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Semigroup compactifications and chaotic flows

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Abstract. We use the concept of enveloping Ellis semigroups in order to classify the behaviour of dynamical systems on intervals. Several notions of chaos and non-chaos are expressed by means of algebraic properties of the corresponding Ellis semigroups.

It has become evident in the last few decades that many apparently simple dynamical systems exhibit surprisingly complicated behaviour and show various forms of mixing properties or 'chaos'. To study such qualitative behaviour there seem to be at least two different and quite unrelated approaches. Ellis [**3**], Auslander [**1**] and many others have looked at the continuous action of a group G on some compact space X and characterized the dynamical behaviour by abstract algebraic-topological properties of G, for example, compactifications of the group G. Others, like, for example, Block and Coppel [**2**], have studied the iterates of a continuous map $\varphi : [0, 1] \to [0, 1]$ using subtle definitions and methods based on the topological nature of the unit interval. There seems to be little interaction between the two fields and we will try to connect them in the following.

In the approach to dynamical systems initiated by Ellis [**3**] and others, one considers *flows*, i.e. groups of continuous self-maps on arbitrary compact spaces. Then several classes of flows such as equicontinuous flows can be described in detail.

A technique widely used in Ellis' work is the description of dynamical properties of a flow in algebraic terms. To this purpose he introduced the following notion, which we use here for semigroups rather than for groups of continuous maps.

Definition. Let X be a compact topological space and S a semigroup of continuous self-maps of X. Then the *enveloping Ellis semigroup* $\Sigma(X, S)$ of the flow (X, S) is the closure of S in the topology of pointwise convergence on X^X.

It is easy to see that $\Sigma(X, S)$ is in fact a compact, right topological semigroup (where right topological means that for every $\eta \in \Sigma(X, S)$ the map $\Sigma(X, S) \to \Sigma(X, S) : \psi \mapsto \psi \circ \eta$ is continuous). By Ruppert's structure theorem for compact, right topological semigroups [**9**, Ch. I.3], $\Sigma(X, S)$ contains at least one idempotent and has a minimal ideal which is a paragroup (see [**9**] for the details). The Ellis semigroup has been extensively used in topological dynamics, for example in [**1**, **3**, **4**, **7**, **8**], and some algebraic properties of enveloping Ellis semigroups, such as the existence and the cardinality of certain subsemigroups, ideals and sets of idempotents, correspond to dynamical properties of the underlying flows. An example for this connection is the following result, which has been proved in [**3**] for groups of continuous maps.

Definition. Let (X, S) be a flow. A pair of points $x, y \in X$ with $x \neq y$ is called *proximal* if there is a point $z \in X$ and a net $(\varphi_\alpha) \subset S$ such that

$$\lim_\alpha \varphi_\alpha(x) = \lim_\alpha \varphi_\alpha(y) = z.$$

The flow (X, S) is called *distal* if no proximal pairs exist.

PROPOSITION. *A flow is distal if and only if its Ellis semigroup is a group whose neutral element is the identity.*

Proof. Let (X, S) be a distal flow. Since no proximal pairs exist, there cannot be any idempotent other than the identity in $\Sigma(X, S)$. By Ruppert's structure theorem there is an idempotent $e \in \Sigma(X, S)$ such that $e\Sigma(X, S)e$ is a group. Now by the distality e must be the identity and $\Sigma(X, S) = e\Sigma(X, S)e$ is a group.

Conversely, let $\Sigma(X, S)$ be a group whose neutral element is the identity map. Then every map in $\Sigma(X, S)$ is invertible, and hence injective, so there cannot be any proximal pairs in X. □

Other theorems of this kind have been proved, for example in [**1**] and [**3**]. However, the assumptions are so general that many of the properties of dynamical systems on simple topological spaces cannot be described by them. In particular, on the unit interval there are many standard examples whose complicated behaviour has been known for a long time. The bifurcation points and mixing properties of such maps have been thoroughly investigated, but many of the notions used in this context cannot be generalized to maps on arbitrary compact spaces and are therefore not considered in topological dynamics.

Let φ be a continuous self-map of the unit interval. In this case the Ellis semigroup of the iterates of φ (which will be denoted by $\Sigma(X, \varphi)$) is a semigroup compactification of the natural numbers. In the category of compactifications of $\mathbb{N}$ there is a universal element, namely the Stone–Čech compactification $\beta\mathbb{N}$ (see [**11**] for details). Hindman has shown in [**5**] that there is a natural semigroup structure on $\beta\mathbb{N}$ which is right topological and extends the usual addition of natural numbers. It follows directly from the universal property that $\beta\mathbb{N}$, equipped with this non-commutative addition, is universal for the enveloping Ellis semigroups in the sense that every Ellis semigroup of the iterates of a single map must be a factor of the semigroup $\beta\mathbb{N}$. This gives us an upper bound for the complexity of enveloping Ellis semigroups. The cardinality of $\beta\mathbb{N}$ is 2^c, and its algebraic structure is quite complicated since $\beta\mathbb{N}$ has 2^c minimal right ideals and 2^c minimal left ideals, and each of these contains 2^c idempotents (see [**6**, Corollary 2.6]).

For the iteration of a single map on the unit interval several different notions of chaos and regularity have been introduced by various authors. Our aim is to show how these notions are reflected by properties of the corresponding Ellis semigroups and that chaos can thus be studied via semigroup compactifications of the natural numbers.

For this purpose we need the following striking theorem due to Šarkovskii [**10**].

ŠARKOVSKII'S THEOREM. *Let φ be a continuous map from $[0, 1]$ into itself. Let the natural numbers be ordered in the following way:*

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec \cdots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

If φ has a periodic orbit of period n and if $n \prec m$ in the order given above, then φ also has a periodic orbit of period m.

One expects that dynamical systems with simpler behaviour have smaller Ellis semigroups than more complicated and 'chaotic' systems since, for example, all orbits are convergent if and only if the Ellis semigroup is the one-point compactification of the natural numbers. We therefore begin with the simplest case in which every orbit converges to a periodic orbit.

THEOREM 1. *Let φ be a continuous self-map of the unit interval I. Then the following assertions are equivalent:*
(i) *φ has periodic points of only finitely many different periods;*
(ii) $\Sigma(I, \varphi) \cong \mathbb{N} \cup \{\omega_1, \dots, \omega_k\}$.
In this case the number of accumulation points in $\Sigma(I, \varphi)$ is a power of two and coincides with the greatest period of a periodic point in I.

Proof. (i) $\Rightarrow$ (ii): If (I, φ) has finitely many different periods, these are by Šarkovskii's Theorem exactly the numbers $1, 2, 4, \dots, 2^d$ for some number $d \in \mathbb{N}$. The map $\eta := \varphi^{2^d}$ has no periodic points apart from its fixed points, hence the sequence $(\eta^n)_{n\in\mathbb{N}}$ converges pointwise on I to a map $\psi \in \Sigma(I, \varphi)$ which maps I into the set of all fixed points of φ (see [**2**, VI.1.1]). Thus the sequence (φ^n) is the disjoint union of 2^d convergent subsequences, namely $\eta^n \to \psi$, $\eta^n\varphi \to \psi\varphi$ etc., so

$$\Sigma(I, \varphi) = \{\varphi^n \mid n \in \mathbb{N}\} \cup \{\psi, \psi\varphi, \psi\varphi^2, \dots, \psi\varphi^{2^d-1}\}.$$

(ii) $\Rightarrow$ (i): Let m be the number of accumulation points of $(\varphi^n)_{n\in\mathbb{N}}$ in $\Sigma(I, \varphi)$. We assume that (I, φ) has periodic points of infinitely many different periods. Then there is a periodic point $x \in I$ whose period is $l > m$. Each of the subsequences

$$(\varphi^{ln})_{n\in\mathbb{N}}, (\varphi^{ln+1})_{n\in\mathbb{N}}, \dots, (\varphi^{ln+(l-1)})_{n\in\mathbb{N}}$$

is constant on x, and their values on x are pairwise distinct. Let $\psi_0, \psi_1, \dots, \psi_{(l-1)}$ be pointwise accumulation points of the respective subsequences. These accumulation points are pairwise distinct, since they differ on the point x. Hence $\Sigma(I, \varphi)$ has at least l elements, which contradicts the assumption $|\Sigma(I, \varphi) \setminus \{\varphi^n \mid n \in \mathbb{N}\}| = m$. □

Examples for the situation of the above theorem are given by maps such as $x \mapsto x^2$ and $x \mapsto 1 - x^2$ on the interval. In fact, for every $d \in \mathbb{N}$ there are examples of maps for which periodic points of exactly the periods $1, 2, 4, \dots, 2^d$ exist (see [**2**, Example I.2.13]).

We show next that chaotic flows have large and complicated Ellis semigroups. We will use the following notion of chaos which can be found in [**2**, Ch. II]. One should, however, keep in mind that various authors have given slightly different definitions of the term 'chaos'.

Definition. Take $Y = \{0,1\}^{\mathbb{N}}$ with its usual product topology, which is induced by the metric d with

$$d(a,b) := \sum_{n\in\mathbb{N}} 2^{-n}|a_n - b_n| \quad \text{for } a = (a_n)_{n\in\mathbb{N}}, \quad b = (b_n)_{n\in\mathbb{N}} \in Y$$

and the shift map $\tau : Y \to Y : (a_n)_{n\in\mathbb{N}} \mapsto (a_{n+1})_{n\in\mathbb{N}}$. Let I be the unit interval. A continuous map $\varphi : I \to I$ is called *chaotic* if there is a positive integer m, a closed, φ-invariant subset $X \subseteq I$ and a continuous, surjective map $h : X \to Y$ such that

$$h \circ \varphi^m(x) = \tau \circ h(x) \quad \text{for all } x \in X,$$

i.e. if the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi^m} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow[\tau]{} & Y \end{array}$$

It has been shown in [**2**] that a continuous self-map of the unit interval is chaotic if and only if it has a periodic point whose period is not a power of two. In order to show that the Ellis semigroup is very large in these cases, we need the following lemma.

LEMMA 1. *Let S_1 and S_2 be compact, monothetic, right topological semigroups, and let $\Pi_i : \beta\mathbb{N} \to S_i$ be the canonical epimorphisms for $i = 1, 2$. Then there exists an epimorphism $\Psi : S_1 \to S_2$ satisfying $\Psi \circ \Pi_1 = \Pi_2$ if and only if $\Pi_1(p) = \Pi_1(q)$ implies $\Pi_2(p) = \Pi_2(q)$ for all $p, q \in \beta\mathbb{N}$.*

Proof. If the epimorphism Ψ with the required property exists, then $\Pi_1(p) = \Pi_1(q)$ implies that $\Pi_2(p) = \Psi(\Pi_1(p)) = \Psi(\Pi_1(q)) = \Pi_2(q)$ for all $p, q \in \beta\mathbb{N}$.

Conversely, assume that $\Pi_1(p) = \Pi_1(q)$ implies $\Pi_2(p) = \Pi_2(q)$ for all $p, q \in \beta\mathbb{N}$. For every $s_1 \in S_1$ take any $p \in \Pi_1^{-1}$ and define $\Psi(s_1) := \Pi_2(p)$. This yields a map $\Psi : S_1 \to S_2$ which is well defined since $\Pi_1^{-1}(s_1) \neq \emptyset$ and since two different elements $p, q \in \Pi_1^{-1}(s_1)$ are by assumption mapped onto the same element of S_2. It is clear from the definition that Ψ satisfies $\Psi \circ \Pi_1 = \Pi_2$.

Furthermore, Ψ is continuous. In fact, if $U \subseteq S_2$ is a closed set, then $\Pi_2^{-1}(U)$ is closed by the continuity of Π_2 and therefore compact in $\beta\mathbb{N}$, and thus $\Pi_1(\Pi_2^{-1}(U)) = \Psi^{-1}(U)$ must be compact and hence closed in S_1.

It is clear that Ψ canonically maps $\mathbb{N} \subset S_1$ onto $\mathbb{N} \subset S_2$, so by continuity it must be a homomorphism of the right topological semigroups. Furthermore, for every $s_2 \in S_2$ there exists $p \in \beta\mathbb{N}$ with $\Pi_2(p) = s_2$, and Ψ maps $\Pi_1(p)$ onto s_2, hence Ψ is onto. □

THEOREM 2. *Let $\varphi : I \to I$ be continuous and chaotic. Then there is an integer $m \in \mathbb{N}$ such that*

$$\overline{\{\varphi^{nm} \mid n \in \mathbb{N}\}}^{I^I} \cong \beta\mathbb{N},$$

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where the closure is taken in the topology of pointwise convergence on I. In particular, $\beta\mathbb{N}$ is a subsemigroup of $\Sigma(I, \varphi)$.

Proof. (a) For the flow (Y, τ) the Ellis semigroup $\Sigma(Y, \tau)$ is homomorphic to $\beta\mathbb{N}$. In fact, by the universal property the map $\mathbb{N} \to \Sigma(Y, \tau) : n \mapsto \tau^n$ can be extended to a continuous, surjective map $\Phi : \beta\mathbb{N} \to \Sigma(Y, \tau)$, which is by its continuity automatically a homomorphism of the right topological semigroups. So it remains to show that Φ is injective. Let $s \neq t \in \beta\mathbb{N}$. Then there are disjoint sets $A, B \subset \mathbb{N}$ such that $s \in \overline{A}$ and $t \in \overline{B} \subset \beta\mathbb{N}$. Define a sequence $x \in Y$ by

$$x_n := \begin{cases} 0 & \text{for } n \in A, \\ 1 & \text{for } n \notin A. \end{cases}$$

The zeroth component of $\tau^n(x) = \Phi(n)(x)$ is zero for every $n \in A$ and one for every $n \in B$. Therefore, the zeroth component is zero in $\Phi(s)(x)$ and one in $\Phi(t)(x)$, so $\Phi(s) \neq \Phi(t)$, i.e. Φ is injective.

(b) Let φ be a chaotic self-map of the interval. By definition there is an integer $m \in \mathbb{N}$, a closed subset $X \subseteq I$ and a continuous map $h : X \to Y$ such that $\varphi^m(X) \subseteq X$ and $h \circ \varphi^m = \tau \circ h$ on X. Let $\eta := \varphi^m$, let $\Pi : \beta\mathbb{N} \to \Sigma(X, \eta)$ and $\Phi : \beta\mathbb{N} \to \Sigma(Y, \tau)$ be the canonical maps, and let $y \in Y$ and $p, q \in \beta\mathbb{N}$ with $\Pi(p) = \Pi(q)$. If $y = h(x)$, then

$$\Phi(p)(y) = \Phi(p)(h(x)) = h(\Pi(p)(x)) = h(\Pi(q)(x)) = \Phi(q)(h(x)) = \Phi(q)(y),$$

so $\Phi(p) = \Phi(q)$. By Lemma 1 there is an epimorphism $\Sigma(X, \eta) \to \Sigma(Y, \tau) \cong \beta\mathbb{N}$. Since Π is the inverse of this epimorphism, $\Sigma(X, \eta)$ must be isomorphic to $\beta\mathbb{N}$, which implies $\Sigma(I, \eta) \cong \beta\mathbb{N}$ since $\Sigma(X, \eta)$ is a factor of $\Sigma(I, \eta)$. □

COROLLARY *Let φ be a chaotic, continuous map from the unit interval into itself. Then the enveloping Ellis semigroup $\Sigma(I, \varphi)$ has the cardinality 2^c and contains 2^c idempotents.*

Proof. By the above theorem $\beta\mathbb{N}$ is a subsemigroup of $\Sigma(I, \varphi)$, and there are 2^c idempotents in the minimal ideal of $\beta\mathbb{N}$ (see [**6**, Corollary 2.6]). □

Theorems 1 and 2 describe in some sense the least and the most chaotic maps, and hence the smallest and the largest possible Ellis semigroup. We will look at another notion of dynamical behaviour that lies between these extreme cases, again using the terminology according to [**2**].

Definition. Let $\varphi : I \to I$ be a continuous map. A point $y \in I$ is called *asymptotically periodic* if there is a periodic point $x \in I$ such that $|\varphi^n(y) - \varphi^n(x)| \to 0$ for $n \to \infty$. The dynamical system (I, φ) is called *strongly non-chaotic* if every point in I is asymptotically periodic.

The class of strongly non-chaotic dynamical systems obviously includes the cases considered in Theorem 1, but there are examples of interval maps for which every power of two occurs as the period of a periodic point (see [**2**, Example I.14]). On the other hand, it has been shown in [**2**, VI.3], that all strongly non-chaotic maps are non-chaotic, so there cannot be any periods other than the powers of two.

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In the following we will fully characterize the enveloping Ellis semigroups of the strongly non-chaotic systems for which every power of two occurs as the period of a periodic point and show that they are much less complicated than the Ellis semigroups of chaotic systems. For this, we will establish three lemmas.

LEMMA 2. *Let φ be a continuous, strongly non-chaotic map that has periodic points of infinitely many different periods. Then the Ellis semigroup contains exactly one idempotent χ which maps I onto the set of all periodic points of φ. Furthermore, the minimal ideal of the Ellis semigroup $\Sigma(I,\varphi)$ is a group, and $\Sigma(I,\varphi)$ is the disjoint union of the minimal ideal and $\{\varphi^n \mid n \in \mathbb{N}\}$.*

Proof. It is clear that an idempotent χ in $\Sigma(I,\varphi)$ acts as the identity on the periodic points. However, since φ is strongly non-chaotic, the behaviour of the sequence (φ^n) is fully determined by the behaviour on the periodic points. In particular, a net (φ^{n_α}) that converges to a projection χ satisfies $\varphi^{n_\alpha}(x) \to x$ for every periodic point x and hence $\varphi^{n_\alpha}(y) \to x$ for every point y whose orbit approximates the orbit of x. Therefore, the idempotent is uniquely determined and it follows from Ruppert's structure theorem for compact, right topological semigroups that the minimal ideal $\chi\Sigma(I,\varphi)\chi = \Sigma(I,\varphi)\chi$ is in fact a group whose unit element is χ. The last statement is obvious since the iterates of φ are the only elements of $\Sigma(I,\varphi)$ that do not map all points in I onto periodic points. □

LEMMA 3. *Let $\varphi : I \to I$ be a continuous, strongly non-chaotic map that has periodic points of infinitely many different periods. Let $\Pi : \beta\mathbb{N} \to \Sigma(I,\varphi) \setminus \{\varphi^n \mid n \in \mathbb{N}\}$ be the canonical continuous extension of the map $n \mapsto \varphi^n\chi$, where χ is the unique idempotent in $\Sigma(I,\varphi)$. Let $p, q \in \beta\mathbb{N}$. Then the following assertions are pairwise equivalent:*
(i) *$\Pi(p) = \Pi(q)$;*
(ii) *$\Pi(p)(x) = \Pi(q)(x)$ for every periodic point x;*
(iii) *for every $d \in \mathbb{N}$ there is an open neighbourhood U of p and q in $\beta\mathbb{N}$ such that $n \equiv m \mod 2^d$ for all $n, m \in U$.*

Proof. (i) ⇒ (ii) is trivial.

(ii) ⇒ (i): Since φ is strongly non-chaotic, all points are asymptotically periodic. Let y be any point in I, let x be a periodic point whose orbit approximates the orbit of y, and let $\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that

$$|\varphi^n(y) - \varphi^n(x)| < \varepsilon \quad \text{for all } n \geq n_\varepsilon.$$

If $\Pi(p) = \psi$ and $\Pi(q) = \eta$, then we have

$$|\psi(y) - \eta(y)| \leq |\psi(y) - \psi(x)| + |\psi(x) - \eta(x)| + |\eta(x) - \eta(y)|.$$

By the assumption $|\psi(x) - \eta(x)| = 0$. Furthermore,

$$|\psi(y) - \psi(x)| \leq |\psi(y) - \varphi^n(y)| + |\varphi^n(y) - \varphi^n(x)| + |\varphi^n(x) - \psi(x)|.$$

Since ψ is a pointwise accumulation point of the sequence (φ^n), we can choose the number $n \geq n_\varepsilon$ such that $|\psi(y) - \varphi^n(y)| < \varepsilon$ and $|\varphi^n(x) - \psi(x)| < \varepsilon$. Hence $|\psi(y) - \psi(x)| < 3\varepsilon$.

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The same argument shows that $|\eta(x) - \eta(y)| < 3\varepsilon$, so we get $|\psi(y) - \eta(y)| < 6\varepsilon$. Since y and ε were chosen arbitrarily, this yields $\psi = \eta$, i.e. $\Pi(p) = \Pi(q)$.

(ii) $\Rightarrow$ (iii): Let $d \in \mathbb{N}$ and let x be a periodic point with period 2^d. By the assumption, $\Pi(p)(x) = \Pi(q)(x) = \varphi^k(x)$, where $k \in \{0, 1, 2, \ldots, 2^d - 1\}$. The set $\{\psi \in \Sigma(I, \varphi) \mid \psi(x) = \varphi^k(x)\}$ is open in $\Sigma(I, \varphi)$, so its counterimage U is open in $\beta\mathbb{N}$ and contains p and q. For all $n \in \mathbb{N} \cap U$ we have $\varphi^n(x) = \varphi^k(x)$, i.e. all elements of $\mathbb{N} \cap U$ are congruent to each other modulo 2^d.

(iii) $\Rightarrow$ (ii): Let x be a periodic point. The period of x must be 2^d for some $d \in \mathbb{N}$, since other periods occur only in chaotic dynamical systems. By (iii) there is an open neighbourhood U of p and q such that $n \equiv m \equiv k \mod 2^d$ for all $n, m \in \mathbb{N} \cap U$, where $k \in \{0, 1, 2, \ldots, 2^d - 1\}$. Then we have $\Pi(n) = \varphi^k(x)$ for every $n \in \mathbb{N} \cap U$ and hence $\Pi(p)(x) = \Pi(q)(x) = \varphi^k(x)$. □

LEMMA 4. *Let $\sigma : \beta\mathbb{N} \to \{0, 1\}^{\mathbb{N}}$ be the canonical, continuous extension of the map $\mathbb{N} \to \{0, 1\}^{\mathbb{N}} : n \mapsto (c(n)(k))_{k\in\mathbb{N}}$ with $n = \sum_{k=0}^{\infty} c(n)(k)2^k$. Then for $p, q \in \beta\mathbb{N}$ the following assertions are equivalent:*

(i) *$\sigma(p) = \sigma(q)$;*

(ii) *assertion* (iii) *of Lemma 3 holds.*

Proof. (i) $\Rightarrow$ (ii): Let $\sigma(p) = \sigma(q)$. Let $d \in \mathbb{N}$, and for all $k \leq d$ let $\sigma(p)(k) = \sigma(q)(k) = z_k \in \{0, 1\}$. The set $U_k = \{r \in \beta\mathbb{N} \mid \sigma(r)(k) = z_k\}$ is open in $\beta\mathbb{N}$ for all $k \leq d$, so the intersection $U := \bigcap_{k=0}^{d} U_k$ of these sets is open in $\beta\mathbb{N}$ and contains p and q by definition. For all $n, m \in \mathbb{N} \cap U$ we have $\sigma(n)(k) = \sigma(m)(k) = z_k \, \forall k \in d$, i.e. $n \equiv m \mod 2^d$.

(ii) $\Rightarrow$ (i): Let $\sigma(p) \neq \sigma(q)$. Then there exists a number $d \in \mathbb{N}$ such that $\sigma(p)(d) \neq \sigma(q)(d)$, say, $\sigma(p)(d) = 0$ and $\sigma(q)(d) = 1$. In every neighbourhood of p there is an $n \in \mathbb{N}$ with $\sigma(n)(d) = 0$, and in every neighbourhood of q there is an $m \in N$ with $\sigma(m)(d) = 1$, i.e. $n \not\equiv m \mod 2^d$. □

We can now characterize the Ellis semigroup of this class of dynamical systems.

THEOREM 3. *Let $\varphi : I \to I$ be a continuous, strongly non-chaotic map that has periodic points of infinitely many different periods. Then $\Sigma(I, \varphi) \setminus \{\varphi^n \mid n \in \mathbb{N}\}$ is isomorphic to the group of the 2-adic integers.*

Proof. The group of the 2-adic integers is just the set $\{0, 1\}^{\mathbb{N}}$, equipped with the addition of binary numbers and the usual compact, metrizable topology.

Let $\Pi : \beta\mathbb{N} \to \Sigma(I, \varphi) \setminus \{\varphi^n \mid n \in \mathbb{N}\}$ be the epimorphism of Lemma 3 and $\sigma : \beta\mathbb{N} \to \{0, 1\}^{\mathbb{N}}$ the epimorphism of Lemma 4. It follows from Lemmas 3 and 4 that $\sigma(p) = \sigma(q)$ if and only if $\Pi(p) = \Pi(q)$. By Lemma 1 there is an epimorphism from $\Sigma(I, \varphi) \setminus \{\varphi^n \mid n \in \mathbb{N}\}$ onto $\{0, 1\}^{\mathbb{N}}$ and an inverse of this epimorphism. Hence the two semigroups must be isomorphic. □

COROLLARY. *Let $\varphi : I \to I$ be a continuous, strongly non-chaotic map that has periodic points of infinitely many different periods. Then the enveloping Ellis semigroup has the cardinality c, and all pointwise accumulation points of the sequence $(\varphi^n)_{n\in\mathbb{N}}$ are the limits of convergent subsequences. Furthermore, $\Sigma(I, \varphi) \setminus \{\varphi^n \mid n \in \mathbb{N}\}$ is an abelian group.*

COROLLARY. *Let φ be a continuous self-map of the unit interval I. Then $\Sigma(I,\varphi) \setminus \{\varphi^n \mid n \in \mathbb{N}\}$ is either finite or uncountable.*

Proof. If φ has only periodic points of finitely many different periods, the sequence $(\varphi^n)_{n\in\mathbb{N}}$ has by Theorem 1 only finitely many accumulation points in $\Sigma(I,\varphi)$. So assume that infinitely many periods occur.

If there is a periodic point whose period is not a power of two, then the system is chaotic and by Theorem 2 the cardinality of $\Sigma(I,\varphi)$ is 2^c.

So it remains to consider the dynamical systems whose periods are exactly the powers of two. It can be seen in the proofs of the above lemmas that for such systems $\Pi(p) = \Pi(q)$ always implies $\sigma(p) = \sigma(q)$ (since we have actually used the assumption of 'strong non-chaos' only to prove (ii) $\Rightarrow$ (i) in Lemma 3). Hence by Lemma 1 there exists an epimorphism from $\Sigma(I,\varphi) \setminus \{\varphi^n \mid n \in \mathbb{N}\}$ onto the 2-adic numbers, which shows that $\Sigma(I,\varphi)$ must at least have the cardinality c of $\{0,1\}^{\mathbb{N}}$. □

These results justify the study of chaos by means of compactifications. We expect that many more connections between the dynamics of a system and the algebraic structure of the corresponding Ellis semigroup exist. In particular, it should be possible to classify algebraically the flows that are non-chaotic without being strongly non-chaotic, for example by the number of idempotents or the ideal structure of their Ellis semigroups.

REFERENCES

[1] J. Auslander. *Minimal Flows and their Extensions.* North Holland, Amsterdam, 1988.

[2] L. S. Block and W. A. Coppel. *Dynamics in One Dimension (Lecture Notes in Mathematics, 1513).* Springer, Berlin, Heidelberg, 1992.

[3] R. Ellis. *Lectures on Topological Dynamics.* Benjamin, New York, 1969.

[4] R. Ellis and M. Nerurkar. Weakly almost periodic flows. *Trans. Amer. Math. Soc.* **313** (1989), 103–119.

[5] N. Hindman. Ultrafilters and combinatorial number theory. *Number Theory, Carbondale 1979, Proc.* Springer, Berlin, Heidelberg, New York, 1979.

[6] N. Hindman and J. S. Pym. Free groups and semigroups in $\beta\mathbb{N}$. *Semigroup Forum* **30** (1984), 177–193.

[7] J. D. Lawson and A. Lisan. Transitive flows: a semigroup approach. *Mathematika* **38** (1991), 348–361.

[8] J. D. Lawson. Flows and compactifications. *J. London Math. Soc.* **46** (1992), 349–363.

[9] W. A. F. Ruppert. *Compact Semitopological Semigroups: An Intrinsic Theory (Lecture Notes in Mathematics, 1079).* Springer, Berlin, Heidelberg, New York, 1984.

[10] A. N. Šarkovskii. Coexistence of cycles of a continuous mapping of the line into itself. *Ukrain. Math. Ž.* **16** (1964), 61–71.

[11] R. Walker. *The Stone–Čech Compactification.* Springer, Berlin, Heidelberg, New York, 1974.

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