1}{2}\right)\Gamma\left(s-\delta\cdot\frac{\sigma-2}{2}\right). \tag{10}$$

Let $(s,\sigma) \in \mathbf{C}^2$ with $\Re s$ sufficiently large. We define the series consisting of positive discriminant index by

$$\Omega^-(s,\sigma) := G_2^-(s,\sigma)\cdot \sum_{d<0,-d\neq\square} \frac{L_{-d}(\sigma-1)\cdot |d|^{1/2}L_{-d}(1)}{|d|^{s-(\sigma/2)+1}}$$

$$+\zeta(\sigma-1)\frac{\zeta(2s-\sigma+1)\zeta(2s+\sigma-2)}{\zeta(2s)}G_2^-(s,\sigma)\left(\frac{\zeta'}{\zeta}(2s+\sigma-1)\right.$$

$$\left.+\frac{\zeta'}{\zeta}(2s-\sigma+2)-\frac{\zeta'}{\zeta}(2s+\sigma-2)-\frac{\zeta'}{\zeta}(2s-\sigma+1)+P(s,\sigma)\right). \tag{11}$$

Here

$$P(s,\sigma) := \sum_{\text{prime } p} \frac{(p^{-2s-1}-p^{-2s})\log p}{(1-p^{-2s-\sigma+1})(1-p^{-2s+\sigma-2})}, \tag{12}$$

where the sum is taken over all primes. While, we define the series consisting of negative discriminant index by

$$\Omega^+(s,\sigma) := G_2^+(s,\sigma)\cdot \frac{1}{2\pi}\sum_{d>0} \frac{L_{-d}(\sigma-1)\cdot d^{1/2}L_{-d}(1)}{|d|^{s-(\sigma/2)+1}}, \tag{13}$$

and also we put

$$\mathcal{G}(s,\sigma) := \frac{-\pi}{\sin\pi\left(s-\frac{\sigma}{2}\right)\cos\pi\left(s+\frac{\sigma}{2}\right)} + \frac{\Gamma'}{\Gamma}\left(s+\frac{\sigma-1}{2}\right)$$

$$-\frac{\Gamma'}{\Gamma}\left(s-\frac{\sigma-1}{2}\right)-\frac{\Gamma'}{\Gamma}\left(s+\frac{\sigma-2}{2}\right)+\frac{\Gamma'}{\Gamma}\left(s-\frac{\sigma-2}{2}\right). \tag{14}$$

Using Theorem 1, we prove

THEOREM 4. *The Dirichlet series $\Omega^\pm(s,\sigma)$ can be meromorphically continued to the whole $(s,\sigma)\in\mathbf{C}^2$, and satisfy the functional equations*

$$\Omega^-(1-s,\sigma) = \Omega^-(s,\sigma) - \frac{2^2\pi\cos(\frac{\pi\sigma}{2})\cos(\pi s)}{\sin\pi\left(s-\frac{\sigma}{2}\right)\cos\pi\left(s+\frac{\sigma}{2}\right)}\frac{G_2^-(s,\sigma)}{G_2^+(s,\sigma)}\Omega^+(s,\sigma)$$

$$+2^{-1}\zeta(\sigma-1)\frac{\zeta(2s-\sigma+1)\zeta(2s+\sigma-2)}{\zeta(2s)}G_2^-(s,\sigma)\mathcal{G}(s,\sigma),$$

$$\Omega^+(1-s,\sigma) = \Omega^+(s,\sigma) + \frac{\zeta(\sigma-1)\sin\left(\frac{\pi\sigma}{2}\right)\cos(\pi s)}{2\cos\pi(s-\frac{\sigma}{2})\sin\pi(s+\frac{\sigma}{2})}$$

$$\times G_2^+(s,\sigma)\frac{\zeta(2s+\sigma-2)\zeta(2s-\sigma+1)}{\zeta(2s)}.$$

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By Böcherer [3], KMDS for the positive-definite Fourier coefficients is essentially the series $\Omega^+(s,\sigma)$. In fact, for $s,\sigma\in\mathbf{C}$ with $\Re s\gg 0$, one has

$$\begin{aligned}\xi_2^+(s,\sigma) &:= \pi^{\sigma-2}\Omega^+\left(s+\frac{\sigma}{2}-1,\sigma\right)\\ &= (2\pi)^{-2s}\zeta(\sigma)\zeta(2\sigma-2)\Gamma\left(s+\sigma-\frac{3}{2}\right)\Gamma(s)\\ &\quad\times\sum_{T\in L_2^+/\mathrm{SL}_2(\mathbf{Z})}\frac{b(T,\sigma)}{\sharp E(T)(\det T)^s}.\end{aligned}\tag{15}$$

Let us consider the indefinite Fourier coefficients case. Recall that $\mu(T)=\int_{E(T)\backslash S_T}ds_T$ is an indefinite analog of $(\sharp E(T))^{-1}$, but is not finite if $-\det(2T)$ is a square of a natural number. This gives a nontrivial question to define the associated Dirichlet series in the indefinite case. One can formally define KMDS by omitting these terms. Then, any analytic continuation and any functional equation cannot be expected for the series defined in this manner. Hence some exceptional treatment is required. Indeed, if the terms such that $-d=\square$ are involved, then $L_{f^2}(\sigma)$ has the pole at $\sigma=1$, and thus we cannot put $n=2$ in $\Omega_n^+(s,\sigma)$ or $\Lambda_n^-(s,\sigma)$.

However, in view of the Böcherer type formula together with Theorem 4, it is quite suitable to define KMDS for the indefinite Fourier coefficients by ($s,\sigma\in\mathbf{C}$ with $\Re s\gg 0$)

$$\begin{aligned}\xi_2^-(s,\sigma) &:= 2\pi^{\sigma-2}\Omega^-\left(s+\frac{\sigma}{2}-1,\sigma\right)\\ &= (2\pi)^{-2s}\zeta(\sigma)\zeta(2\sigma-2)\Gamma\left(s-\frac{1}{2}\right)\Gamma(s+\sigma-2)\\ &\quad\times\sum_{T\in(L_2^-)'/\mathrm{SL}_2(\mathbf{Z})}\frac{\mu(T)b(T,\sigma)}{|\det T|^s}\\ &\quad+2\pi^{-2s}\Gamma\left(s-\frac{1}{2}\right)\Gamma(s+\sigma-2)\,\zeta(2s-1)\zeta(2s+2\sigma-4)\\ &\quad\times\left(\frac{\zeta'}{\zeta}(2s+2\sigma-3)+\frac{\zeta'}{\zeta}(2s)-\frac{\zeta'}{\zeta}(2s+2\sigma-4)-\frac{\zeta'}{\zeta}(2s-1)\right.\\ &\quad\left.+P\left(s+\frac{\sigma}{2}-1,\sigma\right)\right)\zeta(\sigma-1),\end{aligned}\tag{16}$$

where $P(s,\sigma)$ is defined by (12). By Theorem 4 we obtain

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THEOREM 5. *The Koecher–Maass series $\xi_2^{\pm}(s,\sigma)$ can be meromorphically continued to the whole $(s,\sigma)\in\mathbf{C}^2$. They satisfy functional equations similar to the one given in Theorem 4.* □

The positive definite case was first proved in [24]. The only special case for degree 2 indefinite Fourier coefficients was treated in [25], which can be proved again from Theorem 4. Theorem 5 gives a complete solution defining Koecher–Maass series for degree 2 indefinite case from the view point of the theory of explicit forms initiated by Ibukiyama–Saito–Katsurada. Here we should be noted that the definition considered by Arakawa, Suzuki, Ibukiyama–Katsurada and also in the present paper is not by means of integral transforms but formal-algebraic in some sense.

Our proof of Theorem 1 gives informations of poles and residues. For example, for any fixed $\sigma\geqslant 0$, one has

$$\sum_{0<d\leqslant X} L_{-d}(\sigma+1)H(d)\sim\frac{\alpha_\sigma}{3}X^{3/2},$$
$$\sum_{0<d\leqslant X} d^{(\sigma+1)/2}L_{-d}(\sigma+1)H(d)\sim\frac{\alpha_\sigma}{\sigma+4}X^{2+\sigma/2},$$

namely,

$$\sum_{0<d\leqslant X} H(d)^2\sim\frac{\pi^4}{2^7\cdot 3^3\cdot\zeta(3)}X^2,$$

as $X\to\infty$, where

$$H(d):=\sum_{T\in L_2^+/\mathrm{SL}_2(\mathbf{Z}),\det(2T)=d}(\sharp E(T))^{-1}, \tag{17}$$
$$\alpha_\sigma:=\frac{\pi}{12}\frac{\zeta(\sigma+2)\zeta(2\sigma+2)}{\zeta(\sigma+3)}.$$

The first asymptotic is consistent with that of [3, p. 27] treating the case $\sigma=k-2$, $k\geqslant 4$ even. These asymptotic formulas follow from a Tauberian theorem stated in Corollary [28, p. 121] together with $L_{-d}(\sigma+1)\geqslant 0$ for any $d>0$, $\sigma\geqslant 0$, which follows from the Euler product expression (cf. [17, proof of Lemma 10]),

$$\mathrm{Res}_{s=(\sigma+3)/2}S^+(s,-3,\sigma,0)=\pi^{\sigma/2+1}2^{-3-\sigma}\frac{\Gamma(\frac{\sigma+2}{2})}{\Gamma(\frac{3}{2})\Gamma(\sigma+2)}\frac{\zeta(\sigma+2)}{\zeta(\sigma+3)}$$

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and the fact that $H(d) = O_\epsilon(d^{1/2+\epsilon})$ for any $1/4 \geqslant \epsilon > 0$ (see Section 2.5 of this paper). By [27, Theorem 5.2, p. 76], we can discuss about remainder terms.

The proof of Theorem 2 will be omitted, since it is the same and easier than that of Theorem 1. Theorems 3–5 follow from Theorem 1 by a specialization of k and η, or computing the Laurent expansion around a suitable pole with respect to the variable η. This approach owes to Ibukiyama and Saito [13] in their treatment of the Shintani zeta functions. See also Sturm [38] for a previous research and Diamantis and Goldfeld [8] for a more recent work on the Shintani zeta functions. We also should mention that there is an unpublished work on $\Omega^+(s,2)$ and $\sum_{0<d\leqslant X} H(d)^2$ by Arakawa as informed to this author from Sato.

§2. Maass forms and Eisenstein series of half-integral weight

2.1 Maass forms

Let $H = \{\tau = u + iv; v > 0\}$ be the upper half-plane. The action of $\mathrm{SL}_2(\mathbf{R})$ on H is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = (a\tau + b)/(c\tau + d)$. For $z \in \mathbf{C} \setminus \{0\}$, the branch of z^α is taken so that $-\pi < \arg z \leqslant \pi$. Let k be any odd integer. For any function $f : H \to \mathbf{C}$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$, put

$$(f|_{-k/2}g)(\tau) := ((c\tau + d)/|c\tau + d|)^{k/2} f(g\tau).$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ 4c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, set $\chi(\gamma) := \epsilon_d^k (4c/d)$, where $\epsilon_d = 1$ if $d \equiv 1 \pmod 4$, while $\epsilon_d = i$ if $d \equiv 3 \pmod 4$, and $(4c/d)$ is as defined in [32, p. 442]. This χ is a multiplier system for $\Gamma_0(4)$ and weight $-k/2$.

We call a smooth function $f(\tau)$ on H to be a Maass form of weight $-k/2$ on $\Gamma_0(4)$, when $f(\tau)$ satisfies

$$f|_{-k/2}\gamma = \chi(\gamma) f \quad (\forall \gamma \in \Gamma_0(4)), \quad (\Delta_{-k/2} + \lambda) f = 0,$$

and $f(\tau)$ has polynomial growth at every cusps of $\Gamma_0(4)$. Here λ is some complex number and $\Delta_{-k/2} = v^2(\partial_u^2 + \partial_v^2) + i(k/2)v\partial_u$. See Definition 2.1 [27, p. 51] and [29, p. 94]. The space of all such functions is denoted by $\mathcal{F}(\Gamma_0(4), \chi, \lambda, -k/2)$ following [27, p. 52].

2.2 Fourier expansion

Any $f(\tau) \in \mathcal{F}(\Gamma_0(4), \chi, \lambda, -k/2)$ has the Fourier expansion of the form ([29, p. 94] and [27, p. 53])

$$f(\tau) = A_0(v) + \sum_{d \neq 0} a_d W_{-\mathrm{sgn}(d)k/4, \rho}(4\pi|d|v) e(du), \tag{18}$$

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where $e(x)=e^{2\pi ix}$, ρ is chosen by $\lambda=1/4-\rho^2$, the constant term has the form

$$A_0(v)=\begin{cases}a_0v^{1/2+\rho}+b_0v^{1/2-\rho}, & \rho\neq 0,\\ a_0v^{1/2}+b_0v^{1/2}\log v, & \rho=0,\end{cases}$$

the Whittaker functions $\omega(v;\alpha,\beta)$ and $W_{\alpha,\beta}(v)$ for $v>0, \alpha,\beta\in\mathbf{C}$ are defined by

$$W_{\alpha,\beta}(v):=v^{\alpha}e^{-v/2}\omega(v;1/2+\alpha+\beta,1/2-\alpha+\beta)\quad(\Re(\beta-\alpha+1/2)>0),$$
$$\omega(v;\alpha,\beta):=v^{\beta}\Gamma(\beta)^{-1}\int_0^{\infty}(1+u)^{\alpha-1}u^{\beta-1}e^{-vu}du\quad(\Re\beta>0).$$

The functions $\omega(v;\alpha,\beta)$ and $W_{\alpha,\beta}(v)$ have holomorphic continuations for all $(\alpha,\beta)\in\mathbf{C}^2$. For any compact set K of $\mathbf{C}^2$, there exist constants $A, B>0$ such that $|\omega(v;\alpha,\beta)|\leqslant A(1+v^{-B})$ for all $v>0$ and $(\alpha,\beta)\in K$. See Lemma 4 [33, p. 90] and [22, Section 7.2]. More facts about $W_{\alpha,\beta}(v)$ are summarized in Section 3.1 of the present paper.

2.3 Plus space

Put $\epsilon=(-1)^{(k+1)/2}$. Pitale [29, p. 94] has defined the plus space by

$$\begin{aligned}&\mathcal{F}^{+}(\Gamma_0(4),\chi,\lambda,-k/2)\\ &\quad:=\{f\in\mathcal{F}(\Gamma_0(4),\chi,\lambda,-k/2); a_d=0, \text{whenever } \epsilon d\equiv 2,3 \pmod 4\}.\end{aligned}$$

If the zeroth Fourier coefficient vanishes, namely $a_0=0$, we call its element by a Maass cusp form in the plus space (as in Theorem 2).

For $f(\tau)\in\mathcal{F}^{+}(\Gamma_0(4),\chi,\lambda,-k/2)$ with the Fourier expansion (18), put

$$f^{(\mu)}(\tau)=\delta_{0\mu}\cdot A_0(v/4)+\sum_{\substack{d\neq 0\\ d\equiv\epsilon\mu \pmod 4}}a_dW_{-\mathrm{sgn}(d)k/4,\rho}(4\pi|d|v/4)e(du/4),$$

where $\mu=0$ or 1, and $\delta_{0\mu}$ is the Kronecker delta. Proposition 4.2 [29, p. 96] tells us that

$$\begin{pmatrix}f^{(0)}(\tau+1)\\ f^{(1)}(\tau+1)\end{pmatrix}=\begin{pmatrix}1&0\\0&\epsilon i\end{pmatrix}\begin{pmatrix}f^{(0)}(\tau)\\ f^{(1)}(\tau)\end{pmatrix},$$
$$\begin{pmatrix}f^{(0)}(-1/\tau)\\ f^{(1)}(-1/\tau)\end{pmatrix}=\left(\frac{\tau}{|\tau|}\right)^{-k/2}\frac{1-\epsilon i}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}f^{(0)}(\tau)\\ f^{(1)}(\tau)\end{pmatrix}.$$

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2.4 Differential operator

The differential operator $E_{-k/2} = v(i\partial_u - \partial_v) - k/4$ is a linear map from $\mathcal{F}(\Gamma_0(4), \chi, \lambda, -k/2)$ to $\mathcal{F}(\Gamma_0(4), \chi, \lambda, -k/2-2)$. We refer the reader to [27, pp. 52, 53]. The action on the Fourier expansion can be seen from the following formula; for $c > 0$ one has

$$E_{-k/2}(W_{-k/4,\rho}(4\pi cv)e(cu)) = \gamma(-k/4, \rho)W_{-k/4-1,\rho}(4\pi cv)e(cu),$$

where $\gamma(\alpha, \rho) = \rho^2 - (\alpha - 1/2)^2$, and for $c < 0$ one has

$$E_{-k/2}(W_{k/4,\rho}(4\pi|c|v)e(cu)) = W_{1+k/4,\rho}(4\pi|c|v)e(cu).$$

Hence, if $f(\tau) \in \mathcal{F}^+(\Gamma_0(4), \chi, \lambda, -k/2)$, then

$$(E_{-k/2}f)(\tau) \in \mathcal{F}^+(\Gamma_0(4), \chi, \lambda, -k/2-2).$$

In particular for $f(\tau) \in \mathcal{F}^+(\Gamma_0(4), \chi, \lambda, -k/2)$ having the expansion (18), we see

$$\begin{aligned}(E_{-k/2}f)(\tau) = (E_{-k/2}A_0)(v) &+ \gamma(-k/4, \rho)\sum_{d>0} a_d W_{-k/4-1,\rho}(4\pi|d|v)e(du)\\ &+ \sum_{d<0} a_d W_{1+k/4,\rho}(4\pi|d|v)e(du),\end{aligned}$$

and thus we can associate

$$\begin{aligned}(E_{-k/2}f)^{(\mu)}(\tau) = \delta_{0\mu} \cdot (E_{-k/2}A_0)(v/4) &+ \gamma(-k/4, \rho)\sum_{\substack{d>0\\ d\equiv\epsilon\mu \pmod 4}} a_d W_{-k/4-1,\rho}(4\pi|d|v/4)e(du/4)\\ &+ \sum_{\substack{d<0\\ d\equiv\epsilon\mu \pmod 4}} a_d W_{1+k/4,\rho}(4\pi|d|v/4)e(du/4)\end{aligned}$$

for $\mu = 0, 1$. Similarly to Section 2.3, we obtain

$$\begin{pmatrix}(E_{-k/2}f)^{(0)}(\tau+1)\\ (E_{-k/2}f)^{(1)}(\tau+1)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \epsilon i\end{pmatrix}\begin{pmatrix}(E_{-k/2}f)^{(0)}(\tau)\\ (E_{-k/2}f)^{(1)}(\tau)\end{pmatrix},$$

$$\begin{pmatrix}(E_{-k/2}f)^{(0)}(-1/\tau)\\ (E_{-k/2}f)^{(1)}(-1/\tau)\end{pmatrix} = \left(\frac{\tau}{|\tau|}\right)^{-k/2-2}\frac{1-\epsilon i}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}(E_{-k/2}f)^{(0)}(\tau)\\ (E_{-k/2}f)^{(1)}(\tau)\end{pmatrix}.$$

2.5 The quadratic L-function

For any fundamental discriminant d_K (including 1) and any natural number f, we define (see (3) for the notations)

$$\Upsilon^s_{d_K}(f) := \sum_{a|f} \mu(a)\chi_K(a)a^{-s}\sigma_{1-2s}(f/a).$$

The quadratic L-function $L_D(s)$ defined in (3) for any $D \neq 0, D \equiv 0, 1 \pmod 4$ is

$$L_D(s) = L(s, \chi_K)\Upsilon^s_{d_K}(f), \quad D = d_K f^2, \quad K = \mathbf{Q}(\sqrt{D}).$$

It is defined by $L_0(s) = \zeta(2s-1)$ and $L_D(s) = 0$ for $D \equiv 2, 3 \pmod 4$. Here we recall the convention that

$$L_{f^2}(s) = \zeta(s) \sum_{a|f} \mu(a)a^{-s}\sigma_{1-2s}(f/a) = \zeta(s)\Upsilon^s_1(f).$$

For $D \neq 0$, the function $L_D(s)$ satisfies (Proposition 3 [41, p. 130])

$$L_D(s) = |D|^{1/2-s}\gamma_{\mathrm{sgn}(D)}(s)L_D(1-s),$$

$$\gamma_{\mathrm{sgn}(D)}(s) = \begin{cases} \pi^{-(1/2)+s}\dfrac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}, & D > 0, \\[2ex] \pi^{-(1/2)+s}\dfrac{\Gamma(\frac{2-s}{2})}{\Gamma(\frac{s+1}{2})}, & D < 0. \end{cases}$$

The following Lemma 2 is due to pp. 198–199 and [30, Theorem 3].

LEMMA 2. *Let $k > 1$ be an integer and χ a primitive Dirichlet character mod k.*

(1) *Suppose $\eta > 0$. One has $|L(s, \chi)| \leqslant \zeta(1+\eta)$ for $s = 1+\eta+it$ with any $t \in \mathbf{R}$.*

(2) *Suppose $0 < \eta \leqslant 1/2$. Then, for $s = -\eta + it$ with any $t \in \mathbf{R}$, one has*

$$|L(s, \chi)| \leqslant (k/(2\pi))^{\eta+(1/2)}|1+s|^{\eta+(1/2)}\zeta(1+\eta).$$

(3) *Suppose $0 < \eta \leqslant 1/2$. Then, for all $s = \sigma + it$ with $-\eta \leqslant \sigma \leqslant 1+\eta$, $t \in \mathbf{R}$, one has*

$$|L(s, \chi)| \leqslant (k|1+s|/(2\pi))^{(1+\eta-\sigma)/2}\zeta(1+\eta). \qquad \square$$

Similarly, we have the following lemma.

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LEMMA 3. *We have* $\Upsilon_{d_K}^{1-s}(f) = \Upsilon_{d_K}^{s}(f) f^{2s-1}$. *Moreover, the following estimations hold true;*

(1) *Suppose* $\eta > 0$. *One has* $|\Upsilon_{d_K}^{s}(f)| \leqslant \zeta(1+\eta)\zeta(1+2\eta)$ *for* $s = 1+\eta+it$ *with any* $t \in \mathbf{R}$.
(2) *Suppose* $0 < \eta \leqslant 1/2$. *One has* $|\Upsilon_{d_K}^{s}(f)| \leqslant f^{1+2\eta}\zeta(1+\eta)\zeta(1+2\eta)$ *for* $s = -\eta + it$ *with any* $t \in \mathbf{R}$.
(3) *Suppose* $0 < \eta \leqslant 1/2$. *One has* $|\Upsilon_{d_K}^{s}(f)| \leqslant f^{1+\eta-\sigma}\zeta(1+\eta)\zeta(1+2\eta)$ *for all* $s = \sigma + it$ *in the vertical strip* $-\eta \leqslant \sigma \leqslant 1+\eta$, $t \in \mathbf{R}$.

Proof. The first statement follows from $n^{\beta}\sigma_{-\beta}(n) = \sigma_{\beta}(n)$ $(n \in \mathbf{N}, \beta \in \mathbf{C})$. If $\Re\alpha > 1$, then $|\sigma_{-\alpha}(n)| \leqslant \zeta(\Re\alpha)$. Hence, for $\Re s > 1$,

$$|\Upsilon_{d_K}^{s}(f)| \leqslant \sum_{a|f} a^{-\Re s}|\sigma_{1-2s}(f/a)| \leqslant \sum_{a|f} a^{-\Re s}\zeta(2\Re s - 1) \leqslant \zeta(\Re s)\zeta(2\Re s - 1).$$

This confirms (1). By $\Upsilon_{d_K}^{s}(f) = \Upsilon_{d_K}^{1-s}(f) f^{1-2s}$, we deduce (2) from (1) as

$$|\Upsilon_{d_K}^{-\eta+it}(f)| = |\Upsilon_{d_K}^{1+\eta-it}(f) f^{1+2\eta-2it}| \leqslant \zeta(1+\eta)\zeta(1+2\eta) f^{1+2\eta}.$$

The statements (1), (2) combined with Rademacher's Phragmen–Lindelöf theorem implies (3). In the notation of Theorem 2 [30, p. 195], we take

$$Q = 1, \quad \alpha = \beta = 0, \quad a = -\eta, \quad b = 1+\eta,$$
$$A = f^{1+2\eta}\zeta(1+\eta)\zeta(1+2\eta), \quad B = \zeta(1+\eta)\zeta(1+2\eta),$$
$$C = 1 + \max\left\{\sum_{a|f} a^{-x}\sigma_{1-2x}(f/a) : x \in [-\eta, 1+\eta]\right\}, \quad c = 1.$$

Using these lemmas, we obtain

PROPOSITION 1. *Suppose* $D \neq 0$ *and* $s \in \mathbf{C}$.

(1) *If* $\Re s > 1$, *one has* $|L_D(s)| \leqslant \zeta(\Re s)^2\zeta(2\Re s - 1)$.
(2) *If* $\Re s < 0$, *one has*

$$|L_D(s)| \leqslant |D|^{(1/2)-\Re s}\zeta(1-\Re s)^2\zeta(1-2\Re s)|\gamma_{\mathrm{sgn}(D)}(s)|.$$

(3) *For any fixed real number* $0 < \xi \leqslant 1/2$, *put* $S(-\xi, 1+\xi) := \{s \in \mathbf{C} : -\xi \leqslant \Re s \leqslant 1+\xi\}$. *The following estimations of* $|L_D(s)|$ *for any* $s \in S(-\xi, 1+\xi)$ *holds;*

(3-1) *If* $D \neq \square$, *one has*

$$|L_D(s)| \leqslant \left(\frac{|D|}{2\pi}\right)^{(1+\xi-\Re s)/2} |1+s|^{(1+\xi-\Re s)/2}\zeta(1+\xi)^2\zeta(1+2\xi).$$

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(3-2) *If* $D = \square$, *let* $D = f^2$. *One has* $L_D(s) = \zeta(s)\Upsilon_1^s(f)$ *and*

$$|(s-1)L_D(s)| \leqslant |(s-1)\zeta(s)| f^{1+\xi-\Re s}\zeta(1+\xi)\zeta(1+2\xi).$$

REMARK 1. The use of Rademacher's Phragmen–Lindelöf theorem to estimate $L_D(s)$ owes to [11]. We also applied this idea to get Lemma 10 [17, p. 198]. Note that in [17, Lemma 10(2)(ii) and Proposition 3(1)(ii)], $f_*^{1+\eta}$ should be $f_*^{1+2\eta}$ as we can see from the above discussion.

REMARK 2. For any discriminant $-d<0$, by taking the residue at $\sigma=1$ of the both sides of the identity given in **(B)** Section 1.1, one has $L_{-d}(1) = 2\pi d^{-1/2}H(d)$, where $H(d)$ is defined in (17). This together with Proposition 1(3-1) yields the estimation $H(d) = O_\epsilon(d^{1/2+\epsilon})$ for any $1/4 \geqslant \epsilon > 0$, which was used in Section 1 of this paper and also in [26, p. 95]. To refer Kitaoka's paper is not correct, because matrix size is assumed to be greater than 2 in Kitaoka's paper.

2.6 Eisenstein series of half-integral weight

In this section, we summarize facts about nonholomorphic Cohen's Eisenstein series introduced by Ibukiyama and Saito [13, p. 276]. Let k be an odd integer and σ a complex number such that $-k+2\Re\sigma-4>0$. Let $\tau = u+iv \in H$. The Eisenstein series is defined by

$$\begin{aligned} &F(k,\sigma,\tau) \\ &\quad = E(k,\sigma,\tau) + 2^{k/2-\sigma}(e(k/8)+e(-k/8))E(k,\sigma,-1/(4\tau))(-2i\tau)^{k/2}, \end{aligned}$$

where

$$E(k,\sigma,\tau) = v^{\sigma/2}\sum_{d=1:\mathrm{odd}}^{\infty}\sum_{c=-\infty}^{\infty}\left(\frac{4c}{d}\right)\epsilon_d^{-k}(4c\tau+d)^{k/2}|4c\tau+d|^{-\sigma}$$

with ϵ_d and $(4c/d)$ being the same as in Section 2.1. Put

$$f(k,\sigma,\tau) = v^{-k/4}F(k,\sigma,\tau),$$
$$\lambda = (\sigma/2-k/4)(1-\sigma/2+k/4), \quad \rho = \sigma/2-k/4-1/2.$$

We have

$$f(k,\sigma,\tau) \in \mathcal{F}^+(\Gamma_0(4),\chi,\lambda,-k/2),$$

and it has the dth Fourier coefficient of the form

$$c(d,\sigma,k) = 2^{k+3/2-2\sigma}e^{\epsilon\pi i/4}\frac{L_{\epsilon d}(\sigma-\frac{k+1}{2})}{\zeta(2\sigma-k-1)} \quad (\epsilon = (-1)^{(k+1)/2}).$$

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More precisely,

$$f(k,\sigma,\tau)=A_0(k,\sigma,v)+\sum_{d\neq 0}a_d(k,\sigma)W_{-\mathrm{sgn}(d)k/4,\rho}(4\pi|d|v)e(du),$$

where the constant term is given by

$$A_0(k,\sigma,v)=v^{\sigma/2-k/4}+b_0(k,\sigma)v^{1-\sigma/2+k/4}, \tag{19}$$

$$b_0(k,\sigma)=2^{3k/2-3\sigma+7/2}(-1)^{(k^2-1)/8}\pi\frac{\Gamma(\sigma-k/2-1)}{\Gamma(\sigma/2-k/2)\Gamma(\sigma/2)}\frac{\zeta(2\sigma-k-2)}{\zeta(2\sigma-k-1)}, \tag{20}$$

and the dth coefficient is given by

$$\begin{aligned}a_d(k,\sigma)&=c(d,\sigma,k)\cdot i^{k/2}\pi^{\sigma/2-k/4}|d|^{\sigma/2-k/4-1}\\&\quad\times\begin{cases}\Gamma((\sigma-k)/2)^{-1}, & d>0,\\ \Gamma(\sigma/2)^{-1}, & d<0.\end{cases}\end{aligned}$$

By definition, one has $c(d,\sigma,k)=0$ for d with $\epsilon d\equiv 2,3 \pmod 4$.

In view of the above Fourier expansion, $f(k,\sigma,\tau)$ has a meromorphic continuation to the whole $\sigma\in\mathbf{C}$ (cf. Proposition 3 [33, p. 91]). Indeed, by the estimation of $L_D(\sigma)$ in Proposition 1 and $W_{\alpha,\mu}(v)$ in Section 3.1 (26), we find that $(\sigma-(k+2)/2)(\sigma-(k+3)/2)\zeta(2\sigma-k-1)f(k,\sigma,\tau)$ is holomorphic for all $\sigma\in\mathbf{C}$. We refer the reader to [7, 10, 19, 40] for other previous researches.

§3. Dirichlet series of 3 variables

3.1 Gamma factors

We follow [27, Section 3]. We also refer the reader to Appendix A.3 [34, pp. 131–136] and [16, 22, 33] for the Whittaker function. The Whittaker function $W_{\alpha,\mu}(v)$ $(v>0)$ in Section 2.2 can be continued to a holomorphic function for all $(\alpha,\mu)\in\mathbf{C}^2$. It satisfies the differential equations with respect to v as

$$v^2W''_{\alpha,\mu}(v)=\left(\tfrac{1}{4}v^2-\alpha v+\mu^2-\tfrac{1}{4}\right)W_{\alpha,\mu}(v), \tag{21}$$

$$vW'_{\alpha,\mu}(v)=\left(\alpha-\tfrac{1}{2}v\right)W_{\alpha,\mu}(v)-\gamma(\alpha,\mu)W_{\alpha-1,\mu}(v), \tag{22}$$

$$vW'_{\alpha,\mu}(v)=-\left(\alpha-\tfrac{1}{2}v\right)W_{\alpha,\mu}(v)-W_{\alpha+1,\mu}(v), \tag{23}$$

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and satisfies the relation $W_{\alpha,-\mu}(v) = W_{\alpha,\mu}(v)$. Here recall from Section 2.4 that $\gamma(\alpha, \rho) = \rho^2 - (\alpha - 1/2)^2$. The behavior is known to be

$$W_{\alpha,\mu}(v) \sim v^{\alpha} e^{-v/2} \quad \text{as } v \to \infty, \tag{24}$$

$$W_{\alpha,\mu}(v) = \begin{cases} O(v^{1/2-|\Re\mu|}), & \mu \neq 0 \\ O(v^{1/2}|\log v|), & \mu = 0 \end{cases} \quad \text{as } v \to 0. \tag{25}$$

Shimura [33] established the following uniform estimation; for any compact set K of $\mathbf{C}^2$, there exist some positive constants $A, B > 0$ such that

$$|W_{\alpha,\mu}(v)| \leqslant A v^{\Re\alpha} e^{-v/2}(1 + v^{-B}) \quad (\forall v > 0, \forall(\alpha, \mu) \in K). \tag{26}$$

The differential equations (22), (23) and the above properties of $W_{\alpha,\mu}(v)$ tell us a similar behavior and estimation for $W'_{\alpha,\mu}(v)$, $W''_{\alpha,\mu}(v)$.

Fix any real number $\alpha \in \mathbf{R}$. For any complex numbers $s, \rho, \kappa \in \mathbf{C}$ with $\Re s > |\Re\rho| + |\Re\kappa|$, we define

$$G_{\alpha,\rho,\kappa}(s) := \int_0^{\infty} v^{s-2} W_{\alpha,\rho}(v) W_{\alpha,\kappa}(v)\, dv. \tag{27}$$

LEMMA 4. *Put $t_1 = \rho + \kappa$ and $t_2 = \rho - \kappa$.*

(1) *The integral defining $G_{\alpha,\rho,\kappa}(s)$ is absolutely convergent for $(s, \rho, \kappa) \in \mathbf{C}^3$ in the region $\Re s > |\Re\rho| + |\Re\kappa|$. It is holomorphic with respect to the variable s in the same region. The function $G_{\alpha,\rho,\kappa}(s)$ satisfies the recurrence*

$$\begin{aligned} &s(s+1)G_{\alpha,\rho,\kappa}(s+2) \\ &\quad = 2\alpha s(2s+1)G_{\alpha,\rho,\kappa}(s+1) + (s^2 - t_1^2)(s^2 - t_2^2)G_{\alpha,\rho,\kappa}(s). \end{aligned}$$

(2) *For any nonnegative integer M, there exist polynomials $p_{\alpha,M}(s, x_1, x_2)$ and $q_{\alpha,M}(s, x_1, x_2) \in \mathbf{R}[s, x_1, x_2]$ satisfying*

$$\begin{aligned} G_{\alpha,\rho,\kappa}(s) \prod_{j=0}^{M} \prod_{l=1}^{2} \{(s+j)^2 - t_l^2\} &= p_{\alpha,M}(s, t_1^2, t_2^2) G_{\alpha,\rho,\kappa}(s+M+1) \\ &\quad + q_{\alpha,M}(s, t_1^2, t_2^2) G_{\alpha,\rho,\kappa}(s+M+2). \end{aligned} \tag{28}$$

This gives a meromorphic continuation of $G_{\alpha,\rho,\kappa}(s)$ to the whole $(s, \rho, \kappa) \in \mathbf{C}^3$. ∎

We need to evaluate the following functions explicitly;

$$D_{\rho,\kappa}(s) := G_{\alpha,\rho,\kappa}(s)G_{1-\alpha,\rho,\kappa}(s) - \gamma(\alpha, \rho)\gamma(\alpha, \kappa)G_{\alpha-1,\rho,\kappa}(s)G_{-\alpha,\rho,\kappa}(s),$$

$$\mathcal{V}^-_{\alpha,\rho,\kappa}(s) := G_{1-\alpha,\rho,\kappa}(s)G_{-\alpha,\rho,\kappa}(1-s) - G_{1-\alpha,\rho,\kappa}(1-s)G_{-\alpha,\rho,\kappa}(s),$$

$$\mathcal{V}^+_{\alpha,\rho,\kappa}(s) := G_{1-\alpha,\rho,\kappa}(s)G_{\alpha,\rho,\kappa}(1-s) - \gamma(\alpha, \rho)\gamma(\alpha, \kappa)G_{\alpha-1,\rho,\kappa}(1-s)G_{-\alpha,\rho,\kappa}(s).$$

The results are stated as Lemma 1 in Section 1.2.

Proof of Lemma 4. We follow the proof of Lemma 3.1 [27, p. 55]. See also Lemma 2.2 [21, p. 159]. By the asymptotics (24), (25) of $W_{\alpha,\mu}(v)$, the integral defining $G_{\alpha,\rho,\kappa}(s)$ is absolutely convergent for $(s, \rho, \kappa) \in \mathbf{C}^3$ in the region $\Re s > |\Re\rho| + |\Re\kappa|$, and it is holomorphic for such s. By integration by parts and (22), (24), (25), we have

$$-(s-1)G_{\alpha,\rho,\kappa}(s) = \int_0^\infty v^{s-1}\omega_{\alpha,\rho,\kappa}(s)\, dv, \tag{29}$$

$$\omega_{\alpha,\rho,\kappa}(s) := W_{\alpha,\rho}(v)W'_{\alpha,\kappa}(v) + W_{\alpha,\kappa}(v)W'_{\alpha,\rho}(v). \tag{30}$$

Applying integration by parts again to (29) implies

$$s(s-1)G_{\alpha,\rho,\kappa}(s) = H_{\alpha,\rho,\kappa}(s) + H_{\alpha,\kappa,\rho}(s) + 2I_{\alpha,\rho,\kappa}(s), \tag{31}$$

where

$$H_{\alpha,\rho,\kappa}(s) := \int_0^\infty v^s W''_{\alpha,\rho}(v)W_{\alpha,\kappa}(v)\, dv,$$
$$I_{\alpha,\rho,\kappa}(s) := \int_0^\infty v^s W'_{\alpha,\rho}(v)W'_{\alpha,\kappa}(v)\, dv.$$

By (21),

$$H_{\alpha,\rho,\kappa}(s) = \tfrac{1}{4}G_{\alpha,\rho,\kappa}(s+2) - \alpha G_{\alpha,\rho,\kappa}(s+1) + \left(\rho^2 - \tfrac{1}{4}\right)G_{\alpha,\rho,\kappa}(s). \tag{32}$$

To evaluate $I_{\alpha,\rho,\kappa}(s)$, use integration by parts and (22), (24), (25) to get

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$$(33)\qquad -(s+1)I_{\alpha,\rho,\kappa}(s)=J_{\alpha,\rho,\kappa}(s)+K_{\alpha,\rho,\kappa}(s),$$
$$J_{\alpha,\rho,\kappa}(s):=\int_0^\infty v^{s-1}\left(\frac{1}{4}v^2-\alpha v-\frac{1}{4}\right)\omega_{\alpha,\rho,\kappa}(s)\,dv,$$
$$K_{\alpha,\rho,\kappa}(s):=\int_0^\infty v^{s-1}(\rho^2 W_{\alpha,\rho}(v)W'_{\alpha,\kappa}(v)+\kappa^2 W_{\alpha,\kappa}(v)W'_{\alpha,\rho}(v))\,dv.$$

See (30) for $\omega_{\alpha,\rho,\kappa}(s)$. By (29), we have

$$4J_{\alpha,\rho,\kappa}(s)=-(s+1)G_{\alpha,\rho,\kappa}(s+2)+4\alpha sG_{\alpha,\rho,\kappa}(s+1)+(s-1)G_{\alpha,\rho,\kappa}(s),$$

and integration by parts implies

$$sK_{\alpha,\rho,\kappa}(s)=-(\rho^2+\kappa^2)I_{\alpha,\rho,\kappa}(s)-\rho^2H_{\alpha,\kappa,\rho}(s)-\kappa^2H_{\alpha,\rho,\kappa}(s).$$

Substituting these into (33), one deduces

$$\begin{aligned}&4\{s(s+1)-(\rho^2+\kappa^2)\}I_{\alpha,\rho,\kappa}(s)\\&\quad=4\rho^2H_{\alpha,\kappa,\rho}(s)+4\kappa^2H_{\alpha,\rho,\kappa}(s)\\&\qquad+s(s+1)G_{\alpha,\rho,\kappa}(s+2)-4\alpha s^2G_{\alpha,\rho,\kappa}(s+1)-s(s-1)G_{\alpha,\rho,\kappa}(s).\end{aligned}$$

Substituting this expression of $I_{\alpha,\rho,\kappa}(s)$ into (31) and then using the formula (32) of $H_{\alpha,\rho,\kappa}(s)$, we get the recurrence relation of $G_{\alpha,\rho,\kappa}(s)$ given in Lemma 4(1) after some computations.

Applying the recurrence repeatedly (M times), for $\Re s>|\Re\rho|+|\Re\kappa|$ and any nonnegative integer M, we find polynomials $p_{\alpha,M}$, $q_{\alpha,M}\in\mathbf{R}[s,x_1,x_1]$ satisfying (28). The uniform estimate (26) yields the holomorphy of the integrals defining $G_{\alpha,\rho,\kappa}(s+M+1)$ and $G_{\alpha,\rho,\kappa}(s+M+2)$ for $(s,\rho,\kappa)\in\mathbf{C}^3$ on the region $\Re s>|\Re\rho|+|\Re\kappa|$, if we chose M being sufficiently large. Accordingly, the holomorphy of $G_{\alpha,\rho,\kappa}(s)$ for $(s,\rho,\kappa)\in\mathbf{C}^3$ on the same region follows.

The identity (28) gives a meromorphic continuation of $G_{\alpha,\rho,\kappa}(s)$ to the whole $(s,\rho,\kappa)\in\mathbf{C}^3$. The possible polar divisors are given by $\{(s+j)^2-t_1^2\}$ $\{(s+j)^2-t_2^2\}$ $(j=0,1,2,\ldots)$. In fact, the above argument tells us that the right-hand side of (28) is holomorphic for all $(s,\rho,\kappa)\in\mathbf{C}^3$ on the region $\Re s+M+1>|\Re\rho|+|\Re\kappa|$, where M can be arbitrary large. □

Proof of Lemma 1 in Section 1.2. We shall prove

$$D_{\rho,\kappa}(s)\prod_{t\in\mathcal{J}}\Gamma(s+t)^{-2}=\Gamma(s)^{-2}\prod_{t\in\mathcal{J}}\Gamma(s+t)^{-1},$$

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$$\mathcal{V}^{-}_{\alpha,\rho,\kappa}(s)\prod_{t\in\mathcal{J}}\sin\pi(s+t)=E(\alpha,\rho,\kappa)\sin(2\pi s),$$

$$\begin{aligned}&\mathcal{V}^{+}_{\alpha,\rho,\kappa}(s)\prod_{t\in\mathcal{J}}\sin\pi(s+t)\\&\quad=\pi\sin(\pi s)\{\cos(\pi s)\cos\pi(s+2\alpha)+\cos\pi(\rho+\kappa)\cos\pi(\rho-\kappa)\}.\end{aligned}$$

Let $H_{\rho,\kappa}(s)$ be any one of the six functions on the both sides of the above three identities. Note that any $H_{\rho,\kappa}(s)$ is holomorphic for all $(s,\rho,\kappa)\in\mathbf{C}^3$. In addition, $H_{x+iy,x-iy}(s)$ is holomorphic for all $(s,x,y)\in\mathbf{C}^3$. These statements follow from the definitions of $D_{\rho,\kappa}(s)$, $\mathcal{V}^{\pm}_{\alpha,\rho,\kappa}(s)$ and the way of the meromorphic continuation of $G_{\alpha,\rho,\kappa}(s)$.

Put $\rho=x+iy$ and $\kappa=x-iy$ with $x,y\in\mathbf{R}$. In this case, Müller proved the above three identities that we are going to establish. We refer the reader to Lemma 3.3 (p. 58), Lemma 3.4 (p. 60) and Lemma 3.5 (p. 62) of [27] Section 3. Fix $s\in\mathbf{C}$, $y\in\mathbf{R}$. Since, any $H_{x+iy,x-iy}(s)$ is holomorphic for all $x\in\mathbf{C}$ as the function of x, the identities are also true for $\rho=x+iy$ and $\kappa=x-iy$ with any $s,x\in\mathbf{C}$, $y\in\mathbf{R}$ by means of analytic continuation. By a similar consideration with any fixed $s\in\mathbf{C}$ and $x\in\mathbf{C}$, the identities hold for $\rho=x+iy$ and $\kappa=x-iy$ with any $s,x\in\mathbf{C}$, $y\in\mathbf{C}$. □

REMARK. The multiset $\mathcal{J}$ in Lemma 3.3 [27, p. 58] should be

$$\mathcal{J}=\{2\Re(\rho),-2\Re(\rho),2i\Im(\rho),-2i\Im(\rho)\}.$$

COROLLARY. *Let $\alpha\in\mathbf{R}$. The function $G_{\alpha,\rho,\kappa}(s)/(\Gamma(s)^2D_{\rho,\kappa}(s))$ is holomorphic for all $(s,\rho,\kappa)\in\mathbf{C}^3$.* □

3.2 Definition and convergence

Since the case $k\equiv 3 \pmod 4$ can be treated by the same manner, we suppose that $k\equiv 1 \pmod 4$, which is sufficient for our applications. The Dirichlet series (6) is the convolution product of the functions $f(k,\sigma,\tau)$ and $f(k,\eta,\tau)$ in Section 2.6, namely, for $s,\sigma,\eta\in\mathbf{C}$ with $\Re s\gg 0$ and for any sign $\delta=+$ or $\delta=-$, it is

$$S^{\delta}(s,k,\sigma,\eta)=\sum_{d\in\mathbf{Z},\delta d>0}a_d(k,\sigma)\overline{a_d(k,\overline{\eta})}|d|^{-(s-1)}. \tag{34}$$

By the estimations in Proposition 1 of Section 2.5, one has the following convergence of the series.

PROPOSITION 2. *Suppose that* $(s, \rho, \kappa) \in \mathbf{C}^3$ *satisfy* $\Re s > 3/2 + |\Re\rho| + |\Re\kappa|$. *Then, the following partial series defining* $S^\delta(s, k, \sigma, \eta)$ *in (6) are absolutely convergent and holomorphic for the three variables on the region* $\Re s > 3/2 + |\Re\rho| + |\Re\kappa|$;

$$\sum_{\substack{d\in\mathbf{Z},\delta d>0\\ -d\neq\square}} \frac{L_{-d}(2\rho+\frac{1}{2})L_{-d}(2\kappa+\frac{1}{2})}{|d|^{s-\rho-\kappa}},$$

$$\left(2\rho-\tfrac{1}{2}\right)\left(2\kappa-\tfrac{1}{2}\right)\sum_{d\in\mathbf{Z},-d=\square} \frac{L_{-d}(2\rho+\frac{1}{2})L_{-d}(2\kappa+\frac{1}{2})}{|d|^{s-\rho-\kappa}}.$$

REMARK. Any polynomial growth estimate of $L_{-d}(s)$ with respect to $|d|$, which can be deduced from Lemma 5 [33, p. 90], is sufficient for our purpose.

3.3 Analytic properties

Let $k \equiv 1 \pmod 4$ be an integer and $\sigma, \eta \in \mathbf{C}$. Denote by $\mathcal{H}_{k,\sigma,\eta}(\tau)$ one of the following two functions defined by

$$\mathcal{F}_{k,\sigma,\eta}(\tau) := \sum_{\mu=0,1} f^{(\mu)}(k,\sigma,\tau)\overline{f^{(\mu)}(k,\overline{\eta},\tau)},$$

$$\mathcal{G}_{k,\sigma,\eta}(\tau) := \sum_{\mu=0,1} (E_{-k/2}f)^{(\mu)}(k,\sigma,\tau)\overline{(E_{-k/2}f)^{(\mu)}(k,\overline{\eta},\tau)}.$$

See Sections 2.3 and 2.4 for the definitions of $f^{(\mu)}$ and $(E_{-k/2}f)^{(\mu)}$ respectively. This satisfies

$$\mathcal{H}_{k,\sigma,\eta}(\gamma\tau) = \mathcal{H}_{k,\sigma,\eta}(\tau) \quad \text{for all } \gamma \in \mathrm{SL}_2(\mathbf{Z})$$

in view of the transformation formulas in Sections 2.3 and 2.4. Following Zagier (15) [42, p. 419], its integral transform is defined by

$$R(\mathcal{H}_{k,\sigma,\eta}, s) := \int_0^\infty \int_0^1 [\mathcal{H}_{k,\sigma,\eta}(\tau) - \psi_{\mathcal{H}_{k,\sigma,\eta}}(v/4)]v^{s-2}\,du\,dv,$$

where $s, \sigma, \eta \in \mathbf{C}$ with sufficiently large $\Re s$, and

$$\psi_{\mathcal{F}_{k,\sigma,\eta}}(v) := A_0(k,\sigma,v)\overline{A_0(k,\overline{\eta},v)},$$

$$\psi_{\mathcal{G}_{k,\sigma,\eta}}(v) := (E_{-k/2}A_0)(k,\sigma,v)\overline{(E_{-k/2}A_0)(k,\overline{\eta},v)}.$$

See (19) for $A_0(k,\sigma,v)$. We have seen in Section 2.6 that $z(k,\sigma,\eta)\mathcal{H}_{k,\sigma,\eta}(\tau)$ is a holomorphic function for all $s, \sigma, \eta \in \mathbf{C}$. Here $z(k,\sigma,\eta)$ is defined by (8) in Section 1.2 Theorem 1.

Proposition 3.

(1) *Let* $\rho = \sigma/2 - k/4 - 1/2, \kappa = \eta/2 - k/4 - 1/2$ *be as in Theorem 1. The integral defining* $z(k, \sigma, \eta)R(\mathcal{H}_{k,\sigma,\eta}, s)$ *is absolutely convergent for* $\Re s > 2 + |\Re\rho| + |\Re\kappa|$. *In this region, the following relations hold;*

$$\pi^{s-1}R(\mathcal{F}_{k,\sigma,\eta}, s) = G_{-k/4,\rho,\kappa}(s)S^{+}(s, k, \sigma, \eta) + G_{k/4,\rho,\kappa}(s)S^{-}(s, k, \sigma, \eta),$$

$$\begin{aligned}\pi^{s-1}R(\mathcal{G}_{k,\sigma,\eta}, s) &= \gamma(-k/4, \rho, \kappa)G_{-k/4-1,\rho,\kappa}(s)S^{+}(s, k, \sigma, \eta)\\ &\quad + G_{1+k/4,\rho,\kappa}(s)S^{-}(s, k, \sigma, \eta),\end{aligned}$$

where $G_{\alpha,\rho,\kappa}(s)$ *is defined in (27), and* $\gamma(\alpha, \rho, \kappa) = \gamma(\alpha, \rho)\gamma(\alpha, \kappa)$ *with* $\gamma(\alpha, \rho) = \rho^2 - (\alpha - 1/2)^2$.

(2) *The function defined by*

$$R^{*}(\mathcal{H}_{k,\sigma,\eta}, s) = \zeta^{*}(2s)R(\mathcal{H}_{k,\sigma,\eta}, s)$$

with $\zeta^{*}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ *can be meromorphically continued to the whole* $(s, \sigma, \eta) \in \mathbf{C}^3$, *and satisfies the functional equation*

$$R^{*}(\mathcal{H}_{k,\sigma,\eta}, s) = R^{*}(\mathcal{H}_{k,\sigma,\eta}, 1 - s).$$

Proof. First of all, note that $\psi_{\mathcal{H}_{k,\sigma,\eta}}(v/4)$ has the form

$$\psi_{\mathcal{H}_{k,\sigma,\eta}}(v/4) = \sum_{j=1}^{4} c_j v^{\alpha_j},$$

where α_j are as in (9) and c_j are functions of σ, η defined as follows; if $\mathcal{H}_{k,\sigma,\eta} = \mathcal{F}_{k,\sigma,\eta}$, then we put

$$c_1 = 2^{k-\sigma-\eta}, \quad c_2 = 2^{\eta-\sigma-2}b_0(k, \eta),$$
$$c_3 = 2^{\sigma-\eta-2}b_0(k, \sigma), \quad c_4 = 2^{\sigma+\eta-k-4}b_0(k, \sigma)b_0(k, \eta),$$

while if $\mathcal{H}_{k,\sigma,\eta} = \mathcal{G}_{k,\sigma,\eta}$, then we put

$$c_1 = 2^{k-\sigma-\eta-2}\sigma\eta, \quad c_2 = 2^{\eta-\sigma-4}\sigma(2 - \eta + k)b_0(k, \eta),$$
$$c_3 = 2^{\sigma-\eta-4}\eta(2 - \sigma + k)b_0(k, \sigma),$$
$$c_4 = 2^{\sigma+\eta-k-6}(2 - \eta + k)(2 - \sigma + k)b_0(k, \sigma)b_0(k, \eta).$$

See (20) for the definition of $b_0(k, \sigma)$.

Since each α_j in (9) has the form $\alpha_j = \pm\rho \pm \kappa + 1$, we have $|\Re\alpha_j| \leqslant |\Re\rho| + |\Re\kappa| + 1$. Proposition 1, (26) and Section 2.6 imply that $z(k, \sigma, \eta)[\mathcal{H}_{k,\sigma,\eta}(\tau) - \psi_{\mathcal{H}_{k,\sigma,\eta}}(v/4)]$ is of rapid decay and $z(k, \sigma, \eta)\mathcal{H}_{k,\sigma,\eta}(\tau) = O(v^{|\Re\rho|+|\Re\kappa|+1})$ as $v \to \infty$, where $z(k, \sigma, \eta)$ is as in (8). This combined with the argument given in [42, p. 421] yields the behavior of $z(k, \sigma, \eta)\mathcal{H}_{k,\sigma,\eta}(\tau)$ as $v \to 0$, and we obtain the region of the absolutely convergent for the integral defining $z(k, \sigma, \eta)R(\mathcal{H}_{k,\sigma,\eta}, s)$ as stated in Proposition 3(1). The identities in Proposition 3(1) follow from a direct computation together with (34), (27). By the equation (27) of [42, p. 424], we have the integral representation

$$\begin{aligned} R^*(\mathcal{H}_{k,\sigma,\eta}, s) = &\int\int_{D-D_T} [\mathcal{H}_{k,\sigma,\eta}(\tau)E^*(\tau, s) - \psi_{\mathcal{H}_{k,\sigma,\eta}}(v/4)e(v, s)]\frac{du\, dv}{v^2} \\ &+ \int\int_{D_T} \mathcal{H}_{k,\sigma,\eta}(\tau)E^*(\tau, s)\frac{du\, dv}{v^2} \\ &- \zeta^*(2s)h_{T,\mathcal{H}_{k,\sigma,\eta}}(s) - \zeta^*(2s-1)h_{T,\mathcal{H}_{k,\sigma,\eta}}(1-s). \end{aligned} \tag{35}$$

Here we put $D = \{\tau = u + iv \in H;\, |\tau| \geqslant 1, |u| \leqslant 1/2\}$, $D_T = \{u + iv \in D;\, v \leqslant T\}$ with any fixed $T \gg 0$, $E^*(\tau, s) = \zeta^*(2s)E(\tau, s)$ is the Eisenstein series (5) initially defined for $\Re s > 1$ and its meromorphic continuation to the whole $s \in \mathbf{C}$, $e(v, s) = \int_0^1 E^*(\tau, s)\, du$ is the constant term of $E^*(\tau, s)$, and $h_{T,\mathcal{H}_{k,\sigma,\eta}}(s)$ is given by (cf. (26) in [42, p. 423])

$$h_{T,\mathcal{H}_{k,\sigma,\eta}}(s) = \sum_{j=1}^{4} c_j \frac{T^{s+\alpha_j-1}}{s + \alpha_j - 1}$$

with the same α_j and c_j as above.

In view of (35) and the well-known properties of $E^*(\tau, s)$ (which can be found in [42, pp. 415, 416, 422]) together with analytic properties of $z(k, \sigma, \eta)\mathcal{H}_{k,\sigma,\eta}(\tau)$ including its behavior as $v \to \infty$ (which can be deduced from Proposition 1, (26) and Section 2.6), we conclude Proposition 3(2). ∎

Proof of Theorem 1. It follows from Proposition 3 (1) that

$$\begin{aligned} &\pi^{s-1}\begin{pmatrix} R(\mathcal{F}_{k,\sigma,\eta}, s) \\ R(\mathcal{G}_{k,\sigma,\eta}, s) \end{pmatrix} \\ &= \begin{pmatrix} G_{-k/4,\rho,\kappa}(s) & G_{k/4,\rho,\kappa}(s) \\ \gamma(-k/4, \rho, \kappa)G_{-k/4-1,\rho,\kappa}(s) & G_{1+k/4,\rho,\kappa}(s) \end{pmatrix} \begin{pmatrix} S^+(s, k, \sigma, \eta) \\ S^-(s, k, \sigma, \eta) \end{pmatrix}, \end{aligned}$$

$$\begin{pmatrix} S^+(s, k, \sigma, \eta) \\ S^-(s, k, \sigma, \eta) \end{pmatrix} = \frac{\pi^{s-1}}{\mathcal{D}_{\rho,\kappa}(s)}$$

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$$\times \begin{pmatrix} G_{1+k/4,\rho,\kappa}(s) & -G_{k/4,\rho,\kappa}(s) \\ -\gamma(-k/4,\rho,\kappa)G_{-k/4-1,\rho,\kappa}(s) & G_{-k/4,\rho,\kappa}(s) \end{pmatrix} \begin{pmatrix} R(\mathcal{F}_{k,\sigma,\eta},s) \\ R(\mathcal{G}_{k,\sigma,\eta},s) \end{pmatrix}.$$

Notice from Section 2.6 that $(\sigma-(k+2)/2)(\sigma-(k+3)/2)\zeta(2\sigma-k-1)$ $f(k,\sigma,\tau)$ is entire for $\sigma \in \mathbf{C}$. This combined with (35) and Corollary in Section 3.1 implies that the function defined by (7) is holomorphic for all $(s,\sigma,\eta) \in \mathbf{C}^3$. The functional equation stated in Theorem 1 follows from Proposition 3(2) and the above matrix relations. Note that we use

$$\gamma(\alpha,\rho)\gamma(\alpha,\kappa)\mathcal{V}^-_{\alpha,\rho,\kappa}(s) = \mathcal{V}^-_{\alpha-1,\rho,\kappa}(s), \quad \mathcal{V}^+_{\alpha,\rho,\kappa}(s) = \mathcal{V}^+_{\alpha-1,\rho,\kappa}(s),$$

which follows from Lemma 1 in Section 1.2 or can be proved by the same way as in [27, Section 3], to rewrite (2, 1) and (2, 2) components of the matrix. Theorem 2 can be treated in the same manner and we omit its proof. □

3.4 Partial series

Suppose that $(s,\sigma,\eta) \in \mathbf{C}^3$ satisfies $\Re s \gg 0$ $(\sigma \neq 0, \eta \neq 0)$. By a simple computation, one finds that

$$\sum_{d \in \mathbf{Z}, -d = \square} L_{-d}(\eta+1)L_{-d}(\sigma+1)d^{(\eta+\sigma-s)/2} = \zeta(\eta+1)\zeta(\sigma+1)$$
$$\times \frac{\zeta(-\eta-\sigma+s)\zeta(\sigma-\eta+s+1)\zeta(\eta-\sigma+s+1)\zeta(\eta+\sigma+s+2)}{\zeta(s+1)}$$
$$\times \prod_{\text{prime } p} \{(1+p^{-s-1})(1+p^{-s-2}) - (p^{\sigma-s-1}+p^{\eta-s-1})(1+p^{-\eta-\sigma-1})\}.$$

Here the product is taken over all primes.

§4. Degree 2 and indefinite Fourier coefficients

Theorem 3 and the statement about $\Omega^+(s,\sigma)$ in Theorem 4 are simple specializations of Theorem 1. So, we omit the proof. In this section, we prove the statement about $\Omega^-(s,\sigma)$ in Theorem 4.

4.1 Definitions of $\mathcal{A}^*(s,\sigma,\eta), \mathcal{B}^*_j(s,\sigma,\eta)$

Put $k=1$. The parameters ρ, κ in Theorem 1 and the related quantities in Lemma 1 are given by

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$$\rho = \sigma/2 - 3/4, \quad \kappa = \eta/2 - 3/4,$$
$$\mathcal{J} = \{\pm t_1, \pm t_2\}, \quad t_1 = (\sigma + \eta - 3)/2, \quad t_2 = (\sigma - \eta)/2.$$

For $\Re s > 3/2 + |\Re\rho| + |\Re\kappa|$ and for any sign $\delta = +$ or $\delta = -$, we have defined the Dirichlet series $S^{\delta}(s, 1, \sigma, \eta)$ in (6) by

$$S^{\delta}(s, 1, \sigma, \eta) = C_{1,\sigma,\eta} \cdot \sum_{d \in \mathbf{Z}, \delta d > 0} \frac{L_{-d}(\sigma - 1) L_{-d}(\eta - 1)}{|d|^{s-(\sigma+\eta)/2+(3/2)}} \cdot \begin{cases} \Gamma(\frac{\sigma-1}{2})^{-1}\Gamma(\frac{\eta-1}{2})^{-1}, & \delta = +, \\ \Gamma(\frac{\sigma}{2})^{-1}\Gamma(\frac{\eta}{2})^{-1}, & \delta = -. \end{cases}$$

Of course the variables σ, η should not be in the set of poles of each summand, for example, the poles of $C_{1,\sigma,\eta}$ and $L_{f^2}(\sigma - 1)L_{f^2}(\eta - 1)$, although we indicated only $\Re s > 3/2 + |\Re\rho| + |\Re\kappa|$. The same remark should also be applied to trivial zeros of $\zeta(2 - 2s)$ in $\mathcal{B}^*_j(s, \sigma, \eta)$ defined below for example. In the following argument, we will not mention about these.

By Theorem 1, the Dirichlet series $S^{\pm}(s, 1, \sigma, \eta)$ can be continued meromorphically to the whole $(s, \sigma, \eta) \in \mathbf{C}^3$, in the sense that the function defined by (7) is holomorphic for all $(s, \sigma, \eta) \in \mathbf{C}^3$. The second component of the functional equation in Theorem 1 yields

$$S^{-}(1 - s, 1, \sigma, \eta) = \mathcal{A}^*(s, \sigma, \eta) + \mathcal{B}^*(s, \sigma, \eta),$$

where $\mathcal{A}^*(s, \sigma, \eta)$ and $\mathcal{B}^*(s, \sigma, \eta)$ are given by

$$\mathcal{A}^*(s, \sigma, \eta) := -2\pi^{-4s-1}\cos(\pi s)\frac{\zeta(2s)}{\zeta(2 - 2s)}E(1/4, \rho, \kappa) \times \prod_{t \in \mathcal{J}} \Gamma(s + t) \cdot S^{+}(s, 1, \sigma, \eta), \tag{36}$$

$$\mathcal{B}^*(s, \sigma, \eta) := \frac{\pi^{-4s}\zeta(2s)}{2\zeta(2 - 2s)}\left\{2\sin\pi\left(s - \frac{\sigma}{2}\right)\cos\pi\left(s + \frac{\sigma}{2}\right) - \sin(\pi\eta)\right\} \times \prod_{t \in \mathcal{J}} \Gamma(s + t) \cdot S^{-}(s, 1, \sigma, \eta) \tag{37}$$

with

$$E(1/4, \rho, \kappa) = \frac{-\pi^3}{\Gamma(\frac{\sigma}{2})\Gamma(\frac{3-\sigma}{2})\Gamma(\frac{\eta}{2})\Gamma(\frac{3-\eta}{2})}.$$

Here we used $\Gamma(s)\Gamma(1-s)=\pi/\sin\pi s$, $2\cos\pi s\cos\pi(s-1/2)=\sin 2\pi s$,

$$\begin{aligned}2\cos\pi t_1\cos\pi t_2 &= \cos\pi(t_1+t_2)+\cos\pi(t_1-t_2)\\ &= \cos\pi(\sigma-3/2)+\cos\pi(\eta-3/2),\end{aligned}$$

and $\sin 2\pi s+\cos\pi(\sigma-3/2)=2\sin\pi(s-\sigma/2)\cos\pi(s+\sigma/2)$.

Let $s,\sigma,\eta\in\mathbf{C}$ and $\Re s$ be sufficiently large (as usual, we denote this by $\Re s\gg 0$). We decompose the Dirichlet series into two parts

$$S^-(s,1,\sigma,\eta)=S_1^-(s,1,\sigma,\eta)+S_2^-(s,1,\sigma,\eta),$$

where $S_1^-(s,1,\sigma,\eta)$ is the sum over $d<0$ with $-d\neq\square$, and $S_2^-(s,1,\sigma,\eta)$ is the sum over $-d=\square$. Hence, $\mathcal{B}^*(s,\sigma,\eta)=\mathcal{B}_1^*(s,\sigma,\eta)+\mathcal{B}_2^*(s,\sigma,\eta)$ for $s,\sigma,\eta\in\mathbf{C}$ with $\Re s\gg 0$, where

$$\begin{aligned}\mathcal{B}_j^*(s,\sigma,\eta) := \frac{\pi^{-4s}\zeta(2s)}{2\zeta(2-2s)}&\left\{2\sin\pi\left(s-\frac{\sigma}{2}\right)\cos\pi\left(s+\frac{\sigma}{2}\right)-\sin(\pi\eta)\right\}\\ &\times\prod_{t\in\mathcal{J}}\Gamma(s+t)\cdot S_j^-(s,1,\sigma,\eta).\end{aligned}$$

In view of our analysis in Section 3, the functions $S^\pm(s,1,\sigma,\eta)$ are meromorphic for all $(s,\sigma,\eta)\in\mathbf{C}^3$ with the explicit possible polar divisors, in the sense that the functions defined by (7) are holomorphic for all $(s,\sigma,\eta)\in\mathbf{C}^3$. It follows that they have the Laurent expansions around $\eta=2$ and the Laurent coefficients are meromorphic functions of (s,σ) on $\mathbf{C}^2$. We are going to compute the Laurent coefficients. For convenience, for any function $h(s,\sigma,\eta)$, we denote by $h(s,\sigma,\eta)_j$ the jth Laurent coefficient around $\eta=2$, that is, $h(s,\sigma,\eta)=\sum_{j\in\mathbf{Z}}h(s,\sigma,\eta)_j\cdot(\eta-2)^j$, whenever it is meaningful. Note that $h(s,\sigma,\eta)_j$ is a function of s,σ. By this convention, it holds for all $(s,\sigma)\in\mathbf{C}^2$ that

$$S^-(1-s,1,\sigma,\eta)_j=\mathcal{A}^*(s,\sigma,\eta)_j+\mathcal{B}^*(s,\sigma,\eta)_j. \tag{38}$$

4.2 Definitions of $\mathcal{P}(s,\sigma,\eta),\mathcal{M}(s,\sigma,\eta),\mathcal{N}^*(s,\sigma,\eta)$

For $s,\sigma,\eta\in\mathbf{C}$ with $\Re s\gg 0$, we define

$$\mathcal{P}(s,\sigma,\eta):=\prod_{\text{prime } p}p(s,\sigma,\eta),$$

$$p(s,\sigma,\eta):=(1+p^{-2s})(1+p^{-2s-1})-(p^{-2s+\sigma-2}+p^{-2s+\eta-2})(1+p^{3-\eta-\sigma}),$$

$$\mathcal{M}(s,\sigma,\eta) := \frac{2^{5-2\sigma-2\eta}\pi^{(\sigma+\eta-1)/2}}{\zeta(2\eta-2)\Gamma(\frac{\eta}{2})}\mathcal{P}(s,\sigma,\eta)$$
$$\times\,\zeta(2s-\eta-\sigma+3)\zeta(2s-\eta+\sigma)\zeta(2s+\eta-\sigma)\zeta(2s+\eta+\sigma-3),$$

$$\mathcal{N}^*(s,\sigma,\eta) : = \frac{\pi^{-4s}\zeta(2s)}{2\zeta(2-2s)}\left\{2\sin\pi\left(s-\frac{\sigma}{2}\right)\cos\pi\left(s+\frac{\sigma}{2}\right)-\sin(\pi\eta)\right\}$$
$$\times\prod_{t\in\mathcal{J}}\Gamma(s+t)\cdot\mathcal{M}(s,\sigma,\eta).$$

For $s,\sigma,\eta\in\mathbf{C}$ with $\Re s\gg 0$, the formula in Section 3.4 implies

$$S_2^-(s,1,\sigma,\eta) = \frac{\zeta(\sigma-1)\zeta(\eta-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\mathcal{M}(s,\sigma,\eta)}{\zeta(2s)}, \tag{39}$$

$$\mathcal{B}_2^*(s,\sigma,\eta) = \frac{\zeta(\sigma-1)\zeta(\eta-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\mathcal{N}^*(s,\sigma,\eta)}{\zeta(2s)}. \tag{40}$$

See Section 4.1 for the definitions of $S_2^-(s,1,\sigma,\eta)$ and $\mathcal{B}_2^*(s,\sigma,\eta)$.

4.3 Main part of our Dirichlet series

As mentioned in Section 4.1, the function $S^+(s,1,\sigma,\eta)_0$ is a meromorphic function of (s,σ) on $\mathbf{C}^2$. By taking into account of the aside computations given in the remark below, for $s,\sigma\in\mathbf{C}$ with $\Re s\gg 0$, one has

$$S^+(s,1,\sigma,\eta)_0 = \frac{2^{2-2\sigma}\pi^{(\sigma/2)+1}}{\zeta(2)\zeta(2\sigma-2)\Gamma(\frac{\sigma-1}{2})}\frac{\Omega^+(s,\sigma)}{G_2^+(s,\sigma)},$$

$$\mathcal{A}^*(s,\sigma,\eta)_0 = \frac{-2^{3-2\sigma}\pi^{(\sigma+3)/2}\cos(\frac{\pi\sigma}{2})}{\zeta(2)\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}$$
$$\times\frac{\cos(\pi s)}{\sin\pi\left(s-\frac{\sigma}{2}\right)\cos\pi\left(s+\frac{\sigma}{2}\right)}\frac{G_2^-(s,\sigma)}{G_2^-(1-s,\sigma)}\frac{\Omega^+(s,\sigma)}{G_2^+(s,\sigma)}.$$

Here $G_2^\delta(s,\sigma)$ is defined by (10) and $\Omega^+(s,\sigma)$ is defined by (13). On the other hand, one has

$$S_1^-(s,1,\sigma,\eta)_0 = \frac{2^{1-2\sigma}\pi^{(\sigma+1)/2}}{\zeta(2)\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\sum_{d<0,-d\neq\square}\frac{L_{-d}(\sigma-1)\cdot|d|^{1/2}L_{-d}(1)}{|d|^{s-(\sigma/2)+1}},$$

$$\mathcal{B}_1^*(s,\sigma,\eta)_0 = \frac{G_2^-(s,\sigma)}{G_2^-(1-s,\sigma)}S_1^-(s,1,\sigma,\eta)_0$$

for $s, \sigma \in \mathbf{C}$ with $\Re s \gg 0$. We find that the series $S_1^-(s, 1, \sigma, \eta)_0$ can be regarded as the main part of $\Omega^-(s, \sigma)$ defined by (11).

Remark on aside computations. We put $\eta = 2$. Then, $t_1 = (\sigma - 1)/2$, $t_2 = (\sigma - 2)/2$. Using the formula $\Gamma(x)\Gamma(x + 1/2) = \pi^{1/2} 2^{1-2x} \Gamma(2x)$, we see $\prod_{t \in \mathcal{J}} \Gamma(s + t) = \pi 2^{-4s+3} \Gamma(2s - \sigma + 1)\Gamma(2s + \sigma - 2)$ and

$$\mathcal{N}^*(s, \sigma, 2) = \frac{2^{-4s+3}\pi^{1-4s}\zeta(2s)}{\zeta(2-2s)} \sin \pi \left(s - \frac{\sigma}{2}\right) \cos \pi \left(s + \frac{\sigma}{2}\right) \times \Gamma(2s - \sigma + 1)\Gamma(2s + \sigma - 2)\mathcal{M}(s, \sigma, 2).$$

Since $G_2^-(s, \sigma) = \pi^{-2s}\zeta(2s)\Gamma(s - t_1)\Gamma(s + t_2)$, it follows from $\Gamma(s)\Gamma(1 - s) = \pi / \sin \pi s$ that

$$\frac{G_2^-(s, \sigma)}{G_2^-(1 - s, \sigma)} = \frac{\mathcal{N}^*(s, \sigma, 2)}{\mathcal{M}(s, \sigma, 2)}.$$

4.4 Laurent expansion around $\eta = 2$

Recall $\zeta(\eta - 1) = (\eta - 2)^{-1} + \gamma + O(\eta - 2)$ around $\eta = 2$, where γ is Euler's constant. For $s, \sigma, \eta \in \mathbf{C}$ with $\Re s \gg 0$, let us consider the Laurent expansion around $\eta = 2$ of $\mathcal{M}(s, \sigma, \eta)$ and $S_2^-(s, 1, \sigma, \eta)$ defined in (39);

$$\mathcal{M}(s, \sigma, \eta) = c_0 + c_1(\eta - 2) + O((\eta - 2)^2), \quad c_0 = \mathcal{M}(s, \sigma, 2),$$

$$c_1 = \mathcal{M}_\eta(s, \sigma, 2),$$

$$S_2^-(s, 1, \sigma, \eta) = \frac{\zeta(\sigma - 1)}{\zeta(2\sigma - 2)\Gamma(\frac{\sigma}{2})\zeta(2s)} \left\{ \frac{c_0}{\eta - 2} + (c_1 + \gamma c_0) + O(\eta - 2) \right\}.$$

Here $\mathcal{M}_\eta = \partial \mathcal{M} / \partial \eta$. It follows for $s, \sigma \in \mathbf{C}$ with $\Re s \gg 0$ that $S_2^-(s, 1, \sigma, \eta)$ in (39) has the Laurent coefficients

$$S_2^-(s, 1, \sigma, \eta)_{-1} = \frac{\zeta(\sigma - 1)}{\zeta(2\sigma - 2)\Gamma(\frac{\sigma}{2})} \frac{\mathcal{M}(s, \sigma, 2)}{\zeta(2s)}, \tag{41}$$

$$S_2^-(s, 1, \sigma, \eta)_0 = \frac{\zeta(\sigma - 1)}{\zeta(2\sigma - 2)\Gamma(\frac{\sigma}{2})} \frac{\mathcal{M}_\eta(s, \sigma, 2) + \gamma \cdot \mathcal{M}(s, \sigma, 2)}{\zeta(2s)}. \tag{42}$$

Similarly, we obtain for $s, \sigma \in \mathbf{C}$ with $\Re s \gg 0$ that $\mathcal{B}_2^*(s, \sigma, \eta)$ in (40) has the Laurent coefficients

$$\mathcal{B}_2^*(s, \sigma, \eta)_{-1} = \frac{\zeta(\sigma - 1)}{\zeta(2\sigma - 2)\Gamma(\frac{\sigma}{2})} \frac{\mathcal{N}^*(s, \sigma, 2)}{\zeta(2s)}, \tag{43}$$

$$\mathcal{B}_2^*(s,\sigma,\eta)_0 = \frac{\zeta(\sigma-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\mathcal{N}_\eta^*(s,\sigma,2)+\gamma\cdot\mathcal{N}^*(s,\sigma,2)}{\zeta(2s)}. \tag{44}$$

Here $\mathcal{N}_\eta^* = \partial\mathcal{N}^*/\partial\eta$. See Section 4.1 for the symbol indicated by the lower indexes -1 and 0.

4.5 Residues at $\eta = 2$

The functions $S^{\pm}(s,1,\sigma,\eta)_{-1}$ are meromorphic functions of (s,σ) on $\mathbf{C}^2$. The equation (38) gives $S^-(1-s,1,\sigma,\eta)_{-1} = \mathcal{A}^*(s,\sigma,\eta)_{-1} + \mathcal{B}^*(s,\sigma,\eta)_{-1}$. For $s,\sigma \in \mathbf{C}$ with $\Re s \gg 0$, one has

$$S^-(s,1,\sigma,\eta)_{-1} = S_2^-(s,1,\sigma,\eta)_{-1},$$

$$\mathcal{A}^*(s,\sigma,\eta)_{-1} = 0, \quad \mathcal{B}^*(s,\sigma,\eta)_{-1} = \mathcal{B}_2^*(s,\sigma,\eta)_{-1}.$$

In view of (41) and (37), (43), the functions $\mathcal{M}(s,\sigma,2)$ and $\mathcal{N}^*(s,\sigma,2)$ initially defined for $s,\sigma \in \mathbf{C}$ with $\Re s \gg 0$ have meromorphic continuation to the whole $(s,\sigma) \in \mathbf{C}^2$, and satisfy the functional relation

$$\frac{\mathcal{M}(1-s,\sigma,2)}{\zeta(2-2s)} = \frac{\mathcal{N}^*(s,\sigma,2)}{\zeta(2s)}. \tag{45}$$

Remark. The equation (45) is consistent with the functional equation of $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \zeta^*(1-s)$. Indeed,

$$p(s,\sigma,2) = (1-p^{-2s+\sigma-2})(1-p^{-2s-\sigma+1}),$$

$$\mathcal{P}(s,\sigma,2) = \zeta(2s-\sigma+2)^{-1}\zeta(2s+\sigma-1)^{-1},$$

$$\mathcal{M}(s,\sigma,2) = \frac{2^{1-2\sigma}\pi^{(\sigma+1)/2}}{\zeta(2)}\zeta(2s-\sigma+1)\zeta(2s+\sigma-2),$$

$$\mathcal{N}^*(s,\sigma,2) = \frac{G_2^-(s,\sigma)}{G_2^-(1-s,\sigma)}\mathcal{M}(s,\sigma,2),$$

$$G_2^-(s,\sigma)\frac{\mathcal{M}(s,\sigma,2)}{\zeta(2s)} = \frac{2^{1-2\sigma}\pi^{\sigma/2}}{\zeta(2)}\zeta^*(2s-\sigma+1)\zeta^*(2s+\sigma-2).$$

See Section 4.2 for the definition of each function on the left-hand side.

4.6 Constant terms at $\eta = 2$

The functions $S^{\pm}(s,1,\sigma,\eta)_0$ are meromorphic functions of (s,σ) on $\mathbf{C}^2$. The equation (38) gives $S^-(1-s,1,\sigma,\eta)_0 = \mathcal{A}^*(s,\sigma,\eta)_0 + \mathcal{B}^*(s,\sigma,\eta)_0$. Put

$$L(s,\sigma) := S^-(s,1,\sigma,\eta)_0 - \frac{\zeta(\sigma-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\gamma\cdot\mathcal{M}(s,\sigma,2)}{\zeta(2s)},$$

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$$R(s,\sigma) := \mathcal{B}^*(s,\sigma,\eta)_0 - \frac{\zeta(\sigma-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\gamma\cdot\mathcal{N}^*(s,\sigma,2)}{\zeta(2s)}.$$

By (36), (37), $\mathcal{A}^*(s,\sigma,\eta)_0$ and $\mathcal{B}^*(s,\sigma,\eta)_0$ are meromorphic functions of (s,σ) on $\mathbf{C}^2$. In view of Section 4.5, $L(s,\sigma)$ and $R(s,\sigma)$ are also meromorphic functions of (s,σ) on $\mathbf{C}^2$.

Using (45), the functional equation turns out to be

$$L(1-s,\sigma) = \mathcal{A}^*(s,\sigma,\eta)_0 + R(s,\sigma). \tag{46}$$

By (42), (44) and the decomposition of $S^-(s,1,\sigma,\eta)$ in Section 4.1, one has for $s,\sigma\in\mathbf{C}$ with $\Re s \gg 0$ that

$$L(s,\sigma) = S_1^-(s,1,\sigma,\eta)_0 + \frac{\zeta(\sigma-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\mathcal{M}_\eta(s,\sigma,2)}{\zeta(2s)}, \tag{47}$$

$$R(s,\sigma) = \mathcal{B}_1^*(s,\sigma,\eta)_0 + \frac{\zeta(\sigma-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\mathcal{N}_\eta^*(s,\sigma,2)}{\zeta(2s)}. \tag{48}$$

Summarizing the discussion so far, the right-hand sides of (47), (48), which are initially defined for $s,\sigma\in\mathbf{C}$ with $\Re s\gg 0$, can be continued meromorphically to the whole $(s,\sigma)\in\mathbf{C}^2$. The equation (46) gives the functional equation satisfied by them.

4.7 Dirichlet series $\Omega^-(s,\sigma)$

The Dirichlet series expressions have already been given in Section 4.3 for $S_1^-(s,1,\sigma,\eta)_0, \mathcal{A}^*(s,\sigma,\eta)_0, \mathcal{B}_1^*(s,\sigma,\eta)_0$. In order to relate $L(s,\sigma)$, $R(s,\sigma)$ in (47), (48) and $\Omega^-(s,\sigma)$ in (11), we shall compute $\mathcal{M}_\eta(s,\sigma,2)$ and $\mathcal{N}_\eta^*(s,\sigma,2)$ explicitly.

Suppose that $s,\sigma\in\mathbf{C}$ and $\Re s\gg 0$. By (12) and the definitions of $\mathcal{P}(s,\sigma,\eta)$, $\mathcal{M}(s,\sigma,\eta)$ in Section 4.2, one obtains $\mathcal{P}_\eta(s,\sigma,2)/\mathcal{P}(s,\sigma,2) = P(s,\sigma)$ and

$$\begin{aligned}\frac{\mathcal{M}_\eta(s,\sigma,2)}{\mathcal{M}(s,\sigma,2)} &= c + \frac{\zeta'}{\zeta}(2s+\sigma-1) + \frac{\zeta'}{\zeta}(2s-\sigma+2)\\ &\quad - \frac{\zeta'}{\zeta}(2s+\sigma-2) - \frac{\zeta'}{\zeta}(2s-\sigma+1) + P(s,\sigma),\end{aligned} \tag{49}$$

where $c = -2\log 2 + (\log\pi)/2 - 2(\zeta'/\zeta)(2) - 2^{-1}(\Gamma'/\Gamma)(1)$. For simplicity, we introduce $D^-(s,\sigma)$ for $s,\sigma\in\mathbf{C}$ with $\Re s\gg 0$ by

$$D^-(s,\sigma) = S_1^-(s,1,\sigma,\eta)_0 + \frac{\zeta(\sigma-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\mathcal{M}(s,\sigma,2)}{\zeta(2s)}\left(\frac{\zeta'}{\zeta}(2s+\sigma-1)\right.$$
$$\left.+\frac{\zeta'}{\zeta}(2s-\sigma+2) - \frac{\zeta'}{\zeta}(2s+\sigma-2) - \frac{\zeta'}{\zeta}(2s-\sigma+1) + P(s,\sigma)\right).$$

Then (47) tells us that

$$L(s,\sigma) = D^-(s,\sigma) + c\cdot\frac{\zeta(\sigma-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\mathcal{M}(s,\sigma,2)}{\zeta(2s)}.$$

In view of Sections 4.7 and 4.5, this $D^-(s,\sigma)$, initially defined for $s,\sigma\in\mathbf{C}$ with $\Re s\gg 0$, has a meromorphic continuation to the whole $(s,\sigma)\in\mathbf{C}^2$. Using the formula of $S_1^-(s,1,\sigma,\eta)_0$ in Section 4.3, the Dirichlet series $\Omega^-(s,\sigma)$ initially defined by (11) for $s,\sigma\in\mathbf{C}$ with $\Re s\gg 0$ has the form

$$\Omega^-(s,\sigma) = G_2^-(s,\sigma)\frac{\zeta(2)\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}{2^{1-2\sigma}\pi^{(\sigma+1)/2}}D^-(s,\sigma), \tag{50}$$

and thus this $\Omega^-(s,\sigma)$ also has a meromorphic continuation to the whole $(s,\sigma)\in\mathbf{C}^2$.

By the definition of $\mathcal{N}^*(s,\sigma,\eta)$ in Section 4.2 and that of $\mathcal{G}(s,\sigma)$ in (14), we see

$$\frac{\mathcal{N}_\eta^*(s,\sigma,2)}{\mathcal{N}^*(s,\sigma,2)} = \frac{\mathcal{M}_\eta(s,\sigma,2)}{\mathcal{M}(s,\sigma,2)} + 2^{-1}\mathcal{G}(s,\sigma).$$

The equation (48) combined with the formula of $\mathcal{B}_1^*(s,\sigma,\eta)_0$ in Section 4.3 implies

$$R(s,\sigma) = \frac{G_2^-(s,\sigma)}{G_2^-(1-s,\sigma)}D^-(s,\sigma)$$
$$+\frac{\zeta(\sigma-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\mathcal{N}^*(s,\sigma,2)}{\zeta(2s)}\{c+2^{-1}\mathcal{G}(s,\sigma)\},$$

where c is the same as in (49). In view of (46) and (45), we have

$$D^-(1-s,\sigma) = \mathcal{A}^*(s,\sigma,\eta)_0 + \frac{G_2^-(s,\sigma)}{G_2^-(1-s,\sigma)}D^-(s,\sigma)$$
$$+\frac{\zeta(\sigma-1)}{\zeta(2\sigma-2)\Gamma(\frac{\sigma}{2})}\frac{\mathcal{N}^*(s,\sigma,2)}{\zeta(2s)}2^{-1}\mathcal{G}(s,\sigma),$$

which combined with (50) yields a functional equation of $\Omega^-(s,\sigma)$. This completes the proof of the first functional equation stated in Theorem 4. ∎

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4.8 Explicit forms of the Koecher–Maass series

For any positive discriminant $d>0$ with $d\neq\square$, one has the class number formula

$$L_d(1)=\frac{1}{2d^{1/2}}\sum_{T\in(L_2^-)'/\mathrm{SL}_2(\mathbf{Z}),-\det(2T)=d}\mu(T).$$

This can be seen, for example, by taking the residue at $\sigma=1$ of the both sides of the identity given in **(B)** Section 1.1. In view of the facts $\mu(T[U])=\mu(T)$ $(\forall U\in\mathrm{SL}_2(\mathbf{Z}))$ and $\mu(lT)=\mu(T)$ $(\forall l\in\mathbf{N})$, Böcherer's computation proving Satz 3 (d) [3, p. 20] also works for this indefinite case. This remark together with (4) yields for $s,\sigma\in\mathbf{C}$ with $\Re s\gg 0$ that

$$\begin{aligned}&\sum_{T\in(L_2^-)'/\mathrm{SL}_2(\mathbf{Z})}\frac{\mu(T)b(T,\sigma)}{|\det T|^s}\\&=2^{2s}\frac{\zeta(2s+\sigma-2)}{\zeta(\sigma)\zeta(2\sigma-2)}\sum_{d>0,\,d\neq\square}\left(\sum_{\substack{T\in(L_2^-)'/\mathrm{SL}_2(\mathbf{Z})\\-\det(2T)=d}}\mu(T)\right)\frac{L_d(\sigma-1)}{d^s}\\&=2^{2s+1}\frac{\zeta(2s+\sigma-2)}{\zeta(\sigma)\zeta(2\sigma-2)}\sum_{d>0,\,d\neq\square}L_d(1)L_d(\sigma-1)d^{-s+1/2}.\end{aligned}$$

This and (11) prove the identity (16).

The identity (15) follows from Böcherer's Satz 3 in [3, p. 20] combined with (13) and the formula

$$L_{-d}(1)=\frac{2\pi}{d^{1/2}}\sum_{T\in L_2^+/\mathrm{SL}_2(\mathbf{Z}),\det(2T)=d}(\sharp E(T))^{-1},$$

which holds for any negative discriminant $-d<0$ and can be obtained by taking the residue at $\sigma=1$ of the both sides of the identity given in **(B)** Section 1.1 again. □

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